Brief Reports

Random-matrix description of the distribution of mesoscopic conductance

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Any linear statistic defined on a random-matrix ensemble is shown to be Gaussian distributed. This supports the prediction of weak-disorder perturbation theory in the diffusive, metallic limit for the distribution of conductance, since conductance is a linear statistic on the ensemble of transfer matrices.

Many features of sample-specific conductance fluctuations in small systems[1] have been explained in terms of weak-disorder perturbation theory.[2] For example, such calculations predict $O(1)$ fluctuations (in units of e^2/h) that depend weakly on the shape of the conductor but not on its mean conductance as long as the number of elastic scatterings is large while the number of inelastic scatterings is small. A complementary explanation[3] for the magnitude of the fluctuations and their approximate universality follows from considering conductance in terms of transfer matrices. If many channels contribute to the conductance, the results for ensembles of random, large matrices should apply. In particular, the variance of the conductance should be $O(1)$ and independent of the average conductance or the number of channels.[4]

It is also widely believed that mesoscopic conductance is Gaussian distributed in the metallic regime.[5] This follows immediately from a perusal of the leading scattering diagrams in the limit of weak disorder. (All connected diagrams in two or three dimensions with ladders connecting the external currents, extending the analysis of Ref. 2 to third and higher moments of the conductance distribution, vanish in the limit of weak disorder and large size. Including ladder-ladder vertices[6] allows some contributions to survive in this limit in two dimensions, but the vertices introduce inverse powers of the conductance. It follows that the conductance distribution deviates from a pure Gaussian as the impurity concentration increases and the mean conductance decreases.[5]) The purpose of this note is to show that all linear statistics (defined below) of random-matrix theory are Gaussian distributed. This fact may be viewed as an explanation of this feature of weak-disorder perturbation theory or as an extension of the prediction to more general circumstances.

A "linear statistic," A, on a matrix[4] is a quantity that can be expressed as a sum over the eigenvalues, E_i, of a function of one variable, $\mu(x)$, evaluated at each eigenvalue, i.e., $A \equiv \sum_i \mu(E_i)$. If the transfer matrix T is made up of the coefficients connecting incoming and outgoing waves in the various channels allowed at the Fermi energy by quantization of the transfer momenta due to the finite size, then the conductance g is given by[7]

$$g = \text{tr} \frac{4}{TT^\dagger + (TT^\dagger)^{-1} + 2} ,$$

at least for large g. So g is a linear statistic on the ensemble of T's. The variance of any linear statistic A over an ensemble of matrices can be expressed as an integral involving the function $\mu(x)$ and the eigenvalue two-point correlation function for the ensemble.[4] If there are very many eigenvalues in intervals over which the mean density of eigenvalues is approximately constant, then the eigenvalue correlations approach universal functions. Which of the three known universality classes of functions depends on the nature of the short-range "level repulsion," which, in turn, can be determined from the Haar measure on the space of possible matrices. (Reference 8 discusses the analysis of the Haar measure for transfer matrices.) The universality reflects the insensitivity of the eigenvalue correlations to any further smooth modification of the measure on the space of possible matrices that the physics might impose.[9]

Further characteristics of the distribution of a linear statistic, A, are provided by the higher cumulants, which are measures of the deviations from a Gaussian distribution. An explicit calculation (by the author) of the third and fourth cumulants, i.e., $\langle (A - \langle A \rangle)^3 \rangle$ and $\langle (A - \langle A \rangle)^4 \rangle - 3\langle (A - \langle A \rangle)^2 \rangle^2$ yielded zero for each. In particular, the analysis of Ref. 4 was extended to these higher cumulants for ensembles with a measure on eigenvalues that vanishes like $|E_i - E_j|^2$ for near degeneracies,

for which the level correlations are known in closed form. (Transfer matrices in the presence of an external magnetic field are in this category.) The eigenvalue correlations for such ensembles can be expressed in Fourier space in terms of absolute values—and absolute values of sums and differences of absolute values. The third and fourth cumulants vanish for any smooth $\mu(x)$ when one uses the explicit, universal forms of the eigenvalue correlations. It is because the eigenvalues are correlated (and not Poisson distributed) that the central limit theorem is not applicable. Furthermore, it is because these eigenvalue correlations are non-Gaussian (e.g., the four-eigenvalue correlation is not simply a sum of products of two-eigenvalue correlations) that the calculations are nontrivial. In units for which A is $O(N)$ and the variance of A is $O(1)$, these higher cumulants are down by $1/N$, where N is the number of eigenvalues within the interval on which the expected density of eigenvalues is roughly constant.

A proof (albeit formal) that the full distribution for any linear statistic is Gaussian, i.e., all higher cumulants vanish, goes as follows. The expected density of eigenvalues E_i is given by

$$\rho(x) \equiv \left\langle \sum_i \delta(E_i - x) \right\rangle .$$

In the limit of a large number of eigenvalues, N, contributing to a smooth density $\rho(x)$, the density $\rho(x)$ satisfies an integral equation determined by the underlying probability distribution for the various eigenvalue configurations. This equation can be viewed as a consequence of a mean-field approximation,[10] and it has the following structure: Interpret the E_i as the positions of N particles in one dimension and the probability for each particular set of E_i as the exponential of minus the energy of that configuration of particles. The equation for $\rho(x)$ is a statement of equilibrium, i.e., the force on a particle at x is zero. For example, if the probability of a set of E_i is

$$\prod_{i<j} |E_i - E_j| \left[\exp \sum_k V(E_k) \right] \tag{1}$$

then $\rho(x)$ satisfies

$$\frac{\partial V(x)}{\partial x} + \mathrm{P} \int dy \frac{\rho(y)}{x - y} = 0 , \tag{2}$$

where the first term is the force at x due to the external potential V and the second term is a principal-part integral representing the repulsive force at x due to the density of particles at y. If the probability for sets of E_j's contains terms that can be interpreted as n-body forces with $n \geq 3$, then there will be terms nonlinear in ρ (i.e., $n-1$ factors of ρ) in the generalization of Eq. (2).

Consider, now, a particular random-matrix ensemble, with arbitrarily complicated n-eigenvalue "forces" that satisfy the smoothness criteria needed to ensure the realization of the universal eigenvalue correlation functions,[9] and let ρ be the consequent expected eigenvalue density. It is always possible to replace this ensemble with one of the form of Eq. (1) such that the effective $V(x)$ generates the original $\rho(x)$ (thereby replacing all the original n-eigenvalue "forces"), and all eigenvalue correlations re-

tain their original values—at least to leading order in $1/N$. If the eigenvalue probability distribution is then perturbed by an additional factor of $\exp[\sum_i \mu(E_i)]$, the expected density will change to a new function $\rho + \delta\rho$. Since the original ρ satisfies the original mean-field integral equation (2), $\delta\rho$ satisfies an equation of the same form as Eq. (2) but now linear in $\mu(x)$. The general form of $\rho + \delta\rho$ is

$$\rho(x) + \delta\rho(x) = \frac{\left\langle \sum_i \delta(E_i - x) \exp\left[\sum_i \mu(E_i) \right] \right\rangle}{\left\langle \exp\left[\sum_i \mu(E_i) \right] \right\rangle} , \tag{3}$$

where the brackets $\langle \ \rangle$ refer to averages taken with the unperturbed probability distribution, i.e., without μ. Since $\delta\rho$ is precisely linear in μ, i.e., all higher powers of μ have vanishing coefficients, and using

$$\int dx \, \mu(x) \sum_i \delta(E_i - x) = \sum_i \mu(E_i) \equiv A ,$$

it follows that

$$\frac{\langle A e^A \rangle}{\langle e^A \rangle} - \langle A \rangle - \langle A^2 \rangle + \langle A \rangle^2 = 0 . \tag{4}$$

Equation (4) is a statement that the distribution of A in the original ensemble is Gaussian. One can confirm this either by expanding Eq. (4) in powers of A and deriving the vanishing of the third and higher cumulants or by deriving Eq. (4) as a consequence of Gaussian statistics for a random variable A.

There are two aspects which earn this derivation the somewhat pejorative "formal" epithet. Not all quantities in random-matrix theory admit a useful expansion in powers of a perturbing μ. For example, the eigenvalue correlation functions depend nonlinearly on ρ, and a small change in ρ due to a small change in μ leads to large changes in the functions. However, the cancellations implicit in Eq. (4) presumably persist. A more serious issue is how zero is the left-hand side of Eq. (4) or, equivalently, how accurate is the mean-field integral equation for ρ? If we scale ρ and $\langle A \rangle$ to be $O(N)$, it is not sufficient that the corrections to Eqs. (2) and (4) be $O(1)$ because the variance of A is already $O(1)$, and we wish to show that the higher cumulants are yet smaller. Unfortunately, there exists no systematic $1/N$ expansion to the general random-matrix problem. However, several specific problems of the form of Eq. (1) have been solved exactly for finite N in terms of orthogonal solutions of particular differential equations.[11] By inspection of the asymptotics of these special functions (or using the differential equations themselves) one can show that the corrections to ρ relative to Eq. (2) are indeed $O(1/N)$—as desired.

The applicability of these random-matrix notions to mesoscopic conductance rests on the assumption that the conductance in the diffusive, metallic limit comes from many eigenvalues of the transfer matrix and that their distribution is suitably smooth. Given that, the Gaussian distribution is a necessary consequence. This result may have more general validity than weak-disorder perturba-

tion theory, which corresponds to vanishing disorder in a finite, large volume. For example, a small but finite density of disorder in the limit of increasing volume may be a more accurate description of real mesoscopic systems, which may require a renormalization group analysis rather than direct perturbation theory. Such an analysis is reported in Ref. 12, and it is claimed that much of the universality of the weak-disorder calculations[2] is lost. However, the general results of random-matrix theory as applied to the transfer matrix in the metallic regime should still apply. These results are somewhat weaker than the weak-disorder predictions in that random-matrix theory implies that the conductance is Gaussian distributed with variance of $O(1)$ but does not give the numerical value of the variance nor imply that the shape of the conductor is the only variable that determines that value.

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