92. *Random Functions in Fourier Restriction Algebras*

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We denote by $A(R)$ the Fourier algebra on the real line R. The norm of $\hat{h}$ in $A(R)$ is

$$\|h\|_1=\frac{1}{2\pi}\int_{\hat{R}}|h(r)|\,dr.$$

For a closed subset E of R, set

$$A(E)=\{g|E: g\in A(R)\},$$
$$\|f\|_{A(E)}=\inf\{\|g\|_{A(R)}: g\in A(R), g|E=f\} \qquad (f\in A(E)).$$

Let $E_k=\{x_m^{(k)}: m_k\leqq m<m_k+n_k\}$ $(k=1,2,\cdots)$ be pairwise disjoint finite subsets of R each of which consists of n_k points, where $m_1=0$ and $m_k+n_1=n_2+\cdots+n_{k-1}$ $(k\geqq 2)$. Suppose $x_0 \notin \bigcup_{k=1}^{\infty}E_k$ and $\{E_k\}$ converges to x_0. Put

$$E=\bigcup_{k=1}^{\infty}E_k\cup\{x_0\}.$$

Let $\{c_k\}$ be a sequence of complex numbers and let $\{\varepsilon_m\}$ be the Rademacher sequence. We define a random function $f=f_\omega$ on E by

$$\begin{cases} f(x_m^{(k)})=\varepsilon_m(\omega)c_k & (k=1,2,\cdots,m_k\leqq m<m_k+n_k)\\ f(x_0)=0. \end{cases}$$

We investigate the condition for the function f to belong to $A(E)$. By using Rudin-Shapiro polynomials, we see that if each E_k is an arithmetic progression and $\{c_k\sqrt{n_k}\}$ does not converge to zero, then there exists a function $f\notin A(E)$. The following Theorem asserts that it holds almost surely. This is based on the same idea as Paley-Zygmund theorem, but we use the estimate of the L^1-norm of random trigonometric polynomials which is due to Uchiyama.

Theorem. *Suppose each E_k is an arithmetic progression. If $\{c_k\sqrt{n_k}\}$ does not converge to zero, then $f\notin A(E)$ a.s.*

Proof. Put

$$x_m^{(k)}=a_k+mb_k \qquad (k=1,2,\cdots,m_k\leqq m<m_k+n_k).$$

For each k, let v_k be the function in $L^1(\hat{R})$ such that

$$\hat{v}_k(x)=\hat{K}_\lambda(x-\{a_k+(m_k+p_k)b_k\}) \qquad (x\in R),$$

where $p_k=[n_k/2]$, $\lambda=p_kb_k$ and

$$\hat{K}_\lambda(y)=\max\left(1-\frac{|y|}{\lambda},0\right) \qquad (y\in R).$$

If $h\in L^1(\hat{R})$ and $\hat{h}=f$ on E_k, then

$$\sum_m \widehat{v_k * h}(x_m^{(k)}) \exp(ib_k mr)$$
$$=\frac{1}{b_k}\sum_n (v_k * h)\left(r+\frac{2\pi n}{b_k}\right)\exp\left(-ia_k\left(r+\frac{2\pi n}{b_k}\right)\right) \qquad \text{a.e.}$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_m \hat{v}_k(x_m^{(k)})\hat{h}(x_m^{(k)})e^{imt}\right| dt$$
$$\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} |(v_k * h)(t)|\, dt \leqq \|h\|_1,$$

and so

$$\left\|\sum_{m=m_k}^{m_k+n_k-1} \varepsilon_m(\omega)c_k\hat{v}_k(x_m^{(k)})e^{imt}\right\|_{L^1(T)} \leqq \|f\|_{A(E_k)}. \tag{1}$$

Choose $\eta>0$ so that $K=\{k: |c_k|\sqrt{n_k}>\eta\}$ is infinite. Let A_k be the event that the left side of (1) is not less than

$$\frac{1}{2}|c_k|\left(\sum_{m=m_k}^{m_k+n_k-1}|\hat{v}_k(x_m^{(k)})|^2\right)^{\frac{1}{2}}.$$

By Theorem 1 in [2], the probability $p(A_k)$ of A_k is greater than $1/2$. Since $\{A_k\}_{k\in K}$ is independent, the Borel-Cantelli lemma shows that

$$p\left(\varlimsup_{k\in K} A_k\right)=1.$$

If $\omega \in \varlimsup_{k\in K} A_k$, then for infinitely many k we have

$$\|f\|_{A(E_k)} \geqq \left\|\sum_m \varepsilon_m(\omega)c_k\hat{v}_k(x_m^{(k)})e^{imt}\right\|_{L^1(T)}$$
$$\geqq \frac{1}{2}|c_k|\left(\sum_m |\hat{v}_k(x_m^{(k)})|^2\right)^{\frac{1}{2}} > \frac{\eta}{2\sqrt{6}}.$$

It follows that $f \notin A(E)$ (cf. [1, Theorem 2.6.4.]), and the proof is complete.

Remark 1. Suppose $\{E_k\}$ are arithmetically disjoint; that is to say, there is a constant C such that

$$\sum_{k=1}^N \|\hat{\mu}_k\|_\infty \leqq C\left\|\sum_{k=1}^N \hat{\mu}_k\right\|_\infty$$

for every positive integer N and every measure μ_k supported by E_k ($k=1,2,\cdots,N$). (For example, if each E_k is an arithmetic progression and $\{a_k, b_k\}$ is linearly independent over the rationals, then $\{E_k\}$ are arithmetically disjoint.) If $c_k\sqrt{n_k}\to 0$ ($k\to\infty$), then for all ω we have $f \neq A(E)$. Indeed, if μ is a measure supported by E_k, then

$$\left|\int f d\mu\right| \leqq |c_k|\sqrt{n_k}(\sum|\mu(\{x\})|^2)^{\frac{1}{2}} \leqq |c_k|\sqrt{n_k}\,\|\hat{\mu}\|_\infty.$$

If λ is a measure supported by $\bigcup_n^m E_k$ and $\mu_k=\lambda|E_k$, then

$$\left|\int f d\lambda\right| \leqq \sum_n^m \|\hat{\mu}_k\|_\infty \sup_{n\leqq k\leqq m}|c_k|\sqrt{n_k} \leqq C\,\|\hat{\lambda}\|_\infty \sup_{n\leqq k\leqq m}|c_k|\sqrt{n_k}.$$

This implies that

$$\|f\|_{A\left(\bigcup_{n}^{m} E_k\right)} \leqq C \sup_{n \leqq k \leqq m} |c_k| \sqrt{n_k}. \tag{2}$$

There is a sequence $\{g_n\}$ of functions in $A(E)$ and an increasing sequence $\{p_n\}$ such that $g_n=0$ on $\bigcup_1^n E_k$, $g_n=1$ on $\bigcup_{p_n}^{\infty} E_k \cup \{x_0\}$ and $\|g_n\|_{A(E)}<2$ (cf. [1, Theorem 2.6.3.]). It follows from (2) that $\{f-fg_n\}$ is a Cauchy sequence in $A(E)$, so $f \in A(E)$.

Remark 2. If $\{E_k\}$ diverges to infinity and $E=\bigcup_{k=1}^{\infty} E_k$, then the same conclusions as Theorem and Remark 1 are valid.

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References

[1] W. Rudin: Fourier Analysis on Groups. Interscience, New York (1962).
[2] S. Uchiyama: On the mean modulus of trigonometric polynomials whose coefficients have random signs. Proc. Amer. Math. Soc., **16**, 1185–1190 (1965).