Random complex dynamics and devil's coliseums *

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Abstract

We investigate the random dynamics of polynomial maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of semigroups of polynomial maps on $\hat{\mathbb{C}}$. In particular, the dynamics of a semigroup G of polynomials whose planar postcritical set is bounded and the associated random dynamics are studied. In general, the Julia set of such a G may be disconnected. We show that if G is such a semigroup, then regarding the associated random dynamics, the chaos of the averaged system disappears in the C^0 sense, and the function T_∞ of probability of tending to $\infty \in \hat{\mathbb{C}}$ is Hölder continuous on $\hat{\mathbb{C}}$ and varies only on the Julia set of G. Moreover, the function T_∞ has a kind of monotonicity. It turns out that T_∞ is a complex analogue of the devil's staircase, and we call T_∞ a "devil's coliseum." We investigate the details of T_∞ when G is generated by two polynomials. In this case, T_∞ varies precisely on the Julia set of G, which is a thin fractal set. Moreover, under this condition, we investigate the pointwise Hölder exponents of T_∞.

1 Introduction

Some results of this paper have been announced in [21, 26] without proofs.

In this paper, we simultaneously investigate the random dynamics of polynomial maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps where the semigroup operation is functional composition) on $\hat{\mathbb{C}}$.

The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony ([6]). For the motivations to study random complex dynamics, see [25, 27]. For research on random complex dynamics of quadratic polynomials, see [3, 7]. For research on random dynamics of polynomials (of general degrees) with bounded planar postcritical set, see the author's works [22, 23, 24, 29, 26]. In [25, 27], the author of this paper discussed more general random dynamics of rational maps with a systematic approach.

The first study of dynamics of polynomial semigroups was conducted by A. Hinkkanen and G. J. Martin ([9]), who were interested in the role of the dynamics of polynomial semigroups while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group ([8]), who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ (the set of non-normality) of a finitely generated polynomial semigroup G generated by $\{h_1, \ldots, h_m\}$ has "backward self-similarity," i.e., $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ (see [14,

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Lemma 1.1.4]), the study of the dynamics of rational semigroups can be regarded as the study of "backward iterated function systems," and also as a generalization of the study of self-similar sets in fractal geometry. For recent work on the dynamics of polynomial semigroups, see [14]–[27], [13, 28, 29, 30].

In order to consider the random dynamics of a family of polynomials on $\hat{\mathbb{C}}$, for each $z \in \hat{\mathbb{C}}$, let $T_\infty(z)$ be the probability of tending to $\infty \in \hat{\mathbb{C}}$ starting with the initial value $z \in \hat{\mathbb{C}}$. Note that in the usual iteration dynamics of a single polynomial f with $\deg(f) \geq 2$, the function T_∞ is equal to the constant 1 in the basin of infinity, and T_∞ is equal to the constant 0 in the filled-in Julia set of f. Thus T_∞ is not continuous at any point in the Julia set of f. However, we see the following main results of this paper.

Main Results (rough statements).

(I) If the planar postcritical set (see section 2) of the associated polynomial semigroup G of a random dynamical system of complex polynomials is bounded and the Julia set of G is disconnected, then the "Julia set" and the chaos of the averaged system disappears in the "C^0" sense, the function $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is Hölder continuous on $\hat{\mathbb{C}}$ (i.e., there exist constants $C > 0$ and $0 < \alpha < 1$ such that $|T_\infty(z_1) - T_\infty(z_2)| \leq Cd(z_1, z_2)^\alpha$ for each $z_1, z_2 \in \hat{\mathbb{C}}$), and T_∞ has a kind of monotonicity (e.g., if J_1 and J_2 are two connected components of the Julia set of G and J_1 is included in a bounded connected component of $\mathbb{C} \setminus J_2$, then $\max_{z \in J_1} T_\infty(z) \leq \min_{z \in J_2} T_\infty(z)$). (For the precise statement, see Theorem 2.4.)

(II) Under certain conditions T_∞ has some singular properties (for instance, it varies only on a thin fractal set, the so-called Julia set of the associated polynomial semigroup G, and for almost every point z_0 in the Julia set of G with respect to a nice"invariant measure", the function T_∞ is not differentiable at z_0), and this function is a complex analogue of the devil's staircase (Cantor function) or Lebesgue's singular functions (see Theorems 2.11, 2.12, Example 5.4). (For the definition of the devil's staircase and Lebesgue's singular functions, see [31].) Also, the pointwise Hölder exponents of T_∞ are investigated. Note that we do not assume hyperbolicity in Theorem 2.11, while in the previous result [25, Theorem 3.82], it is assumed that G is hyperbolic.

Graphs of T_∞ are illustrated in [25]. Thus even though the chaos of the averaged system disappears, the system has new kind of complexity. These are new phenomena which cannot hold in the usual iteration dynamics of a single polynomial. Such phenomena in random dynamical systems are called "**randomness-induced phenomena**". To explain the detail of the above result, we first remark that these well-known singular functions (the devil's staircase and Lebesgue's singular functions) defined on $[0,1]$ can be redefined by using random dynamical systems on $\mathbb{R}$ as follows (see [25, 26]). Let $f_1(x) := 3x, f_2(x) := 3(x-1)+1$ $(x \in \mathbb{R})$ and we consider the random dynamical system on $\mathbb{R}$ such that at every step we choose f_1 with probability 1/2 and f_2 with probability 1/2. We set $\hat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We denote by $T_{+\infty}(x)$ the probability of tending to $+\infty \in \hat{\mathbb{R}}$ starting with the initial value $x \in \mathbb{R}$. Then, we can see that the function $T_{+\infty}|_{[0,1]} : [0,1] \to [0,1]$ is equal to the devil's staircase. Similarly, let $g_1(x) := 2x, g_2(x) := 2(x-1)+1$ $(x \in \mathbb{R})$ and let $0 < a < 1$ be a constant. We consider the random dynamical system on $\mathbb{R}$ such that at every step we choose the map g_1 with probability a and the map g_2 with probability $1-a$. Let $T_{+\infty,a}(x)$ be the probability of tending to $+\infty$ starting with the initial value $x \in \mathbb{R}$. Then, we can see that the function $T_{+\infty,a}|_{[0,1]} : [0,1] \to [0,1]$ is equal to Lebesgue's singular function L_a with respect to the parameter a provided $a \neq 1/2$. From the above point of view, the function $T_\infty : \hat{\mathbb{C}} \to [0,1]$ is a complex analogue of the devil's staircase and Lebesgue's singular functions. We call T_∞ a "**devil's coliseum**"([25, 27]) .

We also explain why we focus on polynomial semigroups with bounded planar postcritical set. A polynomial semigroup G is said to be postcritically bounded if the planar postcritical set of G is bounded. It is well-known that if $g \in \mathcal{P}$, where $\mathcal{P}$ denotes the set of polynomials of degree two or more, then $J(g)$ is connected if and only if the semigroup $\{g^n \mid n \in \mathbb{N}\}$ is postcritically bounded. However, we remark that there are many examples of elements of postcritically bounded polynomial semigroup G with $G \subset \mathcal{P}$ such that the Julia set of G is disconnected (see section 5, [22, 24]). In fact, it is easy to construct such examples by using (1) in section 2, and many systematic studies on the dynamics of postcritically bounded polynomial semigroups G with $G \subset \mathcal{P}$ are given in [22, 23, 24, 13, 20]. Thus we are very interested in the new phenomena regarding the dynamics of postcritically bounded polynomial semigroups.

One of the purposes of this paper is to combine the study of the dynamics of postcritically bounded polynomial semigroups with disconnected Julia set and the study of random dynamics of polynomials. To prove Main Result (I) (Theorem 2.4), we need the following result from [25, Theorem 3.15] and [27, Theorem 1.9]: For a random dynamical system of complex polynomials generated by finitely many elements in $\mathcal{P}$, if the kernel Julia set $J_{\ker}(G) := \cap_{g \in G_\tau} g^{-1}(J(G))$ of the associated polynomial semigroup G is empty, then T_∞ is Hölder continuous on $\hat{\mathbb{C}}$ and there exists a finite dimensional subspace U of the space $C(\hat{\mathbb{C}})$ of continuous functions on $\hat{\mathbb{C}}$ with $M(U) = U$, where M denotes the transition operator of the system (see section 2), and a bounded operator $\pi : C(\hat{\mathbb{C}}) \to U$ such that $M^n(\varphi - \pi(\varphi)) \to 0$ in $C(\hat{\mathbb{C}})$ as $n \to \infty$ for each $\varphi \in C(\hat{\mathbb{C}})$. Therefore, to prove Main Result (I) (Theorem 2.4), it is an important key to prove that if G with $G \subset \mathcal{P}$ is postcritically bounded and the Julia set of G is disconnected, then $J_{\ker}(G) = \emptyset$, which is proved in Lemma 4.1 of this paper. In order to prove the monotonicity of T_∞ and statements 4 and 5 of Theorem 2.4, we combine the idea from [25] and new careful observations on the dynamics of postcritically bounded polynomial semigroup G with disconnected Julia set.

Main Result (II) (Theorem 2.11) means that even though the chaos of the averaged system disappears in the C^0 sense as in Main Result (I), it can remain in the C^α sense with some $\alpha \in (0, 1)$, where C^α denotes the space of α-Hölder continuous functions. From these, we can say that we have a gradation between chaos and order. In the proof (section 4) of Main Result (II) (Theorem 2.11), we use Birkhoff's ergodic theorem, potential theory, the Koebe distortion theorem, and some observations ([22]) about the space of all connected components of $J(G)$ and the Julia set of the associated real affine semigroup.

We also prove several results on 2- or 3-generator postcritically bounded polynomial semigroups with disconnected Julia set and associated random dynamics (see Theorem 2.12, 2.15, Corollary 2.16, Remark 2.17). In order to prove the results on 2- or 3-generator postcritically bounded polynomial semigroups G with disconnected Julia set and associated random dynamics, we need the idea of the nerves of backward images of $J(G)$ under elements of G and their inverse limit from [20], which are related to certain kind of cohomology groups introduced by the author.

In section 2, we give the details of the main results. In section 3, we explain the known results and tools to prove the main results. In section 4, we prove the main results. In section 5, we give some examples.

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2 Main results

In this section, we give the details of the main results.

A **polynomial semigroup** is a semigroup generated by a family of non-constant polynomial maps on the Riemann sphere $\hat{\mathbb{C}}$ with the semigroup operation being functional composition([9, 8]). We set $\mathcal{P} := \{g : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid g \text{ is a polynomial}, \deg(g) \geq 2\}$ endowed with the distance κ which is defined by $\kappa(f, g) := \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$, where d denotes the spherical distance on $\hat{\mathbb{C}}$. Note that $g_n \to g$ in $\mathcal{P}$ if and only if (i) $\deg(g_n) = \deg(g)$ for each large n, and (ii) the coefficients of g_n converge appropriately to the coefficients of g ([1]). Also, setting $\mathcal{P}_n := \{g \in \mathcal{P} \mid \deg(g) = n\}$

for each $n \geq 2$, we have that $\mathcal{P}_n$ is a connected, open and closed subset of $\mathcal{P}$, $\mathcal{P}_n$ is a connected component of $\mathcal{P}$, and $\mathcal{P}_n \cong (\mathbb{C}\setminus\{0\})\times\mathbb{C}^n$ ([1]). For a polynomial semigroup G, we denote by $F(G)$ the Fatou set of G, which is defined to be the maximal open subset of $\hat{\mathbb{C}}$ where G is equicontinuous with respect to the spherical distance on $\hat{\mathbb{C}}$ (for the definition of equicontinuity, see [1, Definition 3.11]). We call $J(G) := \hat{\mathbb{C}} \setminus F(G)$ the Julia set of G. For fundamental properties on the Fatou sets and Julia sets, see [9, 16]. The Julia set is backward invariant under each element $h \in G$, but might not be forward invariant. This is a difficulty of the theory of rational semigroups. Nevertheless, we "utilize" this to investigate the associated random complex dynamics. For a non-empty subset Λ of $\mathcal{P}$, we denote by $\langle\Lambda\rangle$ the polynomial semigroup generated by Λ. Thus $\langle\Lambda\rangle = \{h_1 \circ \cdots \circ h_m \mid m \in \mathbb{N}, h_1, \ldots, h_m \in \Lambda\}$. For finitely many polynomial maps $g_1, \ldots, g_m$, we denote by $\langle g_1, \ldots, g_m\rangle$ the polynomial semigroup generated by $\{g_1, \ldots, g_m\}$. For a polynomial map g, we set $F(g) := F(\langle g\rangle)$ and $J(g) := J(\langle g\rangle)$. For a polynomial semigroup G, we set $G^* := G \cup \{\mathrm{Id}\}$, where Id denotes the identity map. For a polynomial semigroup G and a subset A of $\hat{\mathbb{C}}$, we set $G(A) := \bigcup_{g\in G} g(A)$ and $G^{-1}(A) := \bigcup_{g\in G} g^{-1}(A)$.

For a polynomial semigroup G, we set $\hat{K}(G) := \{z \in \mathbb{C} \mid G(\{z\}) \text{ is bounded in } \mathbb{C}\}$. This is called the **smallest filled-in Julia set** of G. For a polynomial $g \in \mathcal{P}$, we set $K(g) := \hat{K}(\langle g\rangle)$. For a polynomial semigroup G, we set $P(G) := \overline{\bigcup_{g\in G}\{z \in \hat{\mathbb{C}} \mid z \text{ is a critical value of } g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}$. This is called the **postcritical set** of G. Note that if $G = \langle\Lambda\rangle$, then

$$P(G) = \overline{G^*(\bigcup_{h\in\Lambda}\{z \in \hat{\mathbb{C}} \mid z \text{ is a critical value of } h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\})}. \tag{1}$$

Thus for each $g \in G$, $g(P(G)) \subset P(G)$. For a polynomial semigroup G, we set $P^*(G) := P(G)\setminus\{\infty\}$. This is called the **planar postcritical set** of G. A polynomial semigroup G is said to be **postcritically bounded** if $P^*(G)$ is bounded in $\mathbb{C}$. We denote by $\mathcal{G}$ the set of all postcritically bounded polynomial semigroups G with $G \subset \mathcal{P}$. Moreover, we set $\mathcal{G}_{dis} := \{G \in \mathcal{G} \mid J(G) \text{ is disconnected}\}$. It is well-known that if $g \in \mathcal{P}$, then $J(g)$ is connected if and only if $\langle g\rangle \in \mathcal{G}$. However, we remark that there are many examples of elements of $\mathcal{G}_{dis}$ (see section 5, [22, 24]). In fact, it is easy to construct such examples by using (1), and many systematic studies on the dynamics of semigroups G in $\mathcal{G}$ or $\mathcal{G}_{dis}$ are given in [22, 23, 24, 13, 20]. Thus we are very interested in the new phenomena on $\mathcal{G}_{dis}$. It is very natural to ask **"what happens for a $G \in \mathcal{G}_{dis}$ and the associated random dynamics?" "How can we classify the elements G in $\mathcal{G}_{dis}$ in terms of the dynamics of G and the associated random dynamics?"**

For a polynomial semigroup G with $\infty \in F(G)$, we denote by $F_\infty(G)$ the connected component of $F(G)$ containing ∞. Note that if G is generated by a compact subset of $\mathcal{P}$, then $\infty \in F(G)$. For a polynomial $g \in \mathcal{P}$, we set $F_\infty(g) := F_\infty(\langle g\rangle)$.

For a non-empty subset A of $\hat{\mathbb{C}}$ and a point $z \in \hat{\mathbb{C}}$, we set $d(z, A) := \inf_{a\in A} d(z, a)$, where d is the spherical distance. For a non-empty subset A of $\hat{\mathbb{C}}$ and a positive number r, we set $B(A, r) := \{z \in \hat{\mathbb{C}} \mid d(z, A) < r\}$. For a non-empty subset A of $\mathbb{C}$, we set $d_e(z, A) := \inf_{a\in A} |z - a|$. For a non-empty subset A of $\mathbb{C}$ and a positive number r, we set $D(A, r) := \{z \in \hat{\mathbb{C}} \mid d_e(z, A) < r\}$.

For a metric space X, let $\mathfrak{M}_1(X)$ be the space of all Borel probability measures on X endowed with the topology induced by the weak convergence (thus $\mu_n \to \mu$ in $\mathfrak{M}_1(X)$ if and only if $\int \varphi d\mu_n \to \int \varphi d\mu$ for each bounded continuous function $\varphi : X \to \mathbb{R}$). Note that if X is a compact metric space, then $\mathfrak{M}_1(X)$ is compact and metrizable. For each $\tau \in \mathfrak{M}_1(X)$, we denote by $\operatorname{supp}\tau$ the topological support of τ. Let $\mathfrak{M}_{1,c}(X)$ be the space of all Borel probability measures τ on X such that $\operatorname{supp}\tau$ is compact.

Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. In the following, we consider the independent and identically-distributed random dynamical system on $\hat{\mathbb{C}}$ such that at every step we choose a polynomial map according to the probability distribution τ ([25]). This determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and each Borel subset A of $\hat{\mathbb{C}}$, the transition probability $p(x,A)$ from x to A is equal to $\tau(\{g \in \mathcal{P} \mid g(x) \in A\})$. We set $\Gamma_\tau := \operatorname{supp}\tau$ and $X_\tau := (\operatorname{supp}\,\tau)^{\mathbb{N}}$. We set $\tilde{\tau} := \otimes_{j=1}^{\infty}\tau$. This is the unique Borel probability measure on $\mathcal{P}^{\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, if $A_1, A_2, \ldots, A_n$ are Borel subsets of $\mathcal{P}$, then $\tilde{\tau}(A_1 \times A_2 \times \cdots \times A_n \times \mathcal{P} \times \mathcal{P} \cdots) = \prod_{j=1}^{n} \tau(A_j)$. Note that $\operatorname{supp}\tilde{\tau} = X_\tau$. Let G_τ be the polynomial semigroup generated by the polynomials contained in supp τ. We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi$ is continuous$\}$ endowed with the supremum norm. We define an operator $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ by $M_\tau(\varphi)(z) := \int_{\mathcal{P}} \varphi(g(z))\, d\tau(g)$. This M_τ is called the transition operator of the random dynamical system associated with τ. Moreover, we denote by $M_\tau^* : \mathcal{M}_1(\hat{\mathbb{C}}) \to \mathcal{M}_1(\hat{\mathbb{C}})$ the dual of M_τ (thus $\int_{\hat{\mathbb{C}}} \varphi(z) d(M_\tau^*(\mu))(z) = \int_{\hat{\mathbb{C}}} M_\tau(\varphi)(z) d\mu(z)$ for each $\mu \in \mathfrak{M}_1(\hat{\mathbb{C}}), \varphi \in C(\hat{\mathbb{C}})$). Note that for each $z \in \hat{\mathbb{C}}$, $M_\tau^*(\delta_z) = \int_{\mathcal{P}} \delta_{g(z)} d\tau(g)$. Hence M_τ^* can be regarded as the **averaged map** of elements of $\operatorname{supp}\tau$ with respect to τ. We denote by $F_{meas}(\tau)$ the set of all $\mu \in \mathcal{M}_1(\hat{\mathbb{C}})$ satisfying the following: There exists a neighborhood B of μ in $\mathcal{M}_1(\hat{\mathbb{C}})$ such that $\{(M_\tau^*)^n : B \to \mathcal{M}_1(\hat{\mathbb{C}})\}_{n \in \mathbb{N}}$ is equicontinuous on B. Moreover, we set $J_{meas}(\tau) := \mathcal{M}_1(\hat{\mathbb{C}}) \setminus F_{meas}(\tau)$. We remark that if $h \in \mathcal{P}$ and $\tau = \delta_h$ (the Dirac measure at h), then $J_{meas}(\tau) \neq \emptyset$. In fact, by embedding $\hat{\mathbb{C}}$ into $\mathcal{M}_1(\hat{\mathbb{C}})$ under the map $z \mapsto \delta_z$, we have $J(h) \subset J_{meas}(\tau)$. However, we will see later that for any $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $G_\tau \in \mathcal{G}_{dis}$, $J_{meas}(\tau) = \emptyset$ (Theorem 2.4).

Let G be a rational semigroup. We say that a non-empty compact subset K of $\hat{\mathbb{C}}$ is a minimal set for $(G, \hat{\mathbb{C}})$ if K is minimal in the space $\{L \mid L$ is a non-empty compact subset of $\hat{\mathbb{C}}, \forall g \in G, g(L) \subset L\}$ with respect to inclusion. We set $\operatorname{Min}(G, \hat{\mathbb{C}}) := \{K \mid K$ is a minimal set for $(G, \hat{\mathbb{C}})\}$. Note that by Zorn's lemma, $\operatorname{Min}(G, \hat{\mathbb{C}}) \neq \emptyset$. For any $\gamma = (\gamma_1, \gamma_2, \ldots,) \in \mathcal{P}^{\mathbb{N}}$ and any $n, m \in \mathbb{N}$ with $n > m$, we set $\gamma_{n,m} := \gamma_n \circ \cdots \circ \gamma_m$. Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ and let A be a non-empty subset of $\hat{\mathbb{C}}$. For any $z \in \hat{\mathbb{C}}$, we set $T_{A,\tau}(z) := \tilde{\tau}(\{\gamma = (\gamma_1, \gamma_2, \ldots) \in \mathcal{P}^{\mathbb{N}} \mid d(\gamma_{n,1}(z), A) \to 0, \text{as } n \to \infty\})$. This is nothing else but the **probability of tending to A starting with the initial value** $z \in \hat{\mathbb{C}}$ regarding the random dynamics on $\hat{\mathbb{C}}$ such that at every step we choose a polynomial according to τ. Moreover, for a point $a \in \hat{\mathbb{C}}$, wet set $T_{a,\tau}(z) := T_{\{a\},\tau}(z)$. Note that if $G \subset \mathcal{P}$, then $\{\infty\}$ is a minimal set for $(G, \hat{\mathbb{C}})$. Note also that by [25, Lemma 5.27], if $\tau \in \mathfrak{M}_1(\mathcal{P})$ and if $\infty \in F(G_\tau)$, then for each connected component U of $F(G_\tau)$, the function $T_{\infty,\tau}|_U$ is constant (the constant value depends on U). The main purpose of this paper is to show that if $\tau \in \mathcal{M}_{1,c}(\mathcal{P})$ satisfies that $G_\tau \in \mathcal{G}_{dis}$, then under certain conditions the function $T_{\infty,\tau}$ can be regarded as a complex analogue of the devil's staircase. The following, which was introduced by the author in [25], is the key to investigating the dynamics of rational semigroups and the random complex dynamics.

Definition 2.1 ([25]). Let G be a rational semigroup. We set $J_{\ker}(G) := \bigcap_{g \in G} g^{-1}(J(G))$ and this is called the **kernel Julia set** of G.

Definition 2.2 ([22]). For any connected sets K_1 and K_2 in $\mathbb{C}$, we write $K_1 \leq_s K_2$ to indicate that $K_1 = K_2$, or K_1 is included in a bounded component of $\mathbb{C} \setminus K_2$. Furthermore, $K_1 <_s K_2$ indicates $K_1 \leq_s K_2$ and $K_1 \neq K_2$. Moreover, $K_2 \geq_s K_1$ indicates $K_1 \leq_s K_2$, and $K_2 >_s K_1$ indicates $K_1 <_s K_2$. Note that $\leq_s$ is a partial order in the space of all non-empty compact connected sets in $\mathbb{C}$. This $\leq_s$ is called the **surrounding order.**

Remark 2.3. For a topological space X, we denote by $\operatorname{Con}(X)$ the set of all connected components of X. Let $G \in \mathcal{G}_{dis}$. In [22], it was shown that $J(G) \subset \mathbb{C}$, $(\operatorname{Con}(J(G)), \leq_s)$ is totally ordered, there exists a unique maximal element $J_{\max} = J_{\max}(G) \in (\operatorname{Con}(J(G)), \leq_s)$, there exists a unique minimal element $J_{\min} = J_{\min}(G) \in (\operatorname{Con}(J(G)), \leq_s)$, each element of $\operatorname{Con}(F(G))$ is either simply connected or doubly connected, and the connected component $F_\infty(G)$ of $F(G)$ with $\infty \in F_\infty(G)$ is simply connected. Moreover, in [22], it was shown that $\mathcal{A} \neq \emptyset$, where $\mathcal{A}$ denotes the set of all doubly connected components of $F(G)$ (more precisely, for each $J, J' \in \operatorname{Con}(J(G))$ with $J <_s J'$, there exists an $A \in \mathcal{A}$ with $J <_s A <_s J'$), $\bigcup_{A \in \mathcal{A}} A \subset \mathbb{C}$, and $(\mathcal{A}, \leq_s)$ is totally ordered. Note that each $A \in \mathcal{A}$ is bounded and multiply connected, while for a single $f \in \mathcal{P}$, we have no bounded multiply connected component of $F(f)$.

We now present the main results of this paper.

Theorem 2.4. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $G_\tau \in \mathcal{G}_{dis}$. Then, all of the following 1–9 hold.*

1. (**Hölder Continuity**) *The function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is Hölder continuous on $\hat{\mathbb{C}}$, $M_\tau(T_{\infty,\tau}) = T_{\infty,\tau}$ and $T_{\infty,\tau}(J(G_\tau)) = [0,1]$.*

2. *For each $U \in \mathrm{Con}(F(G_\tau))$, there exists a constant $C_U \in [0,1]$ such that $T_{\infty,\tau}|_U \equiv C_U$.*

3. (**Monotonicity**) *Let $\mathcal{A} := \{U \in \mathrm{Con}(F(G_\tau)) \mid U$ is doubly connected$\}$.*

   (a) If $A_1, A_2 \in \mathcal{A}$ and $A_1 <_s A_2$, then $C_{A_1} < C_{A_2}$. In particular, all elements of $\{C_A \mid A \in \mathcal{A}\}$ are mutually distinct.

   (b) If $J_1, J_2 \in \mathrm{Con}(J(G_\tau))$ and $J_1 <_s J_2$, then $\sup_{z \in J_1} T_{\infty,\tau}(z) \le \inf_{z \in J_2} T_{\infty,\tau}(z)$.

4. *For each $A \in \mathcal{A}$, $T_{\infty,\tau}|_{\hat{K}(G_\tau)} \equiv 0 < C_A < 1 \equiv C_{F_\infty(G_\tau)}$.*

5. *Let Q be an open subset of $\hat{\mathbb{C}}$. If $Q \bigcap \left(\bigcup_{A \in \mathcal{A}} \partial A \bigcup \partial(F_\infty(G_\tau)) \bigcup \partial(\hat{K}(G_\tau)) \right) \neq \emptyset$, then $T_{\infty,\tau}|_Q$ is not constant.*

6. *We have that $J_{\ker}(G_\tau) = \emptyset$ and $F_{meas}(\tau) = \mathcal{M}_1(\hat{\mathbb{C}})$.*

7. *$\sharp \mathrm{Min}(G_\tau, \hat{\mathbb{C}}) = 2$. More precisely, $\{\infty\}$ is a minimal set for $(G_\tau, \hat{\mathbb{C}})$, and there exists a unique minimal set L_τ for $(G_\tau, \hat{\mathbb{C}})$ such that $L_\tau \subset \hat{K}(G_\tau)$.*

8. *For each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of $\mathcal{P}^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots,) \in \mathcal{A}_z$,* (a) *either $\gamma_{n,1}(z) \to \infty$ or $d(\gamma_{n,1}, L_\tau) \to 0$ as $n \to \infty$, and* (b) *there exists a number $\delta = \delta(z,\gamma) > 0$ such that diam$(\gamma_{n,1}(B(z,\delta))) \to 0$ as $n \to \infty$.*

9. *There exists a unique M_τ^*-invariant Borel probability measure μ_τ on $\hat{K}(G_\tau)$ which satisfies the following $(*)$.*

   $(*)$ For each $\varphi \in C(\hat{\mathbb{C}})$, $M_\tau^n(\varphi)(z) \to T_{\infty,\tau}(z) \cdot \varphi(\infty) + (1 - T_{\infty,\tau}(z)) \cdot (\int_{\hat{\mathbb{C}}} \varphi \, d\mu_\tau)$ $(n \to \infty)$ uniformly on $\hat{\mathbb{C}}$.

   Thus $(M_\tau^*)^n(\nu) \to (\int_{\hat{\mathbb{C}}} T_{\infty,\tau} \, d\nu) \cdot \delta_\infty + (\int_{\hat{\mathbb{C}}} (1 - T_{\infty,\tau}) \, d\nu) \cdot \mu_\tau$ $(n \to \infty)$ uniformly on $\mathcal{M}_1(\hat{\mathbb{C}})$. Also, $\mathrm{supp}\, \mu_\tau = L_\tau$. Moreover, the M_τ-invariant subspace of $C(\hat{\mathbb{C}})$ is two-dimensional and it is spanned by the constant function and $T_{\infty,\tau}$. Moreover, the set of ergodic components of M_τ^*-invariant elements in $\mathfrak{M}_1(\hat{\mathbb{C}})$ is equal to $\{\delta_\infty, \mu_\tau\}$.

Remark 2.5. Let $f \in \mathcal{P}$. Then $J_{\ker}(\langle f \rangle) = J(f) \neq \emptyset$, $T_{\infty,\delta_f}(\hat{\mathbb{C}}) = \{0,1\}$ and T_{∞,δ_f} is not continuous at any point of $J(f)$. Moreover, regarding the dynamics of $f : J(f) \to J(f)$, we have chaos in the sense of Devaney ([1, 4]). Thus Theorem 2.4 describes new phenomena which cannot hold in the usual iteration dynamics of a single polynomial. For a $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $G_\tau \in \mathcal{G}_{dis}$, we sometimes call the function $T_{\infty,\tau}$ a "**devil's coliseum**", especially when $\mathrm{int}(J(G_\tau)) = \emptyset$. This terminology and the study were introduced by the author of this paper in [25]. For the graph of $T_{\infty,\tau}$ and the graphics of $J(G_\tau)$, see figures in [25]. Statement 5 means that $T_{\infty,\tau}$ can detect many parts of $J(G_\tau)$. Thus, by obtaining results about the dynamics of polynomial semigroups, one can correspondingly apply such results to the setting of random complex dynamics. Conversely, studying the level sets of $T_{\infty,\tau}$, we can get much information about $J(G)$. In other words, in order to investigate the dynamics of polynomial semigroups, it is very effective to study the associated random complex dynamics and then apply the results to the original polynomial semigroups. In the proof (section 4) of Theorem 2.4, we combine some results (geometric observations) on the dynamics of a $G \in \mathcal{G}_{dis}$ from [22] and some results on random complex dynamics from [25]. It is critical to know whether or not $J_{\ker}(G_\tau) = \emptyset$. This condition implies that the chaos of the averaged system disappears in the C^0 sense due to the cooperation of many kinds of maps in the system even though each map has a chaotic part. For the details of the study of random dynamics generated by $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $J_{\ker}(G_\tau) = \emptyset$, see [25, 27]. In [25, 27], it is shown that regarding the random dynamics of complex polynomials, for a generic $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$, we have that $J_{\ker}(G_\tau) = \emptyset$, the chaos of the averaged system disappears in the C^0 sense due to the automatic cooperation of many kinds of maps in the system (**cooperation principle**), and $T_{\infty,\tau}$ is Hölder continuous on $\hat{\mathbb{C}}$. We remark that many physicists have observed by numerical experiments that if we add uniform noise to a chaotic map on $\mathbb{R}$, there are many cases in which the chaos of the averaged system disappears. This phenomenon in random dynamics on $\mathbb{R}$ is called the "noise-induced order" ([11]).

We are interested in the pointwise Hölder exponents and (non-)differentiability of $T_{\infty,\tau}$ at points in $J(G_\tau)$. In order to state the result, we need several definitions.

Definition 2.6. Let Γ be a non-empty compact subset of $\mathcal{P}$. We endow $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with the product topology. Thus this is a compact metrizable space. We define a map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ as follows: For a point $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ where $\gamma = (\gamma_1, \gamma_2, \ldots)$, we set $f(\gamma, y) := (\sigma(\gamma), \gamma_1(y))$, where $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ is the shift map, that is, $\sigma(\gamma_1, \gamma_2, \ldots) = (\gamma_2, \gamma_3, \ldots)$. The map $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ is called the **skew product associated with the generator system** Γ. Moreover, we use the following notations. Let $\pi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}}$ and $\pi_{\hat{\mathbb{C}}} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the canonical projections. For each $\gamma \in \Gamma^{\mathbb{N}}$ and $n \in \mathbb{N}$, we set $f_\gamma^n := f^n|_{\pi^{-1}(\{\gamma\})} : \pi^{-1}(\{\gamma\}) \to \pi^{-1}(\{\sigma^n(\gamma)\})$. Moreover, we set $f_{\gamma,n} := \gamma_n \circ \cdots \circ \gamma_1$. We denote by F_γ the set of points $z \in \hat{\mathbb{C}}$ satisfying that there exists a neighborhood U of z such that $\{f_{\gamma,n} : U \to \hat{\mathbb{C}}\}_{n=1}^\infty$ is equicontinuous on U. We set $J_\gamma := \hat{\mathbb{C}} \setminus F_\gamma$. The set F_γ is called the Fatou set of γ, and J_γ is called the Julia set of γ. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $J^\gamma := \{\gamma\} \times J_\gamma$ ($\subset \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$). Moreover, we set $\tilde{J}(f) := \overline{\bigcup_{\gamma \in \Gamma^{\mathbb{N}}} J^\gamma}$, where the closure is taken in the product space $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$. Furthermore, we set $\tilde{F}(f) := (\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}) \setminus \tilde{J}(f)$. For each $\gamma \in \Gamma^{\mathbb{N}}$, we set $\hat{J}^{\gamma,\Gamma} := \pi^{-1}(\{\gamma\}) \cap \tilde{J}(f)$, $\hat{F}^{\gamma,\Gamma} := \pi^{-1}(\{\gamma\}) \setminus \hat{J}^{\gamma,\Gamma}$, $\hat{J}_{\gamma,\Gamma} := \pi_{\hat{\mathbb{C}}}(\hat{J}^{\gamma,\Gamma})$, and $\hat{F}_{\gamma,\Gamma} := \hat{\mathbb{C}} \setminus \hat{J}_{\gamma,\Gamma}$. Note that $J_\gamma \subset \hat{J}_{\gamma,\Gamma}$. For any point $z \in \hat{\mathbb{C}}$, we denote by $T\hat{\mathbb{C}}_z$ the complex tangent space of $\hat{\mathbb{C}}$ at z. For any holomorphic map φ defined on a domain V and for any point $z \in V$, we denote by $D\varphi_z : T\hat{\mathbb{C}}_z \to T\hat{\mathbb{C}}_{\varphi(z)}$ the derivative map at z. For each $z = (\gamma, y) \in \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$, we set $Df_z := (D\gamma_1)_y$. Let U be a domain in $\hat{\mathbb{C}}$ and let $g : U \to \hat{\mathbb{C}}$ be a meromorphic function. For each $z \in U$, we denote by $\|Dg_z\|_s$ the norm of the derivative of g at z with respect to the spherical metric.

Remark 2.7. Under the above notation, let $G = \langle \Gamma \rangle$. Then $\pi_{\hat{\mathbb{C}}}(\tilde{J}(f)) \subset J(G)$ and $\pi \circ f = \sigma \circ \pi$ on $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$. Furthermore, for each $\gamma \in \Gamma^{\mathbb{N}}$, $\gamma_1(J_\gamma) = J_{\sigma(\gamma)}$, $\gamma_1^{-1}(J_{\sigma(\gamma)}) = J_\gamma$, $\gamma_1(\hat{J}_{\gamma,\Gamma}) = \hat{J}_{\sigma(\gamma),\Gamma}$, $\gamma_1^{-1}(\hat{J}_{\sigma(\gamma),\Gamma}) = \hat{J}_{\gamma,\Gamma}$, $f(\tilde{J}(f)) = \tilde{J}(f) = f^{-1}(\tilde{J}(f))$, and $f(\tilde{F}(f)) = \tilde{F}(f) = f^{-1}(\tilde{F}(f))$ (see [17, Lemma 2.4]). We remark that in general, $J_\gamma \subsetneqq \hat{J}_{\gamma,\Gamma}$ ([19, Example 1.7]).

Definition 2.8. Let $m \in \mathbb{N}$. We set $\mathcal{W}_m := \{(p_1, \ldots, p_m) \in (0,1)^m \mid \sum_{j=1}^m p_j = 1\}$. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ be an element such that $h_1, \ldots, h_m$ are mutually distinct. We set $\Gamma := \{h_1, \ldots, h_m\}$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be an f-invariant Borel probability measure. For each $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$, we define a function $\tilde{p} : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$ by $\tilde{p}(\gamma, y) := p_j$ if $\gamma_1 = h_j$ (where $\gamma = (\gamma_1, \gamma_2, \ldots)$), and we set

$$u(h, p, \mu) := \frac{-(\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \tilde{p}(z)\ d\mu(z))}{\int_{\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}} \log \|Df_z\|_s\ d\mu(z)}$$

(when the integral in the denominator is positive and finite). For each $\gamma \in \mathcal{P}^{\mathbb{N}}$, we set $A_{\infty,\gamma} := \{z \in \hat{\mathbb{C}} \mid \gamma_{n,1}(z) \to \infty\ (n \to \infty)\}$ and $K_\gamma := \{z \in \mathbb{C} \mid \{\gamma_{n,1}(z)\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{C}\}$. For any $(\gamma, y) \in \Gamma^{\mathbb{N}} \times \mathbb{C}$, let $G_\gamma(y) := \lim_{n \to \infty} \frac{1}{\deg(\gamma_{n,1})} \log^+ |\gamma_{n,1}(y)|$, where $\log^+ a := \max\{\log a, 0\}$ for each $a > 0$. By the arguments in [12], for each $\gamma \in \Gamma^{\mathbb{N}}$, $G_\gamma(y)$ exists, G_γ is subharmonic on $\mathbb{C}$, and $G_\gamma|_{A_{\infty,\gamma}}$ is equal to the Green's function on $A_{\infty,\gamma}$ with pole at ∞. Moreover, $(\gamma, y) \mapsto G_\gamma(y)$ is continuous on $\Gamma^{\mathbb{N}} \times \mathbb{C}$. Let $\mu_\gamma := dd^c G_\gamma$, where $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$. Note that by the argument in [10], μ_γ is a Borel probability measure on J_γ such that $\operatorname{supp} \mu_\gamma = J_\gamma$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product map associated with Γ. Moreover, let $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma)$. Then, there exists a unique f-invariant ergodic Borel probability measure μ on $\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ such that $\pi_*(\mu) = \tilde{\tau}$ and $h_\mu(f|\sigma) = \max_{\rho \in \mathfrak{E}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}) : f_*(\rho) = \rho, \pi_*(\rho) = \tilde{\tau}} h_\rho(f|\sigma) = \sum_{j=1}^m p_j \log(\deg(h_j))$, where $h_\rho(f|\sigma)$ denotes the relative metric entropy of (f, ρ) with respect to $(\sigma, \tilde{\tau})$, and $\mathfrak{E}_1(\cdot)$ denotes the space of ergodic measures (see [16]). This μ is called the **maximal relative entropy measure** for f with respect to $(\sigma, \tilde{\tau})$. Note that in [25, Lemma 5.51] it was shown that for each continuous function $\varphi : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \mathbb{R}$, $\int \varphi(\gamma, y) d\mu(\gamma, y) = \int d\tilde{\tau}(\gamma) \int \varphi(\gamma, y) d\mu_\gamma(y)$. Thus $(\pi_{\hat{\mathbb{C}}})_*(\mu) = \int_{\Gamma^{\mathbb{N}}} \mu_\gamma d\tilde{\tau}(\gamma)$.

Definition 2.9. Let V be a non-empty open subset of $\mathbb{C}$. Let $\varphi : V \to \mathbb{C}$ be a function and let $y \in V$ be a point. Suppose that φ is bounded around y. Then we set
$\text{Höl}(\varphi, y) := \sup\{\beta \in [0, \infty) \mid \limsup_{z \to y, z \neq y} \frac{|\varphi(z) - \varphi(y)|}{|z - y|^\beta} < \infty\} \in [0, \infty]$. This is called the **pointwise Hölder exponent of φ at y.**

Remark 2.10. If $\text{Höl}(\varphi, y) < 1$, then φ is non-differentiable at y. If $\text{Höl}(\varphi, y) > 1$, then φ is differentiable at y and the derivative at y is equal to 0. See also [27, Remark 3.39].

We now present the results on the pointwise Hölder exponents and (non-)differentiability of $T_{\infty,\tau}$ at points in $J(G_\tau)$.

Theorem 2.11 (Non-differentiability of $T_{\infty,\tau}$ at points in $J(G_\tau)$). *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $h = (h_1, \ldots, h_m) \in \mathcal{P}^m$ such that $h_1, \ldots, h_m$ are mutually distinct and let $\Gamma = \{h_1, h_2, \ldots, h_m\}$. Let $G = \langle h_1, \ldots, h_m \rangle$. Let $p = (p_1, \ldots, p_m) \in \mathcal{W}_m$. Let $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ be the skew product associated with Γ. Let $\tau := \sum_{j=1}^m p_j \delta_{h_j} \in \mathfrak{M}_1(\Gamma) \subset \mathfrak{M}_1(\mathcal{P})$. Let $\mu \in \mathfrak{M}_1(\Gamma^{\mathbb{N}} \times \hat{\mathbb{C}})$ be the maximal relative entropy measure for $f : \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}} \to \Gamma^{\mathbb{N}} \times \hat{\mathbb{C}}$ with respect to $(\sigma, \tilde{\tau})$. Let $\lambda = (\pi_{\hat{\mathbb{C}}})_*(\mu) \in \mathfrak{M}_1(\hat{\mathbb{C}})$. Suppose that $G \in \mathcal{G}$ and $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i, j) with $i \neq j$. Then, we have all of the following.*

1. *$G_\tau = G \in \mathcal{G}_{dis}$, $J_{\ker}(G) = \emptyset$, and all statements in Theorem 2.4 hold for τ. Moreover, $J(G) = \{z \in \hat{\mathbb{C}} \mid$ for any neighborhood U of $z, T_{\infty,\tau}|_U$ is not constant$\}$ and $\operatorname{int}(J(G)) = \emptyset$. Furthermore,* $\operatorname{supp} \lambda = J(G)$ *and for each $z \in J(G)$, $\lambda(\{z\}) = 0$.*
2. *There exists a Borel subset A of $J(G)$ with $\lambda(A) = 1$ such that for each $z_0 \in A$,*
$$\text{Höl}(T_{\infty,\tau}, z_0) \leq u(h, p, \mu) = \frac{-(\sum_{j=1}^m p_j \log p_j)}{\sum_{j=1}^m p_j \log \deg(h_j)} < 1.$$
3. *We have that*
$$\dim_H(\{z \in J(G) \mid \text{Höl}(T_{\infty,\tau}, z) \leq u(h, p, \mu)\}) \geq \frac{\sum_{j=1}^m p_j \log \deg(h_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log \deg(h_j)} > 1,$$
where $\dim_H$ denotes the Hausdorff dimension with respect to the Euclidian distance on $\mathbb{C}$.
4. *For each non-empty open subset U of $J(G)$ there exists an uncountable dense subset A_U of U such that for each $z \in A_U$, $T_{\infty,\tau}$ is non-differentiable at z.*

In [25, Theorem 3.82], it is assumed that G is hyperbolic, i.e., $P(G) \subset F(G)$. However, in Theorem 2.11, we do not assume hyperbolicity of G. Note that there are many examples of (non-hyperbolic) $G = \langle h_1, \ldots, h_m \rangle \in \mathcal{G}_{dis}$ for which $\{h_i^{-1}(J(G))\}_{i=1}^m$ are mutually disjoint (see Theorem 2.12, Propositions 5.2, 5.3, Examples 5.4, 5.6, 5.7, Remark 5.5).

We present a result on 2-generator semigroup $G = \langle h_1, h_2 \rangle \in \mathcal{G}_{dis}$ and the associated random dynamics generated by $\tau = \sum_{j=1}^{2} p_j \delta_{h_j}$ where $(p_1, p_2) \in \mathcal{W}_2$.

Theorem 2.12. *Let $G = \langle h_1, h_2 \rangle \in \mathcal{G}_{dis}$. Let $(p_1, p_2) \in \mathcal{W}_2$ and let $\tau = \sum_{j=1}^{2} p_j \delta_{h_j}$. Let $\Gamma = \{h_1, h_2\}$. Then, we have all of the following.*

1. *$h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$. For $((h_1, h_2), (p_1, p_2))$, all statements 1–4 in Theorem 2.11 hold. For each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma = J_{\gamma,\Gamma} = \bigcap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$. The map $\gamma \mapsto J_\gamma$ is continuous on $\Gamma^{\mathbb{N}}$ with respect to the Hausdorff metric in the space of all non-empty compact sets in $\hat{\mathbb{C}}$.*
2. *For each $J \in \mathrm{Con}(J(G))$, there exists a unique $\gamma \in \Gamma^{\mathbb{N}}$ with $J = J_\gamma$. $\mathrm{Con}(J(G)) = \{J_\gamma \mid \gamma \in \Gamma^{\mathbb{N}}\}$. The map $\gamma \mapsto J_\gamma$ is a bijection between $\Gamma^{\mathbb{N}}$ and $\mathrm{Con}(J(G))$. In particular, there exist uncountably many connected components of $J(G)$.*
3. *There exist infinitely many doubly connected components of $F(G)$.*
4. *For each $J \in \mathrm{Con}(J(G))$, $T_{\infty,\tau}|_J$ is constant.*
5. *Let $J_1, J_2 \in \mathrm{Con}(J(G))$ with $J_1 \neq J_2$. Suppose $T_{\infty,\tau}|_{J_1} = T_{\infty,\tau}|_{J_2}$. Then there exists a doubly connected component A of $F(G)$ such that $\partial A \subset J_1 \cup J_2$.*
6. *Either $J(h_1) <_s J(h_2)$ or $J(h_2) <_s J(h_1)$. Without loss of generality, we may assume that $J(h_1) <_s J(h_2)$. Then $J_{\min}(G) = J(h_1)$ and $J_{\max}(G) = J(h_2)$. Moreover, the map $\zeta : w = (w_1, w_2, \ldots) \in \{1,2\}^{\mathbb{N}} \mapsto J_{\gamma(w)} \in \mathrm{Con}(J(G))$, where $\gamma(w) = (h_{w_1}, h_{w_2}, \ldots) \in \Gamma^{\mathbb{N}}$, is a bijection such that $w^1 <_l w^2$ implies $\zeta(w^1) <_s \zeta(w^2)$, where $<_l$ denotes the lexicographic order in $\{1,2\}^{\mathbb{N}}$, i.e., $(i_1, \ldots, i_n, 1, \ldots) <_l (i_1, \ldots, i_n, 2, \ldots)$.*
7. *Suppose $J(h_1) <_s J(h_2)$. Then $T_{\infty,\tau}^{-1}(\{0\}) = K(h_1)$ and $T_{\infty,\tau}^{-1}(\{1\}) = \overline{F_\infty(h_2)}$. Moreover, for each $t \in (0,1)$, exactly one of the following* (a) *and* (b) *holds.*
   - (a) *There exists a unique $w \in \{1,2\}^{\mathbb{N}}$ such that $T_{\infty,\tau}^{-1}(\{t\}) = J_{\gamma(w)}$. Moreover, $\sharp\{n \in \mathbb{N} \mid w_n = 1\} = \sharp\{n \in \mathbb{N} \mid w_n = 2\} = \infty$. Moreover, there exists exactly one bounded component B_w of $F_{\gamma(w)}$. Furthermore, $\partial B_w = \partial A_{\infty,\gamma(w)} = J_{\gamma(w)}$.*
   - (b) *There exist two elements $\rho, \mu \in \{1,2\}^{\mathbb{N}}$ such that $\rho <_l \mu$, $J_{\gamma(\rho)} <_s J_{\gamma(\mu)}$, and $T_{\infty,\tau}^{-1}(\{t\}) = K_{\gamma(\mu)} \setminus \mathrm{int}(K_{\gamma(\rho)})$. Moreover, either* (i) *$\rho = (1,2,2,2,\ldots)$ and $\mu = (2,1,1,1,\ldots)$, or* (ii) *there exists a finite word $(i_1, \ldots, i_n) \in \{1,2\}^n$ for some $n \in \mathbb{N}$ such that $\rho = (i_1, \ldots, i_n, 1, 2, 2, 2, \ldots)$ and $\mu = (i_1, \ldots, i_n, 2, 1, 1, 1, \ldots)$. Moreover, there exists a doubly connected component A of $F(G)$ such that $\partial A \subset J_{\gamma(\rho)} \cup J_{\gamma(\mu)}$. Furthermore, $J_{\gamma(\rho)}$ is a quasicircle.*

Remark 2.13. We remark that in general, $\gamma \in \Gamma^{\mathbb{N}} \mapsto J_\gamma$ is not continuous ([19, Example 1.7]). Under the assumptions of Theorem 2.12, for the studies of $\{J_\gamma\}_{\gamma \in \Gamma^{\mathbb{N}}}$, see [23, 24]. In [23], under the assumptions of Theorem 2.12 and assuming that h_1 (with $J(h_1) <_s J(h_2)$) is hyperbolic and $P^*(\langle h_2 \rangle) \subset \mathrm{int}(K(h_1))$ (which implies G is hyperbolic, i.e., $P(G) \subset F(G)$, see [22, Theorem 2.36]), a classification of the fiberwise Julia sets J_γ was given. In particular, it was shown that under the assumptions of Theorem 2.12, if the above h_1 is hyperbolic, $P^*(\langle h_2 \rangle) \subset \mathrm{int}(K(h_1))$ and $J(h_1)$ is not a Jordan curve, then for any $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^{\mathbb{N}}$ satisfying that (a) $\sharp\{n \in \mathbb{N} \mid \gamma_n \neq h_1\} = \infty$ and (b) there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that $\sigma^{n_k}(\gamma) \to (h_1, h_1, h_1, \ldots)$ as $k \to \infty$, the Julia set J_γ of γ satisfies that (I) J_γ is a Jordan curve but not a quasicircle, (II) the unbounded component $A_{\infty,\gamma}$ of $\hat{\mathbb{C}} \setminus J_\gamma$ is a John domain, and (III) the bounded component of $\hat{\mathbb{C}} \setminus J_\gamma$ is not a John domain. Note that the above phenomenon is a new one which cannot hold in the usual iteration dynamics of a single polynomial.

Remark 2.14. Under the assumption of Theorem 2.12, suppose that h_1 and h_2 with $J(h_1) <_s J(h_2)$ are real polynomials. Then for each $\gamma \in \Gamma^{\mathbb{N}}$, J_γ is symmetric with respect to the real axis, and $T_{\infty,\tau}$ is symmetric with respect to the real axis. If, in addition to the above assumption, h_1 is hyperbolic, $P^*(\langle h_2 \rangle) \subset \mathrm{int}(K(h_1))$ and 7(a) in Theorem 2.12 holds, then by [23, 24], $J_{\gamma(w)}$ is a Jordan curve and $\sharp(J_{\gamma(w)} \cap \mathbb{R}) = 2$. For the figure of the Julia set of $\langle h_1, h_2 \rangle \in \mathcal{G}_{dis}$ and the graph of $T_{\infty,\tau}$, see [25].

We now present some results on 3-generator semigroups in $\mathcal{G}_{dis}$ and the associated random dynamics.

Theorem 2.15. *Let $G = \langle h_1, h_2, h_3 \rangle \in \mathcal{G}_{dis}$. For each $i = 1, 2, 3$, let $J_i \in \mathrm{Con}(J(G))$ with $J(h_i) \subset J_i$. Suppose without loss of generality (since $(\mathrm{Con}(J(G)), \leq_s)$ is totally ordered), that $J_1 \leq_s J_2 \leq_s J_3$. Then, we have exactly one of the following (1),(2),(3).*

(1) *$\{h_i^{-1}(J(G))\}_{i=1,2,3}$ are mutually disjoint, $J_{\min}(G) = J(h_1)$, $J_{\max}(G) = J(h_3)$, $\hat{K}(G) = K(h_1)$ and $F_\infty(G) = F_\infty(h_3)$.*

(2) *$h_1^{-1}(J(G)) \cap (\bigcup_{i=2,3} h_i^{-1}(J(G))) = \emptyset$, $h_2^{-1}(J(G)) \cap h_3^{-1}(J(G)) \neq \emptyset$, $J_{\min}(G) = J(h_1)$ and $\hat{K}(G) = K(h_1)$.*

(3) *$h_3^{-1}(J(G)) \cap (\bigcup_{i=1,2} h_i^{-1}(J(G))) = \emptyset$, $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) \neq \emptyset$, $J_{\max}(G) = J(h_3)$ and $F_\infty(G) = F_\infty(h_3)$.*

Moreover, we have the following. (a) *If $J_1 = J_2$, then* (3) *holds.* (b) *If $J_2 = J_3$, then* (2) *holds.* (c) *If $h_2^{-1}(J(G)) \cap (\bigcup_{i=1,3} h_i^{-1}(J(G))) = \emptyset$, then* (1) *holds and $J_1 <_s J_2 <_s J_3$.*

Corollary 2.16. *Let $G = \langle h_1, h_2, h_3 \rangle \in \mathcal{G}_{dis}$. Then there exist infinitely many connected components of $J(G)$ and there exist infinitely many doubly connected components of $F(G)$. More precisely, there exists an $i \in \{1, 2, 3\}$ such that (1) $h_i^{-1}(J(G)) \cap (\bigcup_{j:j \neq i} h_j^{-1}(J(G))) = \emptyset$, (2) either $J(h_i) = J_{\max}(G)$ or $J(h_i) = J_{\min}(G)$, and (3) there exists a sequence $\{J_n\}_{n \in \mathbb{N}}$ of mutually different elements in $\mathrm{Con}(J(G))$ and a sequence $\{A_n\}_{n \in \mathbb{N}}$ of mutually different doubly connected components of $F(G)$ such that $J_n \to J(h_i)$ and $\overline{A_n} \to J(h_i)$ as $n \to \infty$ with respect to the Hausdorff metric.*

Remark 2.17. Let $G = \langle h_1, h_2, h_3 \rangle \in \mathcal{G}_{dis}$, $(p_1, p_2, p_3) \in \mathcal{W}_3$ and $\tau = \sum_{i=1}^3 p_i \delta_{h_i}$. Then, by Theorem 2.4 and Corollary 2.16, the continuous function $T_{\infty,\tau}$ can detect the boundaries of infinitely many doubly connected components of $F(G)$. Moreover, it can detect either $J_{\max}(G)$ or $J_{\min}(G)$. There are many examples of each of (1), (2), and (3) of Theorem 2.15 ([22]).

Remark 2.18. In [22], it was shown that there exists a 3-generator semigroup $G = \langle h_1, h_2, h_3 \rangle \in \mathcal{G}_{dis}$ such that $\sharp\mathrm{Con}(J(G)) = \aleph_0$. In [22], it was also shown that for each $n \in \mathbb{N}$ with $n \geq 2$, there exists a $2n$-generator semigroup $G = \langle h_1, \ldots, h_{2n} \rangle \in \mathcal{G}_{dis}$ with $\sharp\mathrm{Con}(J(G)) = n$. By developing the idea in [22], it was shown in [13] that for each $n \in \mathbb{N}$ with $n \geq 2$, there exists a 4-generator semigroup $G = \langle h_1, \ldots, h_4 \rangle \in \mathcal{G}_{dis}$ with $\sharp\mathrm{Con}(J(G)) = n$. Note that in [20], the author of this paper constructed a new cohomology theory for "backward self-similar systems" (backward IFSs), and by using it, for a finitely generated semigroup $G = \langle h_1, \ldots, h_m \rangle \in \mathcal{G}$, we can investigate the cardinality of $\mathrm{Con}(J(G))$ and $\mathrm{Con}(F(G))$. More precisely, we investigate the cohomology groups of the nerve $\mathcal{N}_k$ of $\{(h_{i_1} \cdots h_{i_k})^{-1}(J(G)) \mid (i_1, \ldots, i_k) \in \{1, \ldots, m\}^k\}$ for each $k \in \mathbb{N}$ and their direct limits as $k \to \infty$. In the proofs (section 4) of Theorems 2.12 and 2.15, we use some results (geometric observations on the nerves $\mathcal{N}_k$ and their inverse limit, e.g. $\mathrm{Con}(J(G)) \cong \mathrm{Con}(\varprojlim_k |\mathcal{N}_k|)$) from [20] and some results on the dynamics of $G \in \mathcal{G}_{dis}$ from [22].

Remark 2.19. Let $\tau \in \mathcal{M}_1(\mathcal{P})$. Suppose $G_\tau \in \mathcal{G}_{dis}$ and $\sharp\mathrm{Con}(J(G)) \leq \aleph_0$. Then, by Theorem 2.4-1, $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous and $T_{\infty,\tau}(J(G_\tau)) = [0,1]$. Thus there exists an element $J \in \mathrm{Con}(J(G_\tau))$ such that $T_{\infty,\tau}|_J$ is not constant. This illustrates the difference between 2-generator semigroups in $\mathcal{G}_{dis}$ (see Theorem 2.12) and m-generator semigroups ($m \geq 3$) in $\mathcal{G}_{dis}$ (see Remark 2.18).

3 Background and tools

In this section, we give the known results and tools to prove the main results.

(I) We first explain the known results on general polynomial semigroups. Let G be a polynomial semigroup in $\mathcal{P}$. Then $F(G)$ is an open subset of $\hat{\mathbb{C}}$, $J(G)$ is a compact subset of $\hat{\mathbb{C}}$, and for each $g \in G$, $g(F(G)) \subset F(G)$ and $g^{-1}(J(G)) \subset J(G)$. If H is a subsemigroup of G, then $F(G) \subset F(H)$ and $J(H) \subset J(G)$. We set $E(G) := \{z \in \hat{\mathbb{C}} \mid \sharp G^{-1}(\{z\}) < \infty\}$. Then $\sharp E(G) \leq 2$ and for each $z \in \hat{\mathbb{C}} \setminus E(G)$, $J(G) \subset \overline{G^{-1}(\{z\})}$. In particular, for each $z \in J(G) \setminus E(G)$, $J(G) = \overline{G^{-1}(\{z\})}$. The Julia set $J(G)$ is a perfect set. The Julia set $J(G)$ is the unique minimal element in the space of all compact subsets K of $\hat{\mathbb{C}}$ with $\sharp K \geq 3$ for which $g^{-1}(K) \subset K$ for each $g \in G$. The Julia set $J(G)$ is equal to the closure of the set of repelling fixed points of elements of G. In particular, $J(G) = \overline{\cup_{g\in G} J(g)}$. For the proofs of these results, see [9]. Moreover, if $G = \langle h_1, \ldots, h_m \rangle$, then $J(G) = \cup_{j=1}^m h_j^{-1}(J(G))$ (see [14, Lemma 1.1.4]). Moreover, it is easy to see that if G is generated by a compact subset of $\mathcal{P}$, then $\infty \in F(G)$.

(II) We next explain the known results on the random dynamics of polynomials obtained in [25]. Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose $J_{\ker}(G_\tau) = \emptyset$. Then there exists a non-empty finite dimensional subspace U_τ of $C(\hat{\mathbb{C}})$ with $M_\tau(U_\tau) = U_\tau$ and a bounded operator $\pi_\tau : C(\hat{\mathbb{C}}) \to U_\tau$ such that for each $\varphi \in C(\hat{\mathbb{C}})$, $M_\tau^n(\varphi - \pi_\tau(\varphi)) \to 0$ in $C(\hat{\mathbb{C}})$ as $n \to \infty$. Moreover, $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$. Moreover, there exist at least one and at most finitely many minimal sets of G_τ. Moreover, for each minimal set L of G_τ, the function $T_{L,\tau} : \hat{\mathbb{C}} \to [0,1]$ of probability of tending to L is continuous on $\hat{\mathbb{C}}$ and locally constant on $F(G)$. In particular, the function $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous on $\hat{\mathbb{C}}$ and locally constant on $F(G_\tau)$. Moreover, denoting by S_τ the union of all minimal sets of G_τ, we have that for each $z \in \hat{\mathbb{C}}$, there exists a Borel subset $\mathcal{A}_z$ of $\mathcal{P}^{\mathbb{N}}$ with $\tilde{\tau}(\mathcal{A}_z) = 1$ such that for each $\gamma \in \mathcal{A}_z$, $d(\gamma_{n,1}(z), S_\tau) \to 0$ as $n \to \infty$. For the proofs of these results, see [25, Theorem 3.15].

In the proofs of the main results of this paper, we combine the above results in **(I)(II)** and some new careful observations on the dynamics of $G \in \mathcal{G}_{dis}$ and associated random dynamics.

4 Proofs of the main results

4.1 Proof of Theorem 2.4

In this subsection, we prove Theorem 2.4. We need several lemmas.

Lemma 4.1. *Let $G \in \mathcal{G}_{dis}$ (possibly generated by a non-compact subset of $\mathcal{P}$). Then, $\infty \in F(G)$, $\mathrm{int}(\hat{K}(G)) \neq \emptyset$, $F_\infty(G) \cup \mathrm{int}(\hat{K}(G)) \subset F(G)$, and for each $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ with $g(z) \in F_\infty(G) \cup \mathrm{int}(\hat{K}(G)) \subset F(G)$. In particular, $J_{\ker}(G) = \emptyset$.*

Proof. By [22, Theorem 2.20-1,5], $\infty \in F(G)$ and $\mathrm{int}(\hat{K}(G)) \neq \emptyset$. Moreover, by [22, Proposition 2.19], $\mathrm{int}(\hat{K}(G)) \subset F(G)$. Let $z \in \hat{\mathbb{C}}$ be a point. We consider the following three cases: Case 1: $z \notin \hat{K}(G)$. Case 2: $z \in \mathrm{int}(\hat{K}(G))$. Case 3: $z \in \partial(\hat{K}(G))$. If we have Case 1, then there exists an element $g \in G$ with $g(z) \in F_\infty(G)$. If we have Case 2, then each element $h \in G$ satisfies $h(z) \in \mathrm{int}(\hat{K}(G))$. Suppose we have Case 3. Then, by [22, Theorem 2.20-2], $z \in \partial(\hat{K}(G)) \subset J_{\min}(G)$. By [22, Theorem 2.1], there exists an element $g \in G$ with $J(g) \cap J_{\min}(G) = \emptyset$. By [22, Theorem 2.20-5(b)], $g(J_{\min}(G)) \subset \mathrm{int}(\hat{K}(G))$. Thus $g(z) \in \mathrm{int}(\hat{K}(G))$. Therefore, we obtain that for each $z \in \hat{\mathbb{C}}$, there exists an element $g \in G$ with $g(z) \in F_\infty(G) \cup \mathrm{int}(\hat{K}(G)) \subset F(G)$. Thus, $J_{\ker}(G) = \emptyset$. $\square$

Lemma 4.2. *Under the assumptions of Theorem 2.4, statements 1, 2, 6– 9 in Theorem 2.4 hold.*

Proof. By Lemma 4.1 and [25, Theorem 3.14], we obtain that $J_{\ker}(G_\tau) = \emptyset$ and $F_{meas}(\tau) = \mathfrak{M}_1(\hat{\mathbb{C}})$. Thus statement 6 holds. By [25, Lemmas 5.24, 5.27, Theorem 3.31] and [27, Theorem 1.9], statements 2 and 1 in Theorem 2.4 hold.

We now prove statements 7 and 8 in Theorem 2.4. By [22, Theorem 2.1], there exists an element $g \in \operatorname{supp}\tau$ with $J(g) \cap J_{\min}(G_\tau) = \emptyset$. By [22, Theorem 2.20-4,5], $\operatorname{int}(K(g))$ is connected and there exists an attracting fixed point z_g of g in $\operatorname{int}(\hat{K}(G_\tau))$ such that $\operatorname{int}(K(g))$ is the immediate attracting basin of z_g for the dynamics of g and $\hat{K}(G_\tau) \subset \operatorname{int}(K(g))$. Since $G_\tau(\hat{K}(G_\tau)) \subset \hat{K}(G_\tau)$, Zorn's lemma implies that there exists a minimal set L_0 for $(G_\tau, \hat{\mathbb{C}})$ with $L_0 \subset \hat{K}(G_\tau)$. Considering the dynamics of g in $\hat{K}(G_\tau)$, we see that there exists a unique minimal set L_τ for $(G_\tau, \hat{\mathbb{C}})$ with $L_\tau \subset \hat{K}(G_\tau)$. Therefore $\operatorname{Min}(G_\tau, \hat{\mathbb{C}}) = \{\{\infty\}, L_\tau\}$. Thus statement 7 holds. Statement 8 follows from statements 6, 7 and [25, Theorem 3.15-5,15].

We now prove statement 9. We again use the element g in the previous paragraph. Since $g^n|_{L_\tau} \to z_g$ as $n \to \infty$, [25, Theorem 3.15-12] implies that the number r_{L_τ} in [25, Theorem 3.15-8] is equal to 1. By [25, Theorem 3.15-1,2,9,13,15], it follows that there exist two continuous linear functionals $\rho_1, \rho_2 : C(\hat{\mathbb{C}}) \to \mathbb{C}$ such that for each $\varphi \in C(\hat{\mathbb{C}})$,

$$M_\tau^n(\varphi) \to \rho_1(\varphi) \cdot T_{\infty,\tau} + \rho_2(\varphi) \cdot T_{L_\tau,\tau} \text{ in } C(\hat{\mathbb{C}}) \text{ as } n \to \infty,$$

and such that $\operatorname{supp}\rho_1 = \{\infty\}$ and $\operatorname{supp}\rho_2 = L_\tau$. From this, it is easy to see that $\rho_1 = \delta_\infty$ and ρ_2 is a Borel probability measure on $\hat{\mathbb{C}}$. Moreover, by [25, Theorem 3.15-15], we obtain that $T_{\infty,\tau}(z) + T_{L_\tau,\tau}(z) = 1$ for each $z \in \hat{\mathbb{C}}$. From these arguments, statement 9 holds. □

Lemma 4.3. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose that $\infty \in F(G_\tau)$. Let U be a multiply connected component of $F(G_\tau)$. Let B be a bounded component of $\mathbb{C} \setminus U$. Let $y \in B$ and let $z \in U$. Then, for any $\gamma \in X_\tau$ with $\gamma_{n,1}(y) \to \infty$ as $n \to \infty$, we have $\gamma_{n,1}(z) \to \infty$ as $n \to \infty$. In particular, $T_{\infty,\tau}(y) \le T_{\infty,\tau}(z)$.*

Proof. Suppose that $\gamma_{n,1}(y) \to \infty$ as $n \to \infty$. Let ζ be a Jordan curve (i.e. simple closed curve) in U such that y belongs to the bounded component of $\mathbb{C} \setminus \zeta$. By the maximum principle, [25, Lemma 5.24] and forward invariance of $F(G_\tau)$ under any element of G_τ, we obtain that $\gamma_{n,1} \to \infty$ as $n \to \infty$ on ζ. Hence, $\gamma_{n,1}(z) \to \infty$ as $n \to \infty$. □

Proposition 4.4. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Let U be a multiply connected component of $F(G_\tau)$. Let C be the boundary of a bounded component of $\mathbb{C} \setminus U$. Let V be an open subset of $\hat{\mathbb{C}}$ such that $V \cap C \neq \emptyset$. Then, we have the following.*

1. *If $\infty \in F(G_\tau)$ and $\operatorname{int}(\hat{K}(G_\tau)) \neq \emptyset$, then $T_{\infty,\tau}|_V$ is not constant.*

2. *If $\operatorname{supp}\tau$ is compact, $\sharp\operatorname{supp}\tau \le \aleph_0$ and $\hat{K}(G_\tau) \neq \emptyset$, then $T_{\infty,\tau}|_V$ is not constant.*

Proof. We may assume that V does not meet the unbounded component of $\hat{\mathbb{C}} \setminus U$. We first prove statement 1. Suppose that $\infty \in F(G_\tau)$ and $\operatorname{int}(\hat{K}(G_\tau)) \neq \emptyset$. Let $y \in V \cap C$. Let ζ be a Jordan curve in U such that y belongs to the bounded component A of $\mathbb{C} \setminus \zeta$. Since $C \subset J(G_\tau)$, [9, Corollary 3.1] implies that there exists a $g \in G_\tau$ such that $J(g) \cap V \cap A \neq \emptyset$. Then, $\zeta \subset F_\infty(g)$. For, if $\zeta \subset \operatorname{int}K(g)$, then the maximum principle implies that $A \subset F(g)$, which is a contradiction. Hence, $\zeta \subset F_\infty(g)$. Therefore, $g^n \to \infty$ as $n \to \infty$ on U. Since $J(g) \cap V \cap A \neq \emptyset$ and $\operatorname{int}(\hat{K}(G_\tau)) \neq \emptyset$, there exists a point $y_1 \in V \cap A$ and an $l \in \mathbb{N}$ such that $g^l(y_1) \in \operatorname{int}(\hat{K}(G_\tau))$. Let $y_2 \in U \cap V$ be a point. We may assume that $g^l(y_2) \in F_\infty(G_\tau)$. Let $\{\gamma_j\}_{j=1}^m$ be a finite sequence of elements of Γ_τ such that $g^l = \gamma_m \circ \cdots \circ \gamma_1$. Then, there exists a neighborhood W of $(\gamma_1, \ldots, \gamma_m)$ in Γ_τ^m such that for each $\alpha = (\alpha_1, \ldots, \alpha_m) \in W$, $\alpha_m \cdots \alpha_1(y_1) \in \operatorname{int}(\hat{K}(G_\tau))$ and $\alpha_m \cdots \alpha_1(y_2) \in F_\infty(G_\tau)$. We set $Z := \{\rho = (\rho_1, \rho_2, \ldots) \in X_\tau \mid (\rho_1, \ldots, \rho_m) \in W\}$. Then, for each $\omega = (\omega_1, \omega_2, \ldots) \in Z$, $\{\omega_{r,1}(y_1)\}_{r \in \mathbb{N}}$ is bounded in $\mathbb{C}$ and $\omega_{r,1}(y_2) \to \infty$ as $r \to \infty$. Hence, y_1 belongs to a bounded component B of $\mathbb{C} \setminus U$. By Lemma 4.3, $\{\rho \in X_\tau \mid \rho_{n,1}(y_1) \to \infty\} \subset \{\rho \in X_\tau \mid \rho_{n,1}(y_2) \to \infty\}$. From these arguments, it follows that $T_{\infty,\tau}(y_1) + \tilde{\tau}(Z) \le T_{\infty,\tau}(y_2)$. Since $\tilde{\tau}(Z) > 0$, we obtain that $T_{\infty,\tau}(y_1) < T_{\infty,\tau}(y_2)$. Therefore, $T_{\infty,\tau}|_V$ is not constant. Thus, we have proved statement 1.

We now prove statement 2. Let ζ be a Jordan curve in U such that y belongs to the bounded component A of $\mathbb{C} \setminus \zeta$. We now show the following claim 1:
Claim 1: There exist a $g \in G_\tau$, an $l \in \mathbb{N}$, and a point $y_1 \in V \cap A$ such that $J(g) \cap V \cap A \neq \emptyset$ and $g^l(y_1) \in \hat{K}(G_\tau)$.

In order to show claim 1, we consider the following two cases. Case 1. $\sharp\hat{K}(G_\tau) \ge 2$. Case 2. $\sharp\hat{K}(G_\tau) = 1$.

Suppose that we have case 1. By [9, Corollary 3.1], there exists a $g \in G_\tau$ such that $J(g) \cap V \cap A \neq \emptyset$. Since $\sharp\hat{K}(G_\tau) \ge 2$ and $\cup_{n\in\mathbb{N}} g^n(V \cap A) \subset \mathbb{C}$, Montel's theorem implies that there exists an $l \in \mathbb{N}$ and a point $y_1 \in V \cap A$ such that $g^l(y_1) \in \hat{K}(G_\tau)$. Hence, the statement of claim 1 holds when we have case 1.

Suppose that we have case 2. Let $z_0 \in \mathbb{C}$ be such that $\hat{K}(G_\tau) = \{z_0\}$. By [25, Lemma 5.28], $h(z_0) = z_0$ for each $h \in \Gamma_\tau$ and $z_0 \in J(G_\tau)$. Since Γ_τ is compact, there exists an element $\beta_1 \in \Gamma_\tau$ such that $z_0 \notin E(\beta_1)$, where $E(\beta_1)$ denotes the exceptional set of β_1. Moreover, [9, Corollary 3.1] implies that there exists an element $\beta_2 \in G_\tau$ such that $J(\beta_2) \cap V \cap A \neq \emptyset$. By [19, Proposition 2.2 (3)], there exists a $p \in \mathbb{N}$ such that $J(\beta_1\beta_2^p) \cap V \cap A \neq \emptyset$. Let $g := \beta_1\beta_2^p$. Since $h(z_0) = z_0$ for each $h \in G_\tau$ and $z_0 \notin E(\beta_1)$, we obtain that $z_0 \notin E(g)$. Therefore, there exist an $l \in \mathbb{N}$ and a point $y_1 \in V \cap A$ such that $g^l(y_1) = z_0 \in \hat{K}(G_\tau)$. Thus, we have shown claim 1.

Let (g, l, y_1) be as in claim 1. Let $y_2 \in U \cap V$ be a point. Since $J(g) \cap V \cap A \neq \emptyset$, the maximum principle implies that $g^n \to \infty$ as $n \to \infty$ on U. Hence, we may assume that $g^l(y_2) \in F_\infty(G_\tau)$. Therefore $g^l(y_1) \in \hat{K}(G_\tau)$, $g^l(y_2) \in F_\infty(G_\tau)$ and y_1 belongs to a bounded component B of $\mathbb{C} \setminus U$. Combining this with the method in the proof of statement 1, we obtain that $T_{\infty,\tau}(y_1) < T_{\infty,\tau}(y_2)$. Therefore, $T_{\infty,\tau}|_V$ is not constant. Thus, we have proved statement 2. □

Corollary 4.5. *Let $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$. Suppose that $\hat{K}(G_\tau) \neq \emptyset$ and $J_{\ker}(G_\tau) = \emptyset$. Let U be a multiply connected component of $F(G_\tau)$. Let C be the boundary of a bounded component of $\mathbb{C} \setminus U$. Let V be an open subset of $\hat{\mathbb{C}}$ such that $V \cap C \neq \emptyset$. Then, $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous and $T_{\infty,\tau}|_V$ is not constant.*

Proof. Since $\operatorname{supp}\tau$ is compact, we have $\infty \in F(G_\tau)$. By [25, Theorem 3.31], $\operatorname{int}\hat{K}(G_\tau) \neq \emptyset$. By Proposition 4.4, it follows that $T_{\infty,\tau}|_V$ is not constant. Moreover, by [25, Theorem 3.22], $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous. □

Lemma 4.6. *Let Γ be a subset of $\mathcal{P}$ and let $G = \langle\Gamma\rangle$. Suppose $G \in \mathcal{G}_{dis}$. Then for each $\gamma \in \Gamma^{\mathbb{N}}$, K_γ is a connected compact subset $\mathbb{C}$, $A_{\infty,\gamma}$ is a simply connected domain, and $K_\gamma \cup A_{\infty,\gamma} = \hat{\mathbb{C}}$.*

Proof. Since $G \in \mathcal{G}_{dis}$, by [22, Theorem 2.20] we have $\infty \in F(G)$. For each $r > 0$, we denote by $B_h(\infty, r)$ the ball with center ∞ and radius r with respect to the hyperbolic distance on $F_\infty(G)$. Then $g(B_h(\infty, r)) \subset B_h(\infty, r)$ for each $g \in G$. Let $r > 0$ be small enough such that $B_h(\infty, r)$ is simply connected. Let $B := B_h(\infty, r)$. By [25, Lemma 5.24], for each $\alpha \in \Gamma^{\mathbb{N}}$, $\alpha_{n,1} \to \infty$ uniformly on B as $n \to \infty$. Therefore for each $\gamma \in \Gamma^{\mathbb{N}}, K_\gamma \cup A_{\infty,\gamma} = \hat{\mathbb{C}}$ and $A_{\infty,\gamma}$ is an open neighborhood of ∞. By the maximum principle, $A_{\infty,\gamma}$ is connected. Moreover, $A_{\infty,\gamma} = \cup_{n=1}^{\infty}(\gamma_{n,1})^{-1}(B)$. Since $G \in \mathcal{G}$, each $\gamma_{n,1}^{-1}(B)$ is a simply connected domain. Thus $A_{\infty,\gamma}$ is the union of increasing simply connected domains $\gamma_{n,1}^{-1}(B)$. Therefore $A_{\infty,\gamma}$ is simply connected. Thus K_γ is connected. □

Lemma 4.7. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $G_\tau \in \mathcal{G}_{dis}$. Let A be a doubly connected component of $F(G_\tau)$. Let $y_1 \in A$ and let y_2 be a point in the unbounded component of $\mathbb{C} \setminus A$. Then, we have the following.*

1. *For any $\gamma \in X_\tau$ with $\gamma_{n,1}(y_1) \to \infty$ as $n \to \infty$, we have $\gamma_{n,1}(y_2) \to \infty$ as $n \to \infty$. In particular, $T_{\infty,\tau}(y_1) \le T_{\infty,\tau}(y_2)$.*

2. *In addition to the assumptions of our lemma, suppose $y_2 \in F(G_\tau)$. Let U be the connected component of $F(G_\tau)$ containing y_2. Suppose that either U is doubly connected or $U = F_\infty(G_\tau)$. Then $T_{\infty,\tau}(y_1) < T_{\infty,\tau}(y_2)$.*

Proof. We first prove statement 1. Since $G_\tau \in \mathcal{G}_{dis}$, by [22, Theorem 2.20] we have $\infty \in F(G_\tau)$. Let $\gamma \in X_\tau$ and suppose $\gamma_{n,1}(y_1) \to \infty$ as $n \to \infty$. By [25, Lemma 5.24], $\gamma_{n,1} \to \infty$ locally uniformly on A as $n \to \infty$. Therefore $K_\gamma \subset \mathbb{C} \setminus A$. Thus $\partial \hat{K}(G_\tau) \subset K_\gamma \subset \mathbb{C} \setminus A$. By [22, Theorem 2.20-2], $\partial \hat{K}(G_\tau) \subset J_{\min}(G_\tau)$. Since $J_{\min}(G_\tau)$ is included in the bounded component of $\mathbb{C} \setminus A$, and since K_γ is connected (see Lemma 4.6), it follows that K_γ is included in the bounded component of $\mathbb{C} \setminus A$. Therefore $\gamma_{n,1}(y_2) \to \infty$ as $n \to \infty$. Thus we have proved statement 1.

We now prove statement 2. We prove the following claim:
Claim: There exists a map $g \in G_\tau$ such that $g(y_1) \in \text{int}(\hat{K}(G_\tau))$ and $g(y_2) \in F_\infty(G_\tau)$.

To prove this claim, let B_1 and B_2 be the two connected components of ∂A. We may assume $B_2 <_s B_1$. For each $i = 1, 2$, let $B_i' \in \text{Con}(J(G_\tau))$ with $B_i \subset B_i'$. Then $B_2' <_s B_1'$. Therefore $J_{\min}(G_\tau) \leq_s B_2' <_s B_1'$. Let D be a bounded, doubly connected, and open neighborhood of B_1' such that $J_{\min}(G_\tau) \cup \{y_1\} <_s \overline{D}$ and y_2 belongs to the unbounded component of $\mathbb{C} \setminus \overline{D}$. By [22, Lemma 4.2], there exists an element $h \in G_\tau$ with $J(h) \subset D$. Then $J(h) \cap J_{\min}(G_\tau) = \emptyset$. Moreover, $y_2 \in F_\infty(h)$. By [22, Theorem 2.20-4,5], $\text{int}(K(h))$ is connected and is an immediate basin of an attracting fixed point z_h of h, and $z_h \in \text{int}(\hat{K}(G_\tau))$. Since $\partial \hat{K}(G_\tau) \subset J_{\min}(G_\tau)$ ([22, Theorem 2.20-2]), $\{z_h\} <_s J_{\min}(G_\tau) <_s \overline{D}$. Since z_h belongs to the bounded component of $\mathbb{C} \setminus J(h)$, it follows that y_1 belongs to the bounded component of $\mathbb{C} \setminus J(h)$. Therefore, there exists an $n \in \mathbb{N}$ such that $h^n(y_1) \in \text{int}(\hat{K}(G_\tau))$ and $h^n(y_2) \in F_\infty(G_\tau)$. Thus, we have proved the claim.

Let $g \in G_\tau$ be the element in the above claim. Let $h_1, \ldots, h_n \in \Gamma_\tau$ be some elements such that $g = h_n \circ \cdots \circ h_1$. Then there exists a neighborhood W of $(h_1, \ldots, h_n)$ in Γ_τ^n such that for each $\omega = (\omega_1, \ldots, \omega_n) \in W$, $\omega_n \cdots \omega_1(y_1) \in \text{int}(\hat{K}(G_\tau))$ and $\omega_n \cdots \omega_1(y_2) \in F_\infty(G_\tau)$. Therefore, for each $\gamma = (\gamma_1, \gamma_2, \ldots) \in X_\tau$ with $(\gamma_1, \ldots, \gamma_n) \in W$, we have that $\{\gamma_{r,1}(y_1)\}_{r \in \mathbb{N}}$ is bounded and that $\gamma_{r,1}(y_2) \to \infty$ as $r \to \infty$. Combining it with statement 1, we get $T_{\infty,\tau}(y_1) + \tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_n) \in W\}) \leq T_{\infty,\tau}(y_2)$. Since $\tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_n) \in W\}) > 0$, we obtain $T_{\infty,\tau}(y_1) < T_{\infty,\tau}(y_2)$. Therefore we have proved statement 2. □

Lemma 4.8. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $G_\tau \in \mathcal{G}_{dis}$. Let $J_1, J_2 \in \text{Con}(J(G_\tau))$ with $J_1 <_s J_2$. Then $\sup_{z \in J_1} T_{\infty,\tau}(z) \leq \inf_{z \in J_2} T_{\infty,\tau}(z)$.*

Proof. By [22, Theorem 2.20-1], $\infty \in F(G_\tau)$. By [22, Lemma 4.4], there exists a doubly connected component A of $F(G_\tau)$ with $J_1 <_s A <_s J_2$. By Lemmas 4.3, 4.7, it follows that $\sup_{z \in J_1} T_{\infty,\tau}(z) \leq \inf_{z \in J_2} T_{\infty,\tau}(z)$. □

Lemma 4.9. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ and suppose $\infty \in F(G_\tau)$. Let $A \in \text{Con}(F(G_\tau))$ be multiply connected and let $y_1 \in A$. Then $T_{\infty,\tau}(y_1) > 0$.*

Proof. Let K be a bounded component of $\mathbb{C} \setminus A$ and let $B \in \text{Con}(J(G_\tau))$ such that $\partial K \subset B$. Let D be a bounded neighborhood of B such that y_1 belongs to the unbounded component of $\mathbb{C} \setminus \overline{D}$. By [9, Corollary 3.1], there exists an element $\alpha \in G_\tau$ with $J(\alpha) \cap D \neq \emptyset$. By the maximum principle, $A \subset F_\infty(\alpha)$. Therefore, there exists an $m \in \mathbb{N}$ such that $\alpha^m(y_1) \in F_\infty(G_\tau)$. Let $h_1, \ldots, h_n \in \Gamma_\tau$ be some elements such that $\alpha^m = h_n \circ \cdots \circ h_1$. Then there exists a neighborhood W of $(h_1, \ldots, h_n)$ in Γ_τ^n such that for each $\omega = (\omega_1, \ldots, \omega_n) \in W$, $\omega_n \cdots \omega_1(y_1) \in F_\infty(G_\tau)$. Therefore, for each $\gamma \in X_\tau$ with $(\gamma_1, \ldots, \gamma_n) \in W$, $\gamma_{r,1}(y_1) \to \infty$ as $r \to \infty$. Thus $T_{\infty,\tau}(y_1) \geq \tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_n) \in W\}) > 0$. □

Corollary 4.10. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ and suppose $G_\tau \in \mathcal{G}_{dis}$. Let $A \in \text{Con}(F(G_\tau))$ be doubly connected. Let $y_1 \in A$. Then $T_{\infty,\tau}(y_1) > 0$.*

Proof. By Lemma 4.9 and [22, Theorem 2.20-1], the statement of our lemma holds. □

Lemma 4.11. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $G_\tau \in \mathcal{G}_{dis}$. Let $A \in \mathrm{Con}(F(G_\tau))$ be doubly connected. Let Q be an open subset of $\hat{\mathbb{C}}$ with $Q \cap \partial A \neq \emptyset$. Then $T_{\infty,\tau}|_Q$ is not constant.*

Proof. Let B_1 and B_2 be the two connected components of ∂A. Let $B_2 <_s B_1$. If $Q \cap B_2 \neq \emptyset$, then by Lemma 4.4-1 and [22, Theorem 2.20-1,5], $T_{\infty,\tau}|_Q$ is not constant. Therefore we may assume $Q \cap B_1 \neq \emptyset$. We may also assume that Q is a disk and $Q \cap B_2 = \emptyset$. Since $Q \cap J(G_\tau) \neq \emptyset$, by [9, Corollary 3.1] there exists an element $g \in G_\tau$ such that $J(g) \cap Q \neq \emptyset$. Since $J_{\min}(G_\tau) \leq_s B_2$, $J(g) \cap J_{\min}(G_\tau) = \emptyset$. By [22, Theorem 2.20-4,5], it follows that $J(g)$ is a quasicircle and there exists an attracting fixed point $z_g \in \mathrm{int}(\hat{K}(G_\tau))$ of g. By [22, Theorem 2.20-2], $\partial \hat{K}(G_\tau) \subset J_{\min}(G_\tau) \leq_s B_2 <_s A$. Therefore $\{z_g\} <_s A$. Since $J(g) \cap A = \emptyset$, it follows that $A \subset \mathrm{int} K(g)$. Let $y_1 \in A$ be a point. From the above arguments, we obtain that there exists a number $n_1 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq n_1$, $g^n(y_1) \in \mathrm{int}(\hat{K}(G_\tau))$. Moreover, since $J(g) \cap Q \neq \emptyset$, there exists a point $y_2 \in Q$ and a number $n_2 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq n_2$, $g^n(y_2) \in F_\infty(G_\tau)$. Let $m := \max\{n_1, n_2\}$. Let $\alpha_1, \ldots, \alpha_p \in \Gamma_\tau$ be some elements such that $g^m = \alpha_p \circ \cdots \circ \alpha_1$. Let W be a neighborhood of $(\alpha_1, \ldots \alpha_p)$ in Γ_τ^p such that for each $\omega = (\omega_1, \ldots, \omega_p) \in W$, $\omega_p \cdots \omega_1(y_1) \in \mathrm{int}(\hat{K}(G_\tau))$ and $\omega_p \cdots \omega_1(y_2) \in F_\infty(G_\tau)$. Therefore, for each $\gamma \in X_\tau$ with $(\gamma_1, \ldots, \gamma_p) \in W$, $\{\gamma_{r,1}(y_1)\}_{r \in \mathbb{N}}$ is bounded and $\gamma_{r,1}(y_2) \to \infty$ as $n \to \infty$. Combining this with Lemma 4.7-1, we see that $T_{\infty,\tau}(y_1) < T_{\infty,\tau}(y_1) + \tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_p) \in W\}) \leq T_{\infty,\tau}(y_2)$. Therefore, $T_{\infty,\tau}|_Q$ is not constant. Thus, we have proved our lemma. ☐

Lemma 4.12. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $\mathrm{int}(\hat{K}(G_\tau)) \neq \emptyset$. Then we have the following.*

1. $\mathrm{int}(T_{\infty,\tau}^{-1}(\{1\})) \subset F(G_\tau)$.
2. *If, in addition to the assumption of our lemma, $\infty \in F(G_\tau)$, then for each open subset Q of $\hat{\mathbb{C}}$ with $Q \cap \partial F_\infty(G_\tau) \neq \emptyset$, $T_{\infty,\tau}|_Q$ is not constant.*

Proof. We first prove statement 1. We prove the following claim.
Claim. For each $z_0 \in T_{\infty,\tau}^{-1}(\{1\})$, there exists no $g \in G_\tau$ with $g(z_0) \in \mathrm{int}(\hat{K}(G_\tau))$.

To prove this claim, let $z_0 \in T_{\infty,\tau}^{-1}(\{1\})$ and suppose there exists an element $g \in G_\tau$ with $g(z_0) \in \mathrm{int}(\hat{K}(G_\tau))$. Let $h_1, \ldots, h_m \in \Gamma_\tau$ be some elements with $g = h_m \circ \cdots \circ h_1$. Then there exists a neighborhood W of $(h_1, \ldots, h_m)$ in Γ_τ^m such that for each $\omega = (\omega_1, \ldots, \omega_m) \in W$, $\omega_m \cdots \omega_1(z_0) \in \mathrm{int}(\hat{K}(G_\tau))$. Therefore for each $\gamma \in X_\tau$ with $(\gamma_1, \ldots, \gamma_m) \in W$, $\{\gamma_{n,1}(z_0)\}_{n \in \mathbb{N}}$ is bounded. Thus $T_{\infty,\tau}(z_0) \leq 1 - \tilde{\tau}(\{\gamma \in X_\tau \mid (\gamma_1, \ldots, \gamma_m) \in W\}) < 1$. This is a contradiction. Hence we have proved the claim.

From this claim, $G_\tau(\mathrm{int}(T_{\infty,\tau}^{-1}(\{1\}))) \subset \hat{\mathbb{C}} \setminus \mathrm{int}(\hat{K}(G_\tau))$. Therefore $\mathrm{int}(T_{\infty,\tau}^{-1}(\{1\})) \subset F(G_\tau)$. Thus we have proved statement 1.

We now prove statement 2. Suppose $\infty \in F(G_\tau)$. Let Q be an open subset of $\hat{\mathbb{C}}$ with $Q \cap \partial F_\infty(G_\tau) \neq \emptyset$. By [25, Lemma 5.24], $T_{\infty,\tau}|_{F_\infty(G_\tau)} \equiv 1$. Combining this with statement 1, we obtain that $T_{\infty,\tau}|_Q$ is not constant. Thus we have proved statement 2. ☐

Lemma 4.13. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$. Suppose $\infty \in F(G_\tau)$. Then $\mathrm{int}(T_{\infty,\tau}^{-1}(\{0\}) \subset F(G_\tau)$, and for each open subset Q of $\hat{\mathbb{C}}$ with $Q \cap \partial \hat{K}(G_\tau) \neq \emptyset$, $T_{\infty,\tau}|_Q$ is not constant.*

Proof. We can prove this lemma in the same way as that in the proof of Lemma 4.12. ☐

Theorem 4.14. *Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ (we do not assume that $\mathrm{supp}\,\tau$ is compact). Suppose $G_\tau \in \mathcal{G}_{dis}$. Then statements 2, 3, 4 and 5 in Theorem 2.4 hold.*

Proof. By [22, Theorem 2.20-1,5], $\infty \in F(G_\tau)$ and $\mathrm{int}(\hat{K}(G_\tau)) \neq \emptyset$. By [25, Lemma 5.27], statement 2 holds. Statement 3 follows from Lemmas 4.7 and 4.8. Statement 4 follows from Corollary 4.10 and Lemma 4.7-2. Statement 5 follows from Lemmas 4.11, 4.12 and 4.13. ☐

We now prove Theorem 2.4.
Proof of Theorem 2.4: Theorem 2.4 follows from Lemma 4.2 and Theorem 4.14. ☐

4.2 Proof of Theorem 2.11

In this subsection, we prove Theorem 2.11. We need several lemmas.

Lemma 4.15. *Under the assumptions of Theorem 2.11, statement 1 in Theorem 2.11 holds.*

Proof. Since $J(G)=\cup_{j=1}^m h_j^{-1}(J(G))$ ([16, Lemma 2.4]), we obtain that $J(G)$ is disconnected. Thus $G\in\mathcal{G}_{dis}$. By Theorem 2.4, for the τ, all statements in Theorem 2.4 hold. The rest of statement 1 follows from [25, Lemma 3.75] and [16, Theorem 4.3, Lemma 5.1]. $\square$

Lemma 4.16. *Under the assumptions of Theorem 2.11, we obtain that* (1) $u(h,p,\mu)$ *is well-defined and* $u(h,p,\mu)=\frac{-\sum_{j=1}^m p_j\log p_j}{\sum_{j=1}^m p_j\log(\deg(h_j))}$, (2) *for* λ*-a.e.* $z_0\in J(G)$, $\text{Höl}(T_{\infty,\tau},z_0)\le u(h,p,\mu)$, *and* (3) $\pi_{\hat{\mathbb{C}}}:\tilde{J}(f)\to J(G)$ *is a homeomorphism.*

Proof. Since $\pi_*(\mu)=\tilde{\tau}$, $\int\log\tilde{p}\ d\mu=\sum_{j=1}^m p_j\log p_j$. By [25, Lemma 5.52] and that $G\in\mathcal{G}$, we obtain that $u(h,p,\mu)$ is well-defined and $u(h,p,\mu)=\frac{-\sum_{j=1}p_j\log p_j}{\sum_{j=1}^m p_j\log\deg(h_j)}$.

Since $h_i^{-1}(J(G))\cap h_j^{-1}(J(G))=\emptyset$ for each (i,j) with $i\neq j$, we may assume that $J(h_1)<_s\cdots<_s J(h_m)$. Then, by [22, Proposition 2.24], $J(h_1)\subset J_{\min}(G)$. Since $h_i^{-1}(J(G))\cap h_j^{-1}(J(G))=\emptyset$ for each (i,j) with $i\neq j$, since $J(G)=\bigcup_{j=1}^m h_j^{-1}(J(G))$ ([16, Lemma 2.4]), and since $J(h_j)\subset h_j^{-1}(J(G))$, it follows that for each $j\ge 2$, $J(h_j)\cap J_{\min}(G)=\emptyset$. Hence, by [22, Theorem 2.20-2,5], $h_j^{-1}(J(G))\cap P(G)=\emptyset$ for each $j\ge 2$. Let $A:=\{(\gamma,y)\in J(G)\mid \exists n\in\mathbb{N}\text{ s.t. }\sigma^n(\gamma)=(1,1,1,\ldots)\}$. Since $\pi_*(\mu)=\tilde{\tau}$, and since $\tilde{\tau}(\{(1,1,1,\ldots)\})=0$, it follows that $\mu(A)=0$.

Since $\pi_{\hat{\mathbb{C}}}:\tilde{J}(f)\to J(G)$ is surjective ([25, Lemma 4.5]), and since $h_i^{-1}(J(G))\cap h_j^{-1}(J(G))=\emptyset$ for each (i,j) with $i\neq j$, we obtain that $\pi_{\hat{\mathbb{C}}}:\tilde{J}(f)\to J(G)$ is a homeomorphism. Thus $\lambda(\pi_{\hat{\mathbb{C}}}(A))=0$. Let $\{t_n\}_{n=1}^\infty$ be a decreasing sequence of real numbers such that $t_n>u(h,p,\mu)$ for each $n\in\mathbb{N}$ and such that $t_n\to u(h,p,\mu)$ as $n\to\infty$. By Birkhoff's ergodic theorem and [25, Lemma 5.52], for each $n\in\mathbb{N}$ there exists a Borel subset B_n of $\tilde{J}(f)$ with $\mu(B_n)=1$ such that for each $(\gamma,y)\in B_n$, $\frac{1}{r}\log(\tilde{p}(f^r(\gamma,y))\cdots\tilde{p}(\gamma,y)\|Df^r_{(\gamma,y)}\|_s^{t_n})\to\int_{\Gamma^{\mathbb{N}}\times\hat{\mathbb{C}}}\log\tilde{p}(\gamma,y)d\mu(\gamma,y)+\int_{\Gamma^{\mathbb{N}}\times\hat{\mathbb{C}}}\log\|Df_{(\gamma,y)}\|_s^{t_n}d\mu(\gamma,y)>0$ as $r\to\infty$. Thus for each $(\gamma,y)\in B_n$,

$$\tilde{p}(f^r(\gamma,y))\cdots\tilde{p}(\gamma,y)\|D(\gamma_{r,1})_y\|_s^{t_n}\to\infty\text{ as }r\to\infty. \tag{2}$$

Let $C:=(J(G)\setminus\pi_{\hat{\mathbb{C}}}(A))\cap\bigcap_{n=1}^\infty\pi_{\hat{\mathbb{C}}}(B_n)$. Then $\lambda(C)=1$. Let $z_0\in C$. Let $\gamma\in\Gamma^{\mathbb{N}}$ be the unique element $(\gamma,z_0)\in\tilde{J}(f)$. Since $z_0\in J(G)\setminus\pi_{\hat{\mathbb{C}}}(A)$, there exists a $j\in\{2,\ldots,m\}$ and a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers such that $\gamma_{n_k+1}=j$ for each $k\in\mathbb{N}$. Then for each $k\in\mathbb{N}$, $\gamma_{n_k,1}(z_0)\in\gamma_{n_k+1}^{-1}(J(G))=h_j^{-1}(J(G))$. We may assume that there exists a point $z_1\in h_j^{-1}(J(G))\subset\hat{\mathbb{C}}\setminus P(G)$ such that $\gamma_{n_k,1}(z_0)\to z_1$ as $k\to\infty$. By (2) and [25, Lemma 5.48-1], we obtain that for each $n\in\mathbb{N}$, $\limsup_{z\to z_0,z\neq z_0}\frac{|T_{\infty,\tau}(z)-T_{\infty,\tau}(z_0)|}{d(z,z_0)^{t_n}}=\infty$. Therefore $\text{Höl}(T_{\infty,\tau},z_0)\le u(h,p,\mu)$. Thus we have proved our lemma. $\square$

Definition 4.17 ([22])**.** For a polynomial g, we denote by $a(g)\in\mathbb{C}$ the coefficient of the highest degree term of g. We set $\text{RA}:=\{ax+b\in\mathbb{R}[x]\mid a,b\in\mathbb{R},a\neq 0\}$. The space RA is a semigroup with the semigroup operation being functional composition. Any subsemigroup of RA will be called a *real affine semigroup.* We define a map $\Psi:\mathcal{P}\to\text{RA}$ as follows: For a polynomial $g\in\mathcal{P}$, we set $\Psi(g)(x):=\deg(g)x+\log|a(g)|$. We remark that $\Psi(g\circ h)=\Psi(g)\circ\Psi(h)$. For a polynomial semigroup G, we set $\Psi(G):=\{\Psi(g)\mid g\in G\}$ $(\subset\text{RA})$. Thus $\Psi(G)$ is a real affine semigroup. We set $\hat{\mathbb{R}}:=\mathbb{R}\cup\{\pm\infty\}$ endowed with the topology such that $\{(r,+\infty]\}_{r\in\mathbb{R}}$ makes a fundamental neighborhood system of $+\infty$, and such that $\{[-\infty,r)\}_{r\in\mathbb{R}}$ makes a fundamental neighborhood system of $-\infty$. For a real affine semigroup H, we set
$M(H):=\overline{\{x\in\mathbb{R}\mid\exists h\in H,h(x)=x,|h'(x)|>1\}}$ $(\subset\hat{\mathbb{R}})$, where the closure is taken in the space $\hat{\mathbb{R}}$.

We denote by $\eta : \mathrm{RA} \to \mathcal{P}$ the natural embedding defined by $\eta(x \mapsto ax+b) = (z \mapsto az+b)$, where $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

Lemma 4.18. *Under the assumptions of Theorem 2.11, we get that* (1) $M(\Psi(G))$ *is a Cantor set in* $\mathbb{R}$, (2) $M(\Psi(G)) = \bigcup_{j=1}^m (\Psi(h_j))^{-1}(M(\Psi(G)))$, (3) $(\Psi(h_i))^{-1}(M(\Psi(G))) \cap (\Psi(h_j))^{-1}(M(\Psi(G))) = \emptyset$ *for each* (i,j) *with* $i \neq j$, *and* (4) $\sum_{j=1}^m \frac{1}{\deg(h_j)} < 1$.

Proof. We use the arguments in the proof of [22, Lemma 4.9]. For each $\gamma \in \Gamma^{\mathbb{N}}$, let $J(G)_\gamma := \bigcap_{j=1}^\infty \gamma_{j,1}^{-1}(J(G))$. Since $J(G) = \bigcup_{j=1}^m h_j^{-1}(J(G))$ ([16, Lemma 2.4]) and since $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$, we obtain that $J(G) = \amalg_{\gamma \in \Gamma^{\mathbb{N}}} J(G)_\gamma$ (disjoint union). By [20, Corollary 4.19], for each $\gamma \in \Gamma^{\mathbb{N}}$, $J(G)_\gamma$ is connected. Thus each $J(G)_\gamma$ is a connected component of $J(G)$. By [19, Proposition 2.2(3)], [22, Lemma 4.1] and that $J_\gamma \subset J(G)_\gamma$ for each $\gamma \in \Gamma^{\mathbb{N}}$, it follows that for each $\gamma \in \Gamma^{\mathbb{N}}$, $\sup_{z \in J(\gamma_{n,1})} d(z, J(G)_\gamma) \to 0$ as $n \to \infty$. By [22, Lemma 4.5], $M(\Psi(G)) = J(\eta(\Psi(G))) \subset \mathbb{R}$. Since $J(\eta(\Psi(G))) = \bigcup_{j=1}^m (\eta(\Psi(h_j)))^{-1}(J(\eta(\Psi(G))))$, by [5, Theorem 2.6] it follows that $M(\Psi(G))$ is the self-similar set constructed by contracting similitudes $(\Psi(h_1))^{-1}, \ldots, (\Psi(h_m))^{-1}$ on $\mathbb{R}$. Let $b_{\min} := \min\{\frac{-1}{\deg(h_j)-1} \log|a(h_j)| \mid j = 1, \ldots, m\}$ and $b_{\max} := \max\{\frac{-1}{\deg(h_j)-1} \log|a(h_j)| \mid j = 1, \ldots, m\}$. Note that $\frac{-1}{\deg(g)-1} \log|a(g)|$ is the unique fixed point of $\Psi(g)$ in $\mathbb{R}$. Let $I = [b_{\min}, b_{\max}]$ be the closed interval between $b_{\min}$ and $b_{\max}$. Then we have that $\bigcup_{j=1}^m (\Psi(h_j))^{-1}(I) \subset I$. It follows that $M(\Psi(G)) = \bigcup_{\gamma \in \Gamma^{\mathbb{N}}} \bigcap_{n=1}^\infty (\Psi(\gamma_{n,1}))^{-1}(I)$. Let $\rho : \Gamma^{\mathbb{N}} \to M(\Psi(G))$ be the map defined by $\rho(\gamma) := \bigcap_{n=1}^\infty (\Psi(\gamma_{n,1}))^{-1}(I)$ for each γ. Then $\rho : \Gamma^{\mathbb{N}} \to M(\Psi(G))$ is continuous. For each $\gamma \in \Gamma^{\mathbb{N}}$ and each $n \in \mathbb{N}$, $\frac{-1}{\deg(\gamma_{n,1})} \log|a(\gamma_{n,1})|$ is the fixed point of $\Psi(\gamma_{n,1})$ in I. Therefore $\frac{-1}{\deg(\gamma_{n,1})-1} \log|a(\gamma_{n,1})| = \rho(\omega^{\gamma,n})$, where $\omega^{\gamma,n} \in \Gamma^{\mathbb{N}}$ is the n-periodic point of $\sigma : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ with $((\omega^{\gamma,n})_1, \ldots, (\omega^{\gamma,n})_n) = (\gamma_1, \ldots, \gamma_n)$. Since $\omega^{\gamma,n} \to \gamma$ in $\Gamma^{\mathbb{N}}$ as $n \to \infty$, it follows that for each $\gamma \in \Gamma^{\mathbb{N}}$, $\lim_{n\to\infty} \frac{-1}{\deg(\gamma_{n,1})-1} \log|a(\gamma_{n,1})| = \rho(\gamma)$. For each $\gamma \in \Gamma^{\mathbb{N}}$, let $B_\gamma \in \mathrm{Con}(M(\Psi(G)))$ with $\lim_{n\to\infty} \frac{-1}{\deg(\gamma_{n,1})-1} \log|a(\gamma_{n,1})| \in B_\gamma$. Let $\tilde{\Psi} : \mathrm{Con}(J(G)) \to \mathrm{Con}(M(\Psi(G)))$ be the map defined by $\tilde{\Psi}(J(G)_\gamma) := B_\gamma$ for each $\gamma \in \Gamma^{\mathbb{N}}$. By [22, Claim 2 in the proof of Lemma 4.9], $\tilde{\Psi} : \mathrm{Con}(J(G)) \to \mathrm{Con}(M(\Psi(G)))$ is injective. Therefore, it follows that $\rho : \Gamma^{\mathbb{N}} \to M(\Psi(G))$ is injective. Thus, $\rho : \Gamma^{\mathbb{N}} \to M(\Psi(G))$ is a homeomorphism. In particular, $M(\Psi(G))$ is a Cantor set in I. Let $0 < \epsilon < \min\{|a-b| \mid a \in (\Psi(h_i))^{-1}(M(\Psi(G))), b \in (\Psi(h_j))^{-1}(M(\Psi(G))), i \neq j\}$ and let U be the ϵ-neighborhood of $M(\Psi(G))$ in $\mathbb{R}$. (Thus U is a finite union of bounded open intervals.) Since ρ is a homeomorphism, $(\Psi(h_i))^{-1}(M(\Psi(G))) \cap (\Psi(h_j))^{-1}(M(\Psi(G)) = \emptyset$ for each (i,j) with $i \neq j$. Hence $\bigcup_{j=1}^m (\Psi(h_j))^{-1}(\overline{U}) \subset U$ and $(\Psi(h_i))^{-1}(\overline{U}) \cap (\Psi(h_j))^{-1}(\overline{U}) = \emptyset$ for each (i,j) with $i \neq j$. Thus denoting by l the one-dimensional Lebesgue measure, $\sum_{j=1}^m \frac{1}{\deg(h_j)} l(U) = \sum_{j=1}^m l((\Psi(h_j))^{-1}(U)) < l(U)$. Hence $\sum_{j=1}^m \frac{1}{\deg(h_j)} < 1$. Thus we have proved our lemma. □

We now prove Theorem 2.11.

Proof of Theorem 2.11: Statement 1 follows from Lemma 4.15. By Lemma 4.16, we have $u(h,p,\mu) = \frac{-\sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log(\deg(h_j))}$. It is easy to see that $\min\{\sum_{j=1}^m p_j \log\deg(h_j) + \sum_{j=1}^m p_j \log p_j \mid (p_1, \ldots, p_m) \in \mathcal{W}_m\} = -\log(\sum_{j=1}^m \frac{1}{\deg(h_j)})$. Combining these arguments with Lemmas 4.18 and 4.16, we see that statement 2 follows.

By [16, Theorem 1.3 (f)], $h_\mu(f|\sigma) = \sum_{j=1}^m p_j \log\deg(h_j)$. Hence, $h_\mu(f) = h_\mu(f|\sigma) + h_{\pi_*(\mu)}(\sigma) = \sum_{j=1}^m p_j \log\deg(h_j) - \sum_{j=1}^m p_j \log p_j$, where $h_\mu(f)$ denotes the metric entropy of (f,μ). Combining this with [16, Lemma 7.1], [25, Lemma 5.52], that $\pi_{\hat{\mathbb{C}}} : \tilde{J}(f) \to J(G)$ is a homeomorphism (Lemma 4.16), and that $G \in \mathcal{G}$, we see that $\dim_H(\lambda) = \frac{\sum_{j=1}^m p_j \log\deg(h_j) - \sum_{j=1}^m p_j \log p_j}{\sum_{j=1}^m p_j \log\deg(h_j)} > 1$, where $\dim_H(\lambda) := \inf\{\dim_H(A) \mid A \text{ is a Borel subset of } J(G), \lambda(A) = 1\}$. Hence, we have proved statement 3. Statement 4 follows from statements 1 and 2. Thus we have proved Theorem 2.11. □

4.3 Proof of Theorem 2.12

In this subsection, we prove Theorem 2.12. We need several lemmas and propositions.

Definition 4.19. Let Γ be a non-empty compact subset of $\mathcal{P}$ and suppose $\langle \Gamma \rangle \in \mathcal{G}_{dis}$. We set $\Gamma_{\min} := \{h \in \Gamma \mid J(h) \subset J_{\min}(\langle \Gamma \rangle)\}$. Note that by [22, Proposition 2.24], $\Gamma_{\min} \neq \emptyset$.

Lemma 4.20. *Let $m \in \mathbb{N}$ with $m \geq 2$. Let $\Gamma = \{h_1, \ldots, h_m\} \subset \mathcal{P}$. Let $G = \langle \Gamma \rangle$ and suppose that $G \in \mathcal{G}_{dis}$. Suppose that $\sharp \Gamma_{\min} = 1$. Then, we have the following* (1) *and* (2)*.* (1) *For each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma = \hat{J}_{\gamma,\Gamma} = \bigcap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$.* (2) *The map $\gamma \mapsto J_\gamma$ is continuous on $\Gamma^{\mathbb{N}}$ with respect to the Hausdorff metric in the space of non-empty compact subsets of $\hat{\mathbb{C}}$.*

Proof. We may assume that $\Gamma_{\min} = \{h_1\}$. By [22, Theorem 2.20-5], $\emptyset \neq \text{int}(\hat{K}(G)) \subset \text{int}(K(h_1))$. By [22, Theorem 2.20-5] again, for each $j \geq 2$, $h_j(J(h_1)) \subset h_j(J_{\min}(G)) \subset \text{int}(\hat{K}(G)) \subset \text{int}(K(h_1))$. Therefore for each $j \geq 2$, $h_j(\text{int}(K(h_1))) \subset \text{int}(K(h_1))$. Thus $\text{int}(K(h_1)) \subset F(G)$. Let $\gamma \in \Gamma^{\mathbb{N}}$. Suppose that there exists a point $y_0 \in \hat{J}_{\gamma,\Gamma} \setminus J_\gamma$. We now consider the following two cases. Case 1: $\sharp\{n \in \mathbb{N} \mid \gamma_n \neq h_1\} = \infty$. Case 2: $\sharp\{n \in \mathbb{N} \mid \gamma_n \neq h_1\} < \infty$.

Suppose that we have Case 1. Then there exist an open neighborhood U of y_0 in $\hat{\mathbb{C}}$, a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers, a number $i \in \{2, \ldots, m\}$, and a map $\varphi : U \to \hat{\mathbb{C}}$, such that $\gamma_{n_j+1} = h_i$ for each $j \in \mathbb{N}$, and such that $\gamma_{n_j,1} \to \varphi$ uniformly on U as $j \to \infty$. Since $\gamma_{n_j,1}(y_0) \in J(G)$ for each j, [23, Lemma 5.6] and [22, Proposition 2.19] imply that φ is constant. By [24, Lemma 3.13], it follows that $d(\gamma_{n_j,1}(y_0), P^*(G)) \to 0$ as $j \to \infty$. Moreover, since $\gamma_{n_j+1} = h_i$, we obtain $\gamma_{n_j,1}(y_0) \in h_i^{-1}(J(G))$ for each j. Furthermore, by [22, Theorem 2.20-2,5], $h_i^{-1}(J(G)) \subset \hat{\mathbb{C}} \setminus P^*(G)$. This is a contradiction. Hence, we cannot have Case 1.

Suppose we have Case 2. Let $r \in \mathbb{N}$ be a number such that for each $s \in \mathbb{N}$ with $s \geq r$, $\gamma_s = h_1$. Then $h_1^n(\gamma_{r,1}(y_0)) \in J(G)$ for each $n \geq 0$. Since $y_0 \notin J_\gamma$, we have $\gamma_{r,1}(y_0) \notin J(h_1)$. Moreover, since $\gamma_{r,1}(y_0) \in J(G)$ and $\text{int}(\hat{K}(h_1)) \subset F(G)$, it follows that $\gamma_{r,1}(y_0)$ belongs to $F_\infty(h_1)$. It implies that $h_1^n(\gamma_{r,1}(y_0)) \to \infty$ as $n \to \infty$. However, this contradicts that $h_1^n(\gamma_{r,1}(y_0)) \in J(G)$ for each $n \geq 0$. Therefore, we cannot have Case 2.

Thus, for each $\gamma \in \Gamma$, $J_\gamma = \hat{J}_{\gamma,\Gamma}$. Moreover, by [24, Lemma 3.5], $\hat{J}_{\gamma,\Gamma} = \bigcap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$ for each $\gamma \in \Gamma^{\mathbb{N}}$. Combining the result "$J_\gamma = \hat{J}_{\gamma,\Gamma}$ for each $\gamma \in \Gamma^{\mathbb{N}}$" with [19, Proposition 2.2(3)], we obtain that the map $\gamma \mapsto J_\gamma$ is continuous. □

Proposition 4.21. *Let $m \geq 2$ and let $G = \langle h_1, \ldots, h_m \rangle \in \mathcal{G}$. Let $(p_1, \ldots, p_m) \in \mathcal{W}_m$ and let $\tau = \sum_{j=1}^{m} p_j \delta_{h_j}$. Let $\Gamma = \{h_1, \ldots, h_m\}$. Suppose that $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$. Then we have the following.*

1. *$G \in \mathcal{G}_{dis}$ and $\sharp\Gamma_{\min} = 1$. For each $\gamma \in \Gamma^{\mathbb{N}}$, $J_\gamma = \hat{J}_{\gamma,\Gamma} = \bigcap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_j^{-1}(J(G))$. The map $\gamma \mapsto J_\gamma$ is continuous on $\Gamma^{\mathbb{N}}$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $\hat{\mathbb{C}}$.*

2. *For each $J \in \text{Con}(J(G))$, there exists a unique $\gamma \in \Gamma^{\mathbb{N}}$ with $J = J_\gamma$. $\text{Con}(J(G)) = \{J_\gamma \mid \gamma \in \Gamma^{\mathbb{N}}\}$. The map $\gamma \mapsto J_\gamma$ is a bijection between $\Gamma^{\mathbb{N}}$ and $\text{Con}(J(G))$. In particular, there exist uncountably many connected components of $J(G)$.*

3. *There exist infinitely many doubly connected components of $F(G)$.*

4. *For each $J \in \text{Con}(J(G))$, $T_{\infty,\tau}|_J$ is constant.*

5. *Let $J_1, J_2 \in \text{Con}(J(G))$ with $J_1 \neq J_2$. Suppose $T_{\infty,\tau}|_{J_1} = T_{\infty,\tau}|_{J_2}$. Then there exists a doubly connected component A of $F(G)$ such that $\partial A \subset J_1 \cup J_2$.*

Proof. Since $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ ([16, Lemma 2.4]), $G \in \mathcal{G}_{dis}$. By [22, Proposition 2.24], $\Gamma_{\min} \neq \emptyset$. Without loss of generality, we may assume that $h_1 \in \Gamma_{\min}$. Since $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ again, for each $j \geq 2$, there exists no $J \in \mathrm{Con}(J(G))$ with $J(h_1) \cup J(h_j) \subset J$. Therefore, $\Gamma_{\min} = \{h_1\}$. By Lemma 4.20, it follows that $J_\gamma = \hat{J}_{\gamma,\Gamma} = \bigcap_{j=1}^{\infty} \gamma_1^{-1} \cdots \gamma_n^{-1}(J(G))$ for each $\gamma \in \Gamma^{\mathbb{N}}$, and that the map $\gamma \mapsto J_\gamma$ is continuous. Since $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ and since $h_i^{-1}(J(G)) \cap h_j^{-1}(J(G)) = \emptyset$ for each (i,j) with $i \neq j$, we obtain that $J(G) = \amalg_{\gamma \in \Gamma^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \gamma_1^{-1} \cdots \gamma_n^{-1}(J(G))$. Moreover, by [24, Lemma 3.6], J_γ is connected for each $\gamma \in \Gamma^{\mathbb{N}}$. Therefore J_γ is a connected component of $J(G)$ for each $\gamma \in \Gamma^{\mathbb{N}}$. Moreover, the map $\gamma \in \Gamma^{\mathbb{N}} \mapsto J_\gamma \in \mathrm{Con}(J(G))$ is a bijection. In particular, there exist uncountably many connected components of $J(G)$. Combining this with [22, Theorem 2.7-1, Lemma 4.4], we obtain that there are infinitely many doubly connected components of $F(G)$.

Let $J \in \mathrm{Con}(J(G))$. Then there exists a unique element $\alpha \in \Gamma^{\mathbb{N}}$ such that $J = J_\alpha$. Let $z_0 \in J$ be a point. Let $\gamma \in \Gamma^{\mathbb{N}}$ be an element. Suppose $\gamma_{n,1}(z_0) \to \infty$. Then $\gamma \neq \alpha$. By the uniqueness of α, we obtain $J_\gamma \neq J_\alpha$. By [22, Theorem 2.7] and that $\gamma_{n,1}(z_0) \to \infty$, it follows that $J_\gamma <_s J = J_\alpha$. Therefore, for each $z \in J$, $\gamma_{n,1}(z) \to \infty$. Thus, $T_{\infty,\tau}|_J$ is constant.

We now let $J_1, J_2 \in \mathrm{Con}(J(G))$ with $J_1 \neq J_2$ and suppose $T_{\infty,\tau}|_{J_1} = T_{\infty,\tau}|_{J_2}$. Without loss of generality, we may assume $J_1 <_s J_2$. By [22, Lemma 4.4], there exists a doubly connected component A of $F(G)$ such that $J_1 <_s A <_s J_2$. Let B_1 and B_2 be two connected components of ∂A with $B_1 <_s B_2$. For each $i = 1, 2$, let $J_i' \in \mathrm{Con}(J(G))$ with $B_i \subset J_i'$. Then $J_1 \leq_s J_1' <_s A <_s J_2' \leq_s J_2$. Suppose $J_1 <_s J_1'$. Then by [22, Lemma 4.4], there exists a doubly connected component D_1 of $F(G)$ such that $J_1 <_s D_1 <_s J_1'$. Therefore $J_1 <_s D_1 <_s A <_s J_2$. By Lemma 4.3, Theorem 2.4-3 and Lemma 4.7-1, it follows that $T_{\infty,\tau}|_{J_1} \leq T_{\infty,\tau}|_{D_1} < T_{\infty,\tau}|_A \leq T_{\infty,\tau}|_{J_2}$. However, this contradicts that $T_{\infty,\tau}|_{J_1} = T_{\infty,\tau}|_{J_2}$. Therefore, $J_1 = J_1'$. Similarly, we obtain $J_2 = J_2'$. Therefore, $\partial A \subset J_1 \cup J_2$.

Thus we have proved our proposition. □

We now prove Theorem 2.12.

Proof of 2.12: Let $\Gamma := \{h_1, h_2\}$. By [20, Theorems 3.17, 3.2], $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$. Thus all statements 1–5 in Theorem 2.12 follow from Proposition 4.21 and Theorem 2.11.

We now prove statement 6. By statement 2 and [22, Theorem 2.7], either $J(h_1) <_s J(h_2)$ or $J(h_2) <_s J(h_1)$. We now assume $J(h_1) <_s J(h_2)$. Then, by [22, Proposition 2.24], $J(h_1) \subset J_{\min}(G)$ and $J(h_2) \subset J_{\max}(G)$. By statement 2, it follows that $J(h_1) = J_{\min}(G)$ and $J(h_2) = J_{\max}(G)$. Let $A = K(h_2) \setminus \mathrm{int}(K(h_1))$. We now prove the following claim.

Claim 1. $h_1^{-1}(A) \cup h_2^{-1}(A) \subset A$.

To prove this claim, let $\alpha = (h_2, h_1, h_1, \ldots) \in \Gamma^{\mathbb{N}}$. Then $J_\alpha = h_2^{-1}(J(h_1))$. Since $J(h_1) = J_{\min}(G)$, statement 2 implies that $J(h_1) <_s J_\alpha = h_2^{-1}(J(h_1))$. Therefore $h_2^{-1}(A) \subset A$. Similarly, letting $\beta = (h_1, h_2, h_2, \ldots) \in \Gamma^{\mathbb{N}}$, we have $J_\beta = h_1^{-1}(J(h_2)) <_s J(h_2)$ and $h_1^{-1}(A) \subset A$. Thus we have proved Claim 1.

We have that $h_1^{-1}(A)$ and $h_2^{-1}(A)$ are connected compact sets. We prove the following claim.

Claim 2. $J_\beta = h_1^{-1}(J(h_2)) <_s J_\alpha = h_2^{-1}(J(h_1))$. In particular, $h_1^{-1}(A) <_s h_2^{-1}(A)$.

To prove this claim, suppose that $J_\beta <_s J_\alpha$ does not hold. Then by [22, Theorem 2.7], $J_\alpha <_s J_\beta$. This implies that $A = h_1^{-1}(A) \cup h_2^{-1}(A)$. By [9, Corollary 3.2], we have $J(G) \subset A$. Since $J(G)$ is disconnected (assumption) and since A is connected, $F(G) \cap A \neq \emptyset$. Let $y \in F(G) \cap A$. Since $A = h_1^{-1}(A) \cup h_2^{-1}(A)$, there exists an element $\gamma \in \Gamma^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\gamma_{n,1}(y) \in A$. Since $y \in A \cap F(G)$ and $G(F(G)) \subset F(G)$, $\gamma_{n,1}(y) \in F_\infty(h_1) \cap A$ for each $n \in \mathbb{N}$. Therefore there exists a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ in $\mathbb{N}$ such that for each j, $\gamma_{n_j+1} = h_2$. Since $y \in F_\gamma$, we may assume that there exist an open neighborhood U of y in $\hat{\mathbb{C}}$ and a holomorphic map $\varphi : U \to \hat{\mathbb{C}}$ such that $\gamma_{n_j,1} \to \varphi$ uniformly on U as $j \to \infty$. Since $\gamma_{n_j,1}(y) \in F_\infty(h_1) \cap A \subset (\hat{\mathbb{C}} \setminus \hat{K}(G)) \cap A$ for each j, [23, Lemma 5.6] implies that there exists a constant $c \in \mathbb{C}$ such that $\varphi = c$ on U. By [24, Lemma 3.13], it follows that $c \in P^*(G)$. Since $P^*(G) \subset K(h_1)$ and since $\gamma_{n_j,1}(y) \in F_\infty(h_1)$ for each j, it follows that $d(\gamma_{n_j,1}(y), J(h_1)) \to 0$ as $j \to \infty$. Combining this with that $\gamma_{n_j+1} = h_2$ for each j, we obtain that $d(\gamma_{n_j,1}(y), h_2^{-1}(J(h_1))) \to \infty$. Since $J(h_1) <_s h_2^{-1}(J(h_1))$, it follows that $c \in F_\infty(h_1)$.

However, this is a contradiction, since $c \in P^*(G) \subset K(h_1)$. Therefore, $J_\beta <_s J_\alpha$. Thus we have proved Claim 2.

Let $\theta = (h_2, \theta_2, \theta_3, \ldots) \in \Gamma^{\mathbb{N}}$ and $\xi = (h_1, \xi_2, \xi_3, \ldots) \in \Gamma^{\mathbb{N}}$. Then $J_\theta \subset h_2^{-1}(J(G)) \subset h_2^{-1}(A)$ and $J_\xi \subset h_1^{-1}(J(G)) \subset h_1^{-1}(A)$. By claim 2, statement 2 and [22, Theorem 2.7], we obtain that $J_\xi <_s J_\theta$. Combining this result with statement 2 and [22, Theorem 2.7-3], we see that the map $\zeta : \{1,2\}^{\mathbb{N}} \to \mathrm{Con}(J(G))$ satisfies that if $w^1, w^2 \in \{1,2\}^{\mathbb{N}}$ with $w^1 <_l w^2$, then $\zeta(w^1) <_s \zeta(w^2)$. Moreover, by statement 2, this map $\zeta : \{1,2\}^{\mathbb{N}} \to \mathrm{Con}(J(G))$ is a bijection. Thus we have proved statement 6.

We now prove statement 7. Suppose $J(h_1) <_s J(h_2)$. Then $J_{\min}(G) = J(h_1)$ and $J_{\max}(G) = J(h_2)$. By [22, Theorem 2.20-5], we obtain $h_2(J(h_1)) \subset K(h_1)$. Therefore $\hat{K}(G) = K(h_1)$. Thus $K(h_1) \subset T_{\infty,\tau}^{-1}(\{0\})$. Moreover, for any $y \in F_\infty(h_2)$, there exists an element $g \in G$ with $g(y) \in F_\infty(G)$. Therefore $T_{\infty,\tau}(y) > 0$. It follows that $T_{\infty,\tau}^{-1}(\{0\}) = K(h_1)$. Since $J_{\max}(G) = J(h_2)$, $F_\infty(G) = F_\infty(h_2)$. Since $T_{\infty,\tau} : \hat{\mathbb{C}} \to [0,1]$ is continuous (see Theorem 2.4-1), $\overline{F_\infty(h_2)} \subset T_{\infty,\tau}^{-1}(\{1\})$. By [22, Theorem 2.20-5], $\mathrm{int}(K(h_2))$ is connected, $\mathrm{int}(K(h_2))$ is the immediate basin of an attracting fixed point a of h_2, and $a \in \mathrm{int}(\hat{K}(G))$. Therefore, for any $z \in \mathrm{int}(K(h_2))$, there exists an element $h \in G$ such that $h(z) \in \hat{K}(G)$. Thus $T_{\infty,\tau}(z) < 1$ for any $z \in \mathrm{int}(K(h_2))$. Hence, $T_{\infty,\tau}^{-1}(\{1\}) = \overline{F_\infty(h_2)}$. We now let $w = (w_1, w_2, \ldots) \in \{1,2\}^{\mathbb{N}}$. We first consider the case

$$\sharp\{n \in \mathbb{N} \mid w_n = 1\} = \sharp\{n \in \mathbb{N} \mid w_n = 2\} = \infty. \tag{3}$$

The following claim follows from [23, Theorem 3.11(2)].

Claim 3. There exists exactly one bounded component B_w of $F_{\gamma(w)}$. Moreover, $\partial B_w = \partial A_{\infty,\gamma(w)} = J_{\gamma(w)}$.

By (3), there exists a sequence $\{\lambda^n\}_{n=1}^\infty$ in $\{1,2\}^{\mathbb{N}}$ such that $\lambda^1 <_l \lambda^2 <_l \cdots <_l w$ and $\lambda^n \to w$ as $n \to \infty$. By statements 2, 6, it follows that $J_{\gamma(\lambda^1)} <_s J_{\gamma(\lambda^2)} <_s \cdots <_s J_{\gamma(w)}$ and $J_{\gamma(\lambda^n)} \to J_{\gamma(w)}$ as $n \to \infty$ with respect to the Hausdorff metric. Combining this with [22, Lemma 4.4], Theorem 2.4-3 and Lemmas 4.3, 4.7, we obtain that for each y in the bounded connected component of $\hat{\mathbb{C}} \setminus J_{\gamma(w)}$, $T_{\infty,\tau}(y) < T_{\infty,\tau}|_{J_{\gamma(w)}}$. Similarly, we can obtain that for each y in the unbounded connected component of $\hat{\mathbb{C}} \setminus J_{\gamma(w)}$, $T_{\infty,\tau}(y) > T_{\infty,\tau}|_{J_{\gamma(w)}}$. Therefore letting $t := T_{\infty,\tau}|_{J_{\gamma(w)}} \in (0,1)$, $T_{\infty,\tau}^{-1}(\{t\}) = J_{\gamma(w)}$.

We now consider the case

$$\sharp\{n \in \mathbb{N} \mid w_n = 1\} < \infty, w \neq (2,2,2,\ldots). \tag{4}$$

Let $r \in \mathbb{N}$ be the minimum number such that for each $n \geq r$, $w_n = 2$. Then $r \geq 2$ and $w_{r-1} = 1$. Let $\rho = w$ and let $\mu = (w_1, \ldots w_{r-2}, 2, 1, 1, 1, 1, \ldots) \in \{1,2\}^{\mathbb{N}}$ (if $r = 2$, then let $\mu = (2, 1, 1, 1, 1, \ldots)$). Then there exists no $\lambda \in \{1,2\}^{\mathbb{N}}$ with $\rho <_l \lambda <_l \mu$. By statements 4, 6 and Theorem 2.4-1, we obtain that there exists a doubly connected component A of $F(G)$ with $\partial A \subset J_{\gamma(\rho)} \cup J_{\gamma(\mu)}$, and that there exists a $t \in (0,1)$ with $T_{\infty,\tau}|_{K_{\gamma(\mu)} \setminus \mathrm{int}(K_\rho)} = t$. Moreover, since $(h_{w_{r-1}} \cdots h_{w_1})^{-1}(J(h_2)) = J_{\gamma(\rho)}$, since $J(h_2)$ is a quasicircle ([22, Theorem 2.20-4]), and since $P^*(G) \subset \mathrm{int}(K(h_2))$, we obtain that $J_{\gamma(\rho)}$ is a quasicircle. For the element ρ, there exists a sequence $\{\lambda^n\}_{n=1}^\infty$ in $\{1,2\}^{\mathbb{N}}$ such that $\lambda^1 <_l \lambda^2 <_l \cdots <_l \rho$ and $\lambda^n \to \rho$ as $n \to \infty$. By statements 2, 6, it follows that $J_{\gamma(\lambda^1)} <_s J_{\gamma(\lambda^2)} <_s \cdots$ and $J_{\gamma(\lambda^n)} \to J_{\gamma(\rho)}$ as $n \to \infty$ with respect to the Hausdorff metric. Combining this with [22, Lemma 4.4], Theorem 2.4-3 and Lemmas 4.3, 4.7, we obtain that for each y in the bounded connected component of $\hat{\mathbb{C}} \setminus J_{\gamma(\rho)}$, $T_{\infty,\tau}(y) < T_{\infty,\tau}|_{J_{\gamma(\rho)}}$. Similarly, we can obtain that for each y in the unbounded connected component of $\hat{\mathbb{C}} \setminus J_{\gamma(\mu)}$, $T_{\infty,\tau}(y) > T_{\infty,\tau}|_{J_{\gamma(\mu)}} = T_{\infty,\tau}|_{J_{\gamma(w)}}$. Therefore $T_{\infty,\tau}^{-1}(\{t\}) = K_{\gamma(\mu)} \setminus \mathrm{int}(K_{\gamma(\rho)})$. From these arguments, statement 7 follows.

Thus, we have proved Theorem 2.12. □

4.4 Proofs of Theorem 2.15 and Corollary 2.16

In this subsection, we prove Theorem 2.15 and Corollary 2.16.

Proof of Theorem 2.15: Since $G \in \mathcal{G}_{dis}$, by [20, Theorem 1.7, Theorem 1.5] there exists a number $k \in \{1, 2, 3\}$ such that

$$h_k^{-1}(J(G)) \cap h_j^{-1}(J(G))) = \emptyset \text{ for each } j \text{ with } j \neq k. \tag{5}$$

We set $J_{\min} = J_{\min}(G)$ and $J_{\max} = J_{\max}(G)$. By [22, Proposition 2.24], we have $J_{\min} = J_1$ and $J_{\max} = J_3$. We show the following claim.
Claim 1. $h_1^{-1}(J(G)) \cap h_3^{-1}(J(G)) = \emptyset$.

To prove this claim, we consider the following three cases (i),(ii),(iii). (i) $J_1 = J_2$. (ii) $J_2 = J_3$. (iii) $J_1 <_s J_2 <_s J_3$.

Suppose we have case (i). Since $J(G) = \bigcup_{j=1}^{3} h_j^{-1}(J(G))$ ([16, Lemma 2.4]), we have $J_{\min} = \bigcup_{j=1}^{3}(J_{\min} \cap h_j^{-1}(J(G)))$. Since $J(h_3) \subset J_{\max} \subset \mathbb{C} \setminus J_{\min}$, by [22, Theorem 2.20-5(b)] we obtain that $J_{\min} \cap h_3^{-1}(J(G)) = \emptyset$. Therefore $J_{\min} = \bigcup_{j=1}^{2}(J_{\min} \cap h_j^{-1}(J(G)))$. Moreover, since $J_1 = J_2 = J_{\min}$, and since $h_j^{-1}(J_{\min})$ is connected for each $j = 1, 2$ ([22, Theorem 2.7]), we have that $J_{\min} \cap h_j^{-1}(J(G)) \supset h_j^{-1}(J_{\min}) \neq \emptyset$ for each $j = 1, 2$. Since $J_{\min}$ is connected, it follows that $\bigcap_{j=1}^{2}(J_{\min} \cap h_j^{-1}(J(G))) \neq \emptyset$. In particular $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) \neq \emptyset$. By (5), it follows that $h_3^{-1}(J(G)) \cap (\bigcup_{j=1}^{2} h_j^{-1}(J(G))) = \emptyset$.

We now suppose we have case (ii). By the arguments similar to those in case (i), we obtain that $h_2^{-1}(J(G)) \cap h_3^{-1}(J(G)) \neq \emptyset$ and $h_1^{-1}(J(G)) \cap (\bigcup_{j=2,3} h_j^{-1}(J(G))) = \emptyset$.

We now suppose that we have case (iii). Then by [24, Corollary 3.7], $h_j^{-1}(J(h_1))$ is connected for each $j = 2, 3$. Moreover, since $J(h_j) \cap J_{\min} = \emptyset$ for each $j = 2, 3$ and $\sharp J_{\min} \geq 2$ ([22, Theorem 2.20-5(b)]), we obtain that $h_j^{-1}(J(h_1)) \cap J(h_1) = \emptyset$ for each $j = 2, 3$. By [24, Lemma 3.9], it follows that $J(h_1) <_s h_j^{-1}(J(h_1))$ for each $j = 2, 3$. In particular, $h_j(K(h_1)) \subset \text{int}(K(h_1))$ for each $j = 2, 3$. Therefore, $\hat{K}(G) = K(h_1)$. Similarly, we obtain that for each $i = 1, 2$, $h_i^{-1}(J(h_3))$ is connected, $h_i^{-1}(J(h_3)) <_s J(h_3)$ and $h_i(F_\infty(h_3)) \subset F_\infty(h_3)$. Therefore $F_\infty(G) = F_\infty(h_3)$. Let $A := K(h_3) \setminus \text{int}(K(h_1))$. From the above arguments, $\bigcup_{j=1}^{3} h_j^{-1}(A) \subset A$. Therefore by [9, Corollary 3.2], $J(G) \subset A$. Moreover, since $J_1 \neq J_3$, $\langle h_1, h_3 \rangle \in \mathcal{G}_{dis}$. By Claim 2 in the proof of Theorem 2.12, $h_1^{-1}(A) \cap h_3^{-1}(A) = \emptyset$. Hence, it follows that $h_1^{-1}(J(G)) \cap h_3^{-1}(J(G)) = \emptyset$.

Thus we have proved Claim 1.

By Claim 1 and (5), we obtain that exactly one of the following (I), (II), (III) holds. (I) $\{h_i^{-1}(J(G))\}_{i=1,2,3}$ are mutually disjoint. (II) $h_1^{-1}(J(G)) \cap (\bigcup_{j=2,3} h_j^{-1}(J(G))) = \emptyset$ and $h_2^{-1}(J(G)) \cap h_3^{-1}(J(G)) \neq \emptyset$. (III) $h_3^{-1}(J(G)) \cap (\bigcup_{j=1,2} h_j^{-1}(J(G))) = \emptyset$ and $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) \neq \emptyset$.

Suppose we have Case (I). Then by Proposition 4.21-2, $J_{\min} = J(h_1)$ and $J_{\max} = J(h_3)$. Hence $F_\infty(G) = F_\infty(h_3)$. By [22, Theorem 2.20-5], $h_j(J(h_1)) \subset \text{int}(\hat{K}(G)) \subset \text{int}(K(h_1))$ for each $j = 2, 3$. Therefore $\hat{K}(G) = K(h_1)$. Thus statement (1) of our theorem holds.

Suppose we have Case (III). Since $h_3^{-1}(J(G)) \cap (\bigcup_{j=1}^{2} h_j^{-1}(J(G))) = \emptyset$, by [22, Lemma 4.13-4] and [24, Lemmas 3.5, 3.6] we obtain that $\bigcap_{n=1}^{\infty} h_3^{-n}(J(G))$ is a connected component of $J(G)$. Since $J(h_3) \cap J_{\min} = \emptyset$, by [22, Theorem 2.20-4,5] $\text{int}(K(h_3))$ is connected and there exists an attracting fixed point z_0 of h_3 in $\text{int}(\hat{K}(G))$ such that $\text{int}(K(h_3))$ is the immediate basin of z_0 for the dynamics of h_3. Therefore $\bigcap_{n=1}^{\infty} h_3^{-n}(J(G)) = J(h_3)$. Since $J(h_3) \subset J_{\max}$, we obtain that $J(h_3) = J_{\max}$. Therefore $F_\infty(G) = F_\infty(h_3)$. Thus statement (3) of our theorem holds.

Suppose we have Case (II). By the arguments similar to those in Case (III), we obtain that $\bigcap_{n=1}^{\infty} h_1^{-n}(J(G))$ is a connected component of $J(G)$. Since $J(h_1) \subset J_{\min} \cap \bigcap_{n=1}^{\infty} h_1^{-n}(J(G))$, it follows that $J_{\min} = \bigcap_{n=1}^{\infty} h_1^{-n}(J(G)) \subset K(h_1)$. Moreover, since $(J(h_2) \cup J(h_3)) \cap J_{\min} = \emptyset$, by [22, Theorem 2.20-5] we obtain that $h_j(J(h_1)) \subset \text{int}(\hat{K}(G)) \subset \text{int}(K(h_1))$ for each $j = 2, 3$. Hence $K(h_1) = \hat{K}(G)$ and $\text{int}(K(h_1)) \subset \text{int}(\hat{K}(G)) \subset F(G)$. Therefore $J_{\min} = J(h_1)$. Thus statement (2) of our theorem holds.

Combining all of the above arguments, we obtain that (a) if $J_1 = J_2$, then statement (3) of our theorem holds, and (b) if $J_2 = J_3$, then statement (2) of our theorem holds. We now suppose

$h_2^{-1}(J(G)) \cap (\bigcup_{j=1,3} h_j^{-1}(J(G))) = \emptyset$. Then by Claim 1, Case (I) holds. Therefore statement (1) of our theorem holds. Thus we have proved Theorem 2.15. $\square$

We now prove Corollary 2.16.

Proof of Corollary 2.16: By Theorem 2.15, there exists a number $i \in \{1,2,3\}$ such that $h_i^{-1}(J(G)) \cap (\bigcup_{j:j\neq i} h_j^{-1}(J(G))) = \emptyset$ and either $J(h_i) = J_{\max}(G)$ or $J(h_i) = J_{\min}(G)$.

Suppose $J(h_i) = J_{\min}(G)$. Let $j \in \{1,2,3\}$ be an element with $j \neq i$. By [19, Proposition 2.2(3)], for each $z \in J(h_i)$, $d(z, J(h_j h_i^k)) \to 0$ as $k \to \infty$. For each k, let $I_k \in \mathrm{Con}(J(G))$ with $J(h_j h_i^k) \subset I_k$. Then by the compactness of the space of all non-empty connected compact subsets of $\hat{\mathbb{C}}$ with respect to the Hausdorff metric, we obtain that $I_k \to J(h_i)$ as $k \to \infty$ with respect to the Hausdorff metric. Moreover, for each k, we have $I_k \neq J_{\min}(G)$ since $I_k \subset h_j^{-1}(J(G))$ and $J_{\min}(G) \subset h_i^{-1}(J(G))$. Let $\{J_n\}_{n=1}^{\infty}$ be a subsequence of $\{I_k\}$ such that $J_1 >_s J_2 >_s \cdots >_s J(h_i)$ and $J_n \to J(h_i)$ as $n \to \infty$. By [22, Lemma 4.4], for each n there exists a doubly connected component A_n of $F(G)$ with $J_n >_s A_n >_s J_{n+1}$. Then $\overline{A_n} \to J(h_i)$ as $n \to \infty$.

Suppose $J(h_i) = J_{\max}$. By the arguments similar to those in the previous paragraph, we obtain that there exists a sequence $\{J_n\}$ of mutually distinct elements in $\mathrm{Con}(J(G))$ and a sequence $\{A_n\}$ of mutually distinct doubly connected components of $F(G)$ such that $J_n \to J(h_i)$ and $\overline{A_n} \to J(h_i)$ as $n \to \infty$ with respect to the Hausdorff metric. Thus we have proved Corollary 2.16. $\square$

5 Examples

In this section we give some examples.

Definition 5.1. Let G be a polynomial semigroup. We say that G is semi-hyperbolic if there exists an $N \in \mathbb{N}$ and a $\delta > 0$ such that for each $z \in J(G)$ and for each $g \in G$, $\deg(g : V \to B(z,\delta)) \leq N$ for each $V \in \mathrm{Con}(g^{-1}(B(z,\delta)))$. Here, deg denotes the degree of finite branched covering. We say that G is hyperbolic if $P(G) \subset F(G)$.

Proposition 5.2 (Proposition 2.40 in [22])**.** *Let G be a polynomial semigroup generated by a compact subset Γ of $\mathcal{P}$. Suppose that $G \in \mathcal{G}$ and $\mathrm{int}(\hat{K}(G)) \neq \emptyset$. Let $b \in \mathrm{int}(\hat{K}(G))$. Moreover, let $d \in \mathbb{N}$ be any positive integer such that $d \geq 2$, and such that $(d, \deg(h)) \neq (2,2)$ for each $h \in \Gamma$. Then, there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, there exists a compact neighborhood V of $g_a(z) = a(z-b)^d + b$ in $\mathcal{P}$ satisfying that for any non-empty subset V' of V, the polynomial semigroup $\langle \Gamma \cup V' \rangle$ generated by the family $\Gamma \cup V'$ belongs to $\mathcal{G}_{dis}$ and $\hat{K}(\langle \Gamma \cup V' \rangle) = \hat{K}(G)$. Moreover, in addition to the assumption above, if G is semi-hyperbolic (resp. hyperbolic), then the above $\langle \Gamma \cup V' \rangle$ is semi-hyperbolic (resp. hyperbolic).*

Proposition 5.3 (Proposition 6.1 in [25])**.** *Let $h_1 \in \mathcal{P}$. Suppose that $K(h_1)$ is connected and $\mathrm{int}(K(h_1))$ is not empty. Let $b \in \mathrm{int}(K(h_1))$ be a point. Let d be a positive integer such that $d \geq 2$. Suppose that $(\deg(h_1), d) \neq (2,2)$. Then, there exists a number $c > 0$ such that for each $\lambda \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < c\}$, setting $h_\lambda = (h_{\lambda,1}, h_{\lambda,2}) = (h_1, \lambda(z-b)^d + b)$ and $G_\lambda := \langle h_1, h_{\lambda,2} \rangle$, we have all of the following.*

(a) *$G_\lambda \in \mathcal{G}_{dis}$. Moreover, h_λ satisfies the open set condition with an open subset U_λ of $\hat{\mathbb{C}}$ (i.e., $h_{\lambda,1}^{-1}(U_\lambda) \cup h_{\lambda,2}^{-1}(U_\lambda) \subset U_\lambda$ and $h_{\lambda,1}^{-1}(U_\lambda) \cap h_{\lambda,2}^{-1}(U_\lambda) = \emptyset$), $h_{\lambda,1}^{-1}(J(G_\lambda)) \cap h_{\lambda,2}^{-1}(J(G_\lambda)) = \emptyset$, $\mathrm{int}(J(G_\lambda)) = \emptyset$, $J_{\ker}(G_\lambda) = \emptyset$, $G_\lambda(K(h_1)) \subset K(h_1) \subset \mathrm{int}(K(h_{\lambda,2}))$ and $\emptyset \neq K(h_1) \subset \hat{K}(G_\lambda)$.*

(b) *If h_1 is semi-hyperbolic (resp. hyperbolic), then G_λ is semi-hyperbolic (resp. hyperbolic), $J(G_\lambda)$ is porous (for the definition of porosity, see [19]), and $\dim_H(J(G_\lambda)) < 2$.*

For the dynamics of (semi-)hyperbolic rational semigroups, see [14, 17, 18, 19, 22, 23, 24, 28, 29]. For the study of the Hausdorff dimension of the Julia sets of (semi-)hyperbolic rational semigroups (with open set condition), see [18, 19, 28, 29].

Regarding Proposition 5.3, we can sometimes give the concrete values c.

Example 5.4 (Devil's coliseum)**.** Let $h_1(z) = z^2 - 1$ and let $\lambda \in \mathbb{C}$ with $0 < |\lambda| \leq 0.01$. Let $h_2(z) = \lambda z^3$. Let $G = \langle h_1, h_2 \rangle$ and $\tau := \sum_{i=1}^{2} \frac{1}{2}\delta_{h_i}$. Let $A := K(h_2) \setminus B$ where $B = D(0, 0.4) \cup D(-1, 0.16)$. Then we have $\overline{B} \subset \text{int}(K(h_1)) \subset D(0,2)$ and $h_2(K(h_1)) \subset h_2(D(0,2)) \subset \overline{B} \subset \text{int}(K(h_1))$. Therefore $P^*(G) \subset \text{int}(K(h_1))$. Hence G is hyperbolic and $G \in \mathcal{G}$. Moreover, we have $h_1(B) \subset B$, $h_2(B) \subset B$, $K(h_2) = \overline{D(0, |\lambda|^{-1/2})}$, $h_1^{-1}(K(h_2)) \subset K(h_2)$, and $h_2^{-1}(K(h_2)) \subset K(h_2)$. Hence $h_1^{-1}(A) \cup h_2^{-1}(A) \subset A$. Also, it is easy to see that $h_1^{-1}(A) \cap h_2^{-1}(A) = \emptyset$. Therefore $J(G) \subset A$, $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, $G \in \mathcal{G}_{dis}$ and $\emptyset \neq K(h_1) \subset \hat{K}(G)$. By Theorems 2.4 and 2.11, we obtain that $J_{\text{ker}}(G) = \emptyset$, $T_{\infty,\tau}$ is Hölder continuous on $\hat{\mathbb{C}}$, the set of varying points of $T_{\infty,\tau}$ is equal to $J(G)$, and for each non-empty open subset U of $J(G)$ there exists an uncountable dense subset A_U of U such that for each $z \in A_U$, $T_{\infty,\tau}$ is not differentiable at z. By Theorem 2.11 and [25, Theorem 3.82], there exists a Borel subset A of $J(G)$ with $\dim_H(A) \geq 1 + \frac{2\log 2}{\log 2 + \log 3} \risingdotseq 1.7737$ such that for each $z \in A$, $\text{Höl}(T_{\infty,\tau}, z) = u(h, p, \mu) = \frac{2\log 2}{\log 2 + \log 3} \risingdotseq 0.7737$ and $T_{\infty,\tau}$ is not differentiable at z. Moreover, since G is hyperbolic and $h_1^{-1}(J(G)) \cap h_2^{-1}(J(G)) = \emptyset$, $\dim_H(J(G)) < 2$ (see [15] or [25, Theorem 3.82]). It is easy to see that $\text{Min}(G_\tau, \hat{\mathbb{C}}) = \{\{\infty\}, \{0\}\}$. Thus regarding statements in Theorem 2.4 for τ, $L_\tau = \{0\}$ and $\mu_\tau = \delta_0$. $T_{\infty,\tau}$ is called a devil's coliseum. It is a complex analogue of the devil's staircase.

Remark 5.5. By Proposition 5.2, there exists a 2-generator polynomial semigroup $G = \langle h_1, h_2 \rangle$ in $\mathcal{G}_{dis}$ such that h_1 has a Siegel disk. In fact, Proposition 5.2 implies that for each $h_1 \in \mathcal{P}$ with $\langle h_1 \rangle \in \mathcal{G}$ which has a Siegel disk, there exists an element $h_2 \in \mathcal{P}$ such that $G = \langle h_1, h_2 \rangle$ belongs to $\mathcal{G}_{dis}$. Note that for such a G, we can apply Theorems 2.4, 2.11, 2.12, even though G is not semi-hyperbolic.

Example 5.6. Let $\theta \in \mathbb{R}$ be a Brjuno number ([2, 32]) and let $f_1(z) = e^{2\pi i\theta}z + z^2$ Then f_1 has a Siegel disk with center 0 ([2, 32]). Applying Proposition 5.2 or Proposition 5.3 (with $b = 0$), we obtain that there exists a number $c > 0$ such that for each $\lambda \in \mathbb{C}$ with $0 < |\lambda| < c$, setting $f_2(z) = \lambda z^3$, we have that $G := \langle f_1, f_2 \rangle \in G_{dis}$. Since f_1 is not semi-hyperbolic, G is not semi-hyperbolic. We can apply Theorems 2.4, 2.11, 2.12, to this G, even though G is not semi-hyperbolic.

Example 5.7. Let $f_1, f_2 \in \mathcal{P}$ be two elements such that $G = \langle f_1, f_2 \rangle$ belongs to $\mathcal{G}_{dis}$. Then by [20, Theorems 1.5, 1.7], $f_1^{-1}(J(G)) \cap f_2^{-1}(J(G)) = \emptyset$. Let $n \geq 2$ and let A be a non-empty subset of $\Lambda_n := \{f_{i_1} \circ \cdots \circ f_{i_n} \mid i_1, \ldots, i_n \in \{1, 2\}\}$ with $\sharp A \geq 2$. Let G_A be the polynomial semigroup generated by A. Then $J(G_{\Lambda_n}) = J(G)$ ([9, Theorem 2.4]). Thus for each $(g_1, g_2) \in A^2$ with $g_1 \neq g_2$, we have $g_1^{-1}(J(G_A)) \cap g_2^{-1}(J(G_A)) = \emptyset$. Moreover, $G_A \in \mathcal{G}_{dis}$. For the semigroup G_A, we can apply Theorems 2.4 and 2.11. If f_1 is not semi-hyperbolic and $f_1^n \in A$, then G_A is not semi-hyperbolic.

Example 5.8. Let $f_1, f_2 \in \mathcal{P}$ be two elements such that $\langle f_1, f_2 \rangle \in \mathcal{G}$, $J(f_1) \cap J(f_2) \neq \emptyset$ and $0 \in \text{int}(\hat{K}(\langle f_1, f_2 \rangle))$ (e.g., $f_1(z) = z^2 - 1$ and f_2 is a small perturbation of f_1). Let $d \in \mathbb{N}$ with $d \geq 3$. Then by Proposition 5.2, there exists a number $c > 0$ such that for each $a \in \mathbb{C}$ with $0 < |a| < c$, setting $f_3(z) = az^d$, we have that $G := \langle f_1, f_2, f_3 \rangle$ belongs to $\mathcal{G}_{dis}$. For this G, we can apply Theorems 2.4, 2.15 and Corollary 2.16. Since $J(f_1) \cap J(f_2) \neq \emptyset$, we have $f_1^{-1}(J(G)) \cap f_2^{-1}(J(G)) \neq \emptyset$. Thus Theorem 2.15 implies that statement (3) of Theorem 2.15 holds.

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