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Random Mixture of the Ising Magnets in a Magnetic Field

——*Quenched Site and Bond Problems*——

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A method using projection operators developed in a previous paper [Can. J. Phys. **52** (1974), 120] to obtain the specific heat and the susceptibility of the random mixture of magnets, is applied for the low-field expansion of the free energy and the magnetization. The quartic terms of the free energies of the linear chain and of the infinite Bethe lattice for the site and the bond problems are obtained. The exact solution of the infinite Bethe lattice is equivalent to the Bethe approximation of the ordinary lattices. A divergence of the quartic term of the free energy of the bond problem is discussed in connection with a phase transition relating to the glass-like phase. Transparent formal similarity (which serves as a check and an outlook) between the site and the bond problems is found, and a relation (which serves as an approximation) between the quenched and the annealed systems is discussed.

§ 1. Introduction and conclusion

In a previous paper[1] a method using projection operators to obtain thermodynamic and magnetic properties of the random mixture of the magnets (the site and the bond problems in the quenched Ising spin systems) was presented. The method was applied to the linear chain and to the infinite Bethe lattice (of which the exact solution is equivalent to the Bethe approximation) giving the free energy and the susceptibility at zero field. A remarkable distinction in the phase diagrams between the site and the bond problems was clarified. The method was also applied to the quenched classical Heisenberg model.[2] For the free energy and the magnetization at a finite magnetic field, a concentration expansion was carried out and anomalous behavior in the magnetization process of the dilute linear chain at low temperatures was explained.[3]

In this paper the method of Ref. 1) is applied to the low-field expansion of the free energy and the magnetization of the site and the bond problems for the linear chain and for the infinite Bethe lattice, and the quartic terms with respect to the magnetic field is obtained. The divergence of the second derivative of the susceptibility $(\partial^2\chi/\partial H^2)$ of the bond problem characterizes the appearance of the glass-like phase introduced by Matsubara and Sakata.[4] Transparent formal similarity between the site and the bond problems is found and it serves as a good check. An approximation for the bond model and a relation to the annealed system are discussed.

§ 2. Expansion of the partition function of the quenched site model

We consider a random mixture (site model) of two kinds of spins A and B, the magnetic moments of which are denoted by m_A and m_B, the concentrations p_A and p_B, and the nearest neighbor exchange energies J_{AA}, J_{AB} and J_{BB}. The species A and B are denoted by $s_i=+1$, and -1, respectively. A configuration of spins of the whole crystal is denoted by $\{s_i\}$. The Hamiltonian H of the system is given by

$$-\frac{H\{s_i\}}{kT}=\sum_{\langle ij\rangle} K_{ij}\sigma_i\sigma_j+\sum_i C_i\sigma_i\,, \tag{2·1}$$

where

$$K_{ij}=\begin{bmatrix}\frac{1+s_i}{2} & \frac{1-s_i}{2}\end{bmatrix}\begin{bmatrix}K_{AA} & K_{AB}\\ K_{BA} & K_{BB}\end{bmatrix}\begin{bmatrix}\frac{1+s_j}{2}\\ \frac{1-s_j}{2}\end{bmatrix}, \tag{2·2}$$

$$C_i=\begin{bmatrix}\frac{1+s_i}{2} & \frac{1-s_i}{2}\end{bmatrix}\begin{bmatrix}C_A\\ C_B\end{bmatrix}=[C_A\ \ C_B]\begin{bmatrix}\frac{1+s_i}{2}\\ \frac{1-s_i}{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1+s_i}{2} & \frac{1-s_i}{2}\end{bmatrix}\begin{bmatrix}C_A & 0\\ 0 & C_B\end{bmatrix}\begin{bmatrix}\frac{1+s_i}{2}\\ \frac{1-s_i}{2}\end{bmatrix}, \tag{2·2'}$$

$$K_{AA}=J_{AA}/2kT\,,\quad C_A=m_AH/kT\,,\quad \text{etc.} \tag{2·3}$$

We denote the partition function for the configuration $\{s_i\}$ by $Z\{s_i\}$. The configurational average of $\log Z\{s_i\}$ gives the negative of the free energy divided by kT of the quenched site system.

Expressing $\log Z\{s_i\}$ by the method of high temperature low field expansion, we have

$$\log Z\{s_i\}=\log\Big[\sum_{\sigma_i=\pm1}\prod_i^N \operatorname{ch} C_i(1+\sigma_i \operatorname{th} C_i)\times\sum_{\langle ij\rangle}^{N_B}\operatorname{ch} K_{ij}(1+\sigma_i\sigma_j\operatorname{th} K_{ij})\Big], \tag{2·4}$$

where N and N_B are the numbers of lattice points and bonds, respectively ($N_B=N-1$ for open linear chain and the finite Bethe lattice). We carry out the multiplication, arrange terms in the order of $\operatorname{th} C$, then the summation $\sum_{\{\sigma_i\}}$ gives

$$
\begin{aligned}
\log Z\{s_i\} = &\sum_{\langle ij\rangle} \log \mathrm{ch}\, K_{ij} + \sum_i \log \mathrm{ch}\, C_i + N \log 2 \\
&+ \log\{1 + \sum \mathrm{th}\, K_{i_1 i_2}\, \mathrm{th}\, K_{i_2 i_3} \cdots \mathrm{th}\, K_{i_n i_1} + \sum \mathrm{th}\, C_{i_1}\, \mathrm{th}\, K_{i_1 i_2} \\
&\times \mathrm{th}\, K_{i_2 i_3} \cdots \mathrm{th}\, K_{i_n i_{n+1}}\, \mathrm{th}\, C_{i_{n+1}} + \sum \mathrm{th}\, C_{i_1}\, \mathrm{th}\, K_{i_1 i_2}\, \mathrm{th}\, K_{i_2 i_3} \cdots \mathrm{th}\, K_{i_m i_{m+1}} \\
&\times \mathrm{th}\, C_{i_{m+1}}\, \mathrm{th}\, C_{j_1}\, \mathrm{th}\, K_{j_1 j_2}\, \mathrm{th}\, K_{j_2 j_3} \cdots \mathrm{th}\, K_{j_n j_{n+1}}\, \mathrm{th}\, C_{j_{n+1}} + \sum \mathrm{th}\, C_{i_1} \\
&\times \mathrm{th}\, K_{i_1 i_2} \cdots \mathrm{th}\, K_{i_l i_{l+1}}\, \mathrm{th}\, C_{i_{l+1}}\, \mathrm{th}\, C_{j_1}\, \mathrm{th}\, K_{j_1 j_2} \cdots \mathrm{th}\, K_{j_m j_{m+1}}\, \mathrm{th}\, C_{j_{m+1}} \\
&\times \mathrm{th}\, C_{k_1}\, \mathrm{th}\, K_{k_1 k_2} \cdots \mathrm{th}\, K_{k_n k_{n+1}}\, \mathrm{th}\, C_{k_{n+1}} + \cdots\}. \qquad (2\cdot 5)
\end{aligned}
$$

The coefficient of $O(\mathrm{th}^{2n} C)$ is given[4a)] by summation of all graphs (connected and disconnected) in which the sum of the number of the end points and that of odd junctions is $2n$.

When we expand $\log\{1+X\}$, the terms $O(N^2)$, $O(N^3)$ $\cdots$ cancel out. The configurational average of the terms of $O(N)$ in $\log\{1+X\}$ gives the partition function of the quenched system.

§ 3. Expansion of the order $\mathrm{th}^4 C$ for the linear chain

We consider a linear chain of the quenched site model. The free energy and the susceptibility at zero field together with those of the infinite Bethe lattice were given previously.[1)] In this section the term of $O(\mathrm{th}^4 C)$ is calculated. The fourth term in r.h.s. of (2·5) is expressed as

$$
\log\{\quad\} = \log\{1 + \Omega_0 + \Omega_2(\mathrm{th}^2 C) + \Omega_4(\mathrm{th}^4 C) + \cdots\}. \qquad (3\cdot 1)
$$

If the chain is open, then $\Omega_0 = 0$. If the chain is closed (ring), then $\Omega_0 = \mathrm{th}\, K_{12}\, \mathrm{th}\, K_{23} \cdots \mathrm{th} K_{N-1,N}\, \mathrm{th}\, K_{N1}$. We omit Ω_0 since it does not contribute in the thermodynamic limit even in the ring.

Each term in Ω_2 is represented by a chain of length n. Each term in Ω_4 is represented by a pair of unlinked chains of length m and n. In the expansion $\log\{\quad\}$, the coefficient of $\mathrm{th}^4 C$ is given by $\Omega_4 - \Omega_2^2/2$. Pairs of

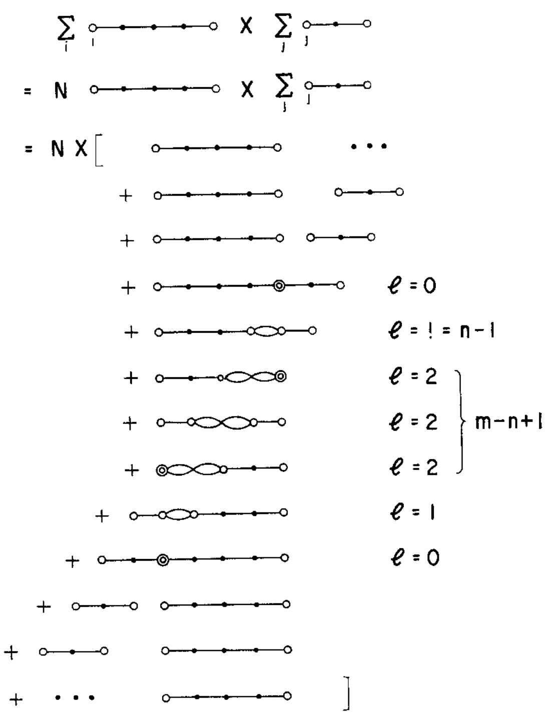

Fig. 1. A term in $\langle \Omega_2^2 \mathrm{th}^4 C \rangle$. $m=4$, $n=2$. Single and double bonds contribute to x and y, respectively. Single and double circles contribute to $\mathrm{th} C$ and $\mathrm{th}^2 C$, respectively. Unlinked graphs are canceled out by the corresponding graph in Ω_4.

unlinked chains in Ω_4 are canceled out by those in $\Omega_2^2/2$, and only pairs of linked chains in $-\Omega_2^2/2$ contribute to $\log Z\{s_{ij}\}$ (see Fig. 1).

$$\Omega_2(\mathrm{th}^2 C)=\sum_{i=1}^{N}\sum_{m=1}^{N-1}\mathrm{th}\,C_i\,\mathrm{th}\,K_{i,i+1}\,\mathrm{th}\,K_{i+1,i+2}\cdots\mathrm{th}\,K_{i+m-1,i+m}\times\mathrm{th}\,C_{i+m}\,.\quad (N+1\equiv 1)\tag{3·2}$$

Then

$$\Omega_2^2(\mathrm{th}^4 C)=\sum_i\sum_j\sum_m\sum_n\mathrm{th}\,C_i\,\mathrm{th}\,K_{i,i+1}\cdots\mathrm{th}\,K_{i+m-1,i+m}\,\mathrm{th}\,C_{i+m}\times\mathrm{th}\,C_j\,\mathrm{th}K_{j,j+1}\cdots\mathrm{th}\,K_{j+n-1,j+n}\,\mathrm{th}\,C_{j+n}\tag{3·3}$$

$$=\sum_i\sum_{n=1}^{N}\Big(\sum_{m=1}^{n-1}+\sum_{m=n}^{N-n-1}\Big)\sum_j\mathrm{th}\,C_i\,\mathrm{th}\,K_{i,i+1}\cdots\mathrm{th}\,K_{i+m-1,i+m}\,\mathrm{th}\,C_{i+m}\times\mathrm{th}\,C_j\,\mathrm{th}\,K_{j,j+1}\cdots\mathrm{th}\,K_{j+n-1,j+n}\,\mathrm{th}\,C_{j+n}\,.\tag{3·4}$$

Let $j=i+m-l$ and we change the summation index j to l. Then the contribution of linked graphs in (3·3) is given by

$$\mathrm{O}(N)\ \text{of}\ \Omega_2^2(\mathrm{th}^4 C)=\sum_i\sum_{n=1}\Big(\sum_{m=1}^{n-1}\sum_{l=0}+\sum_{m=n}^{N-n-1}\sum_{l=0}\Big)\times\mathrm{th}\,C_i\,\mathrm{th}\,K_{i,i+1}\cdots\mathrm{th}\,K_{i+m-1,i+m}\,\mathrm{th}\,C_{i+m}\,\mathrm{th}\,C_{i+m-l}\times\mathrm{th}\,K_{i+m-l,i+m-l+1}\cdots\mathrm{th}\,K_{i+m-l+n-1,i+m-l+n}\,\mathrm{th}\,C_{i+m-l+n}\,.\tag{3·5}$$

For the meaning of Σ_l, see (3·19). The terms for $m+n\geqq N$ do not contribute in the thermodynamic limit.

We substitute (2·2) and (2·2′) into a term in (3·5). Then the configurational average (denoted by $\langle\ \ \rangle$) can be obtained by equating $\langle(1+s_i)/2\rangle=p_A$, $\langle(1-s_i)/2\rangle=p_B$, after the multiplication of s_i at the same site is carried out (cf. [3.8] and [3.9] in Ref. 1)). The average of a term in (3·5) is given by

$$\langle\mathrm{th}\,C_i\,\mathrm{th}\,K_{i,i+1}\cdots\mathrm{th}\,K_{i+m-l,i+m-l+1}\,\mathrm{th}\,C_{i+m-l}\times\mathrm{th}^2K_{i+m-l,i+m-l+1}\,\mathrm{th}^2K_{i+m-l+1,i+m-l+2}\cdots\mathrm{th}^2K_{i+m-1,i+m}\times\mathrm{th}\,C_{i+m}\,\mathrm{th}\,K_{i+m,i+m+1}\cdots\mathrm{th}\,K_{i+m-l+n-1,i+m-l+n}\,\mathrm{th}\,C_{i+m-l+n}\rangle$$

$$=\Big\langle[c_A\ \ c_B]\begin{bmatrix}\frac{1+s_i}{2}&0\\0&\frac{1-s_i}{2}\end{bmatrix}\Big\{\prod_{k=1}^{m-l}\begin{bmatrix}t_{AA}&t_{AB}\\t_{BA}&t_{BB}\end{bmatrix}\begin{bmatrix}\frac{1+s_{i+k}}{2}&0\\0&\frac{1-s_{i+k}}{2}\end{bmatrix}\Big\}$$

$$\times\begin{bmatrix}c_A&0\\0&c_B\end{bmatrix}\Big\{\prod_{k=1}^{l}\begin{bmatrix}t_{AA}^2&t_{AB}^2\\t_{BA}^2&t_{BB}^2\end{bmatrix}\begin{bmatrix}\frac{1+s_{i+m-l+k}}{2}&0\\0&\frac{1-s_{i+m-l+k}}{2}\end{bmatrix}\Big\}\begin{bmatrix}c_A&0\\0&c_B\end{bmatrix}$$

$$\times\left\{\prod_{k=1}^{n-l}\begin{bmatrix} t_{AA} & t_{AB} \\ t_{BA} & t_{BB}\end{bmatrix}\begin{bmatrix} \frac{1+s_{i+m+k}}{2} & 0 \\ 0 & \frac{1-s_{i+m+k}}{2}\end{bmatrix}\right\}\begin{bmatrix} c_A \\ c_B\end{bmatrix}\Bigg\rangle \tag{3·6}$$

$$=\mathrm{Tr}\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix}\begin{bmatrix} c_A & 0 \\ 0 & c_B\end{bmatrix}\begin{bmatrix} p_A & 0 \\ 0 & p_B\end{bmatrix}\left\{\begin{bmatrix} t_{AA} & t_{AB} \\ t_{BA} & t_{BB}\end{bmatrix}\begin{bmatrix} p_A & 0 \\ 0 & p_B\end{bmatrix}\right\}^{m-l}\begin{bmatrix} c_A & 0 \\ 0 & c_B\end{bmatrix}$$

$$\times\left\{\begin{bmatrix} t_{AA}^2 & t_{AB}^2 \\ t_{BA}^2 & t_{BB}^2\end{bmatrix}\begin{bmatrix} p_A & 0 \\ 0 & p_B\end{bmatrix}\right\}^{l}\begin{bmatrix} c_A & 0 \\ 0 & c_B\end{bmatrix}\left\{\begin{bmatrix} t_{AA} & t_{AB} \\ t_{BA} & t_{BB}\end{bmatrix}\begin{bmatrix} p_A & 0 \\ 0 & p_B\end{bmatrix}\right\}^{n-l}\begin{bmatrix} c_A & 0 \\ 0 & c_B\end{bmatrix} \tag{3·7}$$

$$=\mathrm{Tr}'\,\boldsymbol{pc}(\boldsymbol{tp})^{m-l}\boldsymbol{c}(\boldsymbol{t}_2\boldsymbol{p})^{l}\boldsymbol{c}(\boldsymbol{tp})^{n-l}\boldsymbol{c}$$

$$=\mathrm{Tr}'\,\boldsymbol{p}^{1/2}\boldsymbol{cx}^{m-l}\boldsymbol{cy}^{l}\boldsymbol{cx}^{n-l}\boldsymbol{cp}^{1/2}\,, \tag{3·8}$$

where

$$\boldsymbol{p}=\begin{bmatrix} p_A & 0 \\ 0 & p_B\end{bmatrix}, \tag{3·9}$$

$$\boldsymbol{c}=\begin{bmatrix} c_A & 0 \\ 0 & c_B\end{bmatrix}\equiv\begin{bmatrix} \mathrm{th}\,C_A & 0 \\ 0 & \mathrm{th}\,C_B\end{bmatrix}, \tag{3·10}$$

$$\boldsymbol{t}=\begin{bmatrix} t_{AA} & t_{AB} \\ t_{BA} & t_{BB}\end{bmatrix},\quad \boldsymbol{t}_2=\begin{bmatrix} t_{AA}^2 & t_{AB}^2 \\ t_{BA}^2 & t_{BB}^2\end{bmatrix},$$

$$\boldsymbol{x}=\begin{bmatrix} p_A t_{AA} & \sqrt{p_A p_B}\,t_{AB} \\ \sqrt{p_B p_A}\,t_{BA} & p_B t_{BB}\end{bmatrix}, \tag{3·11}$$

$$\boldsymbol{y}=\begin{bmatrix} p_A t_{AA}^2 & \sqrt{p_A p_B}\,t_{AB}^2 \\ \sqrt{p_B p_A}\,t_{BA}^2 & p_B t_{BB}^2\end{bmatrix}, \tag{3·12}$$

$$\mathrm{Tr}'\boldsymbol{A}=\mathrm{Tr}\,\boldsymbol{gA}\,,\qquad \boldsymbol{g}=\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix},$$

$$t_{AA}=\mathrm{th}\,K_{AA}\,,\quad \text{etc.}$$

Here we have used a lemma[1)]

$$f(K_{ij})=\begin{bmatrix} \frac{1+s_i}{2} & \frac{1-s_i}{2}\end{bmatrix}\begin{bmatrix} f(K_{AA}) & f(K_{AB}) \\ f(K_{BA}) & f(K_{BB})\end{bmatrix}\begin{bmatrix} \frac{1+s_j}{2} \\ \frac{1-s_j}{2}\end{bmatrix}$$

for $f(x)$ which is regular at $x=0$.

Using the orthogonal transformations

$$\boldsymbol{uxu}=\boldsymbol{\lambda}\,,\qquad \boldsymbol{vyv}=\boldsymbol{\mu}\,, \tag{3·13}$$

$$\boldsymbol{u}=\boldsymbol{u}^{-1}=\begin{bmatrix} u_{11} & u_{12} \\ u_{12} & -u_{11}\end{bmatrix},\qquad \boldsymbol{v}=\boldsymbol{v}^{-1}=\begin{bmatrix} v_{11} & v_{12} \\ v_{12} & -v_{11}\end{bmatrix},$$

$$\boldsymbol{\lambda}=\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}, \qquad \boldsymbol{\mu}=\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix},$$

we have

$$(3\cdot 8) = \mathrm{Tr}'\,\boldsymbol{p}^{1/2}\boldsymbol{cu\lambda}^{m-l}\boldsymbol{ucv\mu}^{l}\boldsymbol{vcu\lambda}^{n-l}\boldsymbol{ucp}^{1/2}$$

$$=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} x_i^{m-l} y_j^{\,l} x_k^{\,n-l} f_{ijk}\,, \tag{3·14}$$

where

$$f_{ijk}=\sum_{h=1}^{2}\sum_{i'=1}^{2}\sum_{j'=1}^{2}\sum_{k'=1}^{2} p_h^{1/2} c_h u_{hi} u_{ii'} c_{i'} v_{i'j} v_{jj'} \times c_{j'} u_{j'k} u_{kk'} p_{k'}^{1/2} \tag{3·15}$$

and

$$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} = \frac{1}{2}\{p_A t_{AA} + p_B t_{BB} \pm [(p_A t_{AA} - p_B t_{BB})^2 + 4 p_A p_B t_{AB}^2]^{1/2}\}, \tag{3·16}$$

$$\left.\begin{matrix} y_1 \\ y_2 \end{matrix}\right\} = \frac{1}{2}\{p_A t_{AA}^2 + p_B t_{BB}^2 \pm [(p_A t_{AA}^2 - p_B t_{BB}^2)^2 + 4 p_A p_B t_{AB}^4]^{1/2}\}, \tag{3·16′}$$

$$\left.\begin{matrix} u_{11} \\ u_{12} \end{matrix}\right\} = \left\{\frac{1}{2}\left[1 \pm \frac{p_A t_{AA} - p_B t_{BB}}{[(p_A t_{AA} - p_B t_{BB})^2 + 4 p_A p_B t_{AB}^2]^{1/2}}\right]\right\}^{1/2}, \tag{3·17}$$

$$\left.\begin{matrix} v_{11} \\ v_{12} \end{matrix}\right\} = \left\{\frac{1}{2}\left[1 \pm \frac{p_A t_{AA}^2 - p_B t_{BB}^2}{[(p_A t_{AA}^2 - p_B t_{BB}^2) + 4 p_A p_B t_{AB}^4]^{1/2}}\right]\right\}^{1/2}. \tag{3·17′}$$

We substitute (3·14) into (3·5) and rearrange terms of which the configurational averages are equal, then $\sum_i$ gives simply $\times N$, and $\sum_{l=0}$ is divided into three parts. Then

$$(3\cdot 5) = N\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f_{ijk} g_{ijk}\,, \tag{3·18}$$

where

$$g_{ijk}=\frac{1}{2}\sum_{n=1}^{N}\Big\{\sum_{m=1}^{n-1}\Big[\sum_{l=0}^{m-1} x_i^{m-l} y_j^{\,l} x_k^{\,n-l} + \sum_{l=0}^{n-m} x_i^{\,l} y_j^{\,m} x_k^{\,n-m-l} + \sum_{l=0}^{m-1} x_i^{n-l} y_j^{\,l} x_k^{\,m-l}\Big] + \sum_{m=n}^{N-n-1}\Big[\sum_{l=0}^{n-1} x_i^{n-l} y_j^{\,l} x_k^{\,m-l} + \sum_{l=0}^{m-n} x_i^{\,l} y_j^{\,n} x_k^{\,m-n-l} + \sum_{l=0}^{n-1} x_i^{m-l} y_j^{\,l} x_k^{\,n-l}\Big]\Big\}. \tag{3·19}$$

Carrying out the summation $\sum_n\sum_m\sum_l$ and taking the thermodynamic limit, we have

$$g_{ijk}=\frac{1}{2}\,\frac{y_j(x_i+x_k-x_i x_k+1)+2x_i x_k}{(1-x_i)(1-x_k)(1-y_j)}\,. \tag{3·20}$$

Substituting (3·20) and (3·15) into (3·18), and carrying out the summation $\sum_i\sum_j\sum_k$, we transform (3·18) in a simplified matrix form. Thus we have obtained the free energy of the random mixture of Ising spins (quenched site model) up to order $O(\mathrm{th}^4 C)$.

$$\left\langle\frac{1}{N}\log Z\{s_i\}/2\right\rangle=\mathrm{Tr}'\left\{\boldsymbol{p}\begin{bmatrix}\log\mathrm{ch}\,K_{AA} & \log\mathrm{ch}\,K_{AB}\\ \log\mathrm{ch}\,K_{BA} & \log\mathrm{ch}\,K_{BB}\end{bmatrix}\boldsymbol{p}\right.$$

$$+\boldsymbol{p}^{1/2}\begin{bmatrix}\log\mathrm{ch}\,C_A & 0\\ 0 & \log\mathrm{ch}\,C_B\end{bmatrix}\boldsymbol{p}^{1/2}+\boldsymbol{p}^{1/2}\boldsymbol{c}^2(\boldsymbol{1}-\boldsymbol{x})^{-1}\boldsymbol{x}\boldsymbol{p}^{1/2}$$

$$-\frac{1}{2}\boldsymbol{p}^{1/2}(\boldsymbol{1}-\boldsymbol{x})^{-1}[\boldsymbol{xc}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{yc}+\boldsymbol{c}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{ycx}$$

$$-\boldsymbol{xc}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{ycx}+2\boldsymbol{xc}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{cx}$$

$$\left.+\boldsymbol{c}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{yc}](\boldsymbol{1}-\boldsymbol{x})^{-1}\boldsymbol{cp}^{1/2}\right\}, \tag{3·21}$$

where

$$(\boldsymbol{1}-\boldsymbol{x})^{-1}=\frac{1}{\det(\boldsymbol{1}-\boldsymbol{x})}\begin{bmatrix}1-p_Bt_{BB} & \sqrt{p_Ap_B}\,t_{AB}\\ \sqrt{p_Bp_A}\,t_{BA} & 1-p_A\,t_{AA}\end{bmatrix},$$

$$(\boldsymbol{1}-\boldsymbol{y})^{-1}=\frac{1}{\det(\boldsymbol{1}-\boldsymbol{y})}\begin{bmatrix}1-p_Bt_{BB}^2 & \sqrt{p_Ap_B}\,t_{AB}^2\\ \sqrt{p_Bp_A}\,t_{BA}^2 & 1-p_A\,t_{AA}^2\end{bmatrix}. \tag{3·22}$$

Hence the magnetization (per site) is given by

$$\langle m\sigma\rangle=\frac{\partial\langle(1/N)\log Z\{s_i\}/2\rangle}{\partial(H/kT)}$$

$$=\mathrm{Tr}'\{\boldsymbol{p}^{1/2}\boldsymbol{mc}(\boldsymbol{1}-\boldsymbol{x})^{-1}(\boldsymbol{1}+\boldsymbol{x})\boldsymbol{p}^{1/2}-2\boldsymbol{p}^{1/2}\boldsymbol{m}(\boldsymbol{1}-\boldsymbol{x})^{-1}[\boldsymbol{xc}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{cx}$$

$$+\boldsymbol{xc}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{ycx}+\boldsymbol{xc}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{c}+\boldsymbol{c}(\boldsymbol{1}-\boldsymbol{y})^{-1}\boldsymbol{yc}](\boldsymbol{1}-\boldsymbol{x})^{-1}\boldsymbol{cp}^{1/2}\}. \tag{3·23}$$

Here we have used the identity

$$\frac{\partial\boldsymbol{c}^2}{\partial(H/kT)}=2\boldsymbol{m}(\boldsymbol{c}-\boldsymbol{c}^3), \tag{3·24}$$

$$\frac{\partial}{\partial(H/kT)}\boldsymbol{cAcBcDc}=4\boldsymbol{mAcBcDc}+O((H/kT)^5) \tag{3·25}$$

for $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$ which do not contain H/kT, and

$$\boldsymbol{m}=\begin{bmatrix}m_A & 0\\ 0 & m_B\end{bmatrix}.$$

The explicit form of the first term, $\mathrm{Tr}'\,\boldsymbol{p}^{1/2}\boldsymbol{mc}(\boldsymbol{1}-\boldsymbol{x})^{-1}(\boldsymbol{1}+\boldsymbol{x})\boldsymbol{p}^{1/2}$, the susceptibility term, is given in Ref. 1).

In the case of the dilute magnetism ($K_{AA}=K$, $K_{AB}=K_{BB}=0$, $C_A=C$, $C_B=0$, $p_A=p$)

$$\left\langle\frac{1}{N}\log Z\{s_i\}\right\rangle=p^2\log\mathrm{ch}\,K+\log 2+p\log\mathrm{ch}\,C+p\,\mathrm{th}^2 C\frac{p\,\mathrm{th}\,K}{1-p\,\mathrm{th}\,K}$$

$$-\frac{p}{2}\,\mathrm{th}^4 C\frac{p\,\mathrm{th}^2K(2p\,\mathrm{th}\,K-p^2\mathrm{th}^2K+1)+2p^2\mathrm{th}^2K}{(1-p\,\mathrm{th}\,K)^2(1-p\,\mathrm{th}^2K)}. \tag{3·26}$$

In the case $K_{AB}=0$, all f_{ijk} other than $f_{111}(=p_A C_A^4)$ and $f_{222}(=p_B C_B^4)$ vanish, and the free energy is given by simply the sum of those of A and B species with concentration p_A and p_B.

$$\left\langle\frac{1}{N}\log Z\{s_i\}\right\rangle=\log 2+p_A^2\log\text{ch}\,K_{AA}+p_A\log\text{ch}\,C_A+p_A\,\text{th}^2 C_A\frac{p_A t_{AA}}{1-p_A t_{AA}}$$
$$-\frac{p_A}{2}\,\text{th}^4 C\frac{p_A t_{AA}^2(1+2p_A t_{AA}-p_A^2 t_{AA}^2)+2p_A^2 t_{AA}^2}{(1-p_A t_{AA})^2(1-p_A t_{AA}^2)}$$
$$+p_B^2\log\text{ch}\,K_{BB}+p_B\log\text{ch}\,C_B+p_B\,\text{th}^2 C_B\frac{p_B t_{BB}}{1-p_B t_{BB}}$$
$$-\frac{p_B}{2}\,\text{th}^4 C_B\frac{p_B t_{BB}^2(1+2p_B t_{BB}-p_B^2 t_{BB}^2)+2p_B^2 t_{BB}^2}{(1-p_B t_{BB})^2(1-p_A t_{BB}^2)}+O(C^6). \quad (3\cdot 27)$$

When $p_A=1$, $p_B=0$, we have

$$\left\langle\frac{1}{N}\log Z\right\rangle=\log\text{ch}\,K+\log 2+\log\text{ch}\,C+\text{th}^2 C\frac{\text{th}\,K}{1-\text{th}\,K}-\frac{1}{2}\,\text{th}^4 C\frac{3\,\text{th}^2 K-\text{th}^3 K}{(1-\text{th}\,K)^3} \quad (3\cdot 28)$$

which agrees with the corresponding low field expansion of the regular linear chain and serves as a check.

§ 4. Bond model

Similar calculation is also carried out for the bond model of a mixture of α and β bonds, and $s_{ij}=\pm 1$ refer α and β, respectively. The exchanges and concentrations of both species are denoted by J_α, J_β, p_α, p_β, respectively. Then

$$K_{ij}=\left[\frac{1+s_{ij}}{2}\;\;\frac{1-s_{ij}}{2}\right]\left[\begin{matrix}K_\alpha\\ K_\beta\end{matrix}\right], \quad (4\cdot 1)$$
$$K_\alpha=J_\alpha/2kT,\ \text{etc.}$$

In the bond model the configurational average can be taken for each bond separately. For example,

$$\langle\text{th}\,K_{12}\,\text{th}\,K_{23}\,\text{th}\,K_{34}\,\text{th}\,K_{23}\,\text{th}\,K_{34}\,\text{th}\,K_{45}\,\text{th}\,K_{56}\rangle$$
$$=\langle\text{th}\,K_{12}\rangle\langle\text{th}^2 K_{23}\rangle\langle\text{th}^2 K_{34}\rangle\langle\text{th}\,K_{45}\rangle\langle\text{th}\,K_{56}\rangle. \quad (4\cdot 2)$$

The averages of single and double bonds are given by

$$x\equiv\langle\text{th}\,K\rangle=p_\alpha\,\text{th}\,K_\alpha+p_\beta\,\text{th}\,K_\beta,$$
$$y\equiv\langle\text{th}^2 K\rangle=p_\alpha\,\text{th}^2 K_\alpha+p_\beta\,\text{th}^2 K_\beta. \quad (4\cdot 3)$$

Hence

$$\left\langle \frac{1}{N} O(N) \text{ of } \Omega_2^2 \,\text{th}^4 C \right\rangle$$

$$= \text{th}^4 C \sum_n \{ (\sum_{m=1}^{n-1} \sum_{l=0}^{n+m} + \sum_{m=n}^{N-n-1} \sum_{l=0}^{m+n}) x^{m+n-2l} y^l \}$$

$$= \text{th}^4 C \sum_n [\sum_{m=1}^{n-1} (2\sum_{l=0}^{m-1} + \sum_{l=m}^{n}) + \sum_{m=n}^{N-n-1} (2\sum_{l=0}^{n-1} + \sum_{l=n}^{m})] x^{m+n-2l} y^l \,. \tag{4·4}$$

Each term in $2\sum_{l=0}^{m-1}$ (and in $2\sum_{l=0}^{n-1}$) is represented by a linked graph composed of a chain of m bonds and that of n bonds in which l bonds are doubled. Each term in $\sum_{l=m}^{n}$ (or $\sum_{l=n}^{m}$) is represented by a graph composed of a chain of m bonds and that of n bonds in which m bonds (n bonds) are doubled.

$$(4·4) = \text{th}^4 C \sum_n \{ \sum_{m=1}^{n-1} [2 \sum_{l=0}^{m-1} x^{m+n-2l} y^l + (n-m+1) x^{n-m} y^m]$$
$$+ \sum_{m=n}^{N-n-1} [2 \sum_{l=0}^{n-1} x^{m+n-2l} y^l + (m-n+1) x^{m-n} y^n] \} \,. \tag{4·4'}$$

Carrying out the summation $\sum_n \sum_m \sum_l$ and taking the thermodynamic limit, we have

$$\left\langle \frac{1}{N} \log Z\{s_{ij}\} \right\rangle = p_\alpha \log \text{ch}\, K_\alpha + p_\beta \log \text{ch}\, K_\beta + \log 2$$
$$+ \log \text{ch}\, C + \text{th}^2 C \frac{x}{1-x} - \frac{1}{2} \text{th}^4 C \left[\frac{2x^2 + 2xy - x^2 y + y}{(1-x)^2 (1-y)} \right] + \cdots . \tag{4·5}$$

The magnetization is given by

$$\langle \sigma \rangle = \frac{1+x}{1-x} \text{th}\, C - 2\frac{(x+y)(1+x)}{(1-x)^2 (1-y)} \text{th}^3 C + O(\text{th}^5 C) \,. \tag{4·6}$$

Equation (4·6) agrees with the known results.[5),12)]

When $\boldsymbol{p}$, $\boldsymbol{c}$, $\boldsymbol{x}$, $\boldsymbol{y}$, of the corresponding expressions of the site model, in (3·23), are regarded as a scalar, the right-hand side of (3·23) is reduced to (4·6) multiplied by p. This is a remarkable similarity between the site and the bond problems.

§ 5. The Bethe lattice

In this section the method and the results of the preceding sections are generalized to the infinite Bethe lattice, of which the exact solution is equivalent to the solution of the Bethe approximation of the ordinary lattice.[6)~8)]

We consider the infinite Bethe lattice in which the exchange energies and the magnetic moments are denoted by J_{ij} ($=J_{AA}$, J_{BB} or J_{AB}) and m_j ($=m_A$ or m_B). Then the magnetization $\langle m_i \sigma \rangle$ at the site i is determined in terms of the effective field at the nearest neighbors of the i site, and the recurrence relation between effective fields is derived.[8),1)]

$$\langle m_i\sigma\rangle = \langle m_i\, \mathrm{th}[C_i + \sum_j^z \mathrm{th}^{-1} t_{ij} l_j]\rangle\,, \tag{5·1}$$

$$L_i = C_i + \sum_j^{z-1} \mathrm{th}^{-1} t_{ij} l_j\,, \tag{5·2}$$

where

$$l_i = \mathrm{th}\, L_i\,,\ \ L_i = m_i H^*/kT \quad (H^*\text{: effective field}),$$

and $\sum_j^z$ represents the summation for all nearest neighbors j of i, and $\sum_j^{z-1}$ that for which one of them is removed.

Using (5·2), we have transformed (5·1) into

$$\langle m_i\sigma\rangle = \langle m_i\, \mathrm{th}[L_i + \mathrm{th}^{-1} t_{iz} l_z]\rangle$$

$$= \left\langle m_i \frac{l_i + t_{iz} l_z}{1 + l_i t_{iz} l_z}\right\rangle \tag{5·3}$$

$$= \langle m_i (l_i + t_{iz} l_z) \sum_{n=0} (-)^n (l_i t_{iz} l_z)^n\rangle \tag{5·4}$$

$$= \langle m_i [l_i + t_{iz} l_z - l_i^2 t_{iz} l_z - l_i t_{iz}^2 l_z^2 + \cdots]\rangle\,. \tag{5·4'}$$

Taking the hyperbolic tangent of (5·2) and expanding it up to order $\mathrm{O}(c^3, l^3)$, we have

$$l_i = \frac{c_i + \mathrm{th}\ (\sum_j^{z-1} \mathrm{th}^{-1} t_{ij} l_j)}{1 + c_i\, \mathrm{th}\ (\sum_j^{z-1} \mathrm{th}^{-1} t_{ij} l_j)} \tag{5·5}$$

$$\mathrm{th}[\sum_j^{z-1} \mathrm{th}^{-1}(t_{ij} l_j)] = \sum_j^{z-1} t_{ij} l_j + \frac{1}{3}\sum_j^{z-1} t_{ij}^3 l_j^3 - \frac{1}{3}\sum_j^{z-1}\sum_k^{z-1}\sum_m^{z-1} t_{ij} t_{ik} t_{im} l_j l_k l_m + \cdots\,.$$

$$(z-1)^3 \text{ terms} \tag{5·6}$$

Substituting (5·6) into (5·5) we have (up to $\mathrm{O}(c^3, l^3)$)

$$l_i = c_i + \sum_j^{z-1} t_{ij} l_j - c_i^2 \sum_j^{z-1} t_{ij} l_j - c_i (\sum_j^{z-1} t_{ij} l_j)^2$$

$$- \frac{1}{3}\sum_j^{z-1}\sum_k^{z-1} t_{ij}^2 l_j^2 t_{ik} l_k - \frac{1}{3}\sum_j^{z-1}\sum_k^{z-1}\sum_m^{z-1} t_{ij} l_j t_{ik} l_k t_{im} l_m + \cdots\,.$$

$$j \neq k \qquad\qquad (j \neq k \neq m \neq j)$$

$$3(z-1)(z-2) \text{ terms} \qquad (z-1)(z-2)(z-3) \text{ terms} \tag{5·7}$$

The square of (5·7) of up to $\mathrm{O}(c^2, l^2)$ reads

$$l_i^2 = c_i^2 + 2c_i \sum_j^{z-1} t_{ij} l_j + \sum_j^{z-1} t_{ij}^2 l_j^2 + \sum_j \sum_k t_{ij} l_j t_{ik} l_k + \cdots\,.$$

$$(z-1)(z-2) \text{ terms} \tag{5·8}$$

In the paramagnetic state the configurational average is assumed to be independent of the site. Then $\langle t_{ij} l_j\rangle_j$ ($\langle\ \ \rangle_j$ represents the average carried out at the site j),

$\langle t_{ij}l_jt_{ik}l_k\rangle_{jk}\cdots$ are given by

$$\langle t_{ij}l_j\rangle_j=\left\langle\sum_{g_i}\sum_{g_j}\frac{1+s_ig_i}{2}\frac{1+s_jg_j}{2}t_{g_ig_j}\sum_{g_j}\frac{1+s_jg_j}{2}l_{g_j}\right\rangle_j$$

$$=\left\langle\sum_{g_i}\sum_{g_j}\frac{1+s_ig_i}{2}\frac{1+s_jg_j}{2}t_{g_ig_j}l_{g_j}\right\rangle_j$$

$$=\sum_{g_i}\frac{1+s_ig_i}{2}(t_{g_iA}p_Al_A+t_{g_iB}p_Bl_B), \tag{5·9}$$

where

$$g_j=1,\ -1\ (A \text{ or } B),\ t_{11}=t_{AA},\ t_{1,-1}=t_{AB},\ \cdots,\ l_1=l_A\cdots$$

and

$$\langle t_{ij}l_jt_{ik}l_k\rangle_{jk}=\sum_{g_i}\frac{1+s_ig_i}{2}(t_{g_iA}p_Al_A+t_{g_iB}p_Bl_B)^2, \tag{5·10}$$

$$\langle t_{ij}l_jt_{ik}l_kt_{im}l_m\rangle_{jkm}=\sum_{g_i}\frac{1+s_ig_i}{2}(t_{g_iA}p_Al_A+t_{g_iB}p_Bl_B)^3, \tag{5·11}$$

$$\langle t_{ij}^2l_j^2t_{ik}l_k\rangle_{jk}=\sum_{g_i}\frac{1+s_ig_i}{2}(t_{g_iA}^2p_Al_A^2+t_{g_iB}^2p_Bl_B^2)$$

$$\times(t_{g_iA}p_Al_A+t_{g_iB}p_Bl_B). \tag{5·12}$$

Here we have used similar relations to (2·2) and (2·2′) in non-matrix form.

Equations (5·7) and (5·8) form simultaneous equations for l_i and l_i^2. We solve $\overline{l}$ and $\overline{l^2}$ ($\overline{l}^2\neq\overline{l^2}$) using (5·7) and (5·8) up to $O(c^3)$ by iteration, where $\overline{l}$ (denoted by $\boldsymbol{l}$) is the matrix l_i averaged at $i+1$, $i+2$, $\cdots$, and $\overline{l^2}$ (denoted by $\boldsymbol{l}_2$) is the matrix l_i^2 averaged at $i+1$, $i+2$, $\cdots$. The first approximation for l_i, denoted by $\boldsymbol{l}$, from (5·7) is

$$\boldsymbol{l}=[\mathbf{1}-(z-1)\boldsymbol{tp}]^{-1}\boldsymbol{c} \tag{5·13}$$

and the first approximation for l_i^2 from (5·8), denoted by l_2, is given by

$$\boldsymbol{l}_2=[\mathbf{1}-(z-1)\boldsymbol{t}_2\boldsymbol{p}]^{-1}\boldsymbol{c}^2+2(z-1)[\mathbf{1}-(z-1)\boldsymbol{t}_2\boldsymbol{p}]^{-1}\boldsymbol{ctp}$$

$$\times[\mathbf{1}-(z-1)\boldsymbol{tp}]^{-1}\boldsymbol{c}+(z-1)(z-2)[\mathbf{1}-(z-1)\boldsymbol{t}_2\boldsymbol{p}]^{-1}\langle t_{ij}l_jt_{ik}l_k\rangle_{jk}. \tag{5·14}$$

Substituting (5·9) ~ (5·14) into (5·7), we obtain the second approximation for $\boldsymbol{l}$ exactly up to $O(c^3)$.

$$\boldsymbol{l}=[\mathbf{1}-(z-1)\boldsymbol{tp}]^{-1}\{\boldsymbol{c}-(z-1)\boldsymbol{c}^2\boldsymbol{tp}[\mathbf{1}-(z-1)\boldsymbol{tp}]^{-1}\boldsymbol{c}-(z-1)\boldsymbol{ct}_2\boldsymbol{l}_2$$

$$-(z-1)(z-2)\boldsymbol{c}\langle t_{ij}l_jt_{ik}l_k\rangle-z(z-1)(z-2)\langle t_{ij}^2l_jt_{ik}l_k\rangle_{jk}$$

$$-\tfrac{1}{3}(z-1)(z-2)(z-3)\langle t_{ij}l_jt_{ik}l_kt_{im}l_m\rangle_{jkm}\}. \tag{5·15}$$

In r.h.s. of (5·15), l_i is understood as to be its first approximation, (5·13). Equation (5·4) with $\boldsymbol{l}$ and $\boldsymbol{l}_2$ thus obtained ((5·14) and (5·15)), gives the average magnetization of the site model up to $O(c^3)$, i.e.,

$$\langle m\sigma\rangle = \mathrm{Tr}'[\boldsymbol{mp}(\boldsymbol{1}+\boldsymbol{tp})\boldsymbol{l}-\boldsymbol{mpl_2tpl}-\boldsymbol{mplt_2pl_2}+\cdots]. \tag{5·16}$$

The leading term, $\mathrm{Tr}'\,\boldsymbol{mp}(\boldsymbol{1}+\boldsymbol{tp})[\boldsymbol{1}-(z-1)\boldsymbol{tp}]^{-1}\boldsymbol{c}$ expressed the susceptibility given in Ref. 1). When $z=2$, (5·16) with (5·14) and (5·15) reproduces the previous result (3·23) for the linear chain.

In (5·16) with (5·14) and (5·15), the products which are concerned with terms contained in (5·9)～(5·12), are interpreted in the following way: For example,

$$\begin{aligned}&\{\boldsymbol{mp}[\boldsymbol{1}-(z-1)\boldsymbol{t_2p}]^{-1}\boldsymbol{c}^2\boldsymbol{tp}\langle t_{ij}^2 l_j^2 t_{ik} l_k\rangle\}_{n_1n_5}\\ &=\sum_{n_2}\sum_{n_3}\sum_{n_4} m_{n_1n_1}p_{n_1n_1}\{[\boldsymbol{1}-(z-1)\boldsymbol{t_2p}]^{-1}\}_{n_1n_2}\\ &\quad\times c_{n_2n_2}c_{n_2n_2}t_{n_2n_3}p_{n_3n_3}t_{n_3n_4}^2 p_{n_4n_4}l_{n_4n_5}^2 t_{n_3n_4}p_{n_4n_4}l_{n_4n_5}.\end{aligned} \tag{5·17}$$

($n_1, n_2, n_3, n_4, n_5 = A, B$; $\boldsymbol{m}$ and $\boldsymbol{p}$ are diagonal)

Next we consider the bond model on the infinite Bethe lattice. Using the independence of the bond, we have simultaneous equations for $\bar{l}$ and $\overline{l^2}$ from (5·7) and (5·8).

$$\begin{aligned}\bar{l} &= c+(z-1)x\bar{l}-c^2(z-1)x\bar{l}-c(z-1)y\overline{l^2}-c(z-1)(z-2)x^2\bar{l}^2\\ &\quad-(z-1)(z-2)y\overline{l^2}x\bar{l}-\tfrac{1}{3}(z-1)(z-2)(z-3)x^3\bar{l}^3\end{aligned} \tag{5·18}$$

and

$$\overline{l^2}=c^2+2c(z-1)x\bar{l}+(z-1)y\overline{l^2}+(z-1)(z-2)x^2\bar{l}^2, \tag{5·19}$$

where x and y are defined in (4·3). The first approximation of $\bar{l}$ is obtained from (5·18):

$$\bar{l}=\frac{1}{1-(z-1)x}c. \tag{5·20}$$

Substituting (5·20) into (5·19), we have the first approximation of $\overline{l^2}$

$$\overline{l^2}=\frac{1}{1-(z-1)y}\frac{1-(z-1)x^2}{(1-(z-1)x)^2}c^2. \tag{5·21}$$

By using the first approximation of $\bar{l}$ and $\overline{l^2}$, we obtain the second approximation of $\bar{l}$ (up to $O(c^3)$):

$$\begin{aligned}\bar{l}&=\frac{c}{1-(z-1)x}-\frac{c^3}{1-(z-1)x}\Bigg[\frac{(z-1)x}{1-(z-1)x}+(z-1)y\\ &\quad\times\frac{1}{(1-(z-1)x)^2}\frac{1-(z-1)x^2}{1-(z-1)y}+(z-1)(z-2)x^2\frac{1}{(1-(z-1)x)^2}\\ &\quad+(z-1)(z-2)yx\frac{1}{(1-(z-1)x)^3}\frac{1-(z-1)x^2}{1-(z-1)y}\end{aligned}$$

$$+\frac{1}{3}(z-1)(z-2)(z-3)x^3\frac{1}{(1-(z-1)x)^3}\Big]. \tag{5·22}$$

Hence the magnetization up to $O(c^3)$ is obtained from (5·4′) to be

$$\begin{aligned}\langle\sigma\rangle &= (1+x)\bar{l} - (x+y)\bar{l}\,\bar{l^2}\\ &= \frac{(1+x)c}{1-(z-1)x} + \Big\{\frac{z}{(z-1)[1-(z-1)x]^4}\Big[1-(z-1)x+(z-1)(z-3)x^2\\ &\quad -\frac{1}{3}(z-1)^2(z-5)x^3-\frac{1}{3}(z-1)^2(z-2)x^4\Big]\\ &\quad -\frac{z[1-(z-1)x^2]^2}{(z-1)[1-(z-1)x]^4[1-(z-1)y]}\Big\}c^3. \end{aligned} \tag{5·23}$$

The case of $z=2$ reduces to the previous result of the linear chain, (4·6).

Equations (5·16) (with (5·14) and (5·15)) and (5·23) are seen to be valid in the region where $\bar{l}=O(c)$ and $\bar{l^2}=O(c^2)$ from their derivation. They can be said to be valid in the paramagnetic region until the coefficient of c^3 diverges.

The second derivative of the susceptibility with respect to the magnetic field, $\partial^2\chi/\partial H^2$, diverges at a line for which $1-(z-1)y=0$. A phase where $\langle\sigma\rangle=0$ and $\langle\sigma^2\rangle\neq 0$ ($H=0$) ($\bar{l}=0$ and $\bar{l^2}\neq 0$) in the bond problem was regarded (defined microscopically) as a glass-like phase by Matsubara and Sakata,[4] and its phase boundary between the paramagnetic phase was given by $p_\alpha\,\mathrm{th}^2K_\alpha+p_\beta\,\mathrm{th}^2K_\beta=1/(z-1)$. The present result, (5·23), characterizes the appearance of this phase macroscopically as the divergence of the second derivative of the susceptibility. The magnetization in the ferro-, the antiferromagnetic, and the glass-like regions will be studied separately.

§ 6. An approximation to the bond model

The method of § 3 can be generalized to any order of $\mathrm{th}\,C$ of an arbitrary lattice by counting the number of necessary connected multiply bonded graphs for any crystal lattices.*) In the case of the bond model, the results are expressed in terms of $\langle\mathrm{th}\,K\rangle$, $\langle\mathrm{th}^2 K\rangle$, $\cdots$, $\langle\mathrm{th}^m K\rangle$. In the case where $\langle\mathrm{th}^m K\rangle$ is well approximated by $\langle\mathrm{th}\,K\rangle^m$, the results can be simplified. In that case

$$\begin{aligned}\langle\log Z\{s_i\}\rangle &= \sum_{\langle ij\rangle}\langle\log\mathrm{ch}\,K_{ij}\rangle + N\log C + N\log 2\\ &\quad +\text{term of } O(N) \text{ of } \{\sum\langle\mathrm{th}\,K_{i_1i_2}\rangle\langle\mathrm{th}\,K_{i_2i_3}\rangle\cdots\langle\mathrm{th}\,K_{i_ni_1}\rangle\\ &\quad +\mathrm{th}^2 C\sum\langle\mathrm{th}\,K_{i_1i_2}\rangle\langle\mathrm{th}\,K_{i_2i_3}\rangle\cdots\langle\mathrm{th}\,K_{i_ni_{n+1}}\rangle\\ &\quad +\mathrm{th}^4 C\sum\langle\mathrm{th}\,K_{i_1i_2}\rangle\langle\mathrm{th}\,K_{i_2i_3}\rangle\cdots\langle\mathrm{th}\,K_{i_ni_{n+1}}\rangle\end{aligned}$$

*) Recently Pęcalski and Oguchi[9] gave the high temperature expansion of the susceptibility of the bond model of the square lattice.

$$\times\langle \mathrm{th}\, K_{j_1 j_2}\rangle\langle \mathrm{th}\, K_{j_2 j_3}\rangle\cdots\langle \mathrm{th}\, K_{j_n j_{n+1}}\rangle\}$$
$$+\cdots. \tag{6·1}$$

The coefficient of $\mathrm{th}^n C$, the summation $\sum$ in $\{\ \}$, is simply given by $\sum\Omega_{nm}\langle \mathrm{th}\, K\rangle^m$, where Ω_{nm} is the number of graphs which consist of m bonds and in which the sum of the number of end points and that of odd junctions is n.[4a] The number Ω_{nm} is the same as that of the corresponding regular system. Hence (6·1) can be expressed as

$$\langle \log Z\{s_{ij}\}\rangle = N_B(p_\alpha \log \mathrm{ch}\, K_\alpha + p_\beta \log \mathrm{ch}\, K_\beta)$$
$$-N_B \log\{\mathrm{ch}\,[\mathrm{th}^{-1}(p_\alpha\, \mathrm{th}\, K_\alpha + p_\beta\, \mathrm{th}\, K_\beta)]\} + \log Z(K, C),$$
$$K = \mathrm{th}^{-1}(p_\alpha\, \mathrm{th}\, K_\alpha + p_\beta\, \mathrm{th}\, K_\beta). \tag{6·2}$$

Thus the logarithm of the partition function of the quenched bond model is expressed in terms of the corresponding regular system in this approximation.

In particular, the susceptibility and the magnetization-magnetic field characteristic are obtained by the replacement of $\mathrm{th}\, K$ by $\langle \mathrm{th}\, K\rangle = p_\alpha\, \mathrm{th}\, K_\alpha + p_\beta\, \mathrm{th}\, K_\beta$ in the corresponding expressions for the regular system, i.e., the susceptibility of the infinite Bethe lattice:

$$\frac{kT\chi}{Nm^2} = \frac{1 + p_\alpha \mathrm{th}\, K_\alpha + p_\beta \mathrm{th}\, K_\beta}{1 - (z-1)\,(p_\alpha \mathrm{th}\, K_\alpha + p_\beta \mathrm{th}\, K_\beta)}, \tag{6·3}$$

the magnetization-magnetic field characteristic of the linear chain:

$$\frac{M}{Nm} = \left[1 + \frac{1}{\mathrm{sh}^2 C}\left(\frac{1 - p_\alpha \mathrm{th}\, K_\alpha - p_\beta \mathrm{th}\, K_\beta}{1 + p_\alpha \mathrm{th}\, K_\alpha + p_\beta \mathrm{th}\, K_\beta}\right)^2\right]^{-1/2} \tag{6·4}$$

and the spontaneous magnetization of the square lattice:

$$\frac{M}{Nm} = \left[1 - \frac{[1 - (p_\alpha\, \mathrm{th}\, K_\alpha + p_\beta\, \mathrm{th}\, K_\beta)^2]^4}{16\,(p_\alpha\, \mathrm{th}\, K_\alpha + p_\beta\, \mathrm{th}\, K_\beta)^4}\right]^{1/8} \tag{6·5}$$

are obtained straightforwardly.

Since in Ω_2, $\langle \mathrm{th}^2\, \mathrm{K}\rangle$ and higher order averages do not appear but only $\langle \mathrm{th}\, K\rangle$, the susceptibility (6·3) is exact.[1] It is pointed out by Pękalski and Oguchi[9] that the susceptibilities of the infinite Bethe lattice of the bond model are the same for the quenched system and for the annealed system.[10),11]

The approximation $\langle \mathrm{th}\, K\rangle^m \simeq \langle \mathrm{th}^m\, K\rangle$ is well in the ferro-ferro mixture. The worst case arises when $J_\alpha = -J_\beta$ and $p_\alpha = p_\beta$. In this case the most dominant discrepancy between quenched and annealed systems arises.

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