TOKYO J. MATH.
VOL. 5, NO. 2, 1982

Random Iteration of Unimodal Linear Transformations

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Introduction

Starting from the model of May, $f_\lambda(x)=\lambda x(1-x)$, $(0\leqq\lambda\leqq 4, x\in[0, 1])$, many works have been done on the topological and measure theoretical study of one-parameter family of one-dimensional transformations [1]~[8]. Especially, these works treat the phenomena of bifurcation, that is, the change of the behaviour of orbits according to the change of the parameter. In the case of the model of May, it is considered that the parameter λ expresses the characteristics of the species considered and that the value x expresses the population; and also the population of the $(n+1)$-st generation x_{n+1} is determined by the population of the n-th generation x_n by $x_{n+1}=f_\lambda(x_n)$.

In this paper we are concerned with a random family $\{f_\alpha; a\leqq\alpha\leqq b\}$ of transformations of an interval I into itself. On one hand, the random family of transformations may serve as more realistic models, e.g., models of population dynamics, if one takes into account of the randomness of the environment. On the other hand, there may appear some interesting situations. For example, the random system $\{f_\alpha\}$ may be mixing (a fortiori, exact ([10])), although each transformation f_α is not mixing.

We formulate the problem in the following manner. Let $\{f_\alpha; a\leqq\alpha\leqq b\}$ be a one-parameter family of transformations of an interval I into itself, and let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent and identically distributed random variables defined on a probability space (Ω, P) with $a\leqq X_n\leqq b$. Then, for each $\omega\in\Omega$, $x_0\in I$ is transformed to $x_1=f_{X_1(\omega)}(x_0)$, $x_2=f_{X_2(\omega)}(x_1), \cdots, x_n=f_{X_n(\omega)}(x_{n-1}), \cdots$. Our aim is to investigate the behaviour of the orbit $\{x_n; n\geqq 0\}$ for almost all $\omega\in\Omega$.

In this paper we only treat the following simplest case where the one-parameter family of transformations is that of unimodal linear transformations, that is,

Received March 30, 1981

$$f_\alpha(x)=\begin{cases}\alpha x & 0\leqq x\leqq \frac{1}{2}\\ -\alpha(x-1) & \frac{1}{2}<x\leqq 1\ ,\end{cases}$$

where $0<\alpha\leqq 2$, and the sequence of random variables is (1/2, 1/2)-Bernoulli trials with state space $\{a, b\}$ where $0<a<b\leqq 2$, that is, $P\{\omega;\ X_n(\omega)=a\}=P\{\omega;\ X_n(\omega)=b\}=1/2$. We call this case the random iteration of unimodal linear transformations.

If we identify this sequence of random variables with the dyadic transformation on [0, 1), then we can represent this random iteration of transformations by the following two-dimensional skew-product transformations

$$T_{a,b}(x, y)=\begin{cases}(f_a(x), 2y) & 0\leqq y<\frac{1}{2}\\ (f_b(x), 2y-1) & \frac{1}{2}\leqq y<1\ .\end{cases}$$

So, in order to investigate the behaviour of orbit of random iteration, it is sufficient to investigate the behaviour of orbit of skew-product transformation $T_{a,b}$. Our aim here is to answer the following three questions:

Question-1. Does $T_{a,b}$ have an invariant measure which is absolutely continuous with respect to the Lebesgue measure?

Question-2. In the case that $T_{a,b}$ has an invariant measure, is $T_{a,b}$ ergodic with respect to this invariant measure?

Question-3. When is $T_{a,b}$ exact?

About these questions, we get the following results:

Result-1. The necessary and sufficient condition for $T_{a,b}$ to have an invariant finite measure which is absolutely continuous with respect to the Lebesgue measure is that $ab>1$; in this case we get its density function $h_{a,b}(x)$ explicitly. (Theorem 2.8.) This result shows that in the case $ab>1$, even if $a<1$, the orbit of random iteration shows complicated behaviour by the influence of f_b.

Result-2. $T_{a,b}$ is ergodic with respect to this invariant measure. (Theorem 3.4.) This shows that almost all random iterated orbits $\{x_n\colon n\geqq 0\}$ satisfy the relation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} I_A(x_k)=\int_A h_{a,b}(x)dx\ ,\quad \text{a.e. } \omega\in\Omega\ ,$$

for each Borel set A.

Result-3. Under some additional condition, $T_{a,b}$ is exact. (Theorem 4.5.) In particular if f_b is exact, then $T_{a,b}$ is also exact. But we emphasize that $T_{a,b}$ may be exact even if both f_a and f_b are not exact. So it is important to determine completely the pair of parameters (a, b) for which $T_{a,b}$ is exact, and also the shape of the support of $h_{a,b}(x)$ for each (a, b). But we cannot get the complete solution of this problem now.

In concluding these introductory remarks, we would like to express our thanks to professors Y. Takahashi and T. Ohno. Especially we owe to T. Ohno for the idea of proof of ergodicity and exactness of $T_{a,b}$.

§ 1. Definition of random iteration and the case $ab \leqq 1$.

Let a and b be real numbers such that $0<a<b\leqq 2$ and let f_a and f_b be unimodal linear transformations, that is,

$$f_\alpha(x)=\begin{cases}\alpha x & 0\leqq x\leqq \frac{1}{2}\\ -\alpha(x-1) & \frac{1}{2}<x\leqq 1\,.\end{cases} \tag{1}$$

Let $X=[0, 1]$, $Y=[0, 1)$ and let m_1 and m_2 be the Lebesgue measure on X and Y, respectively. And let $m=m_1\times m_2$ be the Lebesgue measure on $X\times Y$. Let $T=T_{a,b}$ be the skew-product transformation on $X\times Y$ defined by

$$T(x, y)=\begin{cases}(f_a(x), 2y) & 0\leqq y<\frac{1}{2}\\ (f_b(x), 2y-1) & \frac{1}{2}\leqq y<1\,.\end{cases} \tag{2}$$

Hereafter we identify the space $Y=[0, 1)$ with the space $\{a, b\}^N$ of the representation of dyadic expansion (here we use the symbol a, b instead of the usual symbol $0, 1$ of dyadic expansion) and represent $y\in Y$ as $y=(y_1, y_2, \cdots, y_n, \cdots)$. Using this representation, T is represented as

$$T(x, y)=(f_{y_1}(x), \sigma y)\;, \tag{3}$$

where σ is the shift operator on $\{a, b\}^N$. And by this identification, the Lebesgue measure m_2 on $Y=[0, 1)$ can be identified with $(1/2, 1/2)$-Bernoulli measure on $\{a, b\}^N$.

LEMMA 1.1. *For each* $y_1, y_2, \cdots, y_n \in \{a, b\}$, *we have*

$$f_{y_n}\cdots f_{y_1}(x) \leqq y_1 \cdots y_n x . \tag{4}$$

PROOF. From the definition of $f_\alpha(x)$, it is clear that $f_{y_n}\cdots f_{y_1}(x)$ is a continuous and piecewise linear function with derivative $\pm y_1 \cdots y_n$ and satisfies $f_{y_n}\cdots f_{y_1}(0)=0$. So lemma is proved.

LEMMA 1.2. *For* m_2-*a.e.* $y \in Y$, *we have that* $(y_1 \cdots y_n)^{1/n} \to \sqrt{ab}$ *as* $n \to \infty$.

PROOF. It is clear from the law of large numbers

$$\frac{1}{n}\sum_{k=1}^{n} \log y_k \to \frac{1}{2}(\log a + \log b) \quad m_2\text{-a.e.} y. \tag{5}$$

PROPOSITION 1.3. *In the case* $ab<1$, *we have that* $f_{y_n}\cdots f_{y_1}(x) \to 0$ *as* $n \to \infty$ *for* m_2-*a.e.* y, *and so* T *has no invariant measure which is absolutely continuous with respect to the Lebesgue measure* m *on* $X \times Y$.

PROOF. From Lemma 1.2, we have that $m_2\{y_1 \cdots y_n \to 0 \text{ as } n \to \infty\}=1$. So, by using Lemma 1.1, we have that $m_2\{f_{y_n}\cdots f_{y_1}(x) \to 0 \text{ as } n \to \infty\}=1$. The rest of the assertion is clear from this.

PROPOSITION 1.4. *In the case* $ab=1$, T *has no invariant probability measure which is absolutely continuous with respect to* m.

PROOF. Assume that T has an absolutely continuous invariant probability measure μ. Let ν be the measure on Y defined by $\nu(F)=\mu(X \times F)$ for Borel set $F \subset Y$. It is clear that ν is absolutely continuous with respect to m_2, and from the relation

$$\nu(\sigma^{-1}F)=\mu(T^{-1}(X \times F))=\mu(X \times F)=\nu(F) , \tag{6}$$

it follows that ν is σ-invariant, and so we obtain $\nu=m_2$. From $ab=1$ it follows that

$$\int_Y \log y_n dm_2(y)=\frac{1}{2}(\log a+\log b)=0 , \tag{7}$$

and so, from the central limit theorem we obtain that

$$m_2\left\{y; \frac{1}{\sqrt{n}}\log(y_1 \cdots y_n)<\beta\right\} \longrightarrow \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi\gamma}} \exp\left(-\frac{x^2}{2\gamma^2}\right)dx \tag{8}$$

for each β. Let us choose $\beta<0$, and let us denote by A_n the set of the left-hand side of (8) and denote by t the limit value of (8). From (8) it follows that $m_2(A_n)>t/2$ for sufficiently large n. Let $c>0$ and let n be large enough to satisfy $\exp(\beta\sqrt{n})<c$. Then from Lemma 1.1, we have that $f_{y_n}\cdots f_{y_1}(x)<c$ for each $y\in A_n$ and each $x\in X$.

For each $y_1, y_2, \cdots, y_n \in \{a, b\}$, let $[y_1, y_2, \cdots, y_n]$ be the cylinder set in Y, that is, the set of y whose first n symbols are equal to $y_1, y_2, \cdots, y_n$. Then it follows that

$$T^{-n}([0, c)\times Y)=\sum_{y_1,\cdots,y_n} f_{y_1}^{-1}\cdots f_{y_n}^{-1}([0, c))\times[y_1, \cdots, y_n]\,, \tag{9}$$

where the symbol $\sum$ means disjoint union. If we choose β, c and n as above, then we have that

$$\begin{aligned}\mu([0, c)\times Y)&=\mu(T^{-n}([0, c)\times Y))\\ &=\sum_{y_1,\cdots,y_n}\mu(f_{y_1}^{-1}\cdots f_{y_n}^{-1}([0, c))\times[y_1, \cdots, y_n])\\ &\geqq\sum_{[y_1\cdots,y_n]\subset A_n}\mu(X\times[y_1, \cdots, y_n])\\ &=m_2(A_n)>\frac{t}{2}\,.\end{aligned} \tag{10}$$

So we have $\mu([0, c)\times Y)>t/2$ for each $c>0$, which contradicts to the absolute continuity of μ.

REMARK 1.5. In the case $ab=1$, T may have m-$a.c.$ invariant measure with infinite total mass. In fact, in the case $a=1/2$ and $b=2$, the following function is the density of T-invariant measure.

$$h(x)=\begin{cases}1 & \frac{1}{2}\leqq x\leqq 1\\ 2^{n+2}-5 & 2^{-(n+1)}\leqq x<2^{-n} \quad (n\geqq 1)\,.\end{cases} \tag{11}$$

§ 2. The invariant measure of T.

In this section, we give the density of m-$a.c.$ invariant measure in the case $ab>1$.

LEMMA 2.1. *In the case $ab>1$, we have that*

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{f_{y_n}\cdots f_{y_1}(x)}{y_1\cdots y_n}<\infty \tag{12}$$

for each $x\in X$.

PROOF. Let $c=(1/2)(\log a+\log b)$, $d=(1/2)(\log b-\log a)$ and

$$B_n=\left\{y\in Y;\ \frac{\sum_{k=1}^{n}\log y_k-nc}{\sqrt{n}\,d}<-\sqrt{4\log n}\right\}\tag{13}$$
$$=\{y\in Y;\ y_1\cdots y_n<\exp(nc-\sqrt{4n\log n}d)\}\ ,$$

then from the central limit theorem we have that

$$m_2(B_n)\leqq e^{-2\log n}=\frac{1}{n^2}\ .\tag{14}$$

The left hand side of (12) is reduced to

$$\sum_{n=1}^{\infty}\int_Y\frac{f_{y_n}\cdots f_{y_1}(x)}{y_1\cdots y_n}dm_2(y)\tag{15}$$
$$=\sum_{n=1}^{\infty}\int_{B_n}\frac{f_{y_n}\cdots f_{y_1}(x)}{y_1\cdots y_n}dm_2(y)+\sum_{n=1}^{\infty}\int_{B_n^c}\frac{f_{y_n}\cdots f_{y_1}(x)}{y_1\cdots y_n}dm_2(y)\ .$$

By using Lemma 1.1 and (14), the first term of the right-hand side of (15) is majorated by $x\sum_{n=1}^{\infty}m_2(B_n)<\infty$. And the second term of the right-hand side of (15) is majorated by $\sum_{n=1}^{\infty}\exp(-nc+\sqrt{4n\log n}\,d)<\infty$. So we obtain Lemma 2.1.

Now let us give the f-expansion formula of x. From now on, we assume $ab>1$. If we write

$$\varepsilon(x)=\begin{cases}+1 & 0\leqq x\leqq\frac{1}{2}\\ -1 & \frac{1}{2}<x\leqq 1\ ,\end{cases}\tag{16}$$

then it follows from (1) that $x=(1-\varepsilon(x))/2+(\varepsilon(x)/\alpha)f_\alpha(x)$. By using this relation successively, we obtain that

$$x=\frac{1}{2}\sum_{k=0}^{n-1}\frac{\varepsilon(x)\varepsilon(f_{y_1}(x))\cdots\varepsilon(f_{y_{k-1}}\cdots f_{y_1}(x))(1-\varepsilon(f_{y_k}\cdots f_{y_1}(x)))}{y_1\cdots y_k}\tag{17}$$
$$+\frac{\varepsilon(x)\varepsilon(f_{y_1}(x))\cdots\varepsilon(f_{y_{n-1}}\cdots f_{y_1}(x))}{y_1\cdots y_n}f_{y_n}\cdots f_{y_1}(x)$$

for each n and each $y_1,\cdots,y_n$. Now if we multiply $1/2^n$ to each side of (17) and sum for all values of $y_1,\cdots,y_n$, then we obtain that

(18)
$$x=\frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{2^k}\sum_{y_1,\cdots,y_k}\frac{\varepsilon(x)\cdots(1-\varepsilon(f_{y_k}\cdots f_{y_1}(x)))}{y_1\cdots y_k}$$
$$+\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{\varepsilon(x)\cdots\varepsilon(f_{y_{n-1}}\cdots f_{y_1}(x))}{y_1\cdots y_n}f_{y_n}\cdots f_{y_1}(x)\ .$$

From Lemma 2.1, the last term of (18) converges to 0 as $n\to\infty$, so we obtain the following.

PROPOSITION 2.2. *We have the following average f-expansion of x:*

(19)
$$x=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{S_x(y_1,\ \cdots,\ y_{n-1})(1-\varepsilon(f_{y_n}\cdots f_{y_1}(x)))}{y_1\cdots y_n}$$
$$=\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{f_{y_n}\cdots f_{y_1}(x)>1/2}\frac{S_x(y_1,\ \cdots,\ y_{n-1})}{y_1\cdots y_n}\ ,$$

where

(20)
$$S_x(y_1,\ \cdots,\ y_{n-1})=\varepsilon(x)\varepsilon(f_{y_1}(x))\cdots\varepsilon(f_{y_{n-1}}\cdots f_{y_1}(x))\ .$$

Let us write simply $S(y_1,\ \cdots,\ y_{n-1})$ for $S_{1/2}(y_1,\ \cdots,\ y_{n-1})$ and define a function $h(x)$ on X by

(21)
$$h(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{S(y_1,\ \cdots,\ y_{n-1})}{y_1\cdots y_n}I_{[0,f_{y_n}\cdots f_{y_1}(1/2)]}(x)\ ,$$

where I_A is the indicator function of the set A. Then we have the following several lemmas.

LEMMA 2.3. *For almost all* $x\in X$, $h(x)$ *converges absolutely and satisfies* $\int_X h(x)dx<\infty$.

PROOF. Define a function $\bar{h}(x)$ by

(22)
$$\bar{h}(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{1}{y_1\cdots y_n}I_{[0,f_{y_n}\cdots f_{y_1}(1/2)]}(x)\ .$$

Then $\bar{h}(x)$ is non-negative and monotone increasing. By using Lemma 2.1, we obtain that

(23)
$$\int_X\bar{h}(x)dx=\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{f_{y_n}\cdots f_{y_1}\left(\frac{1}{2}\right)}{y_1\cdots y_n}<\infty\ .$$

It is clear that $|h(x)|\leqq\bar{h}(x)$, so we obtain Lemma 2.3.

LEMMA 2.4. The function $h(x)$ satisfies

$$\int_{T^{-1}A} h(x)dxdy = \int_A h(x)dxdy \tag{24}$$

for each Borel set A in $X \times Y$.

PROOF. To prove (24), it is sufficient to show

$$\begin{aligned} h(x) &= \frac{1}{2a}\left\{h\left(\frac{x}{a}\right)+h\left(1-\frac{x}{a}\right)\right\} I_{[0,f_a(1/2)]}(x) \\ &\quad + \frac{1}{2b}\left\{h\left(\frac{x}{b}\right)+h\left(1-\frac{x}{b}\right)\right\} I_{[0,f_b(1/2)]}(x) . \end{aligned} \tag{25}$$

It is easy to show that

$$\begin{aligned} &\left\{I_{[0,\xi]}\left(\frac{x}{\alpha}\right)+I_{[0,\xi]}\left(1-\frac{x}{\alpha}\right)\right\} I_{[0,f_\alpha(1/2)]}(x) \\ &\qquad = \varepsilon(\xi) I_{[0,f_\alpha(\xi)]}(x) + (1-\varepsilon(\xi)) I_{[0,f_\alpha(1/2)]}(x) \end{aligned} \tag{26}$$

for each $\xi \in X$, $\alpha \in \{a, b\}$ and $x \neq f_\alpha(\xi)$. The right-hand side of (25) is equal to

$$\begin{aligned} &\sum_\alpha \frac{1}{2\alpha}\left\{h\left(\frac{x}{\alpha}\right)+h\left(1-\frac{x}{\alpha}\right)\right\} I_{[0,f_\alpha(1/2)]}(x) \\ &\quad = \sum_{n=1}^\infty \frac{1}{2^{n+1}} \sum_{y_1,\cdots,y_n,\alpha} \frac{S(y_1, \cdots, y_{n-1})}{y_1\cdots y_n\alpha} \\ &\qquad \times \left\{I_{[0,\xi_n]}\left(\frac{x}{\alpha}\right)+I_{[0,\xi_n]}\left(1-\frac{x}{\alpha}\right)\right\} I_{[0,f_\alpha(1/2)]}(x) , \end{aligned} \tag{27}$$

where $\xi_n = f_{y_n}\cdots f_{y_1}(1/2)$. By using (26) repeatedly, it follows that the right-hand side of (27) is equal to

$$\begin{aligned} &\sum_{n=1}^\infty \frac{1}{2^{n+1}} \sum_{y_1,\cdots,y_n,\alpha} \frac{S(y_1, \cdots, y_n)}{y_1\cdots y_n\alpha} I_{[0,f_\alpha(\xi_n)]}(x) \\ &\quad + \sum_\alpha \frac{1}{\alpha}\left\{\frac{1}{2}\sum_{n=1}^\infty \frac{1}{2^n}\sum_{y_1,\cdots,y_n} \frac{S(y_1, \cdots, y_{n-1})}{y_1\cdots y_n}\left(1-\varepsilon\left(f_{y_n}\cdots f_{y_1}\left(\frac{1}{2}\right)\right)\right)\right\} \\ &\quad \times I_{[0,f_\alpha(1/2)]}(x) \\ &= \sum_{n=1}^\infty \frac{1}{2^{n+1}} \sum_{y_1,\cdots,y_{n+1}} \frac{S(y_1, \cdots, y_n)}{y_1\cdots y_{n+1}} I_{[0,f_{y_{n+1}}\cdots f_{y_1}(1/2)]}(x) \\ &\quad + \frac{1}{2}\sum_{y_1} \frac{1}{y_1} I_{[0,f_{y_1}(1/2)]}(x) \\ &= h(x) \end{aligned} \tag{28}$$

for almost all x. So we obtain (25).

LEMMA 2.5. *For some real number $t<1/2$, $h(x)$ takes a positive value on $(t, 1/2]$*

PROOF. From (19) and (21) it follows that

$$h\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{f_{y_n}\cdots f_{y_1}(1/2)>1/2}\frac{s(y_1, \cdots, y_{n-1})}{y_1\cdots y_n}=\frac{1}{2}. \tag{29}$$

There exists a natural number n_0 which satisfies

$$\sum_{n=n_0+1}^{\infty}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{1}{y_1\cdots y_n}I_{[0,f_{y_n}\cdots f_{y_1}(1/2)]}(x)<\frac{1}{4} \tag{30}$$

for each $x\geqq 1/4$. Here we use the fact that $\bar{h}(1/4)$ converges and that $\bar{h}(x)$ is monotone decreasing. Let

$$t'=\max\left\{f_{y_n}\cdots f_{y_1}\left(\frac{1}{2}\right)<\frac{1}{2};\ 1\leqq n\leqq n_0,\ y_1, \cdots, y_n\in\{a, b\}\right\} \tag{31}$$

and let $t=t'\vee(1/4)$. Then it follows that

$$\begin{aligned}&\sum_{n=1}^{n_0}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{S(y_1, \cdots, y_{n-1})}{y_1\cdots y_n}I_{[0,f_{y_n}\cdots f_{y_1}(1/2)]}(x)\\&=\sum_{n=1}^{n_0}\frac{1}{2^n}\sum_{y_1,\cdots,y_n}\frac{S(y_1, \cdots, y_{n-1})}{y_1\cdots y_n}I_{[0,f_{y_n}\cdots f_{y_1}(1/2)]}\left(\frac{1}{2}\right)\\&>\frac{1}{2}-\frac{1}{4}=\frac{1}{4}\end{aligned} \tag{32}$$

for each $x\in(t, 1/2]$. From (30) and (32) we see that $h(x)>0$ for each $x\in(t, 1/2]$.

The following two lemmas are proved in [6].

LEMMA 2.6. *Suppose that an integrable function $h(x, y)$ on $X\times Y$ satisfies*

$$\int_A h(x, y)dxdy=\int_{T^{-1}A}h(x, y)dxdy \tag{33}$$

for each Borel set $A\subset X\times Y$. Let us denote by $P(N, Z)$ the set of $(x, y)\in X\times Y$ such that $h(x, y)>0(<0, =0$, respectively). Then we have $TP=P$ m-a.e. and $TN=N$ m-a.e.

LEMMA 2.7. Suppose that *$h(x, y)$ satisfies the same assumption of Lemma 2.6., and suppose that a Borel set $B\subset X\times Y$ satisfies for some*

n_0 *that* $T^n B \cap B = \phi$ *m-a.e. for each* $n > n_0$. *Then we have* $h(x, y) = 0$ *m-a.e.*$(x, y) \in B$.

Now we can prove the following

THEOREM 2.8. *Let* μ *be the probability measure on* $X \times Y$ *defined by* $\mu = (1/C)h(x)dxdy$, *where C is the normalizing constant, that is,* $C = \int_X h(x)dx$. *Then* μ *is a T-invariant probability measure, and its support is given by*

$$\Gamma \times Y = \bigcup_{n=0}^{\infty} T^n((\Gamma_a \cup \Gamma_b) \times Y) , \tag{34}$$

where Γ_α *is the support of the* f_α*-invariant measure (whose explicit form is given in* [6]*). In the case* $\alpha \leqq 1$, *we assume that* $\Gamma_\alpha = \phi$.

PROOF. From Lemmas 2.5. and 2.6, we have that the set

$$\bigcup_{n=0}^{\infty} T^n\left(\left(t, \frac{1}{2}\right] \times Y\right) = \bigcup_{n=0}^{\infty} \bigcup_{\nu_1, \cdots, \nu_n} f_{\nu_n} \cdots f_{\nu_1}\left(\left(t, \frac{1}{2}\right]\right) \times Y \tag{35}$$

is contained in P. It is easy to show that $\bigcup_{n=0}^{M} f_\alpha^n(t, 1/2] = \Gamma_\alpha$ for some M, and so it follows that $(\Gamma_a \cup \Gamma_b) \times Y \subset P$, and by using Lemma 2.6. we get $\Gamma \times Y \subset P$. On the other hand we already showed in [6] that in the case $\alpha > 1$, m_1-*a.e.*$x \in \Gamma_\alpha^c$ satisfy $f_\alpha^n x \in \Gamma_\alpha$ for some n, which shows that for *a.e.*$(x, y) \in (\Gamma \times Y)^c$ there is a natural number n_0 such that $T^n(x, y) \in \Gamma \times Y$ for each $n \geqq n_0$. By using Lemma 2.7, it follows that $\Gamma \times Y = P$ and $N = \phi$, so we have Theorem 2.8.

In the rest of this section, we give several comments on the shape of Γ.

(i) The case $b \geqq \sqrt{2}$.

If $1 \leqq a < b$, then $\Gamma = [a(1-b/2), b/2]$, and if $a < 1$, then $\Gamma = [0, b/2]$.

(ii) The case $\sqrt[4]{2} \leqq b < \sqrt{2}$.

Let a_1 be the minimum of a which satisfies $f_a f_b f_a^2 f_b(1/2) \geqq f_a^2 f_b(1/2)$. If $a_1 \leqq a < b$, then $\Gamma = I_1 \cup I_2$, where $I_1 = [a(1-b/2), b(1-(a^2/2)(1-b/2))]$ and $I_2 = [a^2(1-b/2), b/2]$. If $1 < a < a_1$, then $\Gamma = [a(1-b/2), b/2]$. If $a = 1$, then $\Gamma = I_1 \cup I_2 \cup I_3$, where $I_1 = [1-b/2, 1-b^2(1-b/2)]$, $I_2 = [b(1-b/2), b(1-b^2(1-b/2))]$ and $I_3 = [b^2(1-b/2), b/2]$. If $a < 1$, then $\Gamma = [0, b/2]$.

It is complicated to determine the shape of Γ for all case of parameter (a, b), so we only mention the above cases.

§ 3. Ergodicity of T.

In this section we prove the ergodicity of T. Let E be a T-invariant set such that $\mu(E)>0$. Let $E_x=\{y \in Y;\ (x, y) \in E\}$ for each $x \in X$. Then, by Fubini's theorem, we have that $\mu(E)=\int_X m_2(E_x)h(x)dx$. Now let us define the set $E^n_{x,\delta}$ for a positive number δ and a natural number n by

$$E^n_{x,\delta}=\Big(\bigcup_{m_2(E_x|[y_1,\cdots,y_n])>1-\delta}[y_1, \cdots, y_n]\Big)\cap E_x\ . \tag{36}$$

Then we have the following

LEMMA 3.1. *For each $x \in X$ and $\delta>0$, there exists a natural number $n(x)$ which satisfies*

$$m_2(E^n_{x,\delta})>m_2(E_x)-\delta \qquad \text{for each} \quad n \geqq n(x)\ . \tag{37}$$

PROOF. Let $\mathscr{F}_n$ be the σ-field generated by the family of cylinder sets $\{[y_1, \cdots, y_n]\}$. Then, by the martingale convergence theorem, it follows that $E(I_{E_x}|\mathscr{F}_n)\to I_{E_x}$. So there exists a natural number $n(x)$ such that

$$m_2\{y;\ |E(I_{E_x}|\mathscr{F}_n)(y)-I_{E_x}(y)|>\delta\}<\delta \tag{38}$$

for each $n \geqq n(x)$. But it is clear that

$$\{y;\ |E(I_{E_x}|\mathscr{F}_n)(y)-I_{E_x}(y)|>\delta\}\supset E_x\cap\{y;\ m_2(E_x|[y_1, \cdots, y_n])<1-\delta\}\ , \tag{39}$$

so we get (37).

LEMMA 3.2. $E^n_\delta=\bigcup_{x \in X}\{x\}\times E^n_{x,\delta}$. *Then there exists a natural number n_0 which satisfies $\mu(E^n_\delta)>\mu(E)-2\delta$ for each $n \geqq n_0$.*

PROOF. Choose a natural number n_0 which satisfies

$$\frac{1}{C}\int_{\{n(x)>n_0\}} h(x)dx<\delta\ .$$

Then for each $n \geqq n_0$, it follows that

$$\begin{aligned}\mu(E^n_\delta)&=\frac{1}{C}\int_X m_2(E^n_{x,\delta})h(x)dx \\ &\geqq\frac{1}{C}\int_{\{n(x)\leqq n_0\}} m_2(E^n_{x,\delta})h(x)dx \\ &\geqq\frac{1}{C}\int_{\{n(x)\leqq n_0\}}\{m_2(E_x)-\delta\}h(x)dx \\ &>\frac{1}{C}\int_X\{m_2(E_x)-\delta\}h(x)dx-\frac{1}{C}\int_{\{n(x)>n_0\}} h(x)dx \\ &>\mu(E)-2\delta\ ,\end{aligned} \tag{40}$$

so we have Lemma 3.2.

LEMMA 3.3. *We have $E=F\times Y$ $\mu-a.e.$ for some Borel set F in X.*

PROOF. Let $E_\delta=\{(x, y)\in E;\ m_2(E_x)>1-\delta\}$. Then, from the definition of E_δ^n and the T-invariantness of E, it is clear that $E_\delta\supset T^nE_\delta^n$. So it follows that

$$\mu(E_\delta)\geqq\mu(T^nE_\delta^n)=\mu(T^{-n}T^nE_\delta^n)\geqq\mu(E_\delta^n)>\mu(E)-2\delta\ . \tag{41}$$

Let $\delta\to 0$. Then we obtain $\mu\{(x, y)\in E;\ m_2(E_x)=1\}\geqq\mu(E)$, which shows that $E=F\times Y$ μ-*a.e.* for some F.

By this lemma we have that

$$F\times Y=T^{-1}(F\times Y)=(f_a^{-1}F\times[a])\cup(f_b^{-1}F\times[b]) \quad \mu\text{-}a.e\ , \tag{42}$$

so it follows that $f_a^{-1}F=f_b^{-1}F=F$ μ-*a.e.* And the property of Γ_α (in the case $\alpha>1$, almost all $x\in\Gamma_\alpha^c$ satisfy $f_\alpha^n x\in\Gamma_\alpha$ for some n) shows that $m_1(\Gamma_\alpha\cap F)>0$, so, by using ergidicity of each f_α, it follows that $F\supset\Gamma_a\cup\Gamma_b$, which shows $F\supset\Gamma$. So we obtain the following

THEOREM 3.4. *In the case $ab>1$, T is ergodic with respect to the measure μ.*

Because the density $h(x)$ is independent of y, we have following

COROLLARY 3.5. *For m_2-a.e.y,*

$$\frac{1}{n}\sum_{k=0}^{n-1}I_A(f_{y_k}\cdots f_{y_1}(x))\longrightarrow\int_A h(\xi)d\xi \qquad for\quad a.e.x\ . \tag{43}$$

§ 4. Exactness of T.

In this section, we treat the exactness of the transformation T([10]).

LEMMA 4.1. *Let $ab>1$. For each $\varepsilon>0$, there exist a natural number M and a real number $\beta>2$ which satisfy*

$$m_2\{y\in Y;\ y_1\cdots y_{nM}>\beta^n \quad for\ each\quad n\}>1-\varepsilon\ . \tag{44}$$

PROOF. From Lemma 1.2, it follows that, for each $\delta>0$,

$$m_2\Big(\bigcup_{N=1}^{\infty}\bigcap_{n\geqq N}\{y;\ |(y_1\cdots y_n)^{1/n}-\sqrt{ab}|<\delta\}\Big)=1\ . \tag{45}$$

Let us choose δ such that $\sqrt{ab}-\delta>1$. Then there exists a natural number N which satisfies

(46) $$m_2\Big(\bigcap_{n\geq N}\{y\in Y;\ |(y_1\cdots y_n)^{1/n}-\sqrt{ab}|<\delta\}\Big)>1-\varepsilon\ .$$

So, if we choose M satisfying $\beta=(\sqrt{ab}-\delta)^M>2$, then we obtain (44).

Define the subset G_ε of Y by $G_\varepsilon=\{y\in Y;\ y_1\cdots y_{nM}>\beta^n$ for each $n\}$, and define partitions of X, $\mathscr{Q}_{y_1,\cdots,y_M}$, $\mathscr{Q}$, $\mathscr{P}_{y_1,\cdots,y_M}$ and $\mathscr{P}_y^{(k)}$, for each $y\in Y$ and k, as follows:

(47) $\mathscr{Q}_{y_1,\cdots,y_M}=$the partition of X given from extremum values of $f_{(1)}$

(48) $$\mathscr{Q}=\bigvee_{y_1,\cdots,y_M}\mathscr{Q}_{y_1,\cdots,y_M}$$

(49) $$\mathscr{P}_{y_1,\cdots,y_M}=f_{(1)}^{-1}\mathscr{Q}\vee\{\text{monotone intervals of } f_{(1)}\}$$

(50) $$\mathscr{P}_y^{(k)}=\mathscr{P}_{y_1,\cdots,y_M}\vee f_{(1)}^{-1}\mathscr{P}_{y_{M+1},\cdots,y_{2M}}\vee\cdots\vee f_{(k-1)}^{-1}\mathscr{P}_{y_{(k-1)M+1},\cdots,y_{kM}}\ ,$$

where $f_{(k)}$ is defined by putting $f_{(k)}=f_{y_{kM}}\cdots f_{y_1}$ and the symbol $\vee$ means a generated partition. Then we have the following.

LEMMA 4.2. *For each $y\in G_\varepsilon$ and almost all $x\in X$, there exists $Q\in\mathscr{Q}$ such that the element C_n of $\mathscr{P}_y^{(n)}$ containing x satisfies $f_{(n)}(C_n)\supset Q$ for infinitely many n.*

PROOF. Let $\mathscr{Q}=\{Q_1, Q_2, \cdots, Q_q\}$ and $y\in G_\varepsilon$. We show that, for each $\gamma>0$ and n, there exists $m>n$ which satisfies

(51) $$m_1\Big(\bigcup_{i=n}^{m}\ \bigcup_{C\in\mathscr{P}_y^{(i)};\ f_{(i)}(C)\supset Q_j\text{ for some } j} C\Big)>1-\gamma\ .$$

In fact, let us define families $\mathscr{E}_{n+k}$ for $k=0, 1, 2, \cdots$ inductively as follows:

(52) $$\mathscr{E}_n=\{C\in\mathscr{P}_y^{(n)};\ f_{(n)}(C)\not\supset Q_j\quad\text{for any}\quad j\}$$

(53) $\mathscr{E}_{n+k}=\{C\in\mathscr{P}_y^{(n+k)};\ C$ is the subset of some element of $\mathscr{E}_{n+k-1}$ and $f_{(n+k)}(C)\not\supset Q_j$ for any $j\}$ $(k\geq 1)$.

Then it is clear that each element of $\mathscr{E}_{n+k-1}$ contains at most two elements of $\mathscr{E}_{n+k}$ and that the length of each element of $\mathscr{E}_{n+k}$ is not greater than $\beta^{-(n+k)}$. So it follows that

(54) $$m_1\Big(\bigcup_{C\in\mathscr{E}_{n+k}} C\Big)\leq 2^k\#\mathscr{E}_n\beta^{-(n+k)}\ .$$

And so, for sufficiently large $m>n$, we have $m_1(\bigcup_{C\in\mathscr{E}_{n+m}}C)\leq\gamma$, which means (51).

Let us denote the set of (51) by E_n^m, that is,

$$E_n^m=\bigcup_{i=n}^{m}\ \bigcup_{C\in\mathscr{P}_y^{(i)};\ f_{(i)}(C)\supset Q_j \text{ for some } j} C \tag{55}$$

and choose a sequence γ_k which satisfies $\sum_{k=1}^{\infty}\gamma_k<\infty$. Then, using (51) repeatedly, we obtain that $m_1(E_{m_k}^{m_{k+1}})>1-\gamma_k$ for some $m_1<m_2<\cdots$, which implies that $m_1(\bigcup_{i=1}^{\infty}\bigcap_{k\geq i}E_{m_k}^{m_{k+1}})=1$. If $x\in\bigcup_{i=1}^{\infty}\bigcap_{k\geq i}E_{m_k}^{m_{k+1}}$, then, for sufficiently large k, there exists a natural number n between m_k and m_{k+1} such that the element C of $\mathscr{P}_y^{(n)}$ which contains x satisfies $f_{(n)}(C)\supset Q_j$ for some j. Let us denote by Q an element Q_j which appears infinitely many times in above statement. Then we obtain Lemma 4.2.

LEMMA 4.3. *Let $\mu(E)>0$. Then there exists sequence of natural numbers $n_1<n_2<\cdots$ and a subset $B\subset X$ with $m_1(B)>0$ such that*

$$\bigcup_{i=1}^{\infty}\bigcap_{k\geq 1}T^{n_k}E\supset B\times Y \quad m\text{-a.e.} \tag{56}$$

PROOF. Choose $\varepsilon>0$ so as to satisfy $\varepsilon<\mu(E)$. Then, using Lemma 4.1, we get $\mu(E\cap(X\times G_\varepsilon))>0$. So we can assume $E\subset X\times G_\varepsilon$. Denote by $\tilde{E}$ the set of density points of E, that is, the set of such points (x, y) that $(m(I_n\cap E)/m(I_n))\to 1$ for any sequence of neighbourhoods $\{I_n\}$ of (x, y) which satisfies $m(I_n)\to 0$. Then it is well-known that $\mu(\tilde{E})=\mu(E)$. Let $(x, y)\in\tilde{E}$. Then, by Lemma 4.2, there exist a sequence of natural numbers $n_1<n_2<\cdots$ and $Q\in\mathscr{Q}$ such that the set $C_{n_k}\in\mathscr{P}_y^{(n_k)}$ which contains x satisfies $f_{(n_k)}(C_{n_k})\supset Q$. Put $\Delta_{n_k}=C_{n_k}\times[y_1, \cdots, y_{n_k}]$. Then, from the definition of $\tilde{E}$, it follows that $(m(\Delta_{n_k}\cap E)/m(\Delta_{n_k}))\to 1$. Choose a sequence $\{\delta_k\}$ so as to satisfy $\sum_{k=1}^{\infty}\delta_k\leq 1/2$. Then there exists a subsequence of $\{n_k\}$, which we denote by the same notation $\{n_k\}$ again, such that

$$\frac{m(\Delta_{n_k}\cap E)}{m(\Delta_{n_k})}\geq 1-\delta_k^2\,. \tag{57}$$

It is clear that $T^{n_k}\Delta_{n_k}=B_{n_k}\times Y\supset Q\times Y$ where $B_{n_k}=f_{y_{n_k}}\cdots f_{y_1}C_{n_k}$ and that T^{n_k} is a monotone linear map from Δ_{n_k} to $B_{n_k}\times Y$. So, putting $A_{n_k}=T^{n_k}(\Delta_{n_k}\cap E)$, we have that

$$\frac{m(A_{n_k})}{m_1(B_{n_k})}\geq 1-\delta_k^2\,. \tag{58}$$

Define the subsets B'_{n_k} and B''_{n_k} of B_{n_k} to be

$$B'_{n_k}=\{x\in B_{n_k};\ m_2(A_{n_k}^x)<1-\delta_k\} \tag{59}$$

(60) $$B''_{n_k}=B_{n_k}-B'_{n_k}\ .$$

Then we have that

(61) $$\begin{aligned} m(A_{n_k}) &\leqq (1-\delta_k)m_1(B'_{n_k})+m_1(B''_{n_k}) \\ &= m_1(B_{n_k})-\delta_k m_1(B'_{n_k})\ . \end{aligned}$$

From (58) and (61) it follows that $m_1(B'_{n_k})\leqq\delta_k m_1(B_{n_k})$, which means that $m_1(B''_{n_k})\geqq(1-\delta_k)m_1(B_{n_k})\geqq(1-\delta_k)m_1(Q)$. So, putting $B=\bigcap_k B''_{n_k}$, we have

(62) $$m_1(B)\geqq\left(1-\sum_{k=1}^{\infty}\delta_k\right)m_1(Q)>0\ .$$

From the relation

(63) $$A_{n_k}=\bigcup_{x\in B_{n_k}}(\{x\}\times A^x_{n_k})\supset\bigcup_{x\in B}(\{x\}\times A^x_{n_k})$$

it follows that

(64) $$\bigcup_{i=1}^{\infty}\bigcap_{k\geqq i}T^{n_k}E\supset\bigcup_{x\in B}\left(\{x\}\times\left(\bigcup_{i=1}^{\infty}\bigcap_{k\geqq i}A^x_{n_k}\right)\right)\ .$$

And from the definition of B''_{n_k} we have that, for $x\in B$,

(65) $$m_2\left(\bigcup_{i=1}^{\infty}\bigcap_{k\geqq i}A^x_{n_k}\right)\geqq\lim_{i\to\infty}\left(1-\sum_{k=i}^{\infty}\delta_k\right)=1\ ,$$

which means (56).

Now we obtain the following

THEOREM 4.4. *Let $ab>1$. If T satisfies $\mu(T^n(B\times Y))\to 1$ for each $B\subset X$ with $\mu(B\times Y)>0$, then T is exact with respect to the invariant measure μ.*

PROOF. From Lemma 4.3, we have (56), and so, from the assumption of theorem, we have that

(66) $$\mu\left(T^n\left(\bigcup_{i=1}^{\infty}\bigcap_{k\geqq i}T^{n_k}E\right)\right)\longrightarrow 1\ ,$$

which implies that $\mu(T^{m_k}E)\to 1$ for some $m_1<m_2<\cdots$. But it is clear that $\mu(T^nE)$ is increasing in n, so we have that $\mu(T^nE)\to 1$.

REMARK 4.5. From this theorem, it is clear that if f_b is exact, then T is also exact. But T may be exact even if f_a and f_b are not exact. In fact, in the case $\sqrt[4]{2}\leqq b<\sqrt{2}$ and $a<a_1$, T satisfies the

assumption of Theorem 4.4, so T is exact. Especially, in the case $a=1$, T is exact even if Γ is the union of disjoint intervals I_1, I_2 and I_3.

References

[1] R. B. MAY, Simple mathematical models with very complicated dynamics, Nature, **261** (1976), 459-467.

[2] J. MILNOR and W. THURSTON, On iterated maps of the interval, I and II, preprint.

[3] L. JONKER and D. RAND, A lower bound for the entropy of the unit interval, preprint.

[4] J. GUCKENHEIMER, Sensitive dependence on initial conditions for one-dimensional maps, Comm. Math. Phys., **70** (1979), 133-160.

[5] M. JAKOBSON, Absolutely continuous invariant measures for one-dimensional maps, Comm. Math. Phys., **81** (1981), 39-88.

[6] SH. ITO, S. TANAKA and H. NAKADA, On unimodal linear transformations and chaos, I and II, Tokyo J. Math., **2** (1979), 221-259.

[7] M. MORI, The extension of Sarkovskii's results and the topological entropy in unimodal transformations, Tokyo J. Math., **4** (1981) 133-152

[8] Y. TAKAHASHI, A formula for topological entropy of one-dimensional dynamics, Sci. Papers College Gen. Ed. Univ. Tokyo, **30** (1980), 11-22.

[9] T. OHNO, Asymptotic behaviors of dynamical systems with random parameters, to appear.

[10] V. A. ROKHLIN, Exact endomorphisms of a Lebesgue space, Izv. Akad. Nauk SSSR, Ser. Mat., **25** (1961), 499-530.

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