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Random integrals of Banach space valued functions

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Abstract. In this paper we study random integrals of the form $\int f dM$, where f is a deterministic Banach space valued function and M an independently scattered random measure. Random integrals of this type are a natural generalization of random series with Banach space valued coefficients. We prove an analogon of the Ito–Nisio theorem for random integrals, a comparison theorem and some contraction principles. Results are applied for stable measures on Banach spaces.

1. Introduction. The present paper is devoted to a study of random integrals of the form $\int f\,dM$, where f is a deterministic function taking values in a Banach space E and M is an independently scattered random measure. Random integrals of this type are a natural generalization of random series with Banach space valued coefficients. It is well known that the assymptotic behaviour of such series depends also on some geometric properties of the Banach space. Analogously, the existence of certain bounded linear operators on appropriate function spaces which we call random integrals, depends in general on a geometric structure of E. Hoffmann-Jørgensen and Pisier [7] defined Gaussian random integrals for spaces of type 2. Marcus and Woyczyński [14] and Okazaki [15] considered p-stable random integrals assuming that E is of stable type p. Woyczyński [26] investigated Poissonian random integrals for spaces of Rademacher type p.

In this article we define and study random integrals without any restrictions on a geometry of E, per an analogy to the theory of random series with Banach space valued coefficients. Such approach for Gaussian random integrals was presented in [6] and [19] and for stable random integrals in [18]. It permits to have a non-trivial class of integrable functions in each Banach space which was impossible under the classical approach (see [27]). A general theory of bilinear random integrals is developed in [17].

In Section 2 we consider preliminary facts concerning a random integral. An analogon of the well-known Ito–Nisio theorem for symmetric summands (see f.e. [3], Chap. 3, Th. 2.10) is proved in Section 3 (Theorem 3.4). Namely, if for every $x^* \in E^*$ the real random integral $\int \langle x^*, f\rangle\, dM$ exists and there exists a Radon probability measure μ on E such that

$$\hat{\mu}(x^*) = E\exp\{i\int \langle x^*, f\rangle\, dM\}, \qquad x^* \in E^*,$$

then $\int f\,dM$ exists. Urbanik and Woyczyński [24] characterized the spaces of all real-valued M-integrable functions as certain Orlicz spaces. Section 4 begins with a generalization of their result (Theorem 4.1). Unfortunately the full analogon of the Urbanik–Woyczyński result is true only when E is a Hilbert space. Hence we consider the following question: knowing the parameters of two random measures is it possible to compare the spaces of integrable functions? As the main result of Section 4 we prove a comparison theorem for random integrals (Theorem 4.5). Section 5 deals with contractions principles for random integrals. We study two types of contractions. The first is given by the multiplication of M-integrable function by a bounded real-valued function and the second is defined by a conditional integral. In Section 6 we apply the results of previous sections to stable random integrals. We establish an isomorphism between the space of all functions integrable with respect to p-stable random measure and the space of operators generating p-stable measures on E which was introduced and studied by Linde [12], D. H. Thang and N. Z. Tien [22], and in an equivalent form for $1<p<2$, by Linde, Mandrekar and Weron [13]. We prove that the set of all stable measures with the discrete spectral measures lies densely in the set of all stable measures on a Banach space (Theorem 6.6). Every p-stable probability measure on E can be represented as the distribution of a p-stable random integral $\int_0^1 f\,dM_p$, which follows by Theorem 6.7. Thus we get an analogon of the well-known fact that Gaussian measures on E are represented as distributions of random series $\sum x_j\xi_j$, where $x_j \in E$ and ξ_j are independent $N(0,1)$.

2. Random integral. Let (T,Σ) be a measurable space and $(\Omega,\mathscr{F},P)$ be a probability space. A function

$$M\colon \Sigma \to L^0(\Omega,\mathscr{F},P)$$

such that for every pairwise disjoint sets $A_n\in\Sigma$ random variables $M(A_n)$ are independent, $n=1,2,\ldots$, and $M(\bigcup_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty} M(A_n)$ a.s. is called an *independently scattered random measure*. Every independently scattered random measure M may be decomposed into two independent, independently scattered random measures $M=M_a+M_c$, where M_a is pure atomic and M_c is atomless (as vector measures in L^0). M_a is completely described by a sequence $\{\xi_n\}$ of independent random variables such thas $\sum\xi_n$ unconditionaly converges a.s. The atomless part M_c has the property that, for every $A\in\Sigma$, $M_c(A)$ has an infinitely divisible distribution. In this paper we shall consider independently scattered random measures generated by one infinitely divisible law.

DEFINITION 2.1. Let (T,Σ,λ) be a finite measure space and ν be an infinitely divisible distribution on $\boldsymbol{R}$. We say that a set function

$$M\colon \Sigma \to L^0(\Omega,\mathscr{F},P)$$

is a *random measure on* (T,Σ,λ) *generated by* ν if M is independently scattered and, for every $A\in\Sigma$,

$$\mathscr{L}(M(A)) = \nu^{*\lambda(A)},$$

where ν^{*p} denotes the pth convolution power of ν.

The existence of a random measure on every measure space generated by every infinitely divisible law follows from Daniell–Kolmogorov's consistency theorem.

In the sequel we assume that ν is *symmetric*. Thanks to this restriction we avoid some technical difficulties with non-linear centering and we get more clear theory of random integral. A general situation including the case when ν is arbitrary is investigated in [17]. Therefore we can write

$$E\exp(iuM(A)) = \exp[-\lambda(A)K(u)], \tag{2.1}$$

where

$$K(u) = \tfrac{1}{2}\sigma^2u^2 + \int_{-\infty}^{\infty}(1-\cos uv)\,m(dv) \tag{2.2}$$

while m is a symmetric σ-finite measure on $\boldsymbol{R}$ such that $m(\{0\})=0$ and $\int_{-\infty}^{\infty}\min(1,v^2)\,m(dv)<\infty$. We shall write $M\sim[\sigma^2,m]$ if (2.1) and (2.2) hold. $M\sim[1,0]$ is a so-called *white noise* on (T,Σ,λ).

Let E be a Banach space. For every simple measurable function $f\colon T\to E$, $f=\sum_{j=1}^{n}x_j\mathbf{1}_{A_j}$, where $A_j\in\Sigma$ are pairwise disjoint, $x_j\in E$, $j=1,\ldots,n$, we set

$$\int_A f\,dM = \sum_{j=1}^{n} x_jM(A_j\cap A), \qquad A\in\Sigma.$$

DEFINITION 2.2. A function $f\colon T\to E$ is said to be *M-integrable* if there exist simple measurable functions $f_n\colon T\to E$ such that

(i) $f_n\to f$ λ-a.e. as $n\to\infty$;

(ii) for every $A\in\Sigma$ the sequence $\{\int_A f_n\,dM\}$ converges in probability.

If f is M-integrable, then we put

$$\int_A f\,dM = P-\lim_{n\to\infty}\int_A f_n\,dM.$$

Using the Hahn–Saks–Vitali theorem (see [4], Ch. 3.7, Th. 2) we establish that the random integral is well defined. Moreover, $\int_A f\,dM$ has an infinitely divisible law on E with the characteristic functional

$$(2.3)\qquad E\exp\{i\langle x^*, \int_A f\,dM\rangle\} = \exp\{-\int_A K(\langle x^*, f(t)\rangle)\lambda(dt)\},$$

$x^*\in E^*$, $A\in\Sigma$. The symmetry and independence assumptions imply that for every simple measurable function $f\colon T\to E$ and $A\in\Sigma$

$$P\{\|\int_A f\,dM\| > \varepsilon\} \leqslant 2P\{\|\int_T f\,dM\| > \varepsilon\},\qquad \varepsilon > 0.$$

Hence condition (ii) in Definition 2.2 is equivalent to

(iii) the sequence $\{\int_T f_n\,dM\}$ converges in probability.

We note that even in the case when M is a white noise on $T=[0, 1]$ the stochastic integral of Bartle–Ito type, in general, does not exist. This fact, for $E = C[0, 1]$, was observed by Yor [27] (for more information see [6] and [19]). Our approach permit to define and study a random integral for every Banach space.

Let $\mathscr{L}_E(M)$ denote the linear subspace of $L_E^0 = L_E^0(T, \Sigma, \lambda)$ consisting of all M-integrable functions. $\mathscr{L}_E(M)$ is a complete metrizable vector space with an F-norm

$$\|f\|_M = \int_T \min\{1, \|f(t)\|\}\lambda(dt) + E\min\{1, \|\int_T f\,dM\|\}.$$

A random measure $M\sim[\sigma^2, m]$ is said to be a *Poissonian* if $\sigma^2 = 0$ and $m(\boldsymbol{R}) < \infty$.

Proposition 2.3. *If M is a Poissonian random measure on (T, Σ, λ), then*

$$\mathscr{L}_E(M) = L_E^0.$$

Proof. We may assume that $m(\boldsymbol{R}) > 0$. Put

$$H(u) = \int_{-\infty}^{\infty} \min\{u|v|, 1\}\, m(dv),\qquad u\geqslant 0.$$

Then

$$E\min\{u|M(A)|, 1\} = e^{-\lambda(A)m(\boldsymbol{R})}\sum_{k=0}^{\infty}\frac{\lambda^k(A)}{k!}\int_{-\infty}^{\infty}\min\{u|v|, 1\}\, m^{*k}(dv)$$

$$\leqslant H(u)\lambda(A),\qquad u\geqslant 0.$$

Hence, for every simple measurable function $f = \sum_{j=1}^{n} x_j \mathbf{1}_{A_j}$, we have

$$E\min\{\|\int_T f\,dM\|, 1\} \leqslant \sum_{j=1}^{n} E\min\{\|x_j\|\,|M(A_j)|, 1\}$$
$$\leqslant \sum_{j=1}^{n} H(\|x_j\|)\lambda(A_j)$$
$$= \int_T H(\|f(t)\|)\lambda(dt).$$

Since $f\mapsto \int_T H(\|f\|)\,d\lambda$ is an F-norm on L_E^0 norming the convergence in λ, the proposition is proved.

Remark 2.4. Since the statements of this paper do not depend on a concrete form of M but depend, in fact, on finite dimensional distributions of a stochastic process $\{M(A)\colon A\in\Sigma\}$, we can choose such versions of M which are convenient for proofs. Often we shall use the following version of a random measure generated by $[\sigma^2, m]$. Let M_0 and M_1 be independent random measures on (T, Σ, λ) such that $M_0\sim[\sigma^2, m_0]$ and $M_1\sim[0, m_1]$, where $m_0(B) = m(B\cap[-1, 1])$, and $m_1(B) = m(B\cap[-1, 1]^c)$, $B\in\mathscr{B}_{\boldsymbol{R}}$. Define $M(A) = M_0(A) + M_1(A)$, $A\in\Sigma$. Then M is a random measure on (T, Σ, λ) such that $M\sim[\sigma^2, m]$ and its components M_0 and M_1 are such that $M_0(T)$ has all moments and M_1 is a Poissonian random measure.

Distributions of random integrals are infinitely divisible hence we shall need some results on infinitely divisible laws on Banach spaces which can be found in [3] and [2]. The following fact, proved in a more general form in [1], [9] and [11], will be frequently used.

Proposition 2.5. *Let μ be an infinitely divisible probability measure on a Banach space E and ν be its Lévy measure. Then for every $p > 0$ $\int_E \|x\|^p\mu(dx) < \infty$ if and only if $\int_{\{\|x\|>1\}} \|x\|^p\nu(dx) < \infty$.*

3. Characterization of M-integrable functions. Let $f\colon T\to E$ be a strongly measurable function such that for every $x^*\in E^*$ $\int_T K(\langle x^*, f(t)\rangle)\lambda(dt) < \infty$. Since $x^*\mapsto -\int_T K(\langle x^*, f\rangle)\,d\lambda$ is negative definite and continuous on every finite dimensional subspace of E^*, there exists a cylindrical measure ν_f on E such that

$$\hat{\nu}_f(x^*) = \exp\{-\int_T K(\langle x^*, f\rangle)\,d\lambda\},\qquad x^*\in E^*,$$

(see [20] and [25]). In this section we prove that ν_f has the extension to a Radon measure if and only if f is M-integrable. The extension of ν_f is an infinitely divisible probability law on E.

The following lemma establishes the form of prediction in $\{M(A)\colon A \in \Sigma\}$.

LEMMA 3.1. *Assume that $E|M(T)| < \infty$. Then for every $A, B \in \Sigma$, $A \subset B$*

$$E[M(A)|M(B)] = \frac{\lambda(A)}{\lambda(B)} M(B) \quad a.s.$$

Proof. It is enough to prove that

$$EM(A)\, e^{itM(B)} = \frac{\lambda(A)}{\lambda(B)} EM(B)\, e^{itM(B)}$$

for every $t \in \boldsymbol{R}$. We have for every $C \in \Sigma$ and $t \in \boldsymbol{R}$

$$EM(C)\, e^{itM(C)} = -i \frac{dEe^{itM(C)}}{dt} = i\lambda(C)\big[\sigma^2 t + \int_{-\infty}^{\infty} v \sin(tv)\, m(dv)\big] Ee^{itM(C)}.$$

Thus

$$\begin{aligned} EM(A) e^{itM(B)} &= EM(A)\, e^{itM(A)}\, Ee^{itM(B\setminus A)} \\ &= i\lambda(A)\big[\sigma^2 t + \int_{-\infty}^{\infty} v \sin(tv)\, m(dv)\big] Ee^{itM(B)} \\ &= \frac{\lambda(A)}{\lambda(B)} EM(B)\, e^{itM(B)}. \end{aligned}$$

For every sub-σ-field Σ_0 of Σ and $f \in L^1_E$ let $\lambda(f|\Sigma_0)$ denotes a *conditional integral of f given Σ_0*.

LEMMA 3.2. *Assume $E|M(T)|^p < \infty$ for some $p \geqslant 1$ and $f \in L^1_E$. Let $\{\Sigma_n\}$ be an increasing sequence of finite sub-σ-fields of Σ. Define*

$$X_n = \int_T \lambda(f|\Sigma_n)\, dM, \qquad \mathscr{F}_n = \sigma\{M(A)\colon A \in \Sigma_n\}.$$

Then $(X_n, \mathscr{F}_n)$ is a martingale in $L^p_E(\Omega, \mathscr{F}, P)$.

Proof. Since $\lambda(f|\Sigma_n)$ is a simple function, X_n is well defined. Let $\{A_{n,j}\}_{j=1}^{k_n}$ denote the set of all atoms of Σ_n. Decompose the set $\{1, \ldots, k_{n+1}\}$ into non-empty subsets

$$J_{n,k} = \{1 \leqslant j \leqslant k_n\colon A_{n+1,j} \subset A_{n,k}\},$$

$k = 1, \ldots, k_n$. Put

$$x_{n,j} = \frac{1}{\lambda(A_{nj})} \int_{A_{nj}} f(t)\, \lambda(dt)$$

if $\lambda(A_{n,j}) > 0$ and $x_{n,j} = 0$ in another case. Then by Lemma 3.1 we have

$$\begin{aligned} E(X_{n+1}|\mathscr{F}_n) &= \sum_{j=1}^{k_{n+1}} E[x_{n+1,j} M(A_{n+1,j})|\mathscr{F}_n] \\ &= \sum_{k=1}^{k_n} \sum_{j \in J_{n,k}} E[x_{n+1,j} M(A_{n+1,j})|M(A_{n,k})] \\ &= \sum_{k=1}^{k_n} \sum_{j \in J_{n,k}} x_{n+1,j} \frac{\lambda(A_{n+1,j})}{\lambda(A_{n,k})} M(A_{n,k}) = X_n \quad \text{a.s.} \end{aligned}$$

LEMMA 3.3. *Assume that $E|M(T)| < \infty$. Then for every simple measurable function $f\colon T \to \boldsymbol{R}$*

$$E\Big|\int_T f\, dM\Big| \leqslant c_M \big(\int_T f^2\, d\lambda\big)^{1/2},$$

where c_M is a constant.

Proof. In view of Remark 2.4 we may assume that $M = M_0 + M_1$, where M_0 and M_1 were specified in that remark.

Let $f = \sum_{j=1}^{n} a_j 1_{A_j}$, where $A_j \in \Sigma$ are pairwise disjoint. We get

$$\begin{aligned} E\Big|\int_T f\, dM_0\Big| &\leqslant \big[E\Big|\int_T f\, dM_0\Big|^2\big]^{1/2} \\ &= \big(\sum_{j=1}^{n} a_j^2\, EM_0^2(A_j)\big)^{1/2} = c_0 \big(\int_T f^2\, d\lambda\big)^{1/2}, \end{aligned}$$

where $c_0 = [\sigma^2 + \int_{[-1,1]} u^2\, m(du)]^{1/2}$, and

$$E\Big|\int_T f\, dM_1\Big| \leqslant \sum_{j=1}^{n} |a_j|\, E|M_1(A_j)| \leqslant c_1 \int_T |f|\, d\lambda,$$

where $c_1 = \int_{|u|>1} |u|\, m(du) < \infty$. Hence $c_M = c_0 + c_1 \lambda^{1/2}(T)$ satisfies the lemma.

THEOREM 3.4. *$f \in \mathscr{L}_E(M)$ if and only if f is strongly measurable, $\int_T K(\langle x^*, f(t)\rangle)\, \lambda(dt) < \infty$ for every $x^* \in E^*$ and*

$$\varphi_f(x^*) = \exp\{-\int_T K(\langle x^*, f(t)\rangle)\, \lambda(dt)\} \tag{3.1}$$

is a characteristic functional of a Radon probability measure on E. In this case $\mathscr{L}(\int_T f\, dM)^\wedge(x^*) = \varphi_f(x^*)$.

Proof. The necessity follows from (2.3) and Definition 2.2 (i). The proof of sufficiency is divided into three steps, where the assertion is proved under some additional restrictions on M and f.

Step I. Suppose that $E|M(T)| < \infty$ and f is bounded. Let $\{\Sigma_n\}$ be an increasing sequence of finite sub-σ-fields of Σ such the λ-completeness of $\sigma(\bigcup_{n=1}^{\infty} \Sigma_n)$ contains $f^{-1}(\mathscr{B}_E)$. Then

$$\lambda(f\,|\,\Sigma_n) \mapsto f \text{ a.e. and in } L_E^2 \quad \text{if} \quad n \to \infty. \tag{3.2}$$

Put $X_n = \int_T \lambda(f\,|\,\Sigma_n)\,dM$ and $\mathscr{F}_n = \sigma\{M(A)\colon A \in \Sigma_n\}$. We note that $\lambda(f\,|\,\Sigma_n)$ is a simple function. In view of Lemma 3.3 $L(x^*) = \int_T \langle x^*, f\rangle\,dM$ exists for every $x^* \in E^*$ and

$$\langle x^*, X_n\rangle \mapsto L(x^*) \quad \text{in} \quad L^1(\Omega, \mathscr{F}, P). \tag{3.3}$$

By (2.3) (in the case $E = \boldsymbol{R}$) we get

$$E\exp[iL(x^*)] = \varphi_f(x^*), \qquad x^* \in E^*.$$

Hence L is decomposable, i.e. there exists a random vector $X\colon \Omega \mapsto E$ such that for every $x^* \in E^*$

$$\langle x^*, X\rangle = L(x^*) \text{ a.s.}$$

We prove now that $E\|X\| < \infty$. Since the Lévy measure of $\mathscr{L}(X)$ is given by

$$\mu_f(B) = \lambda \times m(\{(t, u)\colon f(t)u \in B \setminus \{0\}\}), \qquad B \in \mathscr{B}_E,$$

by Proposition 2.5 it suffices to check that $\int_{\{\|x\|>1\}} \|x\|\,\mu_f(dx) < \infty$. We have

$$\int_{\{\|x\|>1\}} \|x\|\,\mu_f(dx) = \iint_{\{\|f(t)u\|>1\}} \|f(t)u\|\,\lambda(dt)\,m(du) \leqslant c\lambda(T) \int_{|u|>c^{-1}} |u|\,m(du) < \infty,$$

where $\sup_{t\in T}\|f(t)\| < c < \infty$. Finishing, by (3.3) and Lemma 3.2 we have for every $x^* \in E^*$

$$\langle x^*, E(X\,|\,\mathscr{F}_n)\rangle = E[L(x^*)\,|\,\mathscr{F}_n] = \lim_{m\to\infty} E(\langle x^*, X_m\rangle\,|\,\mathscr{F}_n) = \langle x^*, X_n\rangle \text{ a.s.}$$

Therefore

$$\int_T \lambda(f\,|\,\Sigma_n)\,dM = E(X\,|\,\mathscr{F}_n) \to X$$

a.s. and in $L_E^1(\Omega, \mathscr{F}, P)$. This and (3.2) give that $f \in \mathscr{L}_E(M)$ and $\int_T f\,dM = X$.

Step II. Suppose that $E|M(T)| < \infty$ but let f be arbitrary. Put $A_n = \{\|f(t)\| \leqslant n\}$. Then $\varphi_{f1_{A_n}}$ and $\varphi_{f1_{(T\setminus A_n)}}$ are the characteristic functionals of symmetric cylindrical measures on E such that $\varphi_{f1_{A_n}}\varphi_{f1_{(T\setminus A_n)}} = \varphi_f$ and φ_f

is the characteristic functional of a Radon measure. Hence $\varphi_{f1_{A_n}}$ is the characteristic functional of a Radon measure on E. By Step I, $f\mathbf{1}_{A_n} \in \mathscr{L}_E(M)$ and there exist simple functions $g_n\colon T \mapsto E$ such that

$$\lambda\{\|g_n - f\mathbf{1}_{A_n}\| > 2^{-n}\} < 2^{-n}$$

and

$$P\{\|\int_T (g_n - f\mathbf{1}_{A_n})\,dM\| > 2^{-n}\} < 2^{-n}.$$

Since the sequence $\{\int_T f\mathbf{1}_{A_n}\,dM\}_{n=1}^{\infty}$ has independent symmetric increaments and

$$\mathscr{L}(\int_T f\mathbf{1}_{A_n}\,dM)^\wedge(x^*) = \varphi_{f1_{A_n}}(x^*) \mapsto \varphi_f(x^*),$$

by Ito–Nisio's theorem $\{\int_T f\mathbf{1}_{A_n}\,dM\}_{n=1}^{\infty}$ converges a.s. Therefore the sequence $\{\int_T g_n\,dM\}_{n=1}^{\infty}$ converges a.s. and $g_n \to f$ a.e. This proves that $f \in \mathscr{L}_E(M)$.

Step III. M and f are arbitrary. In view of Remark 2.4 we may assume that $M = M_0 + M_1$, where M_0 and M_1 were specified in that remark. Put

$$K_0(u) = \tfrac{1}{2}\sigma^2u^2 + \int_{[-1,1]} (1 - \cos uv)\,m(dv),$$

$$K_1 = K - K_0 \quad \text{and} \quad \varphi_{i,f}(x^*) = \exp\{-\int_T K_i(\langle x^*, f\rangle)\,d\lambda\}, \quad i = 0, 1.$$

Since $\varphi_{i,f}$ is the characteristic functional of a cylindrical symmetric probability measure on E, $i = 0, 1$, such that $\varphi_{0,f}\varphi_{1,f} = \varphi_f$, we conclude that $\varphi_{0,f}$ is the characteristic functional of a Radon probability measure on E. By Step II, there exist simple measurable functions $f_n\colon T \mapsto E$ such that $f_n \mapsto f$ a.e. and the sequence $\{\int_T f_n\,dM_0\}$ converges in probability. By Proposition 2.3 the sequence $\{\int_T f_n\,dM_1\}$ also converges in probability. Thus $\int_T f_n\,dM = \int_T f_n\,dM_0 + \int_T f_n\,dM_1$ converge in probability as $n \to \infty$.

The proof of the theorem is complete.

Remark 3.5. We proved in fact also that if $f \in \mathscr{L}_E(M)$, then $\int_T f\,dM$ is a $\sigma\{M(A)\colon A \in f^{-1}(\mathscr{B}_E)\}$-measurable random vector.

The next statement is equivalent to Theorem 3.4 but more useful in some cases. The proof follows from Theorem 3.4 and the fact that if the convolution of two cylindrical symmetric measures is a Radon measure, then the components are also Radon measures (see [16] for a generalization of this fact).

THEOREM 3.6. *$f\in\mathscr{L}_E(M)$ if and only if*

(i) *f is strongly measurable;*

(ii) *$\Gamma_f(x^*)=\int_T \sigma^2\langle x^*,f(t)\rangle^2\,\lambda(dt)$, $x^*\in E^*$, is the covariance of a Gaussian measure on E;*

(iii) *$\mu_f(B)=\lambda\times m(\{(t,u)\in T\times \mathbf{R}: f(t)u\in B\setminus\{0\}\})$, $B\in\mathscr{B}_E$, is a Lévy measure on E.*

If f is M-integrable, then $\mathscr{L}(\int_T f\,dM)=\gamma_f * c_1\operatorname{Pois}(\mu_f)$, where γ_f is a Gaussian measure, $\hat{\gamma}_f(x^*)=\exp\{-\frac{1}{2}\Gamma_f(x^*)\}$, $c_1\operatorname{Pois}(\mu_f)$ is the 1-centered Poisson probability measure with Lévy measure μ_f,

$$[c_1\operatorname{Pois}(\mu_f)]^\wedge(x^*)=\exp\{\int_E(\cos\langle x^*,x\rangle-1)\,\mu_f(dx)\},\qquad x^*\in E^*.$$

From Theorem 3.6 and Theorem 6.3 (i) in [3], Chap. 3, follows immediately the following fact:

COROLLARY 3.7. *Assume that $M\sim[0,m]$ and $\int_{-1}^{1}|u|\,m(du)<\infty$. If a strongly measurable function $f\colon T\mapsto E$ satisfies*

$$\int_T\int_{\mathbf{R}}\min\{\|f(t)\|\,|u|,1\}\,m(du)\,\lambda(dt)<\infty,$$

then $f\in\mathscr{L}_E(M)$,

4. Random integral and Orlicz spaces. Urbanik and Woyczyński in [24] studied the spaces of real-valued M-integrable functions as certain Orlicz spaces and found the full characterization of these spaces. Put

$$G_M(x)=\sigma^2+2\int_{[0,x)}\frac{u^2}{1+u^2}\,m(du)$$

if $x>0$, $G_M(0)=0$ and

$$\Phi_M(x)=\int_{1/x}^{\infty}\frac{G_M(u)}{u^3}\,du$$

if $x>0$, $\Phi_M(0)=0$. Then by Theorem 4.1 [24]

$$\mathscr{L}_{\mathbf{R}}(M)=L(\Phi_M).$$

There is a natural way of a generalization of this result onto the Banach space case by the investigation of the relationship between $\mathscr{L}_E(M)$ and the Orlicz space $L_E(\Phi_M)$.

THEOREM 4.1. *If E is of type 2 [cotype 2], then $L_E(\Phi_M)\subset\mathscr{L}_E(M)$ [$\mathscr{L}_E(M)\subset L_E(\Phi_M)$] and the natural embeding is continuous. Conversely, if Σ contains an infinite sequence of non-zero pairwise disjoint sets and $L_E(\Phi_M)\subset\mathscr{L}_E(M)$ [$\mathscr{L}_E(M)\subset L_E(\Phi_M)$] for every M, then E is of type 2 [cotype 2].*

Proof. By Lemma 4.1 [24] $\Phi_M\sim\Psi_M$, where

$$\Psi_M(x)=\int_0^\infty\min\{x^2u^2,1\}\frac{1+u^2}{u^2}\,dG_M(u)=\sigma^2x^2+\int_{-\infty}^{\infty}\min\{x^2u^2,1\}\,m(du).$$

Hence $f\in L_E(\Phi_M)$ if and only if $\int_T\sigma^2\|f\|^2\,d\lambda<\infty$ and

$$\int_E\min\{\|x\|^2,1\}\,\mu_f(dx)=\int_T\int_{-\infty}^{\infty}\min\{\|f(t)u\|^2,1\}\,m(du)\,\lambda(dt)<\infty,$$

where μ_f is appeared in (iii) of Theorem 3.6. These conditions are sufficient in spaces of type 2 [necessary in spaces of cotype 2] to be respectively: f—a pregaussian function and μ_f—a Lévy measure (see Theorems 7.5, 7.6, 8.16, Chap. III, [3]). By Theorem 3.6 the required inclusions hold. Continuity of the natural embedings follows from the Closed Graph Theorem. For the proof of the second part of this theorem it is sufficient to consider only a white noise, i.e. $M\sim[1,0]$ (see [6] or [19]).

The above result shows that except of the Hilbert space case the Banach space analogon of the Urbanik–Woyczyński's characterization does not hold. The problem we are going to consider is: knowing the parameters σ_i^2, m_i of random measures M_i, $i=1,2$, to compare the spaces $\mathscr{L}_E(M_i)$ when E is arbitrary Banach space.

Let $M\sim[\sigma^2,m]$ be a random measure. Put

$$H_m(u)=m\left(\left(-\frac{1}{u},\frac{1}{u}\right)^c\right)\quad\text{if}\quad u>0$$

and

$$H_m(0)=0.$$

H_m is a non-decreasing right-continuous function on $\mathbf{R}_+$. Let $L_E(H_m)$ denote the Orlicz space of all strongly measurable functions $f\colon T\to E$ such that $\int_T H_m(c\|f(t)\|)\,\lambda(dt)<\infty$ for some $c>0$.

PROPOSITION 4.2. *$\mathscr{L}_E(M)\subset L_E(H_m)$. Moreover, if $\int_T f_n\,dM\to 0$ in P, then for every $c>0$ $\int_T H_m(c\|f_n\|)\,d\lambda\to 0$ as $n\to\infty$.*

Proof. μ_f presented in (iii) of Theorem 3.6 is Lévy measure. Hence

$$\int_T H_m(\|f(t)\|)\,\lambda(dt)=\mu_f\{x\in E:\|x\|\geqslant 1\}<\infty.$$

For the proof of the second part we note that by the symmetry argument $c_1 \operatorname{Pois}(\mu_{f_n}) \Rightarrow \delta_0$. Hence for every $c > 0$, $\mu_{f_n}\{\|x\| \geqslant c^{-1}\} \to 0$ (see [2], Th. 1.10).

The next lemma has rather technical character. It bases on the Kingmen construction of a Poisson point process on an abstract measure space (see [10], Th. 7).

LEMMA 4.3. *Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of independent random elements in (T, Σ) with the common distribution $\lambda/\lambda(T)$, $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables with the distribution ν and N be a random variable with the Poisson distribution with the parameter $L > 0$. Assume that $\{\tau_n\}$, $\{\xi_n\}$ and N are independent and put $\xi_0 \equiv 0$. Then*

$$M(A) = \sum_{n=0}^{N} \xi_n \delta_{\tau_n}(A), \quad A \in \Sigma,$$

is a Poissonian random measure on (T, Σ, λ) and $M \sim \left[0, \frac{L}{\lambda(T)} \nu_0\right]$, where $\nu_0(B) = \nu(B \setminus \{0\})$, $B \in \mathscr{B}_{\mathbf{R}}$.

Proof. By a routine computation of characteristic functional of $(M(A_1), \ldots, M(A_n))$, when $A_1, \ldots, A_n \in \Sigma$ are pairwise disjoint.

LEMMA 4.4. *Let $M_i \sim [0, m_i]$ be poissonian random measures, $i = 1, 2$, such that for every $u \geqslant 0$*

$$H_{m_1}(u) \leqslant k H_{m_2}(u),$$

where $k \in \mathbf{N}$ is a constant. Then for every measurable semi-norm $q \colon E \mapsto [0, \infty]$ and $\varepsilon > 0$

$$P\{q(\int_T f dM_1) > \varepsilon\} \leqslant 2k P\{kq(\int_T f dM_2) > \varepsilon\}.$$

Proof. Since the statement does not depend on versions of random measures, by Lemma 4.3 we can assume that

$$M_i = \sum_{n=0}^{N} \xi_n^{(i)} \delta_{\tau_n},$$

where N has the Poisson distribution with the parameter L

$$L = k\lambda(T) m_2(\mathbf{R}), \quad \text{and} \quad \mathscr{L}(\xi_n^{(1)}) = p\frac{m_1}{m_1(\mathbf{R})} + (1-p)\delta_0$$

while

$$p = \frac{m_1(\mathbf{R})}{k m_2(\mathbf{R})} = \frac{H_{m_1}(+\infty)}{k H_{m_2}(+\infty)} \leqslant 1 \quad \text{and} \quad \mathscr{L}(\xi_n^{(2)}) = \frac{1}{k}\frac{m_2}{m_2(\mathbf{R})} + \left(1 - \frac{1}{k}\right)\delta_0.$$

We have for every $\varepsilon > 0$

$$P\{|\xi_n^{(1)}| \geqslant \varepsilon\} = p\frac{m_1((-\varepsilon, \varepsilon)^c)}{m_1(\mathbf{R})} \leqslant \frac{m_2((-\varepsilon, \varepsilon)^c)}{m_2(\mathbf{R})} = kP\{|\xi_n^{(2)}| \geqslant \varepsilon\}.$$

Thus $\{\xi_n^{(1)}\}$ is dominated by $\{\xi_n^{(2)}\}$ with the constant k and, by Theorem 1.3, [21], we get

$$\begin{aligned} P\{q(\int_T f dM_1) > \varepsilon\} &= E[P\{q(\sum_{n=0}^{N} \xi_n^{(1)} f(\tau_n)) > \varepsilon | N, \{\tau_n\}\}] \\ &\leqslant 2kE[P\{kq(\sum_{n=0}^{N} \xi_n^{(2)} f(\tau_n)) > \varepsilon | N, \{\tau_n\}\}] \\ &= 2kP\{kq(\int_T f dM_2) > \varepsilon\}. \end{aligned}$$

Now we prove the main result of this section showing usefulness of H_M to compare spaces $\mathscr{L}_E(M)$. We write $H_{m_1} \prec H_{m_2}$ if H_{m_2} is non-weaker than H_{m_1}, i.e. there exist constants $k, l, u_0 \geqslant 0$ such that $H_{m_1}(u) \leqslant kH_{m_2}(lu)$ for every $u \geqslant u_0$.

THEOREM 4.5. *Let $M_i \sim [\sigma_i^2, m_i]$ be random measures on (T, Σ, λ), $i = 1, 2$. Assume that $H_{m_1} \prec H_{m_2}$ and there exists $c \geqslant 0$ such that $\sigma_1^2 = c\sigma_2^2$. Then for any Banach space E*

$$\mathscr{L}_E(M_2) \subset \mathscr{L}_E(M_1).$$

Proof. By the assumption there exist $k, l, u_0 \geqslant 0$ such that

$$H_{m_1}(u) \leqslant kH_{m_2}(lu) \quad \text{for every } u \geqslant u_0. \tag{4.1}$$

Without loss of generality we may assume that k is a natural number. We shall demonstrate that it sufficies to prove this theorem in the case when instead of (4.1) we have

$$H_{m_1}(u) \leqslant kH_{m_2}(u) \quad \text{for every } u \geqslant 0. \tag{4.2}$$

Clearly, let $M_3 \sim [\sigma_2^2, m_3]$ be a random measure, where $m_3(B) = m_2(l^{-1}B)$, $B \in \mathscr{B}_{\mathbf{R}}$. By Theorem 3.6 $\mathscr{L}_E(M_2) = \mathscr{L}_E(M_3)$ and $H_{m_1}(u) \leqslant kH_{m_3}(u)$ for $u \geqslant u_0$. Hence we can assume that $l = 1$ in (4.1). Now put

$$m_i^0(B) = m_i(B \cap (-1/u_0, 1/u_0)) + p_i\delta_{-1/u_0}(B) + p_i\delta_{1/u_0}(B),$$

$B \in \mathscr{B}_{\mathbf{R}}$ and $p_i = m_i([1/u_0, \infty))$. Considering random measures $M_i^0 \sim [\sigma_i^2, m_i^0]$ observe that $H_{m_1^0}(u) \leqslant kH_{m_2^0}(u)$ for $u \geqslant 0$ and by Theorem 3.6 $\mathscr{L}_E(M_i^0) = \mathscr{L}_E(M_i)$, $i = 1, 2$. Thus we proved that (4.2) can be assumed instead of (4.1).

Let $f \in \mathscr{L}_E(M_2)$ be fixed. Define measures m_i^δ and μ_i^δ by

$$m_i^\delta(B) = m_i(B \cap (-\delta, \delta)^c), \quad B \in \mathscr{B}_{\boldsymbol{R}},$$

$$\mu_i^\delta(B) = \lambda \times m_i^\delta(\{(t, u) \in T \times \boldsymbol{R}\colon f(t)u \in B \setminus \{0\}\}), \quad B \in \mathscr{B}_E,$$

$\delta \geqslant 0$, $i = 1, 2$. In view of Theorem 3.6 μ_2^0 is a Lévy measure and for the proof of this theorem it is enough to demonstrate that μ_1^0 is a Lévy measure.

Let $M_i^\delta \sim [0, m_i^\delta]$ be poissonian random measures on (T, Σ, λ), $\delta > 0$. We have

$$\mathscr{L}\Big(\int_T f\,dM_i^\delta\Big) = \operatorname{Pois}(\mu_i^\delta)$$

and

$$H_{m_1^\delta}(u) \leqslant kH_{m_2^\delta}(u) \quad \text{for every } u \geqslant 0.$$

Let $\varepsilon > 0$ be fixed. Since $\{\operatorname{Pois}(\mu_2^\delta)\colon \delta > 0\}$ is tight, there exists an absolutely convex compact set $C \subset E$ such that

$$[\operatorname{Pois}(\mu_2^\delta)](C^c) < \varepsilon/2k \quad \text{for every } \delta > 0.$$

Denote by q the gauge of C. Then by Lemma 4.4 for every $\delta > 0$

$$[\operatorname{Pois}(\mu_1^\delta)][(kC)^c] = P\Big\{q\Big(\int_T f\,dM_1^\delta\Big) > k\Big\}$$

$$\leqslant 2kP\Big\{q\Big(\int_T f\,dM_2^\delta\Big) > 1\Big\} = 2k[\operatorname{Pois}(\mu_2^\delta)](C^c) < \varepsilon.$$

Hence $\{\operatorname{Pois}(\mu_1^\delta)\colon \delta > 0\}$ is tight and since $\mu_1^\delta \nearrow \mu_1^0$ as $\delta \downarrow 0$, we proved that μ_1^0 is a Lévy measure (see Th. 4.7, Chap. 3, [3]). The proof is complete.

Lemma 4.2 and Theorem 4.5 show the usefulness of H_m to study of spaces $\mathscr{L}_E(M)$. Hence it is interesting to get a characterization of the class of all functions H_m.

PROPOSITION 4.6. *Let $H\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be a non-decreasing right-continuous function and $H(0) = 0$. Then there exists a symmetric Lévy measure m on $\boldsymbol{R}$ such that $H_m = H$ if and only if*

$$\int_1^\infty \frac{H(u)}{u^3}\,du < \infty.$$

Proof. Let m be a symmetric measure on $\boldsymbol{R}$ such that $m((-u, u)^c) = H(1/u)$. We have

$$\int_0^1 u^2\,m(du) = 2\int_0^1\Big(\int_0^u v\,dv\Big)m(du) = 2\int_0^1\Big(\int_v^1 v\,m(du)\Big)dv$$

$$= 2\int_0^1 v\,m([v, 1))\,dv = \int_1^\infty \frac{H(u)}{u^3}\,du - \frac{1}{2}H(1).$$

Hence

$$\int_{\boldsymbol{R}} \min\{1, u^2\}\,m(du) = 2\int_1^\infty \frac{H(u)}{u^3}\,du$$

which ends the proof.

5. Some contraction principles. In this section we consider two forms of contraction of M-integrable functions, the first is given by $f \to \varphi f$, where φ is a bounded real function on T and the second by a conditional integral $f \mapsto \lambda(f|\Sigma_0)$, where Σ_0 is a sub-σ-field of Σ.

THEOREM 5.1. *Let $\varphi\colon T \mapsto \boldsymbol{R}$ be a bounded measurable function such that $\operatorname{ess\,sup}|\varphi| \leqslant 1$ and $f \in \mathscr{L}_E(M)$. Then $\varphi f \in \mathscr{L}_E(M)$ and for every measurable seminorm $q\colon E \mapsto [0, \infty]$ and $\varepsilon > 0$*

$$P\Big\{q\Big(\int_T \varphi f\,dM\Big) > \varepsilon\Big\} \leqslant 2P\Big\{q\Big(\int_T f\,dM\Big) > \varepsilon\Big\}. \tag{5.1}$$

Proof. First we prove (5.1) in the case when $\varphi = \sum_{j=1}^n a_j \mathbf{1}_{A_j}$, where $|a_j| \leqslant 1$ and $A_j \in \Sigma$ are pairwise disjoint. Since $X_j = \int_{A_j} f\,dM$ are symmetric and independent random vectors, by Kwapień's inequality (see [21], Th. 1.2) we obtain

$$P\Big\{q\Big(\int_T \varphi f\,dM\Big) > \varepsilon\Big\} = P\Big\{q\Big(\sum_{j=1}^n a_j X_j\Big) > \varepsilon\Big\}$$

$$\leqslant 2P\Big\{q\Big(\sum_{j=1}^n X_j\Big) > \varepsilon\Big\} = 2P\Big\{q\Big(\int_T f\,dM\Big) > \varepsilon\Big\}.$$

Now let $\{\varphi_n\}$ be a sequence of simple bounded functions such that $\operatorname{ess\,sup}_T|\varphi_n - \varphi| \to 0$ and $\sup_T|\varphi_n - \varphi_m| < \varepsilon_{n,m} \to 0$ as $n, m \to \infty$. Hence we get

$$P\Big\{q\Big(\int_T (\varphi_n - \varphi_m) f\,dM\Big) > \varepsilon\Big\} = P\Big\{q\Big(\int_T \varepsilon_{n,m}^{-1}(\varphi_n - \varphi_m) f\,dM\Big) > \varepsilon\varepsilon_{n,m}^{-1}\Big\}$$

$$\leqslant P\Big\{q\Big(\int_T f\,dM\Big) > \varepsilon\varepsilon_{n,m}^{-1}\Big\} \mapsto 0 \quad \text{as} \quad n, m \to \infty.$$

Since $\mathscr{L}_E(M)$ is complete, we establish that $\varphi f \in \mathscr{L}_E(M)$ and $\int_T \varphi_n f\,dM \mapsto \int_T \varphi f\,dM$ in probability. Thus the theorem is proved.

The next theorem for a white noise was proved by Hoffman-Jørgensen [6] by the Hilbert space technique. The proof of a general case is more complicated.

THEOREM 5.2. *Let Σ_0 be a sub-σ-field of Σ and $M(\Sigma_0) = \sigma\{M(A)\colon A \in \Sigma_0\}$. Suppose that $f \in \mathscr{L}_E(M)$ satisfies $\int_T \|f\|\,d\lambda < \infty$ and*

$E\|\int_T f dM\| < \infty$. *Then a conditional integral* $\lambda(f|\Sigma_0)$ *belongs to* $\mathscr{L}_E(M)$ *and*

$$\int_T \lambda(f|\Sigma_0)\, dM = E\big(\int_T f dM \,|\, M(\Sigma_0)\big) \text{ a.s.} \tag{5.2}$$

The proof is preceded by the following auxiliary lemma.

LEMMA 5.3. *Assume that* $f\colon T \mapsto \mathbf{R}$ *is square-integrable and* $E|M(T)| < \infty$. *Then* $E|\int_T f dM| < \infty$ *and*

$$E\big[\int_T f dM \,|\, M(\Sigma_0)\big] = \int_T \lambda(f|\Sigma_0)\, dM. \tag{5.3}$$

Proof. From Lemma 3.3 it follows that $E|\int_T f dM| < \infty$. Assume that Σ_0 is finite. If f is a simple function, then (5.3) follows from Lemma 3.1, by a simple computation. If f is arbitrary, then one can choose a sequence $\{f_n\}$ of simple functions such that $\int_T |f_n - f|^2 d\lambda \mapsto 0$ as $n \to \infty$. Then by Lemma 3.3. $\int_T f_n dM \mapsto \int_T f dM$ and

$$\int_T \lambda(f_n|\Sigma_0)\, dM \mapsto \int_T \lambda(f|\Sigma_0)\, dM \quad \text{in} \quad L^1(\Omega, \mathscr{F}, P).$$

Hence (5.3) holds true if Σ_0 is finite. Now let Σ_0 be arbitrary. Then for every finite sub-σ-field Σ_1 of Σ_0 we get

$$E\big[\int_T f dM \,|\, M(\Sigma_1)\big] = \int_T \lambda(f|\Sigma_1)\, dM$$
$$= E\big[\int_T \lambda(f|\Sigma_0)\, dM \,|\, M(\Sigma_1)\big] \text{ a.s.}$$

which, using Remark 3.5, ends the proof.

Proof of Theorem 5.2. Clearly we may assume that f is not a.s. equal to zero. We shall prove that $E|M(T)| < \infty$. By the assumption we have $\|f\| > \varepsilon \mathbf{1}_A$ for some $A \in \Sigma$, $\lambda(A) > 0$ and $\varepsilon > 0$. Since $E\big\|\int_T f dM\big\| < \infty$, thus by Proposition 2.5 and Theorem 3.6 we get

$$\int_{|u| > \varepsilon^{-1}} |u|\, m(du) \leqslant \varepsilon^{-1} \lambda^{-1}(A) \iint_{\{\|f(t)u\| > 1\}} \|f(t)u\|\, \lambda(dt)\, m(du)$$
$$= \varepsilon^{-1} \lambda^{-1}(A) \int_{\{\|x\| > 1\}} \|x\|\, \mu_f(dx) < \infty$$

which gives that $E|M(T)| < \infty$.

Let $A_n = \{t\colon \|f(t)\| \leqslant n\}$. Then by Lemma 5.3 for every $x^* \in E^*$ we have

$$\big\langle x^*, E\big[\int_{A_n} f dM \,|\, M(\Sigma_0)\big]\big\rangle = E\big[\int_T \langle x^*, f\mathbf{1}_{A_n}\rangle dM \,|\, M(\Sigma_0)\big]$$
$$= \int_T \lambda(\langle x^*, f\mathbf{1}_{A_n}\rangle|\Sigma_0)\, dM = \int_T \langle x^*, \lambda(f\mathbf{1}_{A_n}|\Sigma_0)\rangle\, dM.$$

Since $\int_{A_n} f dM \mapsto \int_T f dM$ in $L_E^1(\Omega, \mathscr{F}, P)$, letting $n \to \infty$ we get for every $x^* \in E^*$

$$\big\langle x^*, E\big[\int_T f dM | M(\Sigma_0)\big]\big\rangle = \int_T \langle x^*, \lambda(f|\Sigma_0)\rangle\, dM. \tag{5.4}$$

By (2.3) (for $E = \mathbf{R}$) and (5.4) we establish that

$$x^* \mapsto \exp\{-\int_T K(\langle x^*, \lambda(f|\Sigma_0)\rangle)\, d\lambda\}$$

is the characteristic functional of the random vector $E\big[\int_T f dM | M(\Sigma_0)\big]$. By Theorem 3.4 $\lambda(f|\Sigma_0) \in \mathscr{L}_E(M)$ and (5.4) proves (5.2). Hence the proof of this theorem is complete.

The last theorem leads to the following problem. Let us consider the spaces

$$\mathscr{L}_E^p(M) = \mathscr{L}_E(M) \cap L_E^p, \qquad 0 \leqslant p \leqslant \infty.$$

For $p \geqslant 1$ a conditional integral is a continuous operator on L_E^p and $\mathscr{L}_E^p(M) \subset L_E^p$. We ask whether $\mathscr{L}_E^p(M)$ is invariant with respect to a conditional integral operator?

We can answer this problem affirmatively if $p \geqslant 2$ and if $1 \leqslant p < 2$ only in certain cases (see Section 6, Th. 6.4 (vi) and 6.9).

THEOREM 5.4. *Assume that* $2 \leqslant p \leqslant \infty$. *Then* $\mathscr{L}_E^p(M)$ *is invariant with respect to a conditional integral operator* $\lambda(\cdot|\Sigma_0)\colon L_E^p \mapsto L_E^p$.

Proof. Let $f \in \mathscr{L}_E^p(M)$. By Remark 2.4 and Proposition 2.3 it is sufficient to prove that $\lambda(f|\Sigma_0) \in \mathscr{L}_E^p(M_0)$, where M_0 is specified in Remark 2.4. Hence, in view of Theorem 5.2, it is enough to show that $E\|\int_T f dM_0\| < \infty$. We have

$$\iint_{\{\|f(t)u\| > 1\}} \|f(t)u\|\, \lambda(dt)\, m_0(du) \leqslant \iint_{\{\|f(t)u\| > 1\}} \|f(t)u\|^2\, \lambda(dt)\, m_0(du)$$
$$\leqslant \int_{\mathbf{R}} u^2\, m_0(du) \int_T \|f\|^2 d\lambda = \int_{[-1,1]} u^2\, m(du) \int_T \|f\|^2 d\lambda < \infty$$

which by Theorem 3.6 (ii) and Proposition 2.5 gives that $E\|\int_T f dM_0\| < \infty$ and finishes the proof.

6. Applications for stable measures on Banach spaces. In the last few years stable measures on Banach spaces were intensively investigated by many authors from various points of view. As an useful tool for a study of stable measures in [14] and [15] was constructed a random integral of a Banach space valued function with respect to a p-stable random measure. Such random integral was defined as an continuous operator on L_E^p, thus it exists only in the case when E is of stable type p (see [14] and [15]). An investigation of a general theory of stable random integrals of Banach space valued functions was started in [18]. Our approach permit to define such random integrals for every Banach space E. We will now apply the results of the previous sections to stable random integrals.

Recall that a Radon probability measure μ on a Banach space E is *p-stable (symmetric)*, $0 < p \leqslant 2$, if its characteristic functional fulfils the equality

$$\hat{\mu}(ax^*)\,\hat{\mu}(bx^*) = \hat{\mu}\big((a^p+b^p)^{1/p}x^*\big)$$

for every $x^* \in E^*$ and $a, b > 0$. If μ is a p-stable measure, then there exists a finite Radon measure σ on the unit sphere S of E, called the *spectral measure of* μ, such that

$$\hat{\mu}(x^*) = \exp\{-\int_S |\langle x^*, x\rangle|^p\,\sigma(dx)\}, \qquad x^* \in E^*.$$

(see [3], Th. 6.16, Chap. III).

A random measure M on (T, Σ, λ) is said to be *p-stable standard* $(0 < p \leqslant 2)$ if for every $A \in \Sigma$

$$E\exp\{iuM(A)\} = \exp\{-\lambda(A)|u|^p\}, \qquad u \in \boldsymbol{R}.$$

In this section M_p stands for a p-stable standard random measure and we consider only non-gaussian case, i.e. $0 < p < 2$ (see [6] and [19] for $p = 2$). Hence $M_p \sim [0, m_p]$, where

$$m_p\big((a, b]\big) = c_p \int_a^b \frac{dr}{r^{1+p}}, \qquad 0 < a < b,$$

and c_p is a positive constant (see Cor. 6.7, [3], Chap. II). In this case it is easy to compute the function H_{m_p} defined in Section 4:

$$H_{m_p}(u) = m_p\big((-1/u, 1/u)^c\big) = 2c_p u^p, \qquad u > 0.$$

Put

$$[M_p]_E = \{\sum_{j=1}^{n} x_j M_p(A_j):\ x_j \in E,\ A_j \in \Sigma,\ n \geqslant 1\}.$$

$[M_p]_E$ is a linear space consisting of p-stable random vectors. $[M_p]_E$ can be treated as a subspace of $L_E^r(\Omega, \mathscr{F}, P)$ for $0 \leqslant r < p$ and the relative topologies of $L_E^r(\Omega, \mathscr{F}, P)$ coincide on $[M_p]_E$ (see [5], Th. 6.1). Hence the closures of $[M_p]_E$ in $L_E^r(\Omega, \mathscr{F}, P)$, denoted by $\mathrm{cl}_r[M_p]_E$, are identical and the Fréchet spaces $\mathrm{cl}_r[M_p]_E$, consisting of p-stable random vectors, are isomorphic with respect to the identity map, $0 \leqslant r < p$.

THEOREM 6.1. *The spaces $\mathscr{L}_E(M_p)$ and $\mathrm{cl}_r[M_p]_E$, $0 \leqslant r < p$ are isomorphic. An isomorphism is given by the random integral*

$$\mathscr{L}_E(M_p) \ni f \mapsto \int_T f\,dM_p \in \mathrm{cl}_r[M_p]_E.$$

Proof. By Definition 2.2 $\int_T f\,dM_p \in \mathrm{cl}_0[M_p]_E = \mathrm{cl}_r[M_p]_E$ for $0 \leqslant r < p$. Put $I(f) = \int_T f\,dM_p$. $I\colon \mathscr{L}_E(M_p) \mapsto \mathrm{cl}_r[M_p]_E$ is linear, 1-1 and continuous. Hence it is sufficient to prove that I is "onto". Let $X \in \mathrm{cl}_r[M_p]_E$. There exists a sequence $\{X_n\} \subset [M_p]_E$ such that $X_n \mapsto X$ in $L_E^r(\Omega, \mathscr{F}, P)$. Since $X_n = \sum_{j=1}^{k_n} x_{nj} M_p(A_{nj})$, where $x_{nj} \in E$ and $A_{nj} \in \Sigma$ are pairwise disjoint for $1 \leqslant j \leqslant k_n$, we can write $X_n = \int_T f_n\,dM_p$, where $f_n = \sum_{j=1}^{k_n} x_{nj}\mathbf{1}_{A_{nj}}$, and by Proposition 4.2 we get $\lim_{n,m\to\infty}\int_T \|f_n - f_m\|^p\,d\lambda = 0$. Thus there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f\colon T \mapsto E$ such that $f_{n_k} \mapsto f$ a.e. By Definition 2.2 (replace (ii) by (iii)) we get that $f \in \mathscr{L}_E(M_p)$ and $\int_T f\,dM_p = X$ a.s. The proof is complete.

Remark 6.2. The above theorem is not true in the Gaussian case (i.e. $p = 2$) (see Prop. 6.3, [19]).

COROLLARY 6.3. *For every $0 < r < p < 2$ the mapping*

$$f \mapsto (E\|\int_T f\,dM_p\|^r)^{1/r}$$

is a quasi-norm (norm if $r \geqslant 1$) on $\mathscr{L}_E(M_p)$ equivalent to $\|\cdot\|_{M_p}$. Hence, for $p > 1$, $\mathscr{L}_E(M_p)$ is a Banach space.

The following properties of $\mathscr{L}_E(M_p)$ we get from the general theory of random integrals.

THEOREM 6.4. *Let E be a Banach space and M_p be a standard p-stable random measure, $0 < p < 2$. Then*

(i) *$f \in \mathscr{L}_E(M_p)$ if and only if f is strongly measurable and the function*

$$\varphi_f(x^*) = \exp\{-\int_T |\langle x^*, f\rangle|^p\,d\lambda\}$$

is the characteristic functional of a Radon probability measure on E. In this case $\mathscr{L}(\int_T f\,dM_p)^\wedge(x^*) = \varphi_f(x^*)$;

(ii) *$\mathscr{L}_E(M_p) \subset L_E^p$;*

(iii) *$\mathscr{L}_E(M_p) = L_E^p$ for $0 < p < 1$;*

(iv) *$\mathscr{L}_E(M_{p_2}) \subset \mathscr{L}_E(M_{p_1})$ whenever $0 < p_1 \leqslant p_2 < 2$;*

(v) *If* $\varphi \in L^\infty$ *and* $f \in \mathscr{L}_E(M_p)$, *then* $\varphi f \in \mathscr{L}_E(M_p)$;

(vi) *If* $p > 1$, *then* $\mathscr{L}_E(M_p)$ *is an invariant subspace of* L_E^p *with respect to a conditional integral operator. Moreover, for every sub-*σ*-field* Σ_0 *of* Σ,

$$\int_T \lambda(f|\Sigma_0)\,dM_p = E\Big(\int_T f\,dM_p \Big| M_p(\Sigma_0)\Big) \text{ a.s.},$$

where $M_p(\Sigma_0) = \sigma\{M_p(A)\colon A \in \Sigma_0\}$.

Proof. (i): Note that by the continuity of a characteristic functional at 0 we have $\int_T |\langle x^*, f\rangle|^p\,d\lambda < \infty$ for every $x^* \in E^*$. Since in our case $K(u) = |u|^p$, therefore (i) follows by Theorem 3.4.

(ii): By Proposition 4.2.

(iii): If $0 < p < 1$, then we have

$$\int_T \int_R \min\{\|f(t)u\|, 1\}\, m_p(du)\,\lambda(dt) = 2c_p(1/(1-p)+1/p)\int_T \|f(t)\|^p \lambda(dt).$$

Thus (iii) follows by Proposition 3.7 and (ii) of this theorem.

(iv): By Theorem 4.5.

(v): By Theorem 5.1.

(vi): Since $E\big\|\int_T f\,dM_p\big\| < \infty$ and (ii), Theorem 5.2 ends the proof of this theorem.

Theorem 6.4 (i) in the case when E has approximation property or $p > 1$ was proved in [18]. Statements (ii) and (iii) were established in [18], and in an equivalent form in [12] and [22] (see Cor. 6.5 below), by the standard arguments: all Banach spaces are of stable cotype less than 2 and all Banach spaces are of stable type less than 1. Propositions 4.2 and 3.7, used in our proof, permit to extend these arguments to non stable cases.

Linde [12], D. H. Thang and N. Z. Tien [22] considered the class $\Lambda_p(E^*, L_p)$ of linear operators $T\colon E \mapsto L_p$ defined by the following property: $T \in \Lambda_p(E^*, L_p)$ if and only if the function $x^* \mapsto \exp\{-\|Tx^*\|_{L_p}^p\}$ is the characteristic functional of a Radon probability measure on E (see also [13]). Every operator T from $\Lambda_p(E^*, L_p)$ is decomposable (see [12], Th. 5), i.e. there exists a strongly measurable function $f\colon T \mapsto E$ such that $T = T_f$, where $T_f x^* = \langle x^*, f\rangle$, $x^* \in E^*$. Thus by Theorem 6.4 (i) and Corollary 6.3 we obtain that the mapping

$$\mathscr{L}_E(M_p) \ni f \mapsto T_f \in \Lambda_p(E^*, L_p)$$

is an isomorphism of $\mathscr{L}_E(M_p)$ and $\Lambda_p(E^*, L_p)$. In [23] are studied properties of the function space $S_p(E) = \{f\colon T_f \in \Lambda_p(E^*, L_p)\}$, however without any connection with random integral. By Theorem 6.4 (i) $S_p(E) = \mathscr{L}_E(M_p)$. Now we can formulate the following corollary:

COROLLARY 6.5. *The spaces* $\mathrm{cl}_r[M_p]_E$, $\mathscr{L}_E(M_p)$, $S_p(E)$ *and* $\Lambda_p(E^*, L_p)$ *are isomorphic.*

This corollary solves Problem 2 in [23], i.e. the set of all simple functions is dense in $S_p(E)$. At the same time this very corollary proves that the finite rank operators are dense in $\Lambda_p(E^*, L_p)$. Theorem 6.4 (iv) solves Problem 1 in [23]. Now we can prove the following result:

THEOREM 6.6. *Let* μ *be a* p*-stable measure on a Banach space* E *with the spectral measure* σ. *Then there exist* p*-stable measures* μ_n *with the spectral measures* σ_n, $n \geqslant 1$, *such that*

(a) σ_n *has the finite support*,

(b) $\sigma_n \Rightarrow \sigma$,

(c) $\mu_n \Rightarrow \mu$ *as* $n \to \infty$.

Proof. Put $T = S$, $\Sigma = \mathscr{B}_S$, $\lambda = \sigma$ and let M_p be a standard p-stable random measure on (T, Σ, λ). Consider $f\colon T \mapsto E$, $f(x) = x$. By Theorem 6.4 (i) $f \in \mathscr{L}_E(M_p)$ and $\mathscr{L}(\int_T f\,dM_p) = \mu$. Hence there exists a sequence of simple functions $\{g_n\}$ such that $g_n \mapsto f$ a.e. and $\int_T g_n\,dM_p \mapsto \int_T f\,dM_p$ in P. Set $A_n = \{\sup_{k \geqslant n}\|g_k\| \leqslant 3/2\}$ and $f_n = g_n \mathbf{1}_{A_n}$. Then we have $A_n \nearrow A_\infty$, where $\lambda(T \setminus A_\infty) = 0$, and

$$\int_T (f_n - f)\,dM_p = \int_T \mathbf{1}_{A_n}(g_n - f)\,dM_p - \int_{A_n^c} f\,dM_p.$$

By Theorem 5.1, $\int_T \mathbf{1}_{A_n}(g_n - f)\,dM_p \mapsto 0$ in P, and

$$\int_{A_n^c} f\,dM_p = \int_T f\,dM_p - \Big(\int_{A_1} f\,dM_p + \int_{A_2 \setminus A_1} f\,dM_p + \ldots + \int_{A_n \setminus A_{n-1}} f\,dM_p\Big) \mapsto 0$$

a.s. by Ito–Nisio's theorem. Hence we found an uniformly bounded sequence of simple functions $\{f_n\}$ such that $f_n \mapsto f$ a.e. and $\int_T f_n\,dM_p \mapsto \int_T f\,dM_p$ in P. Put $\mu_n = \mathscr{L}(\int_T f_n\,dM_p)$. Then (c) is fulfilled. We prove (b). If $f_n = \sum_{j=1}^{k_n} x_{nj}\mathbf{1}_{A_{nj}}$, where $x_{nj} \in E$, $x_{nj} \neq 0$ and $A_{nj} \in \Sigma$ are pairwise disjoint $1 \leqslant j \leqslant k_n$, then $\sigma_n = \sum_{j=1}^{k_n} p_{nj}\delta_{y_{nj}}$, where $p_{nj} = \|x_{nj}\|^p \sigma(A_{nj})$ and $y_{nj} = \|x_{nj}\|^{-1}x_{nj}$, is the spectral measure of μ_n. Thus for every continuous bounded function $\varphi\colon S \mapsto \boldsymbol{R}$ we have

$$\lim_{n\to\infty}\int_S \varphi(x)\,\sigma_n(dx) = \lim_{n\to\infty}\sum_{j=1}^{k_n}\varphi\Big(\frac{x_{nj}}{\|x_{nj}\|}\Big)\|x_{nj}\|^p\sigma(A_{nj})$$
$$= \lim_{n\to\infty}\int_{\{\|f_n\|>0\}}\varphi\Big(\frac{f_n}{\|f_n\|}\Big)\|f_n\|^p\,d\lambda = \int_S \varphi(x)\,\sigma(dx)$$

by the Dominated Convergence Theorem. The proof is complete.

The next theorem in the case when E has the approximation property or $p>1$ and $T=[0,1]$ with the Lebesgue measure was proved in [18].

THEOREM 6.7. *Let (T, Σ, λ) be an atomless probability space. Then the set*

$$\{\mathcal{L}(\int_T f\,dM_p)\colon f\in \mathcal{L}_E(M_p)\}$$

is equal to the set of all p-stable symmetric measures on E.

Proof. Let σ be the spectral measure of p-stable measure μ. We shall prove that there exists a function $f\in\mathcal{L}_E(M_p)$ such that $\mathcal{L}(\int_T f\,dM_p)=\mu$. We may assume that $\sigma(S)>0$ (in another case we put $f\equiv 0$). Then there exists a strongly measurable function $g\colon T\mapsto E$ such that

$$\lambda\{t\in T\colon g(t)\in B\}=\frac{\sigma(B\cap S)}{\sigma(S)},\qquad B\in\mathcal{B}_E.$$

Put $f(t)=\sigma^{1/p}(S)g(t)$, $t\in T$. By Theorem 6.1(i) f satisfies the required conditions.

Finishing this section we consider two questions concerning 1-stable random integrals. The first is: does there exist a norm on $\mathcal{L}_E(M_1)$ so that $\mathcal{L}_E(M_1)$ is a Banach space? And the second: is $\mathcal{L}_E(M_1)$ and invariant subspace of L^1_E with respect to a conditional integral operator?

THEOREM 6.8. *Let (T, Σ, λ) be a finite measure space such that Σ contains an infinite sequence of non-zero pairwise disjoint sets. Then the following conditions are equivalent:*

(i) *$\mathcal{L}_E(M_1)$ is isomorphic to a Banach space;*

(ii) *$\mathcal{L}_E(M_1)=L^1_E$ and the topologies coincide;*

(iii) *E is of stable type 1.*

Proof. The equivalence of (ii) and (iii) follows by Corollary 2.2 in [14] and Corollary 6.3 (or by Th. 6 in [12] and Cor. 6.5); (i) obviously follows by (ii). We prove that (i) implies (iii). Let $\{\theta_n\}$ be a sequence of standard 1-stable random variables and $\{x_n\}\subset E$ be such that $\sum\|x_n\|<\infty$ and $x_n\neq 0$. It is enough to show that the sequence $\{\sum_{k=1}^n \theta_k x_k\}_{n=1}^{\infty}$ is bounded in $L^r_E(\Omega,\mathcal{F},P)$, $0<r<p$. By the assumption there exists a sequence of pairwise disjoint sets $\{A_n\}\subset\Sigma$ such that $\lambda(A_n)>0$. Define

$$f_n=\frac{x_n}{\|x_n\|\,\lambda(A_n)}\mathbf{1}_{A_n}.$$

Then $\{f_n\}_{n=1}^{\infty}\subset\mathcal{L}_E(M_1)$ is a bounded sequence and by (i) the sequence $\{\sum_{k=1}^n \|x_k\| f_k\}_{n=1}^{\infty}$ is bounded in $\mathcal{L}_E(M_1)$. Hence by Corollary 6.3

$$\sup_n\big(E\|\sum_{k=1}^n\theta_k x_k\|^r\big)^{1/r}=\sup_n\big(E\|\int_T\big(\sum_{k=1}^n\|x_k\|f_k\big)dM_1\|^r\big)^{1/r}<\infty$$

which ends the proof.

The second question remains open. We can prove a weaker result. Let $LLOG(E)$ denote the set of all strongly measurable functions $f\colon T\mapsto E$ such that $\int_T\|f\|\log(1+\|f\|)\,d\lambda<\infty$.

THEOREM 6.9. *The space*

$$\mathcal{L}_E(M_1)\cap LLOG(E)$$

is an invariant subspace of L^1_E with respect to a conditional integral operator.

Proof. By Remark 2.4 and Proposition 2.3 it is enough to prove that $\lambda(f|\Sigma_0)$ is integrable with respect to the random measure $M\sim$ $\sim[0, m_1|\{|u|\leqslant 1\}]$. By Theorem 5.2 it sufficies to show that $E\|\int_T f\,dM\|<\infty$. In view of Theorem 3.6 and Proposition 2.5 the last condition is equivalent to the following:

$$\iint_{\{\|f(t)u\|>1,|u|\leqslant 1\}}\|f(t)u\|\,m_1(du)\,\lambda(dt)<\infty.$$

Since we have

$$\iint_{\{\|f(t)u\|>1,|u|\leqslant 1\}}\|f(t)u\|\,m_1(du)\,\lambda(dt)=2c_p\int_{\{\|f\|>1\}}\|f\|\log(\|f\|)\,d\lambda<\infty,$$

the proof of our theorem is complete.

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On polynomial classification of locally convex spaces

by

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Abstract. The purpose of this article is to develop a polynomial classification of locally convex spaces, analogous to the classical linear theory and to the holomorphic theory proposed recently by Nachbin.

1. Introduction. In this article we consider polynomially bornological, polynomially barreled, polynomially infrabarreled and polynomially Mackey locally convex spaces defined in [1] (see also [2] and [3]). Our purpose is to obtain a polynomial classification of locally convex spaces, analogous to the classical linear theory and to the holomorphic theory proposed by Nachbin in [15] and [16] (see also [4] and [17]). We must emphasize that, besides its intrinsic importance, the polynomial theory can clarify the holomorphic theory as was pointed out by Aragona in [1] (see also [2] and [3]). We now indicate briefly the organization of this article.

In Section 2 we study the $(\theta_1, \ldots, \theta_m)_t$-locally convex topologies in $\mathcal{L}(E_1, \ldots, E_m; F)$ and the θ_t-locally convex topologies in $\mathcal{P}(^mE; F)$ (see [7], Chap. 3, for such a study in the linear case). We obtain an Alaoglu–Bourbaki theorem for homogeneous polynomials (Theorem 2.11) and Theorem 2.12, important tools in the subsequent sections.

In Section 3 we study the relationship among the above-mentioned polynomial concepts. As principal results of this section we obtain Theorem 3.34 and Theorem 3.37, both well known in the linear theory. As an application of such concepts, we prove Theorem 3.17, a generalization of a classical result of Bourbaki (see Remark 3.19).

In Section 4 we mention some examples of locally convex spaces which have such polynomial properties considered in the text, and observe that the linear notions are, really, more general than the corresponding polynomial ones.

This paper is based on part of my doctoral thesis ([18]), written under the guidance of Professor L. Nachbin, to whom I am sincerely indebted.

We shall adopt the notation and terminology of [4], [14], [15] and [16]. We will also use the following conventions. $\boldsymbol{N}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, will denote the systems of natural integers, real numbers and complex numbers, respectively. All topological vector spaces will be assumed to be complex. If $E_1, \ldots, E_m$