Random Parallel Transport on Surfaces of Finite Type, and Relations to Homotopy

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Random Parallel Transport on Surfaces of Finite Type, and Relations to Homotopy

by

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Abstract

For general surfaces of finite type, probability measures for parallel transport are constructed. Relations to the topology of the surface are pointed out. We also discuss possible loop invariants.

1. Introduction.

In recent years there has been an increasing interest in stochastic differential geometry. One reason for this is the construction of quantized Yang-Mills fields, in Euclidean space-time. In the physics literature, see e.g. [DM] and [Sei], one studies lattice gauge theory, and let the lattice spacing tend to zero to obtain the continuum limit.

In [AH-KH], for a two dimensional case, thecontinuum limit is constructed directly as a particular case of Markov cosurfaces. The theory was expanded in [K], and similar ideas later appeared in [D], [GKS], [S1,2]. In [Be], [Br] and [F1,2] expectations involving Wilson loops have been studied. In [W], using some ideas in Atiyah's lecture notes [A], heuristic arguments were employed to point out certain connections with topological invariants.

In the present paper we shall only consider the two dimensional case, presenting the "basic construction" initiated in [AH-KH] from a different, simplicial, point of view. (Simplicial approximations to construct Markov random fields on surfaces were considered in [AZ].) This permits us to relate our work with the results of [W] and others.

2. Random Parallel Transport

Let G be a Lie group and let M be a manifold. The action of a connection (one form) on the product bundle $M \times G$ is infinitesimally given by the associated parallel transport. This can be described by a G-valued function, defined for curves in M, and satisfying

$$m(c_1 c_2) = m(c_1)m(c_2), \quad m(c^{-1}) = m(c)^{-1},$$

where $c_1 c_2$ designates that the curve c_2 is followed by c_1 and c^{-1} denotes c with opposite orientation.

To obtain quantised field theories, one way is to construct random fields satisfying, sample wise, these two conditions. Below we shall sketch a general construction for surfaces M of finite type.

The particularity of gauge theory in 2 dimensions makes this possible. This fact has certainly been known among physicists for quite some time. The mathematical formulation and construction goes back to [A, H-K, H], and was given a detailed treatment in Kaufmann [K]. Some of these ideas have appeared later in several papers [D], [F], [G,K,S], [S]. There, the starting point, and the principal object, is the Yang-Mills action functional, and the Feynman path integral formalism. It also appears, among several other subjects, in the recent paper by Witten [W]. In this connection, see also Atiyah's Lincei Lecture Notes [A].

The construction presented below is based on certain inner invariant, or class, functions, and partitionings of M. One also needs to "measure" the plaquettes. Except for this feature, the theory is really "topological" (combinatorial; cf. the comment about the "Hauptvermutung of topology" in Sengupta [S], and [W]), rather than based on the differentiable structure of the manifold. Essentially, this is lattice gauge theory, but the particular family of class functions (convolution semigroups), yields invariance under subdivision. Hence, in physics terminology, the continuum limit is in fact realised.

We remark, finally, that in our 2-dimensional case, if in addition G is simply connected, then $E = M \times G$ is the general principal G-bundle over M. In the general case, one must take into account the topology of the bundle. As remarked in [S], [W], one has to sum over all topologically inequivalent principal G-bundles E over M. For further discussions on this point, refer to the latter article.

Basic construction Let G be a compact topological group, and let $G/\mathrm{Ad}\, G$ denote the orbits under the inner automorphisms $x \to g^{-1}xg$, $g \in G$. Functions that are invariant under inner automorphisms (class functions) are called inner invariant. For such functions, the convolution is commutative.

Let now M be an oriented surface which admits a finite triangulation K. The sets of (non-oriented) vertices, bonds and plaquettes are denoted by $\mathcal{V}(K)$, $\mathcal{B}(K)$, and $\mathcal{P}(K)$, respectively. It is understood that the plaquettes are polygons (rather than triangles), hence simply connected. Denote by σ an assignment of orientation to each bond.

We start from a probability space $(\Omega, \mathcal{F}, P_0)$ (which may depend on the oriented triangulation (K, σ)) upon which a collection of random variables $\{X^{(K,\sigma)}(c),\, c \in \mathcal{B}(K)\}$ are i.i.d. with respect to the Haar measure on G. Each sample $\omega \in \Omega$ describes a connection on the product bundle $M \times G$ and $X^{(K,\sigma)}(c, \omega)$ describes the parallel transport along the oriented bond (c, σ) with respect to the connection ω. (We may choose Ω as the product of $n = \#\mathcal{B}(K)$ copies of G, P_0 as the n-fold product of Haar measure and realise the Xs canonically.)

The gauge group $G^{\#\mathcal{V}(K)}$ is supposed to act on Ω by a measure preserving transformation as

$$X^{(K,\sigma)}(c, \omega \cdot g) = g(c(1))^{-1} X^{(K,\sigma)}(c, \omega) g(c(0)), \tag{2.1}$$

where $c(0)$ and $c(1)$ denote the initial and endpoint of c as given by σ, and where $g(c(0))$ and $g(c(1))$ are the elements of the $\#\mathcal{V}(K)$-tuple g that correspond to these vertices.

Referring to the orientation of M, we form the oriented boundary $\partial\Delta$ for $\Delta \in \mathcal{P}(K)$. We write, for the appropriate bonds $c_1, \dots, c_k \in \mathcal{B}(K)$,

$$\partial\Delta = \langle c_1 \cdots c_k \rangle$$

to indicate that each bond c_j is given the orientation inherited from $\partial\Delta$, and that no specific initial vertex is preferred. We then consider the $G/\operatorname{Ad} G$-valued random variable

$$[(X^{(K,\sigma)}(c_1))^{\sigma(c_1)} \cdots (X^{(K,\sigma)}(c_k))^{\sigma(c_k)}].$$

Here, $\sigma(c_j) = 1$ if the orientations of c_j given by σ and that of $\partial\Delta$ coincide, whereas $\sigma(c_j) = -1$ otherwise. Furthermore, the brackets indicate the conjugacy class. Since it is independent of the specific starting curve, it is denoted by $Y^{(K,\sigma)}(\Delta)$. Due to (2.1), it is gauge invariant:

$$Y^{(K,\sigma)}(\Delta, \omega \cdot g) = Y^{(K,\sigma)}(\Delta, \omega), \quad \Delta \in \mathcal{P}(K),\ g \in G^{\#\mathcal{V}(K)}.$$

To each plaquette we assign its measure, written $|\Delta|$. This could be the area obtained from a Riemannian metric on M, but this is not at all necessary. We shall assume however, that the measure of the closure and the interior of each plaquette coincide.

Let now $(\mu_t)_{t>0}$ denote a convolution semigroup of inner invariant probability measures on G, absolutely continuous w.r.t. Haar measure, and with continuous densities. The same notation for the densities, which are looked upon as class functions, will be used. We shall make use of the following consequence of our assumptions:

$$\int_G \mu_s(xyz)\mu_t(x'y^{-1}z')dy = \mu_{s+t}(zxz'x'). \tag{2.2}$$

We define the following measure on our probability space:

$$P = P^{(K,\sigma)} = \frac{1}{Z(K,\sigma)} \prod_{\Delta \in \mathcal{P}(K)} \mu_{|\Delta|}(Y^{(K,\sigma)}(\Delta))\, P_0,$$

where $Z(K,\sigma)$ is a normalisation constant, and we put $\mu_{|\Delta|} = 1$ if $|\Delta|$ is infinite.

Remarks 1. If the μ_t are symmetric, i.e. invariant under $x \to x^{-1}$, we can allow non-oriented surfaces M. Then the Ys above take their values in $G/\operatorname{Ad} G/\mathbf{Z}_2$. Abusing notation, this means that $\partial\Delta$ and $(\partial\Delta)^{-1}$ are identified.

2. Consider the case where M is a compact oriented surface of genus $p \geq 1$ from which we have removed s holes. Then

$$Z(K,\sigma) = \int_{G^{2p+s}} \mu_{|M|}(x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_p y_p x_p^{-1} y_p^{-1} z_1 \cdots z_s)\, dx_1 dy_1 \ldots dx_p dy_p\, dz_1 \ldots dz_s.$$

Except for the area of M, this is a topological invariant. It is identically one if $s \geq 1$, i.e. when M has a boundary, but not otherwise.
3. The construction is obviously invariant under area preserving homeomorphisms $M \to M$.

Consistency under subdivision The system is stable under refinements. To explain this, we have to elaborate a bit more. Suppose that K and K' are two triangulations. We write $K \preceq K'$ if each bond in K is a product of bonds in K'. For such ordered pairs, let σ' be an orientation for the bonds of K'. We can construct two systems (Ω, P) and (Ω', P'), one for K and one for K', together with a map $\pi : \Omega' \to \Omega$ so that P' and π induce the measure P, i.e. $P = P' \circ \pi^{-1}$. Assume that the sample spaces are realised as products of G, and the random variables X, X' are realised canonically. Enlargement of the triangulation K can be done inductively by the following two kinds of operations.

First, the number of vertices is unchanged, but we add a bond between two vertices. If the variables in Ω' are $\omega' = (\omega_1, \ldots, \omega_n, \omega_{n+1})$, where ω_{n+1} refers to the new bond, we simply cancel it, so that $\pi(\omega') = (\omega_1, \ldots, \omega_n)$. This is the case if the other bonds have the same orientation in both cases. Otherwise, we map those components ω_j corresponding to bonds where the orientations σ and σ' are oppposite to ω_j^{-1}.

The second case is when a bond, say c_1, corresponding to ω_1, is divided into two new ones: $c_1 = c_{12}c_{11}$. Then, assuming the orientations σ' at c_{12} and c_{11} are inherited from σ at c_1 (it should be clear how to handle the general case), we map $(\omega_{11}, \omega_{12}, \omega_2 \ldots, \omega_n)$ to $(\omega_{12}\omega_{11}, \omega_2 \ldots, \omega_n)$. This way, $X(c_1) \circ \pi = X'(c_{12})X'(c_{11})$, i.e. the π respects multiplication of curves.

In both cases one will find, due to (2.2), that π pushes out P' to P.

Relations to homotopy Simple examples show that in general the distributions defined above are not homotopy invariant. A general question is to what extent the distribution of let's say $(X(c_1), \ldots, X(c_n))$ depends on $c_1, \ldots, c_n$. We are very far from an answer to this, but already in simple cases very interesting features appear, and we shall look at some examples on the torus $M = T^2$. Let α and β be the homotopy generators. Consider two immersed images of S^1, a and b, representatives of α and β, and the law of $(X(a), X(b))$.

If a is a simple loop and b a simple loop onto which a further simple loop is added, it turns out that we get a contribution when the orientation of the extra loop is such that the total curve has to cross itself (Fig 1.a). When the orientation is reversed (Fig. 1.b) we do not get any contribution. In the first, but not in the second, case a "tie" is formed, so the constructions separates untied formations from tied ones. (The curves studied are "plats" in the sense of Jones [J].) To be more precise, we should exclude abelian, to obtain effects of this kind. The situation, when b has one tie whose small loop is oriented counterclockwise and a is simple, is described by Fig. 2.a, which is equivalent to the graph in Fig. 2.b, provided that the small loops inscribe the same area.

In this case

$$E[f(X(a),X(b))] = \frac{1}{Z}\int f(x,y)\mu_t(z)\mu_{|M|-t}(xyz^{-2}x^{-1}zy^{-1})\,dxdydz,$$

and the word $(t;z)(|M|-t;xyz^{-2}x^{-1}zy^{-1})$ corresponds to Fig. 2. If we put the small loop going around clockwise on the other side so that the loop b is tied in the different way (Figs. 3) another word appears, namely, $(t;z^{-1})(|M|-t;xyz^{-1}x^{-1}z^2y^{-1})$. This does not give the same distribution as above, so there is also a difference between left and right ties. We give some further examples with two ties. Figures 4.a ,b correspond to the word $(t_1;z_1)(t_2;z_2)(|M|-t_1-t_2;xyz_2^{-1}z_1^{-2}z_2^{-1}x^{-1}z_1z_2y^{-1})$, in that

$$E[f(X(a),X(b))] = \frac{1}{Z}\int f(x,y)\mu_{t_1}(z_1)\mu_{t_2}(z_2)$$
$$\times\mu_{|M|-t_1-t_2}(xyz_2^{-1}z_1^{-2}z_2^{-1}x^{-1}z_1z_2y^{-1})\,dxdydz_1dz_2.$$

If instead a left tie and a right tie are simultaneously on the loop b, as in Fig. 5, we get the non-equivalent word $(t_1;z_1)(t_2;z_2^{-1})(|M|-t_1-t_2;xyz_2^{-1}z_1^{-2}x^{-1}z_2z_1z_2y^{-1})$. On the other hand if both a and b have ties as in Fig. 6, we obtain the word $(t_1;z_1)(t_2;z_2)(|M|-t_1-t_2;xz_1^{-2}yz_2^{-2}z_1x^{-1}z_2y^{-1})$.

We can also form figures looking like the number eight in various ways. As an example, Fig. 7 corresponds to the word $(t_1;z_1^{-1})(t_2;z_2)(|M|-t_1-t_2;xyz_2^{-1}z_1^{-1}z_2^{-1}z_1x^{-1}z_1z_2y^{-1})$ which is equivalent to $(t_1;z_1^{-1})(t_2;z_2)(|M|-t_1-t_2;xyz_1^{-1}z_2^{-1}z_1z_2^{-1}x^{-1}z_2z_1y^{-1})$. Finally, the nested loops in Fig. 8, where the inner one gives the area t_1, and the area between the inner and the outer loop is t_2, is associated with the word $(t_1;z_1)(t_2;z_1^{-1}z_2)(|M|-t_1-t_2;xyz_1^{-1}z_2^{-2}x^{-1}z_2z_1y^{-1})$.

In general, it seems that we can simplify to certain types of graphs and corresponding words. To find more invariant expressions, it would be desirable to get rid of the area parameters above. We therefore propose to study the corresponding infinitesimal expressions, i.e. to differentiate w.r.t. the area parameters, and put them all equal to zero, to obtain infinitesimal Schwinger functions. Already in the case of $SU(2)$, this leads to complicated expressions involving the generator of the semigroup, and products of at least three group characters. In principle, though, combinatorial invariants can be read off from the Wigner-Racah $6j$ symbol (cf. Witten [W]).

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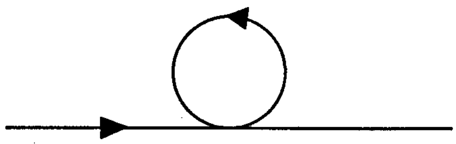

Fig. 1.a Tied loop

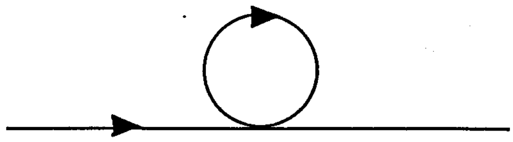

Fig. 1.b Untied loop

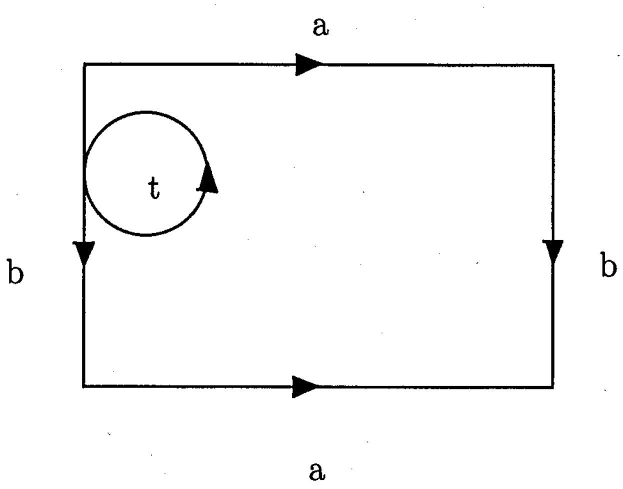

Fig. 2.a

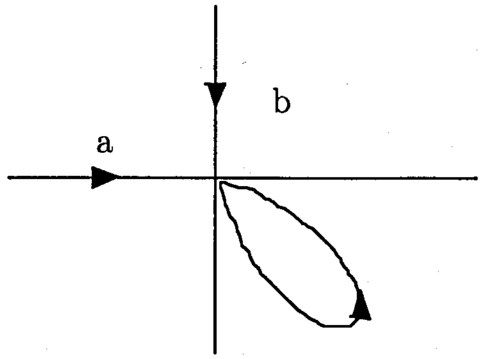

Fig. 2.b

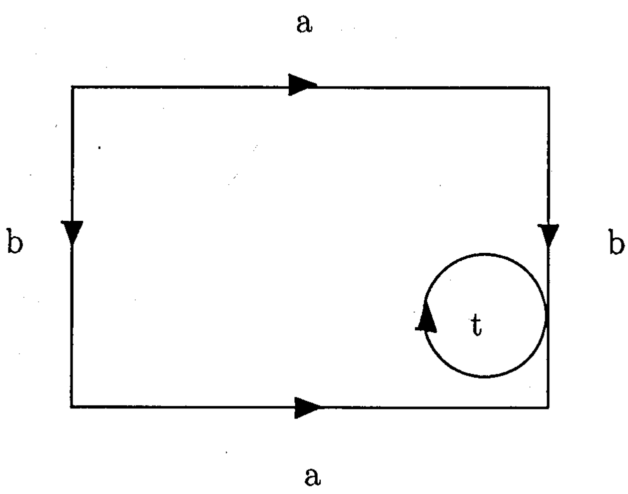

Fig. 3.a

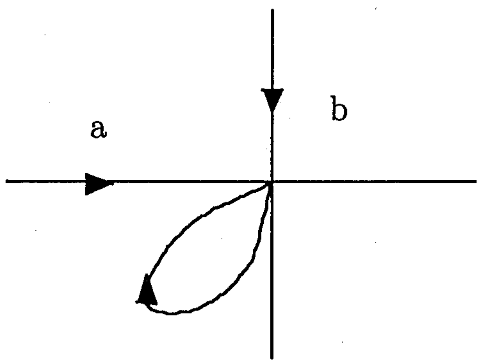

Fig. 3.b

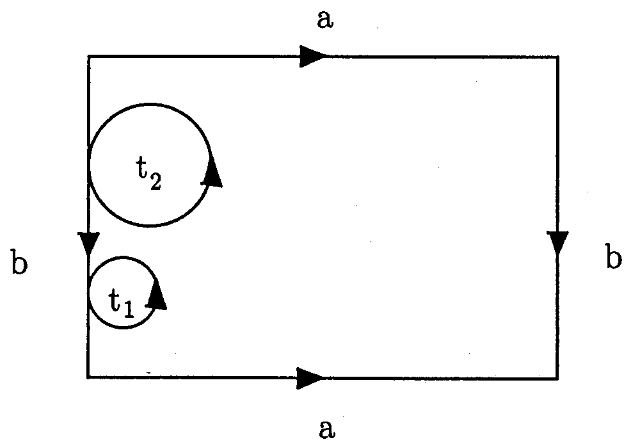

Fig. 4.a

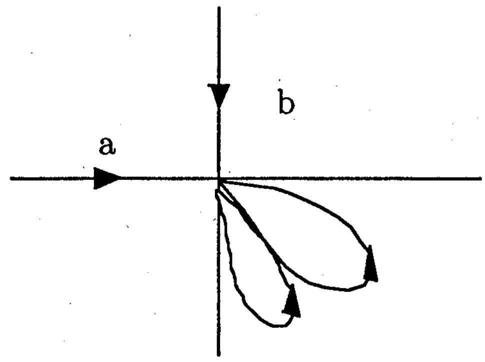

Fig. 4.b

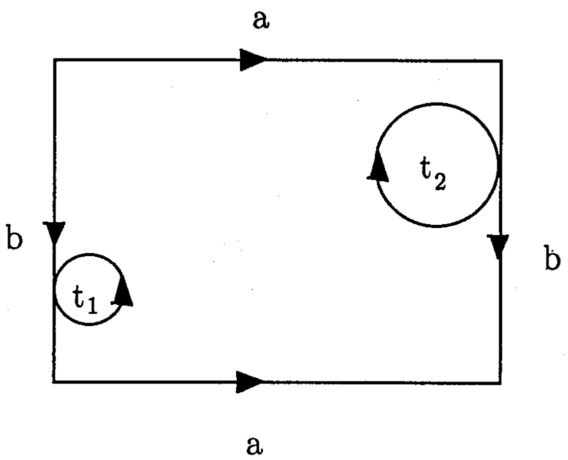

Fig. 5

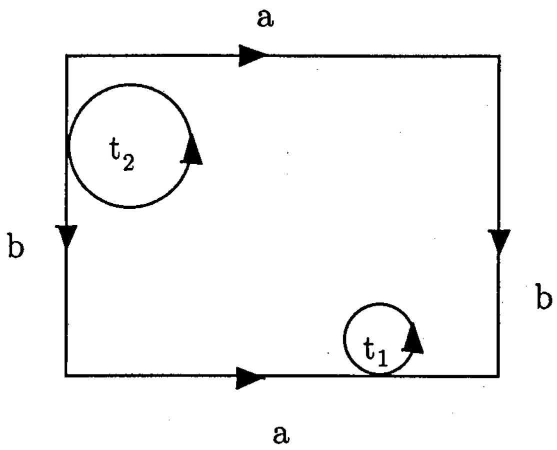

Fig. 6

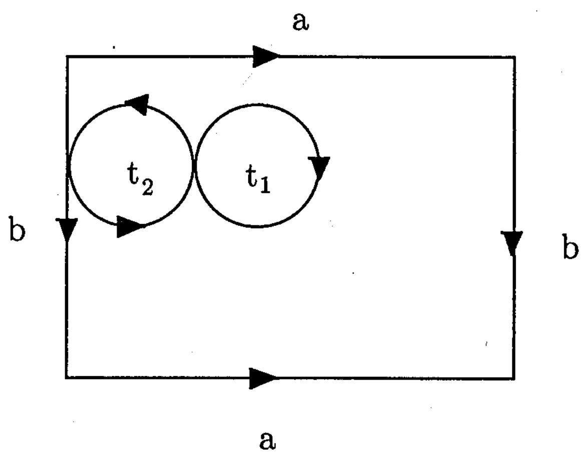

Fig. 7

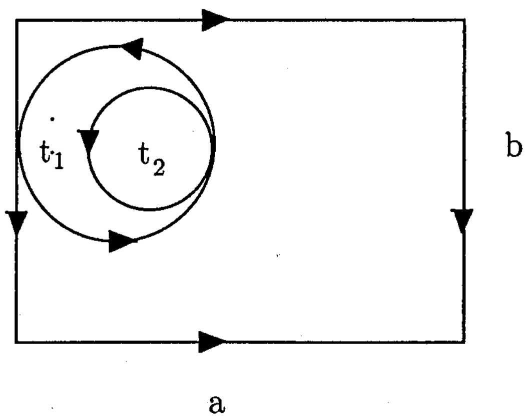

Fig. 8