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RANDOM POINT FIELDS ASSOCIATED WITH CERTAIN FREDHOLM DETERMINANTS II: FERMION SHIFTS AND THEIR ERGODIC AND GIBBS PROPERTIES

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We construct and study a family of probability measures on the configuration space over countable discrete space associated with nonnegative definite symmetric operators via determinants. Under a mild condition they turn out unique Gibbs measures. Also some ergodic properties, including the entropy positivity, are discussed in the lattice case.

1. Introduction. The fermion point processes or point fields have been studied from several viewpoints by many authors, for example, [1, 2, 17, 22, 24] since Macchi [13, 14], and we formulated it in terms of the Laplace transform together with boson point fields and others in our previous papers [17, 18]. Let R be a locally compact Hausdorff space, λ a Radon measure on R and K an integral operator on the L^2-space $L^2(R, \lambda)$. We continue to assume that K is symmetric as in [17, 18] although the non-symmetric case is also studied in [1]. A probability Borel measure μ on the locally finite configuration space Q over R is called a fermion point field or a fermion point process associated with the operator K if its Laplace transform is given by the Fredholm determinant:

$$\int_Q \mu(d\xi)e^{-\langle \xi, f\rangle} = \det\big(I - (1-e^{-f})^{1/2}K(1-e^{-f})^{1/2}\big) \tag{1.1}$$

for every nonnegative continuous function f with compact support where $\langle \xi, f\rangle = \sum_i f(x_i)$ if $\xi = \sum_i \delta_{x_i}$.

In the present paper we focus on the case where R is a countable discrete space, λ is the counting measure and so $L^2(R, \lambda) = \ell^2(R)$ and investigate its basic properties, especially some ergodic properties and the Gibbs property, as we announced in [17]. The condition on K for the existence of μ is the same as before: $\mathrm{Spec}(K) \subset [0, 1]$. But we take another equivalent definition, (1.2) and another method of the proof which fit for the discrete structure. Since the fermion processes have no multiple points, we can take the product space $\{0, 1\}^R$ as the configuration space Q. Moreover, a configuration $\xi \in Q = \{0, 1\}^R$ will often be identified with a countable subset of the base space R and then $\langle \xi, f\rangle = \sum_{x\in\xi} f(x)$.

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THEOREM 1.1. *Let R be a countable discrete space and K be a bounded symmetric operator on $\ell^2(R)$ and assume that its spectrum* $\mathrm{Spec}(K)$ *is contained in the closed unit interval* $[0,1]$. *Then there exists a unique probability Borel measure $\mu = \mu_K$ on $Q = \{0,1\}^R$ such that for any finite subset Λ*

$$\mu(\{\xi \in Q \mid \xi \supset \Lambda\}) = \det(K(x,y))_{x,y\in\Lambda}. \tag{1.2}$$

Moreover, (1.2) *is equivalent to* (1.1) *and the formula*

$$\int_Q \mu(d\xi) e^{-\langle \xi, f\rangle} = \det\bigl(I - (1-e^{-f})K\bigr) \tag{1.3}$$

holds whenever the support of a function f is finite even if it is complex valued.

Theorem 1.1 will be proved in Section 2 by dividing it into Theorems 2.1 and 2.4. Although we only state the case of symmetric operators, the proof works in the case of hermitian operators without any essential changes.

The following fact may justify the naming of the fermion process.

EXAMPLE 1.2. Let H be a bounded symmetric operator on $\ell^2(R)$ with $\mathrm{Spec}(H) \subset [0,1]$. Assume that H belongs to the trace class and let

$$H(x,y) = \sum_{i=1}^{\infty} \lambda_i \psi_i(x)\psi_i(y) \tag{1.4}$$

be its eigenexpansion. Then, it follows from Proposition 2.8 that

$$\begin{aligned} &\mu_H\bigl(\xi(x_i) = 1\ (1 \le i \le k);\ \xi(x) = 0 \text{ otherwise}\bigr) \\ &\quad = \sum_{1\le n_1<\cdots<n_k<\infty} \left(\prod_{i\in\{n_1,\dots,n_k\}} \lambda_i \prod_{j\in\{n_1,\dots,n_k\}^c} (1-\lambda_j) \right) \\ &\qquad \times \bigl|\det\bigl(\psi_{n_i}(x_j)\bigr)_{i,j\in\{1,\dots,k\}}\bigr|^2. \end{aligned} \tag{1.5}$$

If we interpret H as a Hamiltonian and consider the Fermi–Dirac statistics, the wave functions of many body system are given by so-called Slater determinants $\Phi_{n_1,\dots,n_k}(x_1,\dots,x_k) = \det(\psi_{n_i}(x_j))_{i,j=1,\dots,k}$ according to the Pauli exclusion principle. Thus, the fermion process μ_H describes the probability of this many body system.

The discrete and determinantal structure also brings out some interesting properties, as are shown in Section 2. In particular, the image measure of μ under the exchange of 0 and 1 at each site is the fermion process associated with the operator $I - K$ instead of K.

In Sections 3–5 we focus mainly on the case where $R = \mathbf{Z}^d$ and K is a symmetric convolution operator. We consider the shifts σ_i by unit vectors on $\mathbf{Z}^d$

and call the dynamical systems $(Q, \mu, \sigma_i, 1 \le i \le d)$ *fermion shifts* in the present paper. It is always ergodic, a fortiori, mixing. It should be noted that Szegö's first theorem on Toeplitz matrices is a special case of a large deviation result stated as Theorem 3.6.

In Section 4 we give two estimates for the metric entropy or the Kolmogorov–Sinai entropy $h(\mu)$ of fermion shifts (Theorem 4.1). As a consequence, we obtain the following:

THEOREM 1.3. *The metric entropy of a fermion shift is positive unless $K = O$ or I.*

Let us denote by $0^{\Lambda_0}1^{\Lambda_1}$ the cylinder set $\{\xi \in Q \mid \xi(x) = 0 \text{ on } \Lambda_0 \text{ and } \xi(x) = 1 \text{ on } \Lambda_1\}$. In Lemma 5.5 of Section 5 we show a uniform estimate for the ratio of measures:

$$\begin{aligned} &1 - 2\gamma\left(\sum_{i\in\Lambda}\sum_{j\in\Lambda'} k(i-j)^2\right)^{1/2} \\ &\qquad \le \frac{\mu(0^{\Lambda_0\cup\Lambda_0'}1^{\Lambda_1\cup\Lambda_1'})}{\mu(0^{\Lambda_0}1^{\Lambda_1})\mu(0^{\Lambda_0'}1^{\Lambda_1'})} \le \exp\gamma\left(\sum_{i\in\Lambda}\sum_{j\in\Lambda'} k(i-j)^2\right)^{1/2}. \end{aligned} \tag{1.6}$$

The estimate (1.6) follows from an estimate of $\det(I + T)$ for certain trace class operators on Hilbert spaces (Lemma 5.1). As a consequence we get the following:

THEOREM 1.4. *The fermion point process μ is always tail trivial whenever the spectrum of K in contained in the open unit interval* $(0, 1)$.

Also, the weak Bernoulli property of each of the shifts σ_i follows from (1.6) under the condition that $\sum_{n\in\mathbf{Z}^d} |n||k(n)|^2 < \infty$ (Theorem 5.8).

In the final Section 6 we show the Gibbs property (Theorem 6.2) of the fermion point fields.

THEOREM 1.5. *Assume that* $\mathrm{Spec}(K) \subset (0, 1)$. *Set*

$$J = (I - K)^{-1}K \tag{1.7}$$

and denote by J_ξ the restriction of J to the set ξ. Then the fermion process μ is the unique Gibbs measure for the formal Hamiltonian

$$H(\xi) = -\log\det J_\xi. \tag{1.8}$$

Precisely, it is the Gibbs measure for the potential

$$U(\{x_1, \dots, x_n\} \mid \xi) = -\log\det\left(J(x_i, x_j) - \langle J_\xi^{-1} j_\xi^{x_i}, j_\xi^{x_j}\rangle\right)_{i,j=1}^n \tag{1.9}$$

for any $\xi \in Q$ and any finite subset $\{x_1, \dots, x_n\}$ of $R \setminus \xi$ and $j_\xi^x = (J(x, y))_{y\in\xi}$.

Concerning the proof of Theorem 1.5 we would like to emphasize two things.

The Gibbs property will be shown as Theorem 6.2. The crucial key to its proof is the relationship, stated as Theorem 6.3, between the infimum of a quadratic form on a subspace $\{e + f; f \in D_1\}$ of a pre-Hilbert space $\mathbf{R}e \oplus D_1 \oplus D_2$ and the infimum of its dual quadratic form on the subspace $\{e + g; g \in D_2\}$ when e is a unit vector orthogonal to both of D_1 and D_2. This fact together with upper and lower estimates enables us to compute the potential $U(x|\xi)$ as the limit of the ratio of measures of cylinder sets. It seems to the authors that the above fact has been unknown even in finite dimensional cases except for some trivial cases.

The uniqueness of Gibbs measures for U will be shown as Theorem 6.8. In its proof we compute the limits of determinants of infinite matrices by reducing them to the convergence problem of certain quadratic forms thanks to the positive definiteness of those matrices. This idea was suggested by a technique used by Szegö in [25] (cf. [6]).

2. Fermion point field over a discrete space. Let R be a countable discrete set, λ be the counting measure on R, and $Q = Q(R) = \{0, 1\}^R$. We regard an element $\xi \in Q$ as a function $\xi : R \to \{0, 1\}$ and sometimes identify it with the subset $\{x \in R; \xi(x) = 1\}$. Throughout this paper, we assume that the bounded operator $K : \ell^2(R) \to \ell^2(R)$ is symmetric and the spectrum set, denoted by $\operatorname{Spec}(K)$, is contained in the unit closed interval $[0, 1]$. We denote $K \geq O$ if K is nonnegative definite and $K_1 \geq K_2$ if $K_1 - K_2$ is nonnegative definite.

The condition (1.2) is equivalent to the following as is seen from the inclusion–exclusion formula:

$$\mu(0^{\Lambda_0} 1^{\Lambda_1}) = \det\big(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1} K_\Lambda\big) \tag{2.1}$$

for any mutually disjoint finite subsets Λ_0 and Λ_1 with union $\Lambda = \Lambda_0 \cup \Lambda_1$ where P_Λ stands for the orthogonal projection to the subspace $\ell^2(\Lambda)$ and $K_\Lambda = P_\Lambda K P_\Lambda$. The kernel of the operator K will be denoted by $K(x, y)$, $x, y \in R$ and we sometimes identify K_Λ with the matrix $(K(x, y))_{x,y \in \Lambda}$ as is already done in (2.1).

Thus, to prove the former half of Theorem 1.1 it suffices to show the following:

THEOREM 2.1. *Let R be a countable discrete set and K be a bounded symmetric operator on $\ell^2(R)$ with* $\operatorname{Spec}(K) \subset [0, 1]$. *Then there exists a unique Borel probability measure $\mu = \mu_K$ on the configuration space $Q = Q(R) = \{0, 1\}^R$ which satisfies the condition* (2.1).

We need the following simple fact.

LEMMA 2.2. *Let A and B be N by N matrices and P_Λ be the projection to a subset Λ of* $\{1, 2, \dots, N\}$, *that is*, *the N by N diagonal matrix with entries*

$$P_\Lambda(i, i) = \begin{cases} 1, & \text{if } i \in \Lambda, \\ 0, & \text{if } i \in \Lambda^c. \end{cases} \tag{2.2}$$

Then

$$\det(A+B)=\sum_{\Lambda\subset\{1,2,\ldots,N\}}\det(P_\Lambda A+P_{\Lambda^c}B). \tag{2.3}$$

Moreover, if A and B are nonnegative definite matrices and commute each other, then

$$\det(P_\Lambda A+P_{\Lambda^c}B)\geq 0, \tag{2.4}$$

for any $\Lambda\subset\{1,2,\ldots,N\}$.

PROOF. The formula (2.3) is almost trivial, again, by the inclusion–exclusion formula. If A and B commute and are positive definite, then the matrix $AB^{-1}=B^{-1/2}AB^{-1/2}$ is also positive definite and so

$$\begin{aligned}\det(P_\Lambda A+P_{\Lambda^c}B)&=\det B\det(P_\Lambda AB^{-1}+P_{\Lambda^c})\\&=\det B\det(AB^{-1})_\Lambda\\&\geq 0.\end{aligned} \tag{2.5}$$

Hence, the second assertion (2.4) holds by continuity. □

REMARK. In the context of the previous paper [17], the formula (2.3) with $B=I$ may be expressed by using traces of exterior products as

$$\det(I+A)=\sum_{n=0}^{N}\operatorname{tr}(\wedge^n A) \tag{2.6}$$

and

$$\begin{aligned}\operatorname{tr}(\wedge^n A)&=\sum_{|\Lambda|=n}\langle\wedge^n Ae_\Lambda,e_\Lambda\rangle\\&=\sum_{|\Lambda|=n}\det A_\Lambda\end{aligned} \tag{2.7}$$

where e_Λ is defined as follows by taking the canonical basis $e_i=(0,0,\ldots,0,\overset{i}{\check{1}},0,\ldots,0)$:

$$e_\Lambda=e_{i_1}\wedge\cdots\wedge e_{i_k},\qquad \Lambda=\{i_1,\ldots,i_k\},1\leq i_1<\cdots<i_k\leq n. \tag{2.8}$$

Also, whenever the matrix

$$J[\Lambda]=(I_\Lambda-K_\Lambda)^{-1}K_\Lambda \tag{2.9}$$

or its inverse $J[\Lambda]^{-1}=K_\Lambda^{-1}-I_\Lambda$ is well defined, one may write

$$\mu(0^{\Lambda_0}1^{\Lambda_1})=\det(I_\Lambda-K_\Lambda)\det(J[\Lambda])_{\Lambda_1} \tag{2.10}$$

or

$$\mu(0^{\Lambda_0}1^{\Lambda_1}) = \det K_\Lambda \det(J[\Lambda]^{-1})_{\Lambda_0}, \tag{2.11}$$

respectively.

PROOF OF THEOREM 2.1. Define the family of set functions $\{\mu_\Lambda\}_{\Lambda\subset R}$ on $\{0,1\}^\Lambda$ by

$$\mu_\Lambda(\Lambda_0) = \det\big(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1}K_\Lambda\big), \tag{2.12}$$

where $\Lambda = \Lambda_0 \cup \Lambda_1$ with $\Lambda_1 = \Lambda\backslash\Lambda_0$. Since K is a symmetric operator with $0 \le K \le I$, it follows from Lemma 2.2 that the set function μ_Λ defines a probability measure on $\{0,1\}^\Lambda$ for each finite set Λ and $\{\mu_\Lambda\}$ satisfies the Kolmogorov consistency condition. Hence there exists a unique probability measure $\mu = \mu_K$ on Q satisfying (2.1). □

The following is obvious from (2.1).

COROLLARY 2.3. *Let K be a bounded symmetric operator on $\ell^2(R)$ with* $\mathrm{Spec}(K) \subset [0,1]$. *Then the operator $I - K$ defines a probability measure $\overline{\mu}$ which is obtained from μ by interchanging* 0 *and* 1 *at all sites of R.*

Now we show that the Laplace transform of μ is given by (1.1).

THEOREM 2.4. *Let K be a bounded symmetric operator on $\ell^2(R)$ satisfying* $\mathrm{Spec}(K) \subset [0,1]$. *Then, for any nonnegative function f on R with compact support, we have*

$$\int_Q e^{-\langle \xi, f\rangle}\mu(d\xi) = \det(I - K_\varphi), \tag{2.13}$$

where $\langle \xi, f\rangle = \sum_{x\in R}\xi(x)f(x)$ and $K_\varphi = \sqrt{\varphi}K\sqrt{\varphi}$ with $\varphi(x) = 1 - e^{-f(x)}$ as before.

PROOF. Let $\operatorname{supp} f = \Lambda$. Using Lemma 2.2, we get

$$\begin{aligned}
&\int_Q e^{-\langle \xi, f\rangle}\mu(d\xi)\\
&\quad= \sum_{\Lambda_1\subset\Lambda} e^{-\sum_{x\in\Lambda_1} f(x)}\mu(0^{\Lambda_0}1^{\Lambda_1})\\
&\quad= \sum_{\Lambda_1\subset\Lambda} e^{-\sum_{x\in\Lambda_1} f(x)}\det\big(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1}K_\Lambda\big)\\
&\quad= \sum_{\Lambda_1\subset\Lambda} \det\big(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1}e^{-f}K_\Lambda\big)\\
&\quad= \det(I_\Lambda - K_\Lambda + e^{-f}K_\Lambda) = \det(I - K_\varphi).
\end{aligned} \tag{2.14}$$

□

PROOF OF THEOREM 1.1. The former half of Theorem 1.1 is already proved as Theorem 2.1. From Theorem 2.4 we obtain the latter half for nonnegative f's because $\det(I - K_\varphi) = \det(I - (1 - e^{-f})K)$ in (1.3). By the analyticity, (1.3) remains valid for any complex valued f. $\square$

From the formula (2.13) one can easily compute the moments and correlation functions as is done in [17, 18]. The following is the easiest one.

COROLLARY 2.5. *The mean of* $\langle \xi, f\rangle$ *is given by*

$$\int_Q \mu(d\xi)\langle \xi, f\rangle = \sum_{x\in R} K(x,x) f(x). \tag{2.15}$$

In particular, the mean number of 1*'s in a finite set* Λ *equals*

$$\mathrm{tr}(K_\Lambda) = \sum_{x\in\Lambda} K(x,x). \tag{2.16}$$

PROOF. Set tf in place of f in (2.13) and differentiate it at $t = 0$. $\square$

The formula (2.13) also shows that the degeneracy of $I_\Lambda - K_\Lambda$ yields particles in Λ.

COROLLARY 2.6. *If the operator* K_Λ *admits* 1 *as its eigenvalue with multiplicity* $m \geq 1$, *then*

$$\mu_K\{\xi(\Lambda) \geq m\} = 1 \tag{2.17}$$

and

$$\mu_K\{\xi(\Lambda) = m\} = \det\big(I_\Lambda - \big(K_\Lambda - E_\Lambda(1)\big)\big) \tag{2.18}$$

where $E_\Lambda(\lambda)$ *stands for the projection to the eigenspace associated with eigenvalue* λ.

PROOF. Let $\alpha > 0$ and $0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-m} < \lambda_{N-m+1} = \cdots = \lambda_N = 1$ be the eigenvalues of K_Λ counting the multiplicity. Then,

$$\begin{aligned}\int_Q \mu_K(d\xi) e^{-\alpha\xi(\Lambda)} &= \det\big(I - (1 - e^{-\alpha})K_\Lambda\big)\\ &= \prod_{i=1}^N \big(1 - (1 - e^{-\alpha})\lambda_i\big) \\ &\leq \big(1 - (1 - e^{-\alpha})\big)^m = e^{-m\alpha}.\end{aligned} \tag{2.19}$$

Letting $\alpha \to \infty$, one finds $\mu\{\xi(\Lambda) \le m-1\} = 0$. Then it is immediate to see

$$\begin{aligned} &\mu_K\{\xi(\Lambda) = m\} \\ &\quad = \prod_{\lambda=\lambda_1,\dots,\lambda_{N-m}\neq 1} (1-\lambda) = \det\bigl(I_\Lambda - \bigl(K_\Lambda - E_\Lambda(1)\bigr)\bigr). \end{aligned} \tag{2.20}$$ □

Now let us show two of interesting properties brought from the discrete and determinantal structure. The first inequality in (2.21) below indicates that fermion processes are repulsive in contrast with that Ising and similar models are attractive.

PROPOSITION 2.7. *Let K be a bounded symmetric integral operator on $\ell^2(R)$ with* $\mathrm{Spec}(K) \subset [0,1]$. *Then for any mutually disjoint subsets Λ_0 and Λ_1 of R, there hold the inequalities*

$$\mu(0^{\Lambda_0})\mu(1^{\Lambda_1}) \le \mu(0^{\Lambda_0}1^{\Lambda_1}) \le \bigl(\mu(0^{\Lambda_0})\mu(1^{\Lambda_1})\bigr)^{1/2}. \tag{2.21}$$

Besides,

$$\mu(0^{\Lambda_0\cup\Lambda_1}) \le \mu(0^{\Lambda_0})\mu(0^{\Lambda_1}) \quad \textit{and} \quad \mu(1^{\Lambda_0\cup\Lambda_1}) \le \mu(1^{\Lambda_0})\mu(1^{\Lambda_1}) \tag{2.22}$$

and, *in particular*,

$$\mu(0^{\Lambda_0}) \le \prod_{i\in\Lambda_0}\bigl(1-K(i,i)\bigr) \quad \textit{and} \quad \mu(1^{\Lambda_1}) \le \prod_{i\in\Lambda_1} K(i,i). \tag{2.23}$$

PROOF. The left-hand-side inequality in (2.21) follows from the inequality

$$\det\begin{pmatrix} A & B \\ -B^* & C \end{pmatrix} \ge \det A \det C \tag{2.24}$$

which holds if A and C are nonnegative definite symmetric square matrices of size $|\Lambda_0|$ and $|\Lambda_1|$, respectively, and B is a $|\Lambda_0|$ by $|\Lambda_1|$ matrix. This inequality can be proved, for instance, from the direct computation in the special case: the eigenvalues of the matrix

$$\begin{pmatrix} I & B \\ -B^* & I \end{pmatrix}$$

are of the form $1 \pm \sqrt{-1}\beta$ with β real eigenvalues of $(B^*B)^{1/2}$ except possibly for $\beta = 0$.

Note that by a similar argument one may prove the well-known Fischer inequality

$$\det\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \le \det A \det C \tag{2.25}$$

if A and C are nonnegative definite. Indeed, the eigenvalues of

$$\begin{pmatrix} I & B \\ B^* & I \end{pmatrix}$$

are of the form $1 \pm \beta$ with β eigenvalues of $B^* B$ except possibly for $\beta = 0$. Hence, we get (2.22) and also

$$\det \begin{pmatrix} A & B \\ -B^* & C \end{pmatrix} \begin{pmatrix} A & -B \\ -B^* & C \end{pmatrix} \leq \det(A^2 + BB^*) \det(C^2 + B^* B). \tag{2.26}$$

Now set $A = P_{\Lambda_1} K_\Lambda P_{\Lambda_1}$, $B = P_{\Lambda_1} K_\Lambda P_{\Lambda_0} = -P_{\Lambda_1}(I - K_\Lambda) P_{\Lambda_0}$ and $C = P_{\Lambda_0}(I - K_\Lambda) P_{\Lambda_0}$. Then,

$$A^2 + BB^* = P_{\Lambda_1} K_\Lambda^2 P_{\Lambda_1} \leq P_{\Lambda_1} K_\Lambda P_{\Lambda_1} \tag{2.27}$$

and

$$C^2 + B^* B = P_{\Lambda_0}(I - K_\Lambda)^2 P_{\Lambda_0} \leq P_{\Lambda_0}(I - K_\Lambda) P_{\Lambda_0}. \tag{2.28}$$

Consequently, we obtain

$$\det \bigl(P_{\Lambda_0}(I - K_\Lambda) + P_{\Lambda_1} K_\Lambda\bigr)^2 \leq \det(K_{\Lambda_1}) \det(I - K_{\Lambda_0}), \tag{2.29}$$

which is nothing but the desired inequality (2.21).

Finally, (2.23) follows from (2.22) by induction or, directly, from the Hadamard determinant theorem. □

In the above proof, we abused the notation I for I_Λ. Hereafter we omit the suffix Λ for simplicity so far as no confusion occurs.

The next proposition shows that the measure μ_Λ is a mixture of Bernoulli measures defined by eigenvalues of K_Λ. It is also useful to give an estimate on the metric entropy in Section 4. For each finite subset Λ of R let us define a Bernoulli measure $\nu^{(\Lambda)}$ by

$$\nu^{(\Lambda)}(0^{\Lambda_0} 1^{\Lambda_1}) = \prod_{i \in \Lambda_0} (1 - \lambda_i) \prod_{j \in \Lambda_1} \lambda_j, \tag{2.30}$$

where λ_i, $i \in \Lambda$ are the eigenvalues of K_Λ indexed by the set Λ in an arbitrary but fixed manner.

PROPOSITION 2.8. *For each finite subset Λ of R, there exists a $2^{|\Lambda|}$ by $2^{|\Lambda|}$ doubly stochastic matrix $Q = (Q_{I,J})_{I,J \subset \Lambda}$ such that*

$$\mu_\Lambda(0^{\Lambda_0} 1^{\Lambda_1}) = \sum_{\Lambda_1' \subset \Lambda,\ |\Lambda_1'| = |\Lambda_1|} Q_{\Lambda_1, \Lambda_1'} \nu^{(\Lambda)}\bigl(0^{\Lambda_0'} 1^{\Lambda_1'}\bigr) \tag{2.31}$$

for each $\Lambda_1 \subset \Lambda$, where $\Lambda_0' = \Lambda \backslash \Lambda_1'$.

PROOF. By the continuity of determinants it suffices to consider the case where K_Λ does not admit 1 as its eigenvalues. Since $\mu_\Lambda(0^{\Lambda_0} 1^{\Lambda_1}) = \det(I_\Lambda - K_\Lambda) \det(J[\Lambda])_{\Lambda_1}$ by (2.10), it suffices to show the existence of a doubly stochastic matrix Q such that

$$\det(J[\Lambda])_{\Lambda_1} = \sum_{\Lambda_1' \subset \Lambda,\ |\Lambda_1'| = |\Lambda_1|} Q_{\Lambda_1, \Lambda_1'} \prod_{i \in \Lambda_1'} \left(\frac{\lambda_i}{1 - \lambda_i} \right). \qquad \square \tag{2.32}$$

Formula (2.32) is obvious from the following lemma.

LEMMA 2.9. *Let $T = (T_{ij})_{i,j=1}^n$ be an Hermitian matrix with eigenvalues $\alpha_1, \dots, \alpha_n$. Then there exists a 2^n by 2^n doubly stochastic symmetric matrix $Q = (Q_{I,J})$ indexed by subsets I, J of $\{1, \dots, n\}$ such that*

$$\det T_I = \sum_{|J| = |I|} Q_{I,J} \left(\prod_{i \in J} \alpha_i \right) \tag{2.33}$$

for each subset I of $\{1, \dots, n\}$, where $T_I = (T_{ij})_{i,j \in I}$.

PROOF. Take a unitary matrix $P = (p_{ij})_{i,j=1}^n$ which diagonalizes T and write $PTP^* = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. Set

$$Q_{I,J} = \begin{cases} |\det(p_{ij})_{i \in I, j \in J}|^2, & \text{if } |I| = |J|, \\ 0, & \text{otherwise,} \end{cases} \tag{2.34}$$

for each subsets I and J of $\{1, \dots n\}$. Let

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}, \qquad I = \{i_1, \dots, i_k\}, \qquad 1 \le i_1 < \cdots < i_k \le n \tag{2.35}$$

as before. Then if $|I| = k$ one finds

$$\begin{aligned} \det T_I &= \langle \wedge^k T e_I, e_I \rangle \\ &= \langle \wedge^k \left(P^* \operatorname{diag}(\alpha_1, \dots, \alpha_n) P \right) e_I, e_I \rangle \\ &= \langle \wedge^k \operatorname{diag}(\alpha_1, \dots, \alpha_n) (\wedge^k P) e_I, (\wedge^k P) e_I \rangle \\ &= \sum_{J \subset \{1,2,\dots,n\}} \langle \wedge^k \operatorname{diag}(\alpha_1, \dots, \alpha_n) (\wedge^k P) e_I, e_J \rangle \langle e_J, (\wedge^k P) e_I \rangle \\ &= \sum_{J \subset \{1,2,\dots,n\},\ |J| = |I|} \left(\prod_{i \in J} \alpha_i \right) |\langle \wedge^k P e_I, e_J \rangle|^2, \end{aligned} \tag{2.36}$$

where $|\langle \wedge^k P e_I, e_J \rangle|^2$ is nothing but $Q_{I,J}$. Moreover, it is immediate to see the double stochasticity:

$$\sum_{J \subset \{1,\dots,n\}} Q_{I,J} = \sum_{J \subset \{1,\dots,n\}} Q_{J,I} = 1 \tag{2.37}$$

for each $I \subset \{1, \dots, n\}$. Hence we obtain the lemma. $\square$

REMARK. Let $\Lambda = \{1, 2, \dots, n\}$ for simplicity. If $e_1, e_2, \dots, e_n$ are the normalized eigenvectors of K_Λ then the matrix $P = (p_{ij})$ in the proof of Lemma 2.9 can be given by $p_{ij} = e_i(j)$. Hence, (2.31) is rewritten as

$$\tag{2.38} \begin{aligned} &\mu_\Lambda(0^{\Lambda_0} 1^{\Lambda_1}) \\ &\quad = \frac{1}{n_0! n_1!} \sum_\tau \prod_{i \in \Lambda_1} \lambda_{\tau(i)} \prod_{j \in \Lambda_0} (1 - \lambda_{\tau(j)}) \big(\det(e_{\tau(i)}(j))_{i,j \in \Lambda_1}\big)^2 \end{aligned}$$

where τ runs over the permutations of Λ and $n_i = |\Lambda_i|, i = 0, 1$. As easily seen, (2.38) remains valid even if Λ_0 is infinite when K itself is a trace class operator. In particular, formula (1.5) in Example 1.2 holds. Similarly, (2.38) holds even if Λ_1 is infinite when $I - K$ is a trace class operator. Formula (2.38) also gives a refinement of Corollary 2.6.

3. Fermion shifts associated with convolution kernels. Now we restrict ourselves to the lattice case where translations act.

LEMMA 3.1. *Let $R = \mathbf{Z}^d$ and K be a bounded symmetric operator on $\ell^2(\mathbf{Z}^d)$ satisfying* $\mathrm{Spec}(K) \subset [0, 1]$. *When K is a convolution operator, the condition* $\mathrm{Spec}(K) \subset [0, 1]$ *is equivalent to the condition that the convolution kernel k of K is the Fourier transform of an even measurable function $\widehat{k}$ with values in* $[0, 1]$:

$$\tag{3.1} k(n) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \widehat{k}(\theta) e^{in\theta}\, d\theta, \qquad n \in \mathbf{Z}^d, \ \text{with } 0 \le \widehat{k}(\theta) \le 1.$$

The proof is obvious.

From now on we focus on the case where K is a symmetric convolution operator on $\ell^2(\mathbf{Z}^d)$ with kernel k. Then the probability measure $\mu = \mu_K$ defines a shift $(\{0, 1\}^{\mathbf{Z}^d}, \mu, \sigma_i)$, where σ_i's stand for the shift defined by $(\sigma_i \xi)(n) = \xi(n + e_i)$ $(n \in \mathbf{Z}^d)$, where $\{e_i\}_{i=1}^d$ is the standard basis of $\mathbf{Z}^d$.

PROPOSITION 3.2. *Let K be a symmetric convolution operator on $\ell^2(\mathbf{Z}^d)$ with kernel k given by* (3.1). *Then the shift dynamical system $(\{0, 1\}^{\mathbf{Z}^d}, \mu, \sigma_i)$ is mixing.*

PROOF. Let σ^n be the translation by $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$: $\sigma^n = \sigma_1^{n_1} \cdots \sigma_d^{n_d}$. It suffices to show that $\mu(C \cap \sigma^{-n} C') \to \mu(C)\mu(C')$ as $|n| \to \infty$ for any cylinder sets C and C', or equivalently, that

$$\tag{3.2} \begin{aligned} &\det\big(P_{\Lambda_0 \cup (\Lambda_0' + n)}(I - K) P_{\Lambda \cup (\Lambda' + n)} + P_{\Lambda_1 \cup (\Lambda_1' + n)} K P_{\Lambda \cup (\Lambda' + n)}\big) \\ &\qquad \to \det\big(P_{\Lambda_0}(I - K) P_\Lambda + P_{\Lambda_1} K P_\Lambda\big) \det\big(P_{\Lambda_0'}(I - K) P_{\Lambda'} + P_{\Lambda_1'} K P_{\Lambda'}\big) \end{aligned}$$

as $|n| \to \infty$ for any finite subsets Λ_0, Λ_1, Λ_0', and Λ_1' provided that $\Lambda_0 \cap \Lambda_1 = \varnothing$, $\Lambda_0' \cap \Lambda_1' = \varnothing$, $\Lambda = \Lambda_0 \cup \Lambda_1$, and $\Lambda = \Lambda_0' \cup \Lambda_1'$. But since we assume (3.1),

$k(n) \to 0$ as $|n| \to \infty$ by the Riemann–Lebesgue theorem. Hence, it is easy to see that, as $|n| \to \infty$,

$$[P_{\Lambda_0}(I-K)P_{\Lambda\cup(\Lambda'+n)} + P_{\Lambda_1}KP_{\Lambda\cup(\Lambda'+n)}]P_{\Lambda'+n} \to 0 \tag{3.3}$$

and

$$[P_{\Lambda_0'+n}(I-K)P_{\Lambda\cup(\Lambda'+n)} + P_{\Lambda_1'+n}KP_{\Lambda\cup(\Lambda'+n)}]P_{\Lambda} \to 0. \tag{3.4}$$

Consequently, (3.2) holds and the shift is mixing. □

REMARK. The totally mixing property or the mixing property of any multiplicity has been shown for the flow case in [22]. The assertion for the shift is also immediately obtained by the same argument as above.

The following two propositions are discrete versions of the propositions proved in our previous paper [18] and found in a somewhat more general setting in [23]. Our proofs can be done by standard arguments and in a manner easier than the proof of Theorem 3.6 below, so we only state the results here.

PROPOSITION 3.3 (Law of large numbers). *Let f be a bounded continuous function on $\mathbf{R}^d$ with compact support. Then*

$$\left\langle \xi, \frac{f_N}{N^d}\right\rangle \to \int_{\mathbf{R}^d} f(x)k(0)\,dx, \qquad \mu\text{-a.e. and in } L^1(Q,\mu), \tag{3.5}$$

where $f_N(\cdot) = f(\cdot/N)$.

PROPOSITION 3.4 (Central limit theorem). *Let f be a bounded continuous function on $\mathbf{R}^d$ with compact support and $\int_{\mathbf{R}^d} f(x)\,dx = 0$. Then*

$$\lim_{N\to\infty} \int_Q \mu(d\xi)\exp\left(i\left\langle \xi, \frac{f_N}{N^{d/2}}\right\rangle\right) = \exp\left(-\frac{1}{2}\sigma(K)^2\|f\|_2^2\right), \tag{3.6}$$

where we set

$$\sigma(K)^2 = k(0) - \sum_{n\in\mathbf{Z}^d} k(n)^2 \tag{3.7}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \widehat{k}(\theta)\big(1-\widehat{k}(\theta)\big)\,d\theta \tag{3.8}$$

and $f_N(\cdot) = f(\cdot/N)$.

REMARK. The factor $\sigma^2(K)$ in the limiting variance of $\langle \xi, f_N\rangle$ vanishes, for instance, when kernel k is a sine kernel $k(n) = \sin n\alpha/n$ since it is the Fourier transform of the indicator function of an interval in $\mathbf{T}^1$. In this case the variance of $\langle \xi, f_N\rangle$ is of order $\log N$ and the central limit theorem also holds for suitable test functions but under this, appropriate scaling [4, 21]. The shifts associated with such kernels are special among fermion shifts, and similar phenomena appear also in the next large deviation results.

The third of the basic limit theorems is the large deviation principle. Our large deviation result stated below is a natural generalization of the following (multidimensional version of) classical result [9, 26]:

THEOREM 3.5 (Szegö's first theorem). *Let ϕ be a nonnegative continuous function on $\mathbf{T}^d$ and $\widehat{\phi}(n)$ be its Fourier coefficients. Denote the Toeplitz matrices by*

$$T_N[\varphi] = \left(\widehat{\varphi}(n-m)\right)_{0\le n,m\le N}, \qquad N \ge 1. \tag{3.9}$$

Then the following limit exists and in equal to the geometric mean of φ:

$$\lim_{N\to\infty} (\det T_N[\varphi])^{1/(N+1)} = \exp\left\{\left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \log\varphi(\theta)\, d\theta\right\}. \tag{3.10}$$

It might be interesting to note here that Theorem 3.5 and its generalizations are regarded as results of mean field theory by mathematical physicists. It is obvious that Theorem 3.5 follows from the following large deviation result.

THEOREM 3.6 (Large deviation). *Let f be a nonnegative bounded continuous function on $\mathbf{R}^d$ with compact support. Then*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^d} \log \int_Q \mu(d\xi) \exp(-\langle \xi, f_N\rangle) \\
&\quad = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} dx \int_{\mathbf{T}^d} d\theta\, \log\left(1 - \widehat{k}(\theta) + \widehat{k}(\theta) e^{-f(x)}\right),
\end{aligned} \tag{3.11}$$

where $f_N(\cdot) = f(\cdot/N)$.

PROOF. Since formula (3.11) is stable under monotone limits, it suffices to show it in the case where f is continuous. Let $\varphi = 1 - \exp(-f)$, $\varphi_N = 1 - \exp(-f_N)$ and $\Lambda = \operatorname{supp} f$. Note that $K_{\varphi_N} = \sqrt{\varphi_N} K_{N\Lambda} \sqrt{\varphi_N}$, and we obtain

$$\begin{aligned}
\operatorname{tr}(K_{\varphi_N}^n) &\le \|\varphi_N\|_\infty^n \|K_{N\Lambda}\|^{n-1} \operatorname{tr}(K_{N\Lambda}) \\
&\le \|\varphi\|_\infty^n k(0) N^d |\Lambda|
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\operatorname{tr}(K_{\varphi_N}^n) &= \sum_{x_1,\dots,x_n\in\mathbf{Z}^d} k(x_1-x_2)\cdots k(x_n-x_1)\varphi_N(x_1)\cdots\varphi_N(x_n) \\
&= \sum_{y_1,\dots,y_{n-1}\in\mathbf{Z}^d} k(y_1)\cdots k(y_{n-1}) k(-y_1-\cdots-y_{n-1}) \\
&\qquad \times \sum_{x_1\in\mathbf{Z}^d} \varphi\left(\frac{x_1}{N}\right)\varphi\left(\frac{x_1+y_1}{N}\right)\cdots\varphi\left(\frac{x_1+y_1+\cdots+y_{n-1}}{N}\right)
\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&= \sum_{y_1,\dots,y_{n-1}\in\mathbf{Z}^d} k(y_1)\cdots k(y_{n-1})k(-y_1-\cdots-y_{n-1}) \\
&\qquad \times \left\{ N^d \int_{\mathbf{R}^d} \varphi(x)\varphi\left(x+\frac{y_1}{N}\right)\right. \\
&\qquad\qquad \left.\cdots\varphi\left(x+\frac{y_1+\cdots+y_{n-1}}{N}\right)dx + o(N^d)\right\} \\
&= N^d \sum_{y_1,\dots,y_{n-1}\in\mathbf{Z}^d} k(y_1)\cdots k(y_{n-1})k(-y_1-\cdots-y_{n-1}) \\
&\qquad \times \left\{ \int_{\mathbf{R}^d} \varphi(x)^n\,dx + o(1)\right\} \\
&= N^d \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \widehat{k}(\theta)^n\,d\theta \left\{ \int_{\mathbf{R}^d} \varphi(x)^n\,dx + o(1)\right\}.
\end{aligned}
$$

By the dominated convergence theorem, we get

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^d} \log\det\bigl(I - K_{\varphi_N}\bigr) \\
&\qquad = -\lim_{N\to\infty} \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{N^d} \operatorname{tr}\bigl(K_{\varphi_N}^n\bigr) \\
&\qquad = -\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} d\theta\,\widehat{k}(\theta)^n \int_{\mathbf{R}^d} \varphi(x)^n\,dx \\
&\qquad = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} d\theta \int_{\mathbf{R}^d} dx \log\bigl(1-\widehat{k}(\theta)\varphi(x)\bigr).
\end{aligned}
\tag{3.14}
$$

□

REMARK. As a matter of fact, the Hessian of the right-hand side of (3.11) is related to the covariance in the central limit theorem:

$$
\frac{\partial^2}{\partial f^2}[\text{the RHS of (3.11)}]\bigg|_{f=0}[g,g] = \sigma(K)^2\|g\|^2. \tag{3.15}
$$

4. Metric entropy. In this section we continue to assume that K is a symmetric convolution operator on $\ell^2(\mathbf{Z}^d)$ and gives upper and lower estimates for the metric entropy

$$
h(\mu) = \lim_{\Lambda\to\mathbf{Z}^d} \frac{1}{|\Lambda|} H_\Lambda(\mu) \tag{4.1}
$$

where

$$
H_\Lambda(\mu) = -\sum_{\Lambda_1\subset\Lambda} \mu(0^{\Lambda_0}1^{\Lambda_1}) \log\mu(0^{\Lambda_0}1^{\Lambda_1}). \tag{4.2}
$$

Let k be the convolution kernel of K and $\widehat{k}$ be its Fourier transform. Note that k is symmetric: $k(-n) = k(n)$ since K is so. Set

$$H_0 = -k(0)\log k(0) - \big(1-k(0)\big)\log\big(1-k(0)\big) \tag{4.3}$$

and

$$H_1 = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \{-\widehat{k}(\theta)\log\widehat{k}(\theta) - \big(1-\widehat{k}(\theta)\big)\log\big(1-\widehat{k}(\theta)\big)\}\,d\theta. \tag{4.4}$$

Then, by the concavity of $-x\log x - (1-x)\log(1-x)$, there holds the inequality

$$0 \le H_1 \le H_0. \tag{4.5}$$

THEOREM 4.1. *Let K be a symmetric convolution operator on $\ell^2(\mathbf{Z}^d)$ with kernel k and* $\operatorname{Spec}(K) \subset [0,1]$. *Then the following estimates hold for the metric entropy $h(\mu)$*:

$$\max\{\tfrac{1}{2}H_0, H_1\} \le h(\mu) \le H_0. \tag{4.6}$$

Before giving the proof we show an immediate consequence of Theorem 4.1.

COROLLARY 4.2. *The metric entropy of the fermion shift $(\{0,1\}^{\mathbf{Z}^d}, \mu_K, \sigma_i)$ is positive whenever K is nontrivial in the sense that $K \neq O, I$. In the trivial cases the metric entropy is zero.*

PROOF. Theorem 4.1 implies that $h(\mu_K) = 0$ if and only if $H_0 = 0$. And it is immediate to see that $K = O$ or I if and only if $k(0) = 0$ or 1. $\square$

PROOF OF THEOREM 4.1. The upper estimate in (4.6) is obvious since

$$h(\mu) = \inf_{\Lambda\subset\mathbf{Z}^d} \frac{1}{|\Lambda|} H_\Lambda(\mu) \le H_{\{0\}} = H_0. \tag{4.7}$$

Next let us show the inequality

$$h(\mu) \ge \tfrac{1}{2}H_0. \tag{4.8}$$

To this end, we use (2.21) and (2.23) in Proposition 2.7:

$$\begin{aligned}
H_\Lambda(\mu) &= \sum_{\Lambda_1\subset\Lambda} -\mu(0^{\Lambda_0}1^{\Lambda_1})\log\mu(0^{\Lambda_0}1^{\Lambda_1}) \\
&\ge \sum_{\Lambda_1\subset\Lambda} -\mu(0^{\Lambda_0}1^{\Lambda_1})\frac{1}{2}\{\log\mu(0^{\Lambda_0}) + \log\mu(1^{\Lambda_1})\} \\
&\ge -\frac{1}{2}\sum_{\Lambda_1\subset\Lambda} \mu(0^{\Lambda_0}1^{\Lambda_1})\{|\Lambda_0|\log\big(1-k(0)\big) + |\Lambda_1|\log k(0)\} \qquad (4.9)\\
&= -\frac{|\Lambda|}{2}\{\big(1-k(0)\big)\log\big(1-k(0)\big) + k(0)\log k(0)\} \\
&= \frac{|\Lambda|}{2}H_0.
\end{aligned}$$

Hence we obtain (4.8).

Finally, we show the inequality

$$h(\mu) \geq H_1. \tag{4.10}$$

Let $\eta(x) = -x \log x$. Then η is a concave function and it follows from Lemma 2.9 and Jensen's inequality that

$$\begin{aligned}
&- \sum_{\Lambda_1 \subset \Lambda} \mu(0^{\Lambda_0} 1^{\Lambda_1}) \log \mu(0^{\Lambda_0} 1^{\Lambda_1}) \\
&\quad = \sum_{\Lambda_1 \subset \Lambda} \eta\Bigg(\sum_{\Lambda_1' \subset \Lambda} Q_{\Lambda_1, \Lambda_1'} \nu^{(\Lambda)} \big(0^{\Lambda_0'} 1^{\Lambda_1'}\big) \Bigg) \\
&\quad \geq \sum_{\Lambda_1 \subset \Lambda} \sum_{\Lambda_1' \subset \Lambda} Q_{\Lambda_1, \Lambda_1'} \eta\big(\nu^{(\Lambda)} \big(0^{\Lambda_0'} 1^{\Lambda_1'}\big)\big) \\
&\quad = \sum_{\Lambda_1' \subset \Lambda} \eta\big(\nu^{(\Lambda)} \big(0^{\Lambda_0'} 1^{\Lambda_1'}\big)\big),
\end{aligned} \tag{4.11}$$

where $\Lambda_0' = \Lambda \backslash \Lambda_1'$. By the definition of $\nu^{(\Lambda)}$ we get

$$\begin{aligned}
&\sum_{\Lambda_1' \subset \Lambda} \eta\big(\nu^{(\Lambda)} \big(0^{\Lambda_0'} 1^{\Lambda_1'}\big)\big) \\
&\quad = \sum_{i \in \Lambda} \big\{ \eta(\lambda_i(K_\Lambda)) + \eta\big(1 - \lambda_i(K_\Lambda)\big) \big\} \\
&\quad = \operatorname{tr}\big(\eta(P_\Lambda K P_\Lambda)\big) + \operatorname{tr}\big(\eta\big(P_\Lambda (I - K) P_\Lambda\big)\big) \\
&\quad \geq \operatorname{tr}\big(P_\Lambda \eta(K) P_\Lambda\big) + \operatorname{tr}\big(P_\Lambda \eta(I - K) P_\Lambda\big).
\end{aligned} \tag{4.12}$$

Here we used Berezin's inequality [20]: if K is a self-adjoint operator and P is a projection operator on a Hilbert space, then

$$\operatorname{tr}\big(P \varphi(K) P\big) \geq \operatorname{tr}(\varphi(P K P)) \tag{4.13}$$

for each convex function φ. Hence, we obtain

$$\begin{aligned}
h(\mu) &= \lim_{\Lambda \to \mathbf{Z}^d} -\frac{1}{|\Lambda|} \sum_{\Lambda_1 \subset \Lambda} \mu(0^{\Lambda_0} 1^{\Lambda_1}) \log \mu(0^{\Lambda_0} 1^{\Lambda_1}) \\
&\geq \lim_{\Lambda \to \mathbf{Z}^d} \frac{1}{|\Lambda|} \operatorname{tr}\big(P_\Lambda \big(\eta(K) + \eta(I - K)\big) P_\Lambda\big) \\
&= \eta(K)(0,0) + \eta(I - K)(0,0) \\
&= H_1.
\end{aligned} \tag{4.14}$$

Consequently, we get (4.10). $\square$

5. Tail triviality and weak Bernoulli property. In this section we discuss about the tail triviality and a uniform mixing property of the fermion point process. The key lemma is the following.

LEMMA 5.1. *Let T be a bounded operator on a Hilbert space H, P_1 be an orthogonal projection on H and $P_2 = I - P_1$. Assume*

$$P_1 T P_1 = 0, \qquad P_2 T P_2 = 0. \tag{5.1}$$

Moreover, assume $(P_1 T^* P_2 T P_1)^{1/2}$ and $(P_2 T^* P_1 T P_2)^{1/2}$ are trace class operators and

$$\alpha = \sqrt{\operatorname{tr}(P_1 T^* P_2 T P_1)} + \sqrt{\operatorname{tr}(P_2 T^* P_1 T P_2)} < 1. \tag{5.2}$$

Then T is a trace class operator and satisfies the inequalities

$$1 - \alpha \le \det(I + T) \le e^{\alpha/2}. \tag{5.3}$$

PROOF. Write

$$T = \begin{pmatrix} O & B \\ C & O \end{pmatrix}$$

with $B = P_1 T P_2$ and $C = P_2 T P_1$. Since $(B^* B)^{1/2} = (P_2 T^* P_1 T P_2)^{1/2}$ and $(C^* C)^{1/2} = (P_1 T^* P_2 T P_1)^{1/2}$ are of trace class, so is T, and the eigenvalues of T are not greater than 1 in modulus since $\alpha < 1$. Now let

$$A = (I + C^* C)^{-1/2} (B + C^*) (I + B^* B)^{-1/2}. \tag{5.4}$$

Then

$$\begin{aligned} &(I + T)^*(I + T) \\ &\quad = \begin{pmatrix} I + C^* C & O \\ O & I + B^* B \end{pmatrix}^{1/2} \begin{pmatrix} I & A \\ A^* & I \end{pmatrix} \begin{pmatrix} I + C^* C & O \\ O & I + B^* B \end{pmatrix}^{1/2}. \end{aligned} \tag{5.5}$$

Let us show

$$\operatorname{tr}(A^* A) \le \alpha^2. \tag{5.6}$$

Observe that

$$\begin{aligned} &\operatorname{tr}\big((I + B^* B)^{-1/2}\, B^* (I + C^* C)^{-1} B (I + B^* B)^{-1/2}\big) \\ &\qquad = \operatorname{tr}\big((I + C^* C)^{-1} (I + B B^*)^{-1} B B^*\big) \\ &\qquad \le \|(I + C^* C)^{-1}\| \|(I + B B^*)^{-1}\| \operatorname{tr}(B B^*) \\ &\qquad \le \operatorname{tr}(B B^*) = \operatorname{tr}(B^* B) \end{aligned} \tag{5.7}$$

and similarly that

$$\operatorname{tr}\big((I + B^* B)^{-1/2} C (I + C^* C)^{-1} C^* (I + B^* B)^{-1/2}\big) \le \operatorname{tr}(C^* C). \tag{5.8}$$

Then we have, by the Schwarz inequality,

$$
\begin{aligned}
&\big|\operatorname{tr}\big((I+B^*B)^{-1/2}C(I+C^*C)^{-1}B(I+B^*B)^{-1/2}\big)\big|^2\\
&\quad=\big|\operatorname{tr}\big((I+B^*B)^{-1/2}B^*(I+C^*C)^{-1}C^*(I+B^*B)^{-1/2}\big)\big|^2\\
&\quad\le \operatorname{tr}(B^*B)\operatorname{tr}(C^*C).
\end{aligned}
\tag{5.9}
$$

From (5.7)–(5.9), we get

$$
\operatorname{tr}(A^*A)\le \operatorname{tr}(B^*B)+\operatorname{tr}(C^*C)+2\sqrt{\operatorname{tr}(B^*B)\operatorname{tr}(C^*C)}=\alpha^2. \tag{5.10}
$$

Hence (5.6) holds.

Now from (5.5) it follows that

$$
\begin{aligned}
\det(I+T)^2&=\det(I+B^*B)\det(I+C^*C)\det\begin{pmatrix} I & A\\ A^* & I\end{pmatrix}\\
&\le \det(I+B^*B)\det(I+C^*C)\\
&\le \exp\big(\operatorname{tr}(B^*B)+\operatorname{tr}(C^*C)\big)\le e^{\alpha}
\end{aligned}
\tag{5.11}
$$

since $\alpha^2<\alpha<1$. On the other hand, we have

$$
\begin{aligned}
\det(I+T)^2&\ge \det\begin{pmatrix} I & A\\ A^* & I\end{pmatrix}\\
&=\det(I-A^*A)\\
&\ge 1-\operatorname{tr}(A^*A)\ge 1-\alpha^2.
\end{aligned}
\tag{5.12}
$$

Here we used the fact that the inequality

$$
\prod_{i=1}^{\infty}(1-a_i)\ge 1-\sum_{i=1}^{\infty}a_i \tag{5.13}
$$

holds if $0\le a_i<1$ for any $i=1,2,\dots$. Consequently, again by using $\alpha<1$, we obtain

$$
1-\alpha\le\sqrt{1-\alpha^2}\le\det(I+T)\le e^{\alpha/2}. \tag{5.14}
$$

□

We need the following elementary facts.

LEMMA 5.2. *Let T be a bounded operator with bounded inverse T^{-1} on a Hilbert space H, P_1 be an orthogonal projection on H and $P_2=I-P_1$. Write*

$$
T=\begin{pmatrix} A & B\\ C & D\end{pmatrix} \tag{5.15}
$$

if

$$
\begin{aligned}
&A=P_1TP_1:P_1H\to P_1H, && B=P_1TP_2:P_2H\to P_1H,\\
&C=P_2TP_1:P_1H\to P_2H, && D=P_2TP_2:P_2H\to P_2H.
\end{aligned}
\tag{5.16}
$$

Then

$$T^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \tag{5.17}$$

Hereafter, $\|\cdot\|$ stands for the operator norm. Note that the operator norm of a nonnegative definite symmetric operator is equal to the maximum of its spectrum.

COROLLARY 5.3. *Let T be a positive definite bounded symmetric operator on $\ell^2(R)$ with bounded inverse T^{-1} and P_Λ be the orthogonal projection onto $\ell^2(\Lambda)$ for any subset Λ of R. Then,*

$$(P_\Lambda T P_\Lambda)^{-1} \le P_\Lambda T^{-1} P_\Lambda. \tag{5.18}$$

In particular,

$$\|(P_\Lambda T P_\Lambda)^{-1}\| \le \|P_\Lambda T^{-1} P_\Lambda\|. \tag{5.19}$$

PROOF. The proof is obvious from Lemma 5.2. □

From now on we always assume that the spectrum of K is contained in the open interval $(0, 1)$.

LEMMA 5.4. *Let K be a symmetric operator on $\ell^2(R)$ with* $\mathrm{Spec}(K) \subset (0, 1)$, *and set $\gamma = \sup\{\mathrm{Spec}(K^{-1}) \cup \mathrm{Spec}((I - K)^{-1})\}$. Then for any $\Lambda \subset R$ and any choice of $\Lambda_0 \subset \Lambda$, we have*

$$\big\|\big(P_{\Lambda_0}(I - K)P_\Lambda + P_{\Lambda_1} K P_\Lambda\big)^{-1}\big\| \le \gamma. \tag{5.20}$$

PROOF. As is stated in the proof of Proposition 2.7, the eigenvalues of the matrix

$$\begin{pmatrix} I & B \\ -B^* & I \end{pmatrix}$$

are $1 \pm \sqrt{-1}\beta$ with β real eigenvalues of $(B^*B)^{1/2}$. Thus

$$\left\| \begin{pmatrix} I & B \\ -B^* & I \end{pmatrix}^{-1} \right\| \le 1 \tag{5.21}$$

and, hence,

$$\begin{aligned} \left\| \begin{pmatrix} A & B \\ -B^* & C \end{pmatrix}^{-1} \right\| &= \left\| \begin{pmatrix} A^{-1/2} & O \\ O & C^{-1/2} \end{pmatrix} \begin{pmatrix} I & \tilde{B} \\ -\tilde{B}^* & I \end{pmatrix}^{-1} \begin{pmatrix} A^{-1/2} & O \\ O & C^{-1/2} \end{pmatrix} \right\| \\ &\le \max\{\|A^{-1}\|, \|C^{-1}\|\}, \end{aligned} \tag{5.22}$$

where $\tilde{B} = A^{-1/2}BC^{-1/2}$.

From Corollary 5.3, it follows that if T is an invertible symmetric operator on $\ell^2(R)$ with $\mathrm{Spec}(T^{-1}) \subset (0, \gamma]$ then

$$\|(P_\Lambda T P_\Lambda)^{-1}\| \le \gamma \tag{5.23}$$

for any $\Lambda \subset R$. By our assumption and (5.23), we obtain

$$\max\{\|(P_{\Lambda_0}(I-K)P_{\Lambda_0})^{-1}\|, \|(P_{\Lambda_1} K P_{\Lambda_1})^{-1}\|\} \le \gamma \tag{5.24}$$

for any $\Lambda \subset R$ and for any $\Lambda_0, \Lambda_1 \subset \Lambda$. Consequently, we obtain (5.20). $\square$

LEMMA 5.5. *Let K be a symmetric operator with* $\mathrm{Spec}(K) \subset (0,1)$, *and $\Lambda_0, \Lambda_1, \Lambda_0'$ and Λ_1' be mutually disjoint finite subsets of R. Set $\Lambda = \Lambda_0 \cup \Lambda_1$ and $\Lambda' = \Lambda_0' \cup \Lambda_1'$. Then there exists a constant γ depending only on K such that*

$$\begin{aligned} 1 - 2\gamma \left(\sum_{x\in\Lambda}\sum_{y\in\Lambda'} K(x,y)^2 \right)^{1/2} \\ \le \frac{\mu(0^{\Lambda_0\cup\Lambda_0'}1^{\Lambda_1\cup\Lambda_1'})}{\mu(0^{\Lambda_0}1^{\Lambda_1})\mu(0^{\Lambda_0'}1^{\Lambda_1'})} \le \exp\gamma \left(\sum_{x\in\Lambda}\sum_{y\in\Lambda'} K(x,y)^2 \right)^{1/2} \end{aligned} \tag{5.25}$$

whenever $2\gamma(\sum_{x\in\Lambda}\sum_{y\in\Lambda'} K(x,y)^2)^{1/2} < 1$.

PROOF. Given $\Lambda_0, \Lambda_1, \Lambda_0'$ and Λ_1'. Write

$$L = \big(P_{\Lambda_0\cup\Lambda_0'}(I-K) + P_{\Lambda_1\cup\Lambda_1'}K\big)P_{\Lambda\cup\Lambda'} = \begin{pmatrix} L_{\Lambda\Lambda} & L_{\Lambda\Lambda'} \\ L_{\Lambda'\Lambda} & L_{\Lambda'\Lambda'} \end{pmatrix} \tag{5.26}$$

where $L_{\Lambda\Lambda} = (P_{\Lambda_0}(I-K) + P_{\Lambda_1}K)P_\Lambda : \ell^2(\Lambda) \to \ell^2(\Lambda)$, $L_{\Lambda\Lambda'} = (P_{\Lambda_0}(I-K) + P_{\Lambda_1}K)P_{\Lambda'} : \ell^2(\Lambda') \to \ell^2(\Lambda)$ and so on.

Then, we have

$$\frac{\mu(0^{\Lambda_0\cup\Lambda_0'}1^{\Lambda_1\cup\Lambda_1'})}{\mu(0^{\Lambda_0}1^{\Lambda_1})\mu(0^{\Lambda_0'}1^{\Lambda_1'})} = \det\left(I + \begin{pmatrix} O & L_{\Lambda\Lambda}^{-1}L_{\Lambda\Lambda'} \\ L_{\Lambda'\Lambda'}^{-1}L_{\Lambda'\Lambda} & O \end{pmatrix} \right). \tag{5.27}$$

Since $\mathrm{Spec}(K) \cup \mathrm{Spec}(I-K) \subset (0,1)$, by Lemma 5.4 there exists a constant γ such that

$$\|L_{\Lambda\Lambda}^{-1}\| \le \gamma \tag{5.28}$$

for any $\Lambda \subset R$ and for any choice of $\Lambda_0 \subset \Lambda$.

Set $Q_\Lambda = P_{\Lambda_1} - P_{\Lambda_0}$ and $Q_{\Lambda'} = P_{\Lambda_1'} - P_{\Lambda_0'}$. Then,

$$\begin{gathered} Q_\Lambda^2 = P_\Lambda, \qquad Q_{\Lambda'}^2 = P_{\Lambda'}, \\ L_{\Lambda\Lambda'} = Q_\Lambda K_{\Lambda\Lambda'} \quad \text{and} \quad L_{\Lambda'\Lambda} = Q_{\Lambda'}K_{\Lambda'\Lambda}. \end{gathered} \tag{5.29}$$

Thus $L^*_{\Lambda\Lambda'}L_{\Lambda\Lambda'}$ does not depend on Λ_0 and Λ_1 though $L_{\Lambda\Lambda'}$ does (and also $L_{\Lambda\Lambda'}L^*_{\Lambda\Lambda'}$ does):

$$L^*_{\Lambda\Lambda'}L_{\Lambda\Lambda'} = K_{\Lambda'\Lambda}Q_\Lambda Q_\Lambda K_{\Lambda\Lambda'} = K_{\Lambda'\Lambda}K_{\Lambda\Lambda'} \tag{5.30}$$

and so

$$\operatorname{tr}(L^*_{\Lambda\Lambda'}L_{\Lambda\Lambda'}) = \operatorname{tr}(K_{\Lambda'\Lambda}K_{\Lambda\Lambda'}) = \sum_{x\in\Lambda}\sum_{y\in\Lambda'} K(x,y)^2. \tag{5.31}$$

By using an inequality $\operatorname{tr}(ST) \le \|S\| \operatorname{tr}(T)$, we get

$$\operatorname{tr}\big((L^{-1}_{\Lambda\Lambda}L_{\Lambda\Lambda'})^*(L^{-1}_{\Lambda\Lambda}L_{\Lambda\Lambda'})\big) \le \gamma^2 \sum_{x\in\Lambda}\sum_{y\in\Lambda'} K(x,y)^2. \tag{5.32}$$

Similarly,

$$\operatorname{tr}\big((L^{-1}_{\Lambda'\Lambda'}L_{\Lambda'\Lambda})^*(L^{-1}_{\Lambda'\Lambda'}L_{\Lambda'\Lambda})\big) \le \gamma^2 \sum_{x\in\Lambda}\sum_{y\in\Lambda'} K(x,y)^2. \tag{5.33}$$

Consequently, we obtain the desired estimate (5.25) from Lemma 5.1. □

THEOREM 5.6. *Let K be a symmetric operator on $\ell^2(R)$ with kernel k. Assume that* $\operatorname{Spec}(K) \subset (0,1)$. *Then μ is tail trivial.*

PROOF. Let B be a finite set of R and $\mathcal{F}_B$ be the σ-algebra generated by $\{\xi(x); x \in B^c\}$. For the proof it suffices to show that the conditional probability $\mu(0^{\Lambda_0}1^{\Lambda_1} \mid \mathcal{F}_{B^c})(\xi)$ satisfies that

$$\mu(0^{\Lambda_0}1^{\Lambda_1} \mid \mathcal{F}_{B^c})(\xi) \to \mu(0^{\Lambda_0}1^{\Lambda_1}) \qquad \text{as } B \to R \tag{5.34}$$

for any finite subsets Λ_0 and Λ_1 of R and for μ-a.e. $\xi \in \{0,1\}^R$.

By Lemma 5.5 there exists a constant C such that if $B \subset B'$, $\Lambda_0 \cup \Lambda_1 \subset B$ and $\Lambda'_0 \cup \Lambda'_1 \subset B'^c$,

$$\begin{aligned} &\left|\frac{\mu(0^{\Lambda_0\cup\Lambda'_0}1^{\Lambda_1\cup\Lambda'_1})}{\mu(0^{\Lambda_0}1^{\Lambda_1})\mu(0^{\Lambda'_0}1^{\Lambda'_1})} - 1\right|^2 \\ &\qquad \le C \sum_{x\in\Lambda_0\cup\Lambda_1}\ \sum_{y\in\Lambda'_0\cup\Lambda'_1} K(x,y)^2 \le C \sum_{i\in B}\sum_{j\in B'^c} K(x,y)^2. \end{aligned} \tag{5.35}$$

Hence, we obtain

$$\begin{aligned} &\left|\frac{1}{\mu(0^{\Lambda_0}1^{\Lambda_1})}\mu(0^{\Lambda_0}1^{\Lambda_1} \mid \mathcal{F}_{B'^c})(\xi) - 1\right|^2 \\ &\qquad \le C \sum_{x\in B}\sum_{y\in B'^c} K(x,y)^2 \to 0 \qquad (B' \to R). \end{aligned} \tag{5.36}$$

□

REMARK. In Section 6, we will have another proof of Theorem 5.6. Indeed, μ is the unique Gibbs measure with respect to a continuous potential.

In the ergodic-theoretical terminology (e.g., [3, 10]), Theorem 5.6 implies the following.

COROLLARY 5.7. *Let K be a symmetric convolution operator and assume* $\mathrm{Spec}(K) \subset (0,1)$. *Then the shift* (Q, μ_K, σ) *is a K-system. In particular, it has complete positive entropy*: $h(\mu, \sigma, \alpha) > 0$ *for any nontrivial partition* $\alpha = \{A_1, \dots, A_k\}$.

If stronger conditions are imposed on K, one may deduce various uniform mixing properties from Lemma 5.5. For instance, the weak Bernoulli property (cf. [10, 27]) holds under a rather mild summability condition on the kernel.

THEOREM 5.8. *Let K be a symmetric convolution operator on* $\ell^2(\mathbf{Z}^d)$ *with kernel k. Assume that* $\mathrm{Spec}(K) \subset (0,1)$ *and that*

$$\sum_{n \in \mathbf{Z}^d} |n||k(n)|^2 < \infty. \tag{5.37}$$

Then the following uniform mixing property holds:

$$\lim_{|n| \to \infty} \frac{\mu(0^{\Lambda_0 \cup (\Lambda_0' + n)} 1^{\Lambda_1 \cup (\Lambda_1' + n)})}{\mu(0^{\Lambda_0} 1^{\Lambda_1}) \mu(0^{\Lambda_0'} 1^{\Lambda_1'})} = 1 \tag{5.38}$$

uniformly in finite subsets Λ_0, Λ_1, Λ_0' *and* Λ_1' *of* $\mathbf{Z}^d$ *provided that* $\Lambda = \Lambda_0 \cup \Lambda_1 \subset \mathbf{Z}^{k-1} \times \mathbf{Z}^- \times \mathbf{Z}^{d-k}$, $\Lambda' = \Lambda_0' \cup \Lambda_1' \subset \mathbf{Z}^{k-1} \times \mathbf{Z}^+ \times \mathbf{Z}^{d-k}$, $\Lambda_0 \cap \Lambda_1 = \varnothing$ *and* $\Lambda_0' \cap \Lambda_1' = \varnothing$. *Here* $\Lambda_i' + n = \{l + ne_k\ ;\ l \in \Lambda_i'\}$ *for* $i = 0, 1$.

In particular, the shift dynamical system $(\{0,1\}^{\mathbf{Z}^d}, \mu, \sigma_k)$ *satisfies the weak Bernoulli property and so it is isomorphic to a Bernoulli shift.*

PROOF. By Lemma 5.5 it suffices to prove

$$\lim_{n \to \infty} \sum_{i \in \Lambda} \sum_{j \in \Lambda' + n} k(i - j)^2 = 0 \tag{5.39}$$

uniformly in $\Lambda \subset \mathbf{Z}^{k-1} \times \mathbf{Z}^- \times \mathbf{Z}^{d-k}$, $\Lambda' \subset \mathbf{Z}^{k-1} \times \mathbf{Z}^+ \times \mathbf{Z}^{d-k}$. But it is obvious from the assumption (5.37). $\square$

6. Gibbs property. Now we prove that our measures are Gibbs measures with respect to some potentials, which generally have long range.

From now on it is convenient to identify $(\xi(x))_{x \in R} \in \{0,1\}^R$ with the set $\{x \in R;\ \xi(x) = 1\}$, which we also denote by ξ. We employ the following definition in the present paper.

DEFINITION 6.1. A probability measure μ on Q is called a Gibbs measure if there exists a family of continuous functions $U(X \mid \xi)$ defined on the set of pairs of a finite subset X and a configuration ξ with $\xi \cap X = \varnothing$ such that

$$E^{\mu}[F \mid \mathcal{F}_{\Lambda^c}](\xi) = \frac{1}{Z_{\Lambda,\xi}} \sum_{X \subset \Lambda} e^{-U(X \mid \xi_{\Lambda^c})} F(X \cup \xi_{\Lambda^c}) \tag{6.1}$$

for every finite subset Λ of R, where

$$Z_{\Lambda,\xi} = \sum_{X \subset \Lambda} e^{-U(X \mid \xi_{\Lambda^c})} \tag{6.2}$$

is a normalizing constant. The potential U is unique up to an additive constant. In below, we always normalize U so that $U(\varnothing \mid \xi) = 0$.

By (6.1) the functions $U(X \mid \xi)$ necessarily satisfy the relation

$$U(X \cup Y \mid \xi) = U(X \mid \xi) + U(Y \mid X \cup \xi) \tag{6.3}$$

if $\xi \cap (X \cup Y) = \varnothing$ and $X \cap Y = \varnothing$. In particular, if we write $U(x \mid \xi) = U(\{x\} \mid \xi)$ then the continuous function $U(x \mid \xi_{\{x\}^c})$ determines all the values of $U(X \mid \xi)$ and

$$\begin{aligned} U(\{x_1, \dots, x_n\} \mid \xi) &= U(x_n \mid \{x_1, \dots, x_{n-1}\} \cup \xi) \\ &\quad + U(x_{n-1} \mid \{x_1, \dots, x_{n-2}\} \cup \xi) + \cdots + U(x_1 \mid \xi) \end{aligned} \tag{6.4}$$

for any n and any n-point subset $\{x_1, \dots, x_n\}$ of $R \setminus \xi$. Moreover, the function $U(x \mid \xi)$ necessarily satisfies the relation

$$U(x \mid \{y\} \cup \xi) + U(y \mid \xi) = U(y \mid \{x\} \cup \xi) + U(x \mid \xi) \tag{6.5}$$

whenever $x \neq y$ and $\{x, y\} \cap \xi = \varnothing$. Note that under our normalization

$$\frac{\mu(\xi\{x\} = 1 \mid \mathcal{F}_{\{x\}^c})(\xi)}{\mu(\xi\{x\} = 0 \mid \mathcal{F}_{\{x\}^c})(\xi)} = e^{-U(x \mid \xi_{\{x\}^c})} \tag{6.6}$$

and that (6.6) is equal to (6.1) (cf. [16]).

THEOREM 6.2. *Let R be a countable discrete space and K be a symmetric bounded operator on $\ell^2(R)$. Assume*

$$\operatorname{Spec}(K) \subset (0, 1). \tag{6.7}$$

Then μ is a Gibbs measure and the continuous function $U(\cdot|\cdot)$ is given by

$$U(x \mid \xi) = -\log\bigl(J(x,x) - \langle J_\xi^{-1} j_\xi^x, j_\xi^x \rangle\bigr) \tag{6.8}$$

whenever $x \in R$, $\xi \in Q$ and $\{x\} \cap \xi = \varnothing$. Here $J(x, y)$ stands for the kernel of the operator $J = (I - K)^{-1}K$, and we set $J_\xi = (J(y, z))_{y,z \in \xi}$ and $j_\xi^x = (J(x, y))_{y \in \xi}$ for $\xi \in Q$.

REMARK. The unique Gibbs property will be shown as Theorem 6.8 separately. Let $X = \{x_1, \dots, x_n\}$ be a finite subset of R and $\xi \cap X = \varnothing$. Then it turns out that

$$U(X \mid \xi) = -\log\det\bigl(J(x_i, x_j) - \langle J_\xi^{-1} j_\xi^{x_i}, j_\xi^{x_j}\rangle\bigr)_{i,j=1}^n. \tag{6.9}$$

The proof follows immediately from Lemma 6.7 by induction.

For the proof of Theorem 6.2, we need the following fact which may be interesting in itself.

THEOREM 6.3. *Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and A be a positive definite bounded symmetric operator with bounded inverse A^{-1} on H. Let D be a dense linear subspace of H and*

$$D = D_0 \oplus D_1 \oplus D_2 \tag{6.10}$$

be an orthogonal decomposition with $\dim D_0 = 1$. *Take a unit vector e from D_0 and set*

$$\alpha = \inf_{f \in D_1} \bigl(A(e+f), e+f\bigr) \quad \textit{and} \quad \beta = \inf_{g \in D_2} \bigl(A^{-1}(e+g), e+g\bigr). \tag{6.11}$$

Then there holds the identity

$$\alpha\beta = 1. \tag{6.12}$$

PROOF. Let $H_i (i = 0, 1, 2)$ be the completions of $D_i (i = 0, 1, 2)$, respectively. Since both A and A^{-1} are bounded, the quadratic forms $(A\cdot,\cdot)$ and $(A^{-1}\cdot,\cdot)$ are both equivalent to the norm $(\cdot,\cdot)$. Hence there exist the vectors $u \in H_1$ and $v \in H_2$ which attain the minimum of α and β in (6.11). In other words, they satisfy

$$\alpha = \bigl(A(e+u), e+u\bigr) \quad \text{and} \quad \beta = \bigl(A^{-1}(e+v), e+v\bigr). \tag{6.13}$$

Moreover, they necessarily satisfy

$$A(e+u) \in H_1^\perp = H_0 \oplus H_2 \quad \text{and} \quad A^{-1}(e+v) \in H_2^\perp = H_0 \oplus H_1 \tag{6.14}$$

and so

$$\bigl(A(e+u), A^{-1}(e+v)\bigr) = \bigl(A(e+u), e\bigr)\bigl(A^{-1}(e+v), e\bigr). \tag{6.15}$$

Noting that $e + u \in H_0 \oplus H_1$ and $e + v \in H_0 \oplus H_2$, we obtain

$$\begin{aligned}\alpha\beta &= \bigl(A(e+u), e+u\bigr)\bigl(A^{-1}(e+v), e+v\bigr)\\ &= \bigl(A(e+u), e\bigr)\bigl(A^{-1}(e+v), e\bigr)\\ &= \bigl(A(e+u), A^{-1}(e+v)\bigr) = (e,e) = 1.\end{aligned} \tag{6.16}$$

$\square$

The next corollary immediately follows from Theorem 6.3.

COROLLARY 6.4. *Let* $R = \{x\} \cup \Lambda_1 \cup \Lambda_2$ *be a partition and* J *be a bounded symmetric operator such that* $O < c_1 I \le J \le c_2 I$. *Then*,

$$\inf_{f \in \mathcal{D}_x} \sum_{y,z \in \Lambda_1 \cup \{x\}} J(y,z) f(y) f(z) \inf_{f \in \mathcal{D}_x} \sum_{y,z \in \Lambda_2 \cup \{x\}} J^{-1}(y,z) f(y) f(z) = 1 \tag{6.17}$$

where $\mathcal{D}_x = \{f : R \to \mathbf{R};\ f(x) = 1,\ f$ *is of compact support*$\}$.

We recall

$$\mu_\Lambda(0^{\Lambda_0} 1^{\Lambda_1}) = \det(I - K_\Lambda) \det(J[\Lambda])_{\Lambda_1} \tag{6.18}$$

for a finite subset $\Lambda = \Lambda_0 \cup \Lambda_1$. The essential part of the proof of Theorem 6.2 is the following.

LEMMA 6.5. *Suppose* $\mathrm{Spec}(K)$ *is contained in the unit open interval* $(0, 1)$. *Let*

$$\mathcal{D}_x = \{f : R \to \mathbf{R};\ f(x) = 1,\ \operatorname{supp} f \text{ is compact}\}, \tag{6.19}$$

as before. *Then*, *the following hold*:

$$\limsup_{\Lambda \uparrow R} \frac{\mu_\Lambda(\{x\} \cup \xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda)} \le \inf_{f \in \mathcal{D}_x} \sum_{y,z \in \{x\} \cup \xi} J(y,z) f(y) f(z) \tag{6.20}$$

$$= J(x,x) - \sum_{y,z \in \xi} J_\xi^{-1}(y,z) J(x,y) J(z,x) \tag{6.21}$$

for any $\xi \in Q$. *Here* ξ *is identified with the set* $\{x \in R; \xi(x) = 1\}$ *and* $J_\xi = (J(y,z))_{y,z \in \xi}$.

PROOF. Under the assumption above, both $J = (I - K)^{-1} K$ and $J^{-1} = K^{-1}(I - K)$ are bounded operators. We use the following elementary identities for the ratio of determinants of positive definite matrices: if $A = (a(y,z))_{y,z \in I}$ and $\tilde{A} = (a(y,z))_{y,z \in I \cup \{x\}}$, then

$$\begin{aligned} \frac{\det \tilde{A}}{\det A} &= a(x,x) - \sum_{y,z \in I} A^{-1}(y,z) a(x,y) a(z,x) \\ &= \inf_{f(x)=1} \sum_{y,z \in I \cup \{x\}} a(y,z) f(y) f(z). \end{aligned} \tag{6.22}$$

Thus if $\xi \subset R \setminus \{x\}$

$$\frac{\mu_\Lambda(\{x\} \cup \xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda)} = \inf_{f \in \mathcal{D}_x} \sum_{y,z \in \{x\} \cup \xi_\Lambda} J[\Lambda](y,z) f(y) f(z) \tag{6.23}$$

where $J[\Lambda] = (I - K_\Lambda)^{-1} K_\Lambda$. Since $J[\Lambda]_\Lambda \le J_\Lambda$, we have

$$\sum_{y,z \in \{x\} \cup \xi_\Lambda} J[\Lambda](y,z) f(y) f(z) \le \sum_{y,z \in \{x\} \cup \xi} J(y,z) f(y) f(z), \tag{6.24}$$

whenever supp $f \subset \Lambda$. Consequently,

$$\limsup_{\Lambda\uparrow R} \frac{\mu_\Lambda(\{x\}\cup\xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda)} \le \sum_{y,z\in\{x\}\cup\xi} J(y,z)f(y)f(z) \tag{6.25}$$

for any $f \in \mathcal{D}_x$ and we obtain (6.20). It is easy to see

$$\inf_{f\in\mathcal{D}} \sum_{y,z\in\{x\}\cup\xi} J(y,z)f(y)f(z) = J(x,x) - \langle J_\xi^{-1} j_\xi^x, j_\xi^x\rangle. \tag{6.26}$$

□

LEMMA 6.6. *Suppose* Spec(K) *is contained in the unit open interval* $(0,1)$. *Let* $\mathcal{D}_x$ *be as in Lemma* 6.5. *Then, the following inequality holds*:

$$\liminf_{\Lambda\uparrow R} \frac{\mu_\Lambda(\{x\}\cup\xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda)} \ge \left(\inf_{f\in\mathcal{D}_x} \sum_{y,z\in\{x\}\cup\xi^c} J^{-1}(y,z)f(y)f(z)\right)^{-1} \tag{6.27}$$

for any $\xi \in Q$. *Here* ξ *is identified with the set* $\{x \in R; \xi(x) = 1\}$.

PROOF. For a configuration $\xi \in Q$ let $\overline{\xi}$ be the configuration obtained by interchanging 0 and 1 at all sites of R. Recall that the distribution of $\overline{\xi}$ under μ is the fermion process associated with $I - K$ in place of K and the operator J is replaced by

$$\big(I-(I-K)\big)^{-1}(I-K) = K^{-1}(I-K) = J^{-1}. \tag{6.28}$$

Thus, we obtain from Lemma 6.5

$$\limsup_{\Lambda\uparrow R} \frac{\mu_\Lambda(\xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda\cup\{x\})} \le \inf_{f(x)=1} \sum_{y,z\in\{x\}\cup\overline{\xi}} J^{-1}(y,z)f(y)f(z). \tag{6.29}$$

Hence follows the desired lower estimate. □

PROOF OF THEOREM 6.2. Define $U(x \mid \xi)$ by

$$U(x\mid\xi) = J(x,x) - \langle J_\xi^{-1} j_\xi^x, j_\xi^x\rangle. \tag{6.30}$$

Then Lemmas 6.5 and 6.6 above together with Corollary 6.4 show that there exists the limit

$$\lim_{\Lambda\uparrow R} \frac{\mu_\Lambda(\{x\}\cup\xi_\Lambda)}{\mu_\Lambda(\xi_\Lambda)} = e^{-U(x|\xi)}$$

$$= \inf_{f(x)=1} \sum_{y,z\in\{x\}\cup\xi} J(y,z)f(y)f(z) \tag{6.31}$$

$$= \sup_{f(x)=1} \left(\sum_{y,z\in\{x\}\cup\xi^c} J^{-1}(y,z)f(y)f(z)\right)^{-1}. \tag{6.32}$$

The continuity of $U(x|\xi)$ is now obvious. □

Now we proceed to the proof to relation (6.9) stated in the remark after Theorem 6.2.

LEMMA 6.7. *For any mutually disjoint finite subsets X and Y of $R \setminus \xi$, the following relation holds*:

$$\begin{aligned}&\det(J_{X\cup Y} - J_{X\cup Y,\xi} J_\xi^{-1} J_{\xi,X\cup Y})\\ &\quad = \det(J_X - J_{X,\xi\cup Y} J_{\xi\cup Y}^{-1} J_{\xi\cup Y,X}) \det(J_Y - J_{Y,\xi} J_\xi^{-1} J_{\xi,Y}).\end{aligned}\tag{6.33}$$

PROOF. By applying (5.17) of Lemma 5.2 twice, we get

$$\begin{aligned}&(J_X - J_{X,\xi\cup Y} J_{\xi\cup Y}^{-1} J_{\xi\cup Y,X})^{-1}\\ &\quad = (J_{\xi\cup X\cup Y}^{-1})_X\\ &\quad = \big((J_{\xi\cup X\cup Y}^{-1})_{X\cup Y}\big)_X\\ &\quad = \big((J_{X\cup Y} - J_{X\cup Y,\xi} J_\xi^{-1} J_{\xi,X\cup Y})^{-1}\big)_X.\end{aligned}\tag{6.34}$$

Note that formula (5.17) in Lemma 5.2 also implies that if H is finite dimensional or, equivalently, if T is a positive definite symmetric matrix of finite size then there holds the relation

$$\det T = \frac{\det(P_\Lambda T P_\Lambda)}{\det(P_{\Lambda^c} T^{-1} P_{\Lambda^c})}\tag{6.35}$$

for any subset Λ of the suffix set. Hence we get

$$\begin{aligned}&\det(J_{X\cup Y} - J_{X\cup Y,\xi} J_\xi^{-1} J_{\xi,X\cup Y})^{-1}\\ &\quad = \frac{\det\big((J_{X\cup Y} - J_{X\cup Y,\xi} J_\xi^{-1} J_{\xi,X\cup Y})^{-1}\big)_X}{\det(J_{X\cup Y} - J_{X\cup Y,\xi} J_\xi^{-1} J_{\xi,X\cup Y})_Y}\\ &\quad = \det(J_X - J_{X,\xi\cup Y} J_{\xi\cup Y}^{-1} J_{\xi\cup Y,X})^{-1} \det(J_Y - J_{Y,\xi} J_\xi^{-1} J_{\xi,Y})^{-1}.\end{aligned}\tag{6.36}$$

Consequently, we obtain (6.33). □

Now we show the uniqueness of the Gibbs measure. Set

$$q_{\Lambda,\xi}(X) = \frac{e^{-U(X|\xi)}}{Z_{\Lambda,\xi}} = \frac{\det(\Phi(\xi)_X)}{Z_{\Lambda,\xi}} \qquad \text{for } \xi \subset \Lambda^c,\tag{6.37}$$

where $\Phi(\xi)(x, y) = J(x, y) - \langle J_\xi^{-1} j_\xi^x, j_\xi^y\rangle$ for $x, y \in \Lambda$, or equivalently, $\Phi(\xi) = J_\Lambda - J_{\Lambda\xi} J_\xi^{-1} J_{\xi\Lambda}$, and $Z_{\Lambda,\xi}$ is the normalizing constant which can be written by (2.6) and (2.7) as

$$Z_{\Lambda,\xi} = \sum_{X\subset\Lambda} \det\big(\Phi(\xi)_X\big) = \det\big(I + \Phi(\xi)\big).\tag{6.38}$$

We state the uniqueness theorem starting from the operator J.

THEOREM 6.8. *Let R be a countable discrete space and J be a nonnegative definite symmetric bounded operator on $\ell^2(R)$. Assume that*

$$\mathrm{Spec}(J) \subset (0, \infty) \tag{6.39}$$

and define a potential $U(x \mid \xi)$ by

$$U(x \mid \xi) = -\log\bigl(J(x,x) - \langle J_\xi^{-1} j_\xi^x, j_\xi^x\rangle\bigr) \tag{6.40}$$

for $\xi \subset R \setminus \{x\}$. Then the Gibbs measure for U is unique. In other words, if μ is a probability measure on Q such that

$$\mu\bigl(0^{\Lambda\setminus X}1^X \mid \mathcal{F}_{\Lambda^c}\bigr)(\xi) = q_{\Lambda,\xi}(X), \tag{6.41}$$

then μ coincides with the fermion process μ_K with $K = (I+J)^{-1}J$.

PROOF. Let $\Lambda \cap M = \varnothing$, $\widetilde{\Lambda} = \Lambda \cup M$ and define $\Phi(\xi) = J_{\widetilde{\Lambda}} - J_{\widetilde{\Lambda},\xi} J_\xi^{-1} J_{\xi,\widetilde{\Lambda}}$. Then

$$\sum_{Y \subset M} q_{\widetilde{\Lambda},\xi}(X \cup Y) = \frac{\det(P_M + \Phi(\xi)_{X\cup M})}{\det(I + \Phi(\xi))} \tag{6.42}$$

for $\xi \subset \widetilde{\Lambda}^c$.

Let us write

$$I + \Phi(\xi) = \begin{pmatrix} I + \Phi(\xi)_\Lambda & \Phi(\xi)_{\Lambda M} \\ \Phi(\xi)_{M\Lambda} & I + \Phi(\xi)_M \end{pmatrix} \tag{6.43}$$

and

$$A = \begin{pmatrix} I_{\Lambda\setminus X} & O \\ O & \Phi(\xi)_X \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} O \\ \Phi(\xi)_{XM} \end{pmatrix}. \tag{6.44}$$

Then,

$$\det\bigl(P_M + \Phi(\xi)_{X\cup M}\bigr) = \det\begin{pmatrix} A & B \\ \Phi(\xi)_{M\Lambda} & I + \Phi(\xi)_M \end{pmatrix}. \tag{6.45}$$

By Lemma 5.2 we get

$$\begin{aligned}(I + \Phi(\xi))^{-1} \\ = \begin{pmatrix} \Psi(\xi,\Lambda,M)^{-1} & \begin{matrix} -(I+\Phi(\xi)_\Lambda)^{-1} \times \\ \Phi(\xi)_{\Lambda M}\Psi(\xi,M,\Lambda)^{-1}\end{matrix} \\ \begin{matrix} -(I+\Phi(\xi)_M)^{-1} \times \\ \Phi(\xi)_{M\Lambda}\Psi(\xi,\Lambda,M)^{-1}\end{matrix} & \Psi(\xi,M,\Lambda)^{-1}\end{pmatrix},\end{aligned} \tag{6.46}$$

where $\Psi(\xi, \Lambda, M) = I + \Phi(\xi)_\Lambda - \Phi(\xi)_{\Lambda M}(I + \Phi(\xi)_M)^{-1}\Phi(\xi)_{M\Lambda}$. Thus we see

$$\begin{aligned}
&\frac{\det(P_M + \Phi(\xi)_{X\cup M})}{\det(I_\Lambda + \Phi(\xi)_{\widetilde{\Lambda}})} \\
&\qquad = \det\begin{pmatrix} A & B \\ \Phi(\xi)_{M\Lambda} & I + \Phi(\xi)_M \end{pmatrix}\begin{pmatrix} I + \Phi(\xi)_\Lambda & \Phi(\xi)_{\Lambda M} \\ \Phi(\xi)_{M\Lambda} & I + \Phi(\xi)_M \end{pmatrix}^{-1} \\
&\qquad = \det\big(\{A - B\big(I + \Phi(\xi)_M\big)^{-1}\Phi(\xi)_{M\Lambda}\}\Psi(\xi, \Lambda, M)^{-1}\big) \qquad (6.47)\\
&\qquad = \frac{\det(P_{\Lambda\setminus X} + \Phi(\xi)_X - \Phi(\xi)_{XM}(I + \Phi(\xi)_M)^{-1}\Phi(\xi)_{MX})}{\det(I + \Phi(\xi)_\Lambda - \Phi(\xi)_{\Lambda M}(I + \Phi(\xi)_M)^{-1}\Phi(\xi)_{M\Lambda})} \\
&\qquad = \frac{\det(\Phi(\xi)_X - \Phi(\xi)_{XM}(I + \Phi(\xi)_M)^{-1}\Phi(\xi)_{MX})}{\det(I + \Phi(\xi)_\Lambda - \Phi(\xi)_{\Lambda M}(I + \Phi(\xi)_M)^{-1}\Phi(\xi)_{M\Lambda})}.
\end{aligned}$$

Let us show the componentwise convergence of the nonnegative definite symmetric matrix $\Phi(\xi)_\Lambda - \Phi(\xi)_{\Lambda M}(I + \Phi(\xi)_M)^{-1}\Phi(\xi)_{M\Lambda}$ as $M \to \Lambda^c$. It is lucky enough that we have

$$J_{\widetilde{\Lambda}} \geq \Phi(\xi) \geq \Phi(\widetilde{\Lambda}^c) \tag{6.48}$$

because for any $f \in \ell^2(\Lambda)$

$$\langle \Phi(\xi) f, f\rangle = \inf_{g\in\ell^2(\xi)} \langle J(f+g), f+g\rangle. \tag{6.49}$$

Hence, it suffices to consider two cases: $\xi = \varnothing$ and $\xi = \widetilde{\Lambda}^c$.

If $\xi = \varnothing$, then $\Phi(\xi) = J_{\widetilde{\Lambda}}$ and we can show, as $M \to \Lambda^c$,

$$\begin{aligned}
&\big(I + J_\Lambda - J_{\Lambda M}(I + J_M)^{-1}J_{M\Lambda}\big)(x, y) \\
&\qquad \to \big(I + J_\Lambda - J_{\Lambda\Lambda^c}(I + J_{\Lambda^c})^{-1}J_{\Lambda^c\Lambda}\big)(x, y).
\end{aligned} \tag{6.50}$$

Indeed, for any $f \in \ell^2(\Lambda)$,

$$\begin{aligned}
&\big\langle\big(I + J_\Lambda - J_{\Lambda M}(I + J_M)^{-1}J_{M\Lambda}\big)f, f\big\rangle \\
&\qquad = \inf_{g\in\ell^2(M)} \langle (I + J_{\Lambda\cup M})(f+g), (f+g)\rangle \\
&\qquad = \inf_{g\in\ell^2(M)} \langle (I + J)(f+g), (f+g)\rangle \\
&\qquad \to \inf_{g\in\ell^2(\Lambda^c)} \langle (I + J)(f+g), (f+g)\rangle \\
&\qquad = \big\langle\big(I + J_\Lambda - J_{\Lambda\Lambda^c}(I + J_{\Lambda^c})^{-1}J_{\Lambda^c\Lambda}\big)f, f\big\rangle.
\end{aligned} \tag{6.51}$$

Next we show that if $\xi = \widetilde{\Lambda}^c$, as $M \to \Lambda^c$,

$$\begin{aligned}
&\big(I + \Phi_\Lambda - \Phi_{\Lambda M}(I + \Phi_M)^{-1}\Phi_{M\Lambda}\big)(x, y) \\
&\qquad \to \big(I + J_\Lambda - J_{\Lambda\Lambda^c}(I + J_{\Lambda^c})^{-1}J_{\Lambda^c\Lambda}\big)(x, y)
\end{aligned} \tag{6.52}$$

where $\Phi = \Phi(\widetilde{\Lambda}^c)$. Indeed, for any $f \in \ell^2(\Lambda)$, we see that

$$
\begin{aligned}
&\big\langle \big(I + \Phi_\Lambda - \Phi_{\Lambda M}(I + \Phi_M)^{-1}\Phi_{M\Lambda}\big) f, f\big\rangle \\
&\quad = \inf_{g \in \ell^2(M)} \langle (I + \Phi_{\Lambda \cup M})(f+g), f+g\rangle \\
&\quad = \inf_{g \in \ell^2(M)} \big\{\|f+g\|^2 + \langle \Phi_{\Lambda\cup M}(f+g), f+g\rangle\big\} \\
&\quad = \inf_{g \in \ell^2(M)} \inf_{h \in \ell^2(\widetilde{\Lambda}^c)} \big\{\|f+g\|^2 + \langle J(f+g+h), f+g+h\rangle\big\} \\
&\quad = \inf_{k \in \ell^2(\Lambda^c)} \big\{\|f + P_M k\|^2 + \langle J(f+k), f+k\rangle\big\} \qquad (6.53)\\
&\quad = \inf_{k \in \ell^2(\Lambda^c)} \langle (P_{\widetilde{\Lambda}} + J)(f+k), f+k\rangle \\
&\quad = \big\langle \big((P_{\widetilde{\Lambda}} + J)_\Lambda - (P_{\widetilde{\Lambda}} + J)_{\Lambda\Lambda^c}(P_{\widetilde{\Lambda}} + J)^{-1}_{\Lambda^c}(P_{\widetilde{\Lambda}} + J)_{\Lambda^c\Lambda}\big) f, f\big\rangle \\
&\quad = \big\langle \big(I + J_\Lambda - J_{\Lambda\Lambda^c}(P_{\widetilde{\Lambda}} + J)^{-1}_{\Lambda^c} J_{\Lambda^c\Lambda}\big) f, f\big\rangle \\
&\quad \to \big\langle \big(I + J_\Lambda - J_{\Lambda\Lambda^c}(I + J)^{-1}_{\Lambda^c} J_{\Lambda^c\Lambda}\big) f, f\big\rangle
\end{aligned}
$$

as $\widetilde{\Lambda} \to R$ since $P_{\widetilde{\Lambda}}$ converges strongly to I.

From (6.48), (6.50) and (6.52), we conclude, as $M \to \Lambda^c$

$$
\begin{aligned}
&\big(I + \Phi(\xi)_\Lambda - \Phi(\xi)_{\Lambda M}\big(I + \Phi(\xi)_M\big)^{-1}\Phi_{M\Lambda}\big)(x,y) \\
&\qquad \to \big(I + J_\Lambda - J_{\Lambda\Lambda^c}(I + J_{\Lambda^c})^{-1} J_{\Lambda^c\Lambda}\big)(x,y)
\end{aligned} \tag{6.54}
$$

uniformly in ξ provided that $\xi \subset (\Lambda \cup M)^c$, where ξ may vary with M.

Now recall $K = (I + J)^{-1}J$. Thus,

$$
I - K_\Lambda = \big((I+J)^{-1}\big)_\Lambda = [I + J_\Lambda - J_{\Lambda\Lambda^c}(I + J_{\Lambda^c})^{-1} J_{\Lambda^c\Lambda}]^{-1} \tag{6.55}
$$

and so

$$
J[\Lambda] = (I - K_\Lambda)^{-1} - I = J_\Lambda - J_{\Lambda\Lambda^c}(I + J_{\Lambda^c})^{-1} J_{\Lambda^c\Lambda}. \tag{6.56}
$$

Consequently, for any finite subset Λ of R and any subset X of Λ we obtain

$$
\begin{aligned}
\lim_{M \to \Lambda^c} \sum_{Y \subset M} q_{\Lambda\cup M,\xi}(X \cup Y) &= \frac{\det(J[\Lambda]_X)}{\det(I + J[\Lambda])} \\
&= \det(I - K_\Lambda)\det(J[\Lambda]_X) \\
&= \det\big(P_{\Lambda\setminus X}(I - K_\Lambda) + P_X K_\Lambda\big) \\
&= \mu_K(0^{\Lambda\setminus X} 1^X).
\end{aligned} \tag{6.57}
$$

Hence, the Gibbs measure is unique and coincides with μ_K. $\square$

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