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Random polytopes in a convex polytope, independence of shape, and concentration of vertices

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1. Introduction and main results

Write $\mathscr{K}^d$ for the set of all convex bodies (convex compact sets with nonempty interior) in $\mathbb{R}^d$. Define $\mathscr{K}_1^d$ as the set of those $K \in \mathscr{K}^d$ with $\operatorname{vol} K = 1$. Fix $K \in \mathscr{K}_1^d$ and choose points $x_1, \ldots, x_n \in K$ randomly, independently, and according to the uniform distribution on K. Then $K_n = \operatorname{conv}\{x_1, \ldots, x_n\}$ is a random polytope in K. Write $E(K, n)$ for the expectation of the random variable $\operatorname{vol}(K \backslash K_n)$. $E(K, n)$ shows how well K_n approximates K in volume on the average.

Groemer [Gr1] proved that, among all convex bodies $K \in \mathscr{K}_1^d$, the ellipsoids are approximated worst, i.e.

$$E(K, n) \leq E(B, n) \tag{1.1}$$

where B is any ellipsoid of volume one. Equality holds if and only if K is an ellipsoid. Wieacker [Wi] derived that $E(B, n) = \operatorname{const}(d) n^{-2/(d+1)} + o(n^{-2/(d+1)})$. Affentranger [Af1] developed formulae from which $E(B, n)$ can be computed explicitly.

Here we prove that, among all convex bodies $K \in \mathscr{K}_1^d$, the simplices are approximated best in the following sense:

Theorem 1. *Let $K \in \mathscr{K}_1^d$ and $\Delta \in \mathscr{K}_1^d$, Δ a simplex, $d \geq 2$. Then*

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$$\liminf \frac{E(K,n)}{E(\Delta,n)} \geq 1 + \frac{1}{d+1} \tag{1.2}$$

unless K is a simplex.

(1.2) shows that for every $K \in \mathscr{K}_1^d$ different from a simplex there is $n_0(K)$ such that for $n \geq n_0(K)$

$$E(K,n) \geq (1 + \frac{1}{2d})E(\Delta,n).$$

Most probably, for every $K \in \mathscr{K}_1^d$ and $n \geq d+1$

$$E(K,n) \geq E(\Delta,n) \tag{1.3}$$

with equality if and only if K is a simplex. For $d = 2$ and $n = 3$ Blaschke [Bl1], [Bl2] proved (1.1) with equality if and only if K is an ellipse and (1.3) with equality if and only if K is a triangle, but his remark (not repeated in [Bl2]) that the method of proof can be extended without difficulty to all dimensions d and $n = d+1$ appears to be erroneous; cf., e.g., Groemer [Gr2], Schneider [Schn], or Pfiefer [Pf]. Blaschke's result was extended to $n = 4$ by Buchta [Bu1]. For $d = 2$ and $n \geq 3$ Dalla and Larman [DL] proved (1.3) with strict inequality if K is any polygon other than a triangle. Their result was completed by Giannopoulos [Gi] who showed that the inequality is strict whenever K is a plane convex body other than a triangle. The occurring bound was derived by Buchta [Bu2]:

$$E(\text{triangle}, n) = \frac{2}{n+1} \sum_{k=1}^{n} \frac{1}{k}.$$

In higher dimensions, Dalla and Larman [DL] proved (1.3) in the case that K is a d-polytope with at most $d+2$ vertices.

Actually, (1.2) separates the simplices from all other convex bodies. This is due to the fact that for polytopes $P \in \mathscr{K}_1^d$ we can determine $E(P,n)$ up to first order precision. To state this result we call a chain $F_0 \subset F_1 \subset \ldots \subset F_{d-1}$ where F_i is an i-dimensional face of P ($i = 0, 1, \ldots, d-1$) a *tower* of P. (Sometimes this is called a (complete) flag; cf., e.g., Bayer and Lee [BaLe].) Write $T(P)$ for the number of towers of P.

Theorem 2. *Let $P \in \mathscr{K}_1^d$ be a polytope, $d \geq 2$. Then*

$$E(P,n) = \frac{T(P)}{(d+1)^{d-1}(d-1)!} \frac{\log^{d-1} n}{n} + O\left(\frac{\log^{d-2} n \log\log n}{n}\right). \tag{1.4}$$

For a simple polytope P, where $T(P)$ is $d!$ times the number of vertices of P, $\operatorname{vert} P$, Affentranger and Wieacker [AW] recently proved that

$$E(P,n) = \frac{d \operatorname{vert} P}{(d+1)^{d-1}} \frac{\log^{d-1} n}{n} + O\left(\frac{\log^{d-2} n}{n}\right).$$

Before, van Wel [We] deduced for a d-dimensional cube and indicated for any simple polytope P that $E(P,n) \sim \text{const}(d) \operatorname{vert} P\, n^{-1} \log^{d-1} n$ with const(d) expressed by a (d^2-d)-fold integral. In the case that P is a tetrahedron $E(P,n) \sim \frac{3}{4} n^{-1} \log^2 n$ was derived by Buchta [Bu4]. If Efron's identity stated below is taken into consideration, Rényi and Sulanke [RS] much earlier obtained for a polygon P that

$$E(P,n) = \frac{2}{3}\operatorname{vert} P\frac{\log n}{n} + \frac{\operatorname{const}(P)}{n} + o(\frac{1}{n})$$

with explicitly given const(P).

Estimates for $E(P,n)$ were given in the case that P is a d-dimensional cube by Bentley, Kung, Schkolnick and Thompson [BKST] as well as by Devroye [De], in the general case by Dwyer and Kannan [DK], Dwyer [Dw], and Bárány and Larman [BáLa]. The last-mentioned authors proved that $E(P,n)$ is of order $n^{-1}\log^{d-1} n$ for any polytope P.

Denote by $E(\operatorname{vert} K_n)$ the expected number of vertices of K_n. The simple identity due to Efron [Ef]

$$(n+1)E(K,n) = E(\operatorname{vert} K_{n+1}) \text{ when } K \in \mathscr{K}_1^d \tag{1.5}$$

shows that (1.4) is equivalent to

$$E(\operatorname{vert} P_n) = \frac{T(P)}{(d+1)^{d-1}(d-1)!}\log^{d-1} n + O(\log^{d-2} n \log\log n). \tag{1.6}$$

The advantage of this formulation is that the assumption vol $K = 1$ can be dropped. To prove (1.4), or rather (1.6), we will show that the vertices of P_n are "concentrated" in certain simplices associated with towers of P. For the precise statement we need some preparation.

Assume that together with the polytope $P \in \mathscr{K}_1^d$ a *hyperplane selection* $H(\cdot)$ is given. This is a map that associates with every (nontrivial) face F of P a supporting hyperplane $H(F)$ such that

$$H(F) \cap P = F.$$

Given a tower $T = (F_0, F_1, \ldots, F_{d-1})$ we define the simplex $S(T,\varepsilon)$ associated with T for every small enough $\varepsilon > 0$ by induction on d. For $d = 1$, when $P = [0,1]$, say, and $H(\cdot)$ is unique, we set

$$S(0,\varepsilon) = [0,\varepsilon],$$
$$S(1,\varepsilon) = [1-\varepsilon, 1].$$

Assume S has been defined for polytopes $Q \in \mathscr{K}_1^{d-1}$. Let $P \in \mathscr{K}_1^d$, $T = (F_0, \ldots, F_{d-1})$ a tower of P. For notational convenience we assume that $F_0 = \{0\}$. Write cone P for the minimal (convex) cone containing P (with apex at the origin). Set $H_i = H(F_i)$, and consider the hyperplane $H_0(t)$ parallel to H_0 at a distance t and on the same side of H_0 as P. Then

$$Q(t) := \operatorname{cone} P \cap H_0(t) \tag{1.7}$$

is a $(d-1)$-dimensional polytope. Since $\operatorname{vol}_{d-1} Q(t) = \operatorname{const}(P)t^{d-1}$, there is a unique $t_0 > 0$ with $\operatorname{vol}_{d-1} Q(t_0) = 1$. Define

$$Q := Q(t_0) \in \mathscr{K}_1^{d-1}. \tag{1.8}$$

For a face F of P with $0 \in F$ but $F \neq \{0\}$ the set cone $F \cap H_0(t_0)$ is a face of Q. Moreover, all faces of Q are of this form. Correspondingly, the tower $T = T_P$ gives rise to a tower T_Q of Q via

$$T_Q = (\text{cone}\, F_1 \cap H_0(t_0), \text{cone}\, F_2 \cap H_0(t_0), \dots, \text{cone}\, F_{d-1} \cap H_0(t_0)), \tag{1.9}$$

and $H_P(\cdot)$ gives rise to a hyperplane selection $H_Q(\cdot)$ via

$$H_Q(\text{cone}\, F \cap H_0(t_0)) = H_P(F) \cap H_0(t_0) \tag{1.10}$$

where F is a face of P with $0 \in F$, $F \neq \{0\}$. Then, by the induction hypothesis, the simplex $S_Q(T_Q, \varepsilon)$ has been defined. Set

$$S_P(T_P, \varepsilon) = \text{cone}\, S_Q(T_Q, \varepsilon) \cap H_0(0, \varepsilon) \tag{1.11}$$

where $H_0(0, t)$ denotes the slab between the hyperplanes H_0 and $H_0(t)$.

Although $S(T, \varepsilon)$ seems to depend heavily on $H(\cdot)$, it is essentially the same when $\varepsilon \to 0$. More precisely, given another hyperplane selection $H'(\cdot)$, there are constants c_1 and c_2 (independent of ε) such that for all small enough $\varepsilon > 0$

$$S(T, H, c_1\varepsilon) \subset S(T, H', \varepsilon) \subset S(T, H, c_2\varepsilon).$$

This can be proved by induction in an obvious way. We will write $S(T, \varepsilon)$ for $S_P(T, H, \varepsilon)$ as we think of P and $H(\cdot)$ as being fixed.

The notation (vert P_n in A) will denote the number of vertices of P_n in $A \subset \mathbb{R}^d$. The vertices of P_n are concentrated in the simplices $S(T, \varepsilon)$ with $\varepsilon = (\log n)^{-1}$ in the following sense:

Theorem 3. *Let $P \in \mathcal{K}_1^d$, $d \geq 2$, and set $\varepsilon = (\log n)^{-1}$. Then*

$$E(\text{vert}\, P_n \text{ in } P \setminus \bigcup_T S(T, \varepsilon)) \leq \text{const}(P) \log^{d-2} n \log\log n.$$

This is one of the results needed for Theorem 2. The other one is more difficult to prove, and we like to call it "independence of shape".

Theorem 4. *Let $P \in \mathcal{K}_1^d$, $d \geq 2$, and set $\varepsilon = (\log n)^{-1}$. Then for any tower T of P*

$$E(\text{vert}\, P_n \text{ in } S(T, \varepsilon)) = \frac{1}{(d+1)^{d-1}(d-1)!} \log^{d-1} n + O(\log^{d-2} n \log\log n).$$

This shows that $S(T, \varepsilon)$ contains essentially the same number of vertices of P_n no matter what the shape of P is. Actually, we will prove that $E(\text{vert}\, P_n \text{ in } S(T, \varepsilon))$ is the same for all T independently of P up to $O(\log^{d-2} n \log\log n)$. Then this number will be implied from the result of Affentranger and Wieacker.

Theorems 3 and 4 state that the vertices of P_n are concentrated in $\cup_T S(T, \varepsilon)$ and that their number in any particular simplex $S(T, \varepsilon)$ is essentially independent of the shape of P. This is true not only for the vertices but for the k-dimensional faces of P_n as well. Let us write $f_k(P)$ for the number of k-dimensional faces of the polytope P. Then the following analogue of (1.6) holds.

Theorem 5. *For a polytope $P \in \mathcal{K}^d$ and $k = 0, 1, \dots, d-1$*

$$E(f_k(P_n)) = C(d, k) T(P) \log^{d-1} n + O(\log^{d-2} n \log\log n)$$

where $C(d, k)$ is a constant depending only on d and k.

The proof of this theorem is based on statements analogous to Theorems 3 and 4. As it is quite technical and does not require new ideas, we will not present it here.

It can be seen from the work of Affentranger and Wieacker [AW] that

$$C(d,0) = \frac{d^{d-1}}{((d-1)!)^2} M_2(\Delta_{d-1}),$$

$$C(d,d-1) = \frac{d^{d-2}}{((d-1)!)^2} M_1(\Delta_{d-1}),$$

where $M_k(\Delta_{d-1})$ denotes the k-th moment of the volume of the convex hull of d random points in a simplex $\Delta_{d-1} \in \mathscr{K}_1^{d-1}$. Due to Reed [Re],

$$M_2(\Delta_{d-1}) = \frac{(d-1)!}{d^{d-1}(d+1)^{d-1}},$$

whence $C(d,0)$ follows as stated in (1.6). However, $M_1(\Delta_{d-1})$ is not known for $d \geq 5$. ($M_1(\Delta_1) = \frac{1}{3}$, $M_1(\Delta_2) = \frac{1}{12}$, and it was recently proved by Buchta and Reitzner [BR] that $M_1(\Delta_3) = \frac{13}{720} - \frac{\pi^2}{15015}$.)

Since P_n is simplicial with probability 1, for $j = -1, 0, \ldots, d-2$

$$\sum_{k=j}^{d-1} (-1)^k \binom{k+1}{j+1} C(d,k) = (-1)^{d-1} C(d,j) \tag{1.12}$$

with $C(d,-1) = 0$, other than in the usual Dehn–Sommerville equations where the corresponding value is 1. (Euler's theorem $\sum_{k=0}^{d-1}(-1)^k f_k = 1 - (-1)^d$ corresponds to $\sum_{k=0}^{d-1}(-1)^k C(d,k) = 0$.) For example, in the three-dimensional case, (1.12) and $C(3,0) = \frac{1}{32}$ imply $C(3,1) = \frac{3}{32}$, $C(3,2) = \frac{1}{16}$. (The resulting expressions for $E(f_k(P_n))$ can be simplified by observing that $T(P)$ is four times the number of edges for every three-dimensional polytope P.)

The results of this paper were announced in Bárány, Buchta [BB]. For further information about the convex hull of random points and related topics see the section "Random points in a convex body" in the work of Weil and Wieacker [WW] as well as the surveys of Affentranger [Af2], Schneider [Schn], and Buchta [Bu3]. Interesting remarks are also contained in the section "Random polygons and polyhedra" of a new book on unsolved problems in geometry [CFG].

2. Notation, definitions, further results

Given a convex body $K \in \mathscr{K}^d$ and $\theta > 0$, the Macbeath region with centre $x \in K$ is defined as

$$M(x,\theta) = M_K(x,\theta) = x + \theta[(K - x) \cap (x - K)].$$

Sometimes we will write $M(x)$ instead of $M(x,1)$. Macbeath regions were studied in [Ma], [ELR], [BáLa], and [Bá]. Define $u = u_K : K \to \mathbb{R}$ by

$$u(x) = \operatorname{vol} M_K(x).$$

Another function of interest is $v = v_K : K \to \mathbb{R}$ which is defined by

$$v(x) = \min\{\mathrm{vol}(K \cap H^+) : x \in H^+,\ H^+ \text{ a halfspace}\}.$$

It is deduced in [BáLa] that $u(x) \le 2v(x)$ for every $x \in K$ and $v(x) \le (3d)^d u(x)$ if $u(x)$ or $v(x)$ is sufficiently small.

We write $K(u \le \varepsilon)$ for $\{x \in K : u(x) \le \varepsilon\}$; the sets $K(u \ge \varepsilon)$, $K(v \le \varepsilon)$, and $K(v \ge \varepsilon)$ are defined analogously. Macbeath proved that $K(u \ge \varepsilon)$ is convex, see Sections 7 and 11 of [Ma]. Obviously $K(v \ge \varepsilon)$ is convex because it is the intersection of closed halfspaces.

The main result of [BáLa] states that $E(K,n)$ is "essentially the same" as $\mathrm{vol}\, K(v \le \frac{1}{n})$. Precisely, there are constants $c_1(d)$ and $c_2(d)$ such that

$$c_1(d)E(K,n) \le \mathrm{vol}\, K(v \le \frac{1}{n}) \le c_2(d)E(K,n) \tag{2.1}$$

for $K \in \mathscr{K}_1^d$ and $n \ge d+1$. Moreover, $\mathrm{vol}\, K(v \le \frac{1}{n})$ and $\mathrm{vol}\, K(u \le \frac{1}{n})$ are essentially the same, too.

In the case of a polytope we can prove a formula similar to (1.4):

Theorem 6. *Let $P \in \mathscr{K}_1^d$ be a polytope, $d \ge 2$. Then*

$$\mathrm{vol}\, P(u \le \varepsilon) = \frac{T(P)}{2^d d!(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}).$$

Albeit much simpler than Theorem 2 this will be quite useful. Analogously one can show

$$\mathrm{vol}\, P(v \le \varepsilon) = \frac{T(P)}{d^d(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}).$$

This was first proved by Schütt [Schü], we found it independently.

The assumption $\mathrm{vol}\, K = 1$ or $\mathrm{vol}\, P = 1$ in the theorems is made for convenience rather than necessity. What is really needed is $\mathrm{vol}\, K > 0$, and we will have to consider convex bodies with $\mathrm{vol}\, K \ne 1$ as well. In this case it is better to take

$$\frac{\mathrm{vol}\, K(u \le \varepsilon\, \mathrm{vol}\, K)}{\mathrm{vol}\, K}$$

instead of $\mathrm{vol}\, K(u \le \varepsilon)$ because it is affinely invariant. Precisely, let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a nondegenerate affine transformation and $K \in \mathscr{K}^d$. Then, clearly,

$$\frac{\mathrm{vol}\, K(u_K \le \varepsilon\, \mathrm{vol}\, K)}{\mathrm{vol}\, K} = \frac{\mathrm{vol}\, LK(u_{LK} \le \varepsilon\, \mathrm{vol}\, LK)}{\mathrm{vol}\, LK}. \tag{2.2}$$

We mention further that $E(\mathrm{vert}\, K_n)$ does not depend on the volume of K. But Efron's identity (1.5) has to be modified:

$$E(\mathrm{vert}\, K_{n+1}) = \frac{n+1}{\mathrm{vol}\, K}E(K,n) \text{ when } K \in \mathscr{K}^d.$$

Assume $P \in \mathscr{K}^d$ is a polytope and let T be one of its towers. This will define parameters $\tau_0(z), \tau_1(z), \ldots, \tau_{d-1}(z)$ for $z \in P$ in the following way. We use induction, so when $d = 1$, $\tau_0(z)$ is the distance of z from the vertex defining T. When $d > 1$, $\tau_0(z)$ is defined (cf. (1.7) and (1.8)) by

$$z \in H_0(\tau_0(z)).$$

Recall the definitions of $Q, T_Q, S_Q(T_Q, \varepsilon)$ from (1.7), (1.8), (1.9), (1.10), (1.11). Set

$$z_Q := t_0 \tau_0^{-1}(z) z \in Q. \tag{2.3}$$

Define now for $i = 1, 2, \ldots, d-1$

$$\tau_i(z) = \tau_{i-1}(z_Q),$$

where the parameter $\tau_{i-1}(z_Q)$ is meant in Q with respect to the tower T_Q. With this definition we have

$$z \in S(T, \varepsilon) \text{ if and only if } \tau_0(z) \leq \varepsilon \text{ and } z_Q \in S_Q(T_Q, \varepsilon)$$

and, further,

$$z \in S(T, \varepsilon) \text{ if and only if } \tau_i(z) \leq \varepsilon \quad (i = 0, 1, \ldots, d-1).$$

Clearly, for $\alpha > 0$ and $z \in P$

$$\tau_0(\alpha z) = \alpha \tau_0(z), \tag{2.4}$$

but

$$\tau_i(\alpha z) = \tau_i(z) \quad (i = 1, \ldots, d-1). \tag{2.5}$$

In the proof of Theorem 4 we will need the following notation. Again, P is a polytope and $T = (F_0, F_1, \ldots, F_{d-1})$ a tower of P. For $\phi_0, \phi_1, \ldots, \phi_{d-1} > 0$ define

$$\begin{aligned} P(\bar{\phi}_i) &= P(\phi_0, \ldots, \phi_i) \\ &= P(\phi_0, \ldots, \phi_i; F_0, \ldots, F_i) \\ &= \{z \in P : \tau_j(z) \leq \phi_j \ (j = 0, \ldots, i)\}. \end{aligned} \tag{2.6}$$

In particular, if $\varphi_0 = \varphi_1 = \cdots = \varphi_{d-1} = \varepsilon$, then

$$P(\bar{\varphi}_{d-1}) = S(T, \varepsilon).$$

Moreover, we put

$$P(\bar{\phi}_{i-1}) = P \text{ when } i = 0, \tag{2.7}$$

and we set for $i = 0, 1, \ldots, d-1$

$$\begin{aligned} &P(\bar{\phi}_{i-1}, \tau_i \geq \phi_i) \\ &= P(\phi_0, \ldots, \phi_{i-1}, \tau_i \geq \phi_i) \\ &= \{z \in P(\bar{\phi}_{i-1}) : \tau_i(z) \geq \phi_i\}. \end{aligned}$$

Notice that for $i \geq 1$

$$P(\bar{\phi}_{i-1}, \tau_i \geq \phi_i) = \operatorname{cone} Q(\phi_1, \ldots, \phi_{i-1}, \tau_{i-1}^{(Q)} \geq \phi_i) \cap H_0(0, \phi_0) \tag{2.8}$$

where $\tau_{i-1}^{(Q)}$ is the $(i-1)^{\text{st}}$ parameter induced in Q by the tower T.

Finally, we define

$$\operatorname{ray}(x, y) = \{x + t(y - x) : t \geq 0\},$$

and we set

$$u(x, y) = \max\{u(z) : z \in \operatorname{aff}(x, y)\}$$

where $u : \mathbb{R}^d \to \mathbb{R}$ and aff(x, y) denotes the affine hull of $x, y \in \mathbb{R}^d$.

We will use the notation const(P) for different constants. As we think that the hyperplane selection $H(\cdot)$ is given together with the polytope P, we will write const(P) instead of const(P, H).

3. Auxiliary results

For $0 \le \varepsilon \le 1$

$$\mathrm{vol}\{x \in \mathbb{R}^d : \prod_{i=1}^{d} x_i \le \varepsilon,\ 0 \le x_i \le 1\ (i = 1, \ldots, d)\} = \varepsilon \sum_{i=0}^{d-1} \frac{1}{i!} \log^i \frac{1}{\varepsilon}. \tag{3.1}$$

This follows, e.g., from (3.5) and (8.1) in Chapter I of [Fe].

Assume now that P is a polytope with a fixed tower T whose starting vertex is the origin. Then

$$u_P(x) = \int_{\tau_0(x)-\tau_0}^{\tau_0(x)+\tau_0} \mathrm{vol}_{d-1}[M_P(x) \cap H_0(t)]dt \tag{3.2}$$

where $\tau_0 \ge 0$ is defined as the largest t for which the section $M_P(x) \cap H_0(\tau_0(x) - t)$ is nonempty. It is easy to see that the central section $M_P(x) \cap H_0(\tau_0(x))$ coincides with $M_{Q(\tau_0(x))}(x)$. Since $M_P(x)$ is centrally symmetric with centre x, the largest volume section is the central one. Then (3.2) implies

$$u_P(x) \le 2\tau_0 u_{Q(\tau_0(x))}(x). \tag{3.3}$$

On the other hand,

$$\begin{aligned} u_P(x) &= 2 \int_{\tau_0(x)-\tau_0}^{\tau_0(x)} \mathrm{vol}_{d-1}[M_P(x) \cap H_0(t)]dt \\ &\ge 2 \int_{\tau_0(x)-\tau_0}^{\tau_0(x)} \left(\frac{t - \tau_0(x) + \tau_0}{\tau_0}\right)^{d-1} \mathrm{vol}_{d-1} M_{Q(\tau_0(x))}(x)dt \\ &= \frac{2\tau_0}{d} u_{Q(\tau_0(x))}(x). \end{aligned} \tag{3.4}$$

We will often use (3.3) and (3.4) when $\tau_0 = \tau_0(x)$. This happens if x is close enough to the vertex of T, for instance, if the vertex of T is the only vertex of P lying in the slab $H_0(0, 2\tau_0(x))$.

Assume now that $K \in \mathscr{K}^d$ with vol $K = q$. It can be seen from the proof of Theorem 1 in [BáLa] that

$$\mathrm{Prob}(x \notin K_n) \le 2 \sum_{i=0}^{d-1} \binom{n}{i} (\frac{u(x)}{2q})^i (1 - \frac{u(x)}{2q})^{n-i} \tag{3.5}$$

where Prob is meant with $x \in K$ fixed and K_n the random polytope in K varying.

Before stating the first of three lemmata needed in the proof of Theorem 4, we mention a result of Macbeath: Let L be a convex compact subset of K containing interior points of K. Then, according to Lemma 7.1 in [Ma], the maximum value of u_K in L is attained at a unique point of L.

Lemma 1. *Assume $K \in \mathcal{K}^d$, and a and b are points on the boundary of K such that* aff(a,b) *contains interior points of K. Let c be the point where u takes its maximum value on* aff(a,b). *Then, if $u(c)$ is sufficiently small,*

$$\frac{\|a-c\|}{\|b-c\|} \leq (3d)^{d+2}.$$

Lemma 1 says that if H is a hyperplane and $u(c) = \max\{u(x) : x \in H\}$ with $c \in H$, then c is a "$(3d)^{d+2}$-central" point of the section $K \cap H$. Similarly, the v-maximal point on H is the centre of gravity of $K \cap H$ (cf., e.g., the proof of Lemma 4 in [ELR]), whence it is "$(d-1)$-central".

Lemma 2. *Assume $P \in \mathcal{K}_1^d$, T is a tower of P, $\varphi_0 = \varphi_1 = \ldots = \varphi_{d-1} = (\log \frac{1}{\varepsilon})^{-1}$ with $\varepsilon > 0$ small enough, $\phi_0, \phi_1, \ldots, \phi_{d-1} > 0$ are constants, $\theta \geq 1$. Then, for $i = 0, 1, \ldots, d-1$, $x \in P(\bar{\varphi}_i)$ implies*

$$\begin{aligned} &\text{vol}[P(\bar{\phi}_{i-1}, \tau_i \geq \phi_i) \cap M_{P(\bar{\phi}_{i-1})}(x, \theta)] \\ &\quad \leq \text{const}(P)\tau_i(x) \,\text{vol}\, M_{P(\bar{\phi}_{i-1})}(x, \theta). \end{aligned}$$

Lemma 3. *Assume, again, $P \in \mathcal{K}_1^d$, T is a tower of P, $\varphi_0 = \varphi_1 = \ldots = \varphi_{d-1} = (\log \frac{1}{\varepsilon})^{-1}$ with $\varepsilon > 0$ small enough, $\phi_0, \phi_1, \ldots, \phi_{d-1} > 0$ are constants. Then, for $i = 0, 1, \ldots, d-1$,*

$$\begin{aligned} &\text{meas}\{(x,y) \in P(\bar{\varphi}_i) \times P(\bar{\phi}_{i-1}, \tau_i \geq \phi_i) : u_{P(\bar{\phi}_{i-1})}(x,y) \leq \varepsilon\} \\ &\quad \leq \text{const}(P)\varepsilon^2 \log^{d-2} \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon} \end{aligned}$$

where meas *is the product of the Lebesgue measures on $\mathbb{R}^d \times \mathbb{R}^d$.*

The proofs of the lemmata are given in Section 8. In Section 7 we deduce Theorem 1 from Theorem 2. The proof of Theorem 2 consists in proving Theorems 3 and 4 which will be done in Sections 5 and 6. Theorem 6, or rather its proof, turns out to be an important tool for the proofs of Theorem 3 and 4, so we start with Theorem 6.

4. Proof of Theorem 6

For a vertex $v \in P$ define $H_v = H(\{v\})$ and write $H_v(0, \varphi)$ for the slab between H_v and $H_v(\varphi)$. Put $A(\varphi) = P \setminus \cup_v H_v(0, \varphi)$. As a first step in the proof we show

$$\text{vol}[P(u \leq \varepsilon) \cap A(\varphi)] \leq \text{const}(P)\varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \frac{1}{\varphi} \tag{4.1}$$

provided $\varphi^d \geq \text{const}(P)\varepsilon$. (4.1) means that the essential part of $P(u \leq \varepsilon)$ is concentrated near the vertices of P.

When $d = 1$ and $\varphi \geq \varepsilon/2$, the left hand side of (4.1) equals 0. For $d \geq 2$ let $\Delta_1, \ldots, \Delta_m$ be simplices forming a triangulation of P that uses vertices of P only. Clearly,

$$P(u \leq \varepsilon) \subset \bigcup_{i=1}^{m} \Delta_i(u_{\Delta i} \leq \varepsilon).$$

Now for a simplex $\Delta \in \mathscr{K}_1^d$ with hyperplane selection $H(\cdot)$ one can show that

$$\begin{aligned} &\operatorname{vol}\{x \in \Delta : u_\Delta(x) \leq \varepsilon,\ x \notin \bigcup_{v \text{ a vertex of } \Delta} H_v(0, \varphi)\} \\ &\leq \operatorname{const}(d)\varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \frac{1}{\varphi} \end{aligned} \tag{4.2}$$

provided $\varphi^d \geq \operatorname{const}(d)\varepsilon$. The proof of this is a routine calculation using (3.1) and is, therefore, omitted. See [Dw], [AW] for a similar computation.

Using an affine transformation carrying Δ_i into Δ we get by (2.2)

$$\begin{aligned} &\operatorname{vol}\{x \in \Delta_i : u_{\Delta_i}(x) \leq \varepsilon,\ x \notin \bigcup_{v \text{ a vertex of } \Delta_i} H_v(0, \varphi)\} \\ &\leq \operatorname{vol} \Delta_i \operatorname{vol}\{x \in \Delta : u_\Delta(x) \leq \frac{\varepsilon}{\operatorname{vol} \Delta_i},\ x \notin \bigcup_{v \text{ a vertex of } \Delta} H_v(0, \frac{\varphi}{(\operatorname{vol} \Delta_i)^{1/d}})\} \\ &\leq \operatorname{vol} \Delta_i \operatorname{const}(d) \frac{\varepsilon}{\operatorname{vol} \Delta_i} \log^{d-2} \frac{\operatorname{vol} \Delta_i}{\varepsilon} \log \frac{(\operatorname{vol} \Delta_i)^{1/d}}{\varphi} \\ &\leq \operatorname{const}(d)\varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \frac{1}{\varphi} \end{aligned}$$

provided $(\frac{\varphi}{(\operatorname{vol} \Delta_i)^{1/d}})^d \geq \operatorname{const}(d) \frac{\varepsilon}{\operatorname{vol} \Delta_i}$, i.e. $\varphi^d \geq \operatorname{const}(d)\varepsilon$. Summing this for all Δ_i we get (4.1).

It is helpful for the second step in the proof to notice that analogous arguments easily give

$$\operatorname{vol} P(u \leq \varepsilon) \leq \operatorname{const}(P)\varepsilon \log^{d-1} \frac{1}{\varepsilon}. \tag{4.3}$$

This second step consists in showing that $P(u \leq \varepsilon)$ is concentrated in the union of the simplices $S(T, \varphi)$. Setting now $B(\varphi) = P \setminus \cup_T S(T, \varphi)$ we claim

$$\operatorname{vol}[P(u \leq \varepsilon) \cap B(\varphi)] \leq \operatorname{const}(P)\varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \frac{1}{\varphi} \tag{4.4}$$

provided $\varphi^d \geq \operatorname{const}(P)\varepsilon$. We prove (4.4) by induction on d. The case $d = 1$ is trivial. The case $d = 2$ which needs special consideration is quite simple and is left to the reader.

Since $B(\varphi) \supset A(\varphi) = P \setminus \cup_v H_v(0, \varphi)$ we have

$$P(u \leq \varepsilon) \cap B(\varphi) = [P(u \leq \varepsilon) \cap A(\varphi)] \cup \bigcup_v [P(u \leq \varepsilon) \cap B(\varphi) \cap H_v(0, \varphi)],$$

so that

$$\begin{aligned}&\mathrm{vol}[P(u \le \varepsilon) \cap B(\varphi)] \\ &\quad \le \mathrm{vol}[P(u \le \varepsilon) \cap A(\varphi)] + \sum_v \mathrm{vol}[P(u \le \varepsilon) \cap B(\varphi) \cap H_v(0, \varphi)]. \qquad (4.5)\end{aligned}$$

We will estimate

$$\begin{aligned} O(v) &:= \mathrm{vol}[P(u \le \varepsilon) \cap B(\varphi) \cap H_v(0, \varphi)] \\ &= \mathrm{vol}\{x \in P : u(x) \le \varepsilon,\ x \notin \bigcup S(T, \varphi),\ x \in H_v(0, \varphi)\}\end{aligned}$$

separately for each vertex v. We suppose $v = 0$, again. Assume φ is so small that the only vertex lying in $H_0(0, 2\varphi)$ is $v = 0$. Consequently, for $x \in P \cap H_0(0, \varphi)$

$$M_P(x, 1) = M_{\mathrm{conv}(Q \cup \{0\})}(x, 1)$$

where Q is defined in (1.8), cf. (1.7) as well. Then

$$O(v) = \int_0^{\varphi} \mathrm{vol}_{d-1}\{x \in Q(t) : u_P(x) \le \varepsilon,\ x \notin \bigcup S(T, \varepsilon)\} dt. \qquad (4.6)$$

We estimate the integrand in (4.6) using successively (3.4), (2.2), the fact that $\mathrm{vol}_{d-1} Q(t) = c_1(Q) t^{d-1}$, and the induction hypothesis

$$\begin{aligned}&\mathrm{vol}_{d-1}\{x \in Q : u_Q(x) \le \varepsilon,\ x \notin \bigcup S_Q(T_Q, \varphi)\} \\ &\qquad \le \mathrm{const}(Q) \varepsilon \log^{d-3} \frac{1}{\varepsilon} \log \frac{1}{\varphi} \qquad (4.7)\end{aligned}$$

provided $\varphi^{d-1} \ge c_2(Q)\varepsilon$; cf. (4.4) and (1.9). Thus we obtain

$$\begin{aligned}&\mathrm{vol}_{d-1}\{x \in Q(t) : u_P(x) \le \varepsilon,\ x \notin \bigcup S(T, \varphi)\} \\ &\quad \le \mathrm{vol}_{d-1}\{x \in Q(t) : u_{Q(t)}(x) \le \frac{d\varepsilon}{2t},\ x \notin \bigcup S(T, \varphi)\} \\ &\quad = \frac{\mathrm{vol}_{d-1} Q(t)}{\mathrm{vol}\, Q} \mathrm{vol}_{d-1}\{x \in Q : u_Q(x) \le \frac{d\varepsilon \,\mathrm{vol}_{d-1} Q}{2t\, \mathrm{vol}_{d-1} Q(t)},\ x \notin \bigcup S_Q(T_Q, \varphi)\} \\ &\quad = c_1(Q) t^{d-1} \mathrm{vol}_{d-1}\{x \in Q : u_Q(x) \le \frac{d\varepsilon}{2c_1(Q) t^d},\ x \notin \bigcup S_Q(T_Q, \varphi)\} \\ &\quad \le c_1(Q) t^{d-1} \mathrm{const}(Q) \frac{d\varepsilon}{2c_1(Q) t^d} \log^{d-3} \frac{2c_1(Q) t^d}{d\varepsilon} \log \frac{1}{\varphi} \\ &\quad = \mathrm{const}(Q) \frac{\varepsilon}{t} \log^{d-3} \frac{2c_1(Q) t^d}{d\varepsilon} \log \frac{1}{\varphi} \qquad (4.8)\end{aligned}$$

provided $\varphi^{d-1} \ge c_2(Q) \frac{d\varepsilon}{2c_1(Q) t^d}$ and $\frac{d\varepsilon}{2c_1(Q) t^d} \le 1$. Define t_2 and t_1 as the smallest values $t > 0$ such that these inequalities hold, i.e.

$$t_2^d = c_2(Q) \frac{d\varepsilon}{2c_1(Q) \varphi^{d-1}} \text{ and } t_1^d = \frac{d\varepsilon}{2c_1(Q)}.$$

Notice that $t_2 \ge t_1$ as $\frac{c_2(Q)}{\varphi^{d-1}} \ge 1$.

We apply (4.8) when $t_2 \le t \le \varphi$. Observing $\frac{2c_1(Q)\varphi^d}{d} \le 1$ (as the volume of P is 1) we see that

$$\begin{aligned}
&\int_{t_2}^{\varphi} \mathrm{vol}_{d-1}\{x \in Q(t) : u_P(x) \le \varepsilon,\ x \notin \bigcup S(T,\varphi)\}dt \\
&\le \int_{t_2}^{\varphi} \mathrm{const}(Q)\frac{\varepsilon}{t} \log^{d-3} \frac{2c_1(Q)t^d}{d\varepsilon} \log \frac{1}{\varphi} dt \\
&= \mathrm{const}(Q)\frac{1}{d(d-2)}\varepsilon \left(\log^{d-2} \frac{2c_1(Q)\varphi^d}{d\varepsilon} - \log^{d-2} \frac{c_2(Q)}{\varphi^{d-1}}\right) \log \frac{1}{\varphi} \\
&\le \mathrm{const}(Q)\varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \frac{1}{\varphi}.
\end{aligned}$$

For $t_1 \le t \le t_2$ we use

$$\mathrm{vol}_{d-1}\{x \in Q : u_Q(x) \le \varepsilon\} \le \mathrm{const}(Q)\varepsilon \log^{d-2} \frac{1}{\varepsilon}$$

instead of (4.7); cf. (4.3). (Applying (4.3) can be avoided if the whole theorem is proved by induction.) Then

$$\begin{aligned}
&\int_{t_1}^{t_2} \mathrm{vol}_{d-1}\{x \in Q(t) : u_P(x) \le \varepsilon,\ x \notin \bigcup S(T,\varphi)\}dt \\
&\le \int_{t_1}^{t_2} \mathrm{vol}_{d-1}\{x \in Q(t) : u_P(x) \le \varepsilon\}dt \\
&\le \int_{t_1}^{t_2} c_1(Q)t^{d-1} \mathrm{vol}_{d-1}\{x \in Q : u_Q(x) \le \frac{d\varepsilon}{2c_1(Q)t^d}\}dt \\
&\le \int_{t_1}^{t_2} c_1(Q)t^{d-1} \mathrm{const}(Q)\frac{d\varepsilon}{2c_1(Q)t^d} \log^{d-2} \frac{2c_1(Q)t^d}{d\varepsilon} dt \\
&= \int_{t_1}^{t_2} \mathrm{const}(Q)\frac{\varepsilon}{t} \log^{d-2} \frac{2c_1(Q)t^d}{d\varepsilon} dt \\
&= \mathrm{const}(Q)\frac{1}{d(d-1)}\varepsilon \log^{d-1} \frac{c_2(Q)}{\varphi^{d-1}} \\
&\le \mathrm{const}(Q)\varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \frac{1}{\varphi}
\end{aligned}$$

since $1 \le \frac{c_2(Q)}{\varphi^{d-1}} \le \frac{1}{\varepsilon}$.

Finally, for $0 \le t \le t_1$

$$\int_0^{t_1} \mathrm{vol}_{d-1}\{x \in Q(t) : u_P(x) \le \varepsilon,\ x \notin \bigcup S(T,\varphi)\}dt$$

$$\le \int_0^{t_1} \mathrm{vol}_{d-1} Q(t)dt = \int_0^{t_1} c_1(Q)t^{d-1}dt = \frac{\varepsilon}{2}.$$

To summarize, we conclude that

$$O(v) \le \mathrm{const}(Q)\varepsilon \log^{d-2}\frac{1}{\varepsilon}\log\frac{1}{\varphi}.$$

Because of (4.5), this together with (4.1) proves (4.4).

As a third and last step in the proof we compute $\mathrm{vol}[P(u \le \varepsilon) \cap S(T,\varphi)]$. We do this first when $P = C$, the unit cube in $\mathbb{R}^d$. In this case, by symmetry, $C(u \le \varepsilon) \cap S(T,\varphi)$ is the same for all towers T of C. On the other hand,

$$u_C(x) = 2^d x_1 \ldots x_d$$

for those $x = (x_1, \ldots, x_d) \in C$ which satisfy $0 \le x_i \le \frac{1}{2}$ $(i = 1, \ldots, d)$. A routine computation similar to the one needed for (4.2) gives

$$\mathrm{vol}\{x \in C : u_C(x) \le \varepsilon,\ x_i \le \frac{1}{2}\ (i = 1, \ldots, d)\}$$
$$= \frac{1}{2^d(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}).$$

Since there are $d!$ towers and so $d!$ simplices $S(T,\varphi)$ starting with $F_0 = \{0\}$, we get

$$\mathrm{vol}[C(u_C \le \varepsilon) \cap S(T,\varphi)] = \frac{1}{2^d d!(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}\log\frac{1}{\varphi}), \quad (4.9)$$

where we used (4.4) with $P = C$ as well.

Assume now P is a polytope and T is one of its towers. Then one can find two parallelepipeda C_1 and C_2 with towers T_1 and T_2 so that $S_P(T,\varphi) = S_{C_1}(T_1,\varphi) = S_{C_2}(T_2,\varphi)$ and that for x close enough to the origin

$$x \in C_1 \text{ implies } x \in P \text{ and}$$
$$x \in P \text{ implies } x \in C_2.$$

Now if $x \in S(T,\varphi)$ and φ is small enough, then x is close to the origin and so

$$M_{C_1}(x) \subset M_P(x) \subset M_{C_2}(x).$$

Consequently $u_{C_1}(x) \le u_P(x) \le u_{C_2}(x)$. We know from (4.9) and (2.2) that for $i = 1, 2$

$$\mathrm{vol}[C_i(u_{C_i} \le \varepsilon) \cap S(T_i,\varphi)] = \frac{1}{2^d d!(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}\log\frac{1}{\varphi})$$

proving that

$$\mathrm{vol}[P(u_P \le \varepsilon) \cap S(T,\varphi)] = \frac{1}{2^d d!(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}\log\frac{1}{\varphi}). \quad (4.10)$$

Finally, summing (4.10) for all the towers and using (4.4) gives

$$\operatorname{vol} P(u \le \varepsilon) = \frac{T(P)}{2^d d!(d-1)!}\varepsilon \log^{d-1}\frac{1}{\varepsilon} + O(\varepsilon \log^{d-2}\frac{1}{\varepsilon}\log\frac{1}{\varphi})$$

provided $\varphi^d \ge \text{const}(P)\varepsilon$. This certainly holds when φ is a suitable constant and $\varepsilon > 0$ small enough, proving the theorem. □

5. Proof of Theorem 3

Assume $A \subset P$ is measurable. Set $X_n = \{x_1, \dots, x_n\}$. Clearly,

$$E(\operatorname{vert} P_n \text{ in } A) = \sum_{i=1}^{n} \operatorname{Prob}(x_i \in A, x_i \notin \operatorname{conv}(X_n \setminus \{x_i\}))$$
$$= n \int_{x\in A} \operatorname{Prob}(x \notin P_{n-1})dx. \tag{5.1}$$

Here $\operatorname{Prob}(x \notin P_{n-1})$ is meant with x fixed and $P_{n-1} = \operatorname{conv} X_{n-1}$, a random polytope. We apply (5.1) when

$$A = B(\varepsilon) = P \setminus \bigcup_T S(T, \varepsilon)$$

where $\varepsilon = (\log n)^{-1}$. We use the method of [BáLa]. Changing n to $n+1$ and applying (3.5) we get

$$(n+1)\int_{B(\varepsilon)} \operatorname{Prob}(x \notin P_n)dx$$
$$\le (n+1)\int_{B(\varepsilon)} 2\sum_{i=0}^{d-1}\binom{n}{i}(\frac{u(x)}{2})^i(1-\frac{u(x)}{2})^{n-i}dx$$
$$= (n+1)\sum_{\lambda=1}^{n}\int_{\substack{B(\varepsilon)\\ \frac{\lambda-1}{n}\le u(x)\le\frac{\lambda}{n}}} 2\sum_{i=0}^{d-1}\binom{n}{i}(\frac{u(x)}{2})^i(1-\frac{u(x)}{2})^{n-i}dx$$
$$\le 2(n+1)\sum_{\lambda=1}^{n}\sum_{i=0}^{d-1}\binom{n}{i}(\frac{\lambda}{2n})^i(1-\frac{\lambda-1}{2n})^{n-i}\operatorname{vol}\{x\in B(\varepsilon): u(x)\le\frac{\lambda}{n}\}. \tag{5.2}$$

Here $\binom{n}{i}(\frac{\lambda}{2n})^i \le \frac{\lambda^i}{2^i i!}$, $(1-\frac{\lambda-1}{2n})^{-i} \le 2^i$, and $(1-\frac{\lambda-1}{2n})^n \le e^{-(\lambda-1)/2}$ yield

$$\sum_{i=0}^{d-1}\binom{n}{i}(\frac{\lambda}{2n})^i(1-\frac{\lambda-1}{2n})^{n-i} \le \text{const}(d)\lambda^{d-1}e^{-\lambda/2}. \tag{5.3}$$

Moreover, $\operatorname{vol}\{x\in B(\varepsilon): u(x)\le\frac{\lambda}{n}\} \le 1$. Set $\lambda_0 = \lfloor 4\log n\rfloor$. Then

$$2(n+1)\sum_{\lambda=\lambda_0+1}^{n}\sum_{i=0}^{d-1}\binom{n}{i}(\frac{\lambda}{2n})^i(1-\frac{\lambda-1}{2n})^{n-i}\operatorname{vol}\{x\in B(\varepsilon):u(x)\le\frac{\lambda}{n}\}$$
$$\le \text{const}(d)n\sum_{\lambda=\lambda_0+1}^{n}\lambda^{d-1}e^{-\lambda/2}$$
$$\le \text{const}(d)ne^{-\lambda_0/4}\sum_{\lambda=1}^{\infty}\lambda^{d-1}e^{-\lambda/4}$$
$$\le \text{const}(d). \tag{5.4}$$

We know from Theorem 6 or rather from (4.4) that

$$\operatorname{vol}\{x\in B(\varepsilon):u(x)\le\frac{\lambda}{n}\}\le \text{const}(P)\frac{\lambda}{n}\log^{d-2}\frac{n}{\lambda}\log\frac{1}{\varepsilon},$$

since $\varepsilon=(\log n)^{-1}$ satisfies $\varepsilon^d\ge \text{const}(P)\frac{\lambda}{n}$ when $\lambda\le\lambda_0$. So we have

$$2(n+1)\sum_{\lambda=1}^{\lambda_0}\sum_{i=0}^{d-1}\binom{n}{i}(\frac{\lambda}{2n})^i(1-\frac{\lambda-1}{2n})^{n-i}\operatorname{vol}\{x\in B(\varepsilon):u(x)\le\frac{\lambda}{n}\}$$
$$\le \text{const}(d)n\sum_{\lambda=1}^{\lambda_0}\lambda^{d-1}e^{-\lambda/2}\,\text{const}(P)\frac{\lambda}{n}\log^{d-2}\frac{n}{\lambda}\log\log n$$
$$\le \text{const}(P)\sum_{\lambda=1}^{\lambda_0}\lambda^{d}e^{-\lambda/2}\log^{d-2}n\log\log n$$
$$\le \text{const}(P)\log^{d-2}n\log\log n. \qquad \square \tag{5.5}$$

This proof will serve as a model for some proofs to come. In particular, estimations analogous to (5.2), (5.3), (5.4), and (5.5) will frequently be used with reference to this section and without elaboration.

6. Proof of Theorem 4

Again let $X_n=\{x_1,\dots,x_n\}$ be the set of the n random points in P. For $i=0,1,\dots,d-1$ define

$$E(i,n)=E[\text{vert conv}(X_n\cap P(\bar\phi_i))\text{ in }P(\bar\varphi_i)]$$
$$-E[\text{vert conv}(X_n\cap P(\bar\phi_{i-1}))\text{ in }P(\bar\varphi_i)].$$

Here $P(\bar\phi_i)$ and $P(\bar\varphi_i)$ are defined in (2.6), cf. (2.7) as well. We set

$$\varphi_i=(\log n)^{-1},\ \phi_i=\text{const}(P)\quad(i=0,1,\dots,d-1) \tag{6.1}$$

where ϕ_i is chosen so small that the set $\{z\in P:0<\tau_i(z)<2\phi_i\}$ does not contain any vertex of P. We claim that

$$0\le E(i,n)\le \text{const}(P)\log^{d-2}n\log\log n. \tag{6.2}$$

This will prove the theorem in the following way:

$$\begin{aligned}E(\operatorname{vert} P_n) &= \sum_{F_0} E[\operatorname{vert} P_n \text{ in } P(\varphi_0; F_0)] \\ &\quad + O(\log^{d-2} n \log\log n) \\ &= \sum_{F_0} E[\operatorname{vert}\operatorname{conv}(X_n \cap P(\phi_0; F_0)) \text{ in } P(\varphi_0; F_0)] \\ &\quad + O(\log^{d-2} n \log\log n) \\ &= \sum_{F_0, F_1} E[\operatorname{vert}\operatorname{conv}(X_n \cap P(\phi_0; F_0)) \text{ in } P(\varphi_0, \varphi_1; F_0, F_1)] \\ &\quad + O(\log^{d-2} n \log\log n) \\ &= \sum_{F_0, F_1} E[\operatorname{vert}\operatorname{conv}(X_n \cap P(\phi_0, \phi_1; F_0, F_1)) \text{ in } P(\varphi_0, \varphi_1; F_0, F_1)] \\ &\quad + O(\log^{d-2} n \log\log n) \\ &= \dots \\ &= \sum_{T} E[\operatorname{vert}\operatorname{conv}(X_n \cap P(\phi_0, \dots, \phi_{d-1}; T)) \text{ in } P(\varphi_0, \dots, \varphi_{d-1}; T)] \\ &\quad + O(\log^{d-2} n \log\log n) \qquad (6.3)\end{aligned}$$

where the equalities follow from Theorem 3 and (6.2), alternatively. The terms in the last sum are independent of P, they depend only on $\varphi_0, \dots, \varphi_{d-1}$ and $\phi_0, \dots, \phi_{d-1}$. This means that they are the same for every tower of every polytope once these numbers are equal. For a simple polytope Affentranger and Wieacker determined

$$E(\operatorname{vert} P_n) = \frac{d \operatorname{vert} P}{(d+1)^{d-1}} \log^{d-1} n + O(\log^{d-2} n).$$

Since $T(P) = d! \operatorname{vert} P$ for a simple polytope, we get from (6.3) that the expected number of vertices of P_n lying in $S(T, (\log n)^{-1})$ is

$$\frac{1}{(d+1)^{d-1}(d-1)!} \log^{d-1} n + O(\log^{d-2} n \log\log n).$$

But then $E[\operatorname{vert} P_n \text{ in } S(T, (\log n)^{-1})]$ is this very number for every tower T of every polytope, simple or otherwise.

Set $q = \operatorname{vol} P(\bar{\phi}_{i-1})$. Choosing the random n-set X_n from P is the same as the following two-step procedure. First choose $m \in \{0, 1, \dots, n\}$ with probability $\binom{n}{m} q^m (1-q)^{n-m}$, then choose m points $y_1, \dots, y_m$ from $P(\bar{\phi}_{i-1})$ randomly, independently and uniformly, and choose $n-m$ points from $P \backslash P(\bar{\phi}_{i-1})$ randomly, independently and uniformly. Correspondingly,

$$\begin{aligned}E(i,n) = \sum_{m=0}^{n} \binom{n}{m} q^m (1-q)^{n-m} \\ \{E[\operatorname{vert}\operatorname{conv}(X_n \cap P(\bar{\phi}_i)) \text{ in } P(\bar{\varphi}_i) | \operatorname{card}(X_n \cap P(\bar{\phi}_{i-1})) = m] \\ - E[\operatorname{vert}\operatorname{conv}(X_n \cap P(\bar{\phi}_{i-1})) \text{ in } P(\bar{\varphi}_i) | \operatorname{card}(X_n \cap P(\bar{\phi}_{i-1})) = m]\}\end{aligned}$$

$$= \sum_{m=0}^{n} \binom{n}{m} q^m (1-q)^{n-m}$$
$$\{E[\text{vert conv}(Y_m \cap P(\bar{\phi}_i)) \text{ in } P(\bar{\varphi}_i)]$$
$$- E[\text{vert conv}(Y_m \cap P(\bar{\phi}_{i-1})) \text{ in } P(\bar{\varphi}_i)]\} \quad (6.4)$$

with $Y_m = \{y_1, \dots, y_m\}$. Here $\text{conv}(Y_m \cap P(\bar{\phi}_{i-1})) = P(\bar{\phi}_{i-1})_m$ since $Y_m \subset P(\bar{\phi}_{i-1})$, but we cannot use the same notation for $\text{conv}(Y_m \cap P(\bar{\phi}_i))$. So we better leave them as they are. We continue (6.4) using (5.1)

$$E(i,n) = \sum_{m=0}^{n} \binom{n}{m} q^m (1-q)^{n-m}$$
$$m \int_{x \in P(\bar{\varphi}_i)} \text{Prob}[x \notin \text{conv}(Y_{m-1} \cap P(\bar{\phi}_i))$$
$$\text{and } x \in \text{conv}(Y_{m-1} \cap P(\bar{\phi}_{i-1}))]dx. \quad (6.5)$$

So we see that $E(i,n) \geq 0$. We claim now that for $m \geq d+2$

$$E_0 := m \int_{x \in P(\bar{\varphi}_i)} \text{Prob}[x \notin \text{conv}(Y_{m-1} \cap P(\bar{\phi}_i))$$
$$\text{and } x \in \text{conv}(Y_{m-1} \cap P(\bar{\phi}_{i-1}))]dx$$
$$\leq \text{const}(P) \log^{d-2} m \log\log m. \quad (6.6)$$

($E_0 = 0$ clearly for $m \leq d+1$.) This will prove (6.2), since using (6.6) in (6.5) gives

$$E(i,n) \leq \sum_{m=d+2}^{n} \binom{n}{m} q^m (1-q)^{n-m} \text{const}(P) \log^{d-2} m \log\log m$$
$$\leq \text{const}(P) \log^{d-2} n \log\log n \sum_{m=d+2}^{n} \binom{n}{m} q^m (1-q)^{n-m}$$
$$\leq \text{const}(P) \log^{d-2} n \log\log n.$$

As we prove (6.6) we now introduce the notation $K = P(\bar{\phi}_{i-1})$, and we assume that $\text{vol}\, K = \text{vol}\, P(\bar{\phi}_{i-1}) = 1$ since in (6.6) this does not matter. Let us write further

$$K(\tau_i \leq \phi_i) := P(\bar{\phi}_i),$$
$$K(\tau_i \geq \phi_i) := \{z \in K : \tau_i(z) \geq \phi_i\},$$
$$K(\tau_i \geq \varphi_i) := \{z \in K : \tau_i(z) \geq \varphi_i\},$$

but $P(\bar{\varphi}_i) = P(\varphi_0, \dots, \varphi_i)$ as earlier. For the estimation (6.6) we need the simple but important

Proposition 1. *Assume $x, y_1, \dots, y_{m-1}$ are in general position in K. Set $Y_{m-1} = \{y_1, \dots, y_{m-1}\}$ and assume, further, that*

$$x \in P(\bar{\varphi}_i), \; x \in \text{conv}\, Y_{m-1}, \; x \notin \text{conv}(Y_{m-1} \cap K(\tau_i \leq \phi_i)).$$

Then there is a $y_k \in Y_{m-1} \cap K(\tau_i \geq \phi_i)$ such that

$$\mathrm{ray}(x, y_k) \cap \mathrm{conv}[(Y_{m-1}\backslash\{y_k\}) \cap K(\tau_i \geq \varphi_i)] = \emptyset, \tag{6.7}$$

and

$$\mathrm{ray}(x, y_k) \cap \mathrm{conv}[(Y_{m-1}\backslash\{y_k\}) \cap K(\tau_i \leq \phi_i)] = \emptyset. \tag{6.8}$$

Proof. Identify x with the origin for this proof. Then the conditions imply that

$$C_1 := \mathrm{cone}\, Y_{m-1} = \mathbb{R}^d,$$
$$C_2 := \mathrm{cone}(Y_{m-1} \cap K(\tau_i \leq \phi_i)) \neq \mathbb{R}^d,$$
$$C_3 := \mathrm{cone}(Y_{m-1} \cap K(\tau_i \geq \varphi_i)) \neq \mathbb{R}^d.$$

As the sum of the last two cones is C_1, C_3 must have an extreme ray, defined by some $y_k \in Y_{m-1} \cap K(\tau_i \geq \varphi_i)$ that is not in C_2. Then $y_k \notin K(\tau_i \leq \phi_i)$ as well, and $\mathrm{ray}(x, y_k)$ has the claimed properties. □

We rewrite (6.6) using the new notation and Proposition 1.

$$E_0 = m \int_{P(\bar\varphi_i)} \mathrm{Prob}[x \notin \mathrm{conv}(Y_{m-1} \cap K(\tau_i \leq \phi_i)) \text{ and } x \in \mathrm{conv}(Y_{m-1} \cap K)]dx$$
$$\leq m \int_{P(\bar\varphi_i)} \mathrm{Prob}[\exists y_k \in Y_{m-1} \cap K(\tau_i \geq \phi_i) \text{ such that (6.7) and (6.8) hold}]dx$$
$$\leq m \int_{P(\bar\varphi_i)} \sum_{k=1}^{m-1} \mathrm{Prob}[y_k \in K(\tau_i \geq \phi_i) \text{ and (6.7) and (6.8) hold}]dx$$
$$\leq m \int_{x\in P(\bar\varphi_i)} (m-1) \int_{y\in K(\tau_i\geq\phi_i)} \mathrm{Prob}[\mathrm{ray}(x, y) \cap \mathrm{conv}(Y_{m-2} \cap K(\tau_i \geq \varphi_i)) = \emptyset$$
$$\text{and } \mathrm{ray}(x, y) \cap \mathrm{conv}(Y_{m-2} \cap K(\tau_i \leq \phi_i)) = \emptyset]dydx.$$

Now change m to $m+2$ and define the events

$$G1 : \mathrm{ray}(x, y) \cap \mathrm{conv}(Y_m \cap K(\tau_i \geq \varphi_i)) = \emptyset,$$
$$G2 : \mathrm{ray}(x, y) \cap \mathrm{conv}(Y_m \cap K(\tau_i \leq \phi_i)) = \emptyset.$$

Thus, in order to prove (6.6) it will be enough to show that

$$m^2 \iint_{(x,y)\in K^{(i)}} \mathrm{Prob}(G1 \text{ and } G2)dydx \leq \mathrm{const}(P) \log^{d-2} m \log\log m \tag{6.9}$$

where $K^{(i)} = P(\bar\varphi_i) \times K(\tau_i \geq \phi_i)$.

Let z be the point where the function

$$u = u_K (= u_{P(\bar\phi_{i-1})})$$

takes its maximum value on $\mathrm{aff}(x, y)$. It is known that z is unique (cf. Section 3), but we will not need this. We split $K^{(i)}$, the domain of the integration in (6.9), into three parts:

$$K_1^{(i)} = \{(x,y) \in K^{(i)} : \tau_i(z) \geq 2\varphi_i\},$$
$$K_2^{(i)} = \{(x,y) \in K^{(i)} : \tau_i(x) \leq \tau_i(z) \leq 2\varphi_i\},$$
$$K_3^{(i)} = \{(x,y) \in K^{(i)} : \tau_i(z) \leq \tau_i(x)\}.$$

We will estimate the integral (6.9) separately for the three parts.

Case 1: $\tau_i(z) \geq 2\varphi_i$. Set $\bar{u} = u_{K(\tau_i \geq \varphi_i)}$, $\bar{q} = \operatorname{vol} K(\tau_i \geq \varphi_i)$, and recall (3.5).

$$\begin{aligned}
&\text{Prob}(G1 \text{ and } G2) \leq \text{Prob}(G1)\\
&\leq \text{Prob}[z \notin \text{conv}(Y_m \cap K(\tau_i \geq \varphi_i))]\\
&= \sum_{\mu=0}^{m} \binom{m}{\mu} \bar{q}^{\mu}(1-\bar{q})^{m-\mu}\\
&\quad \text{Prob}[z \notin \text{conv}(Y_m \cap K(\tau_i \geq \varphi_i)) | \text{card}(Y_m \cap K(\tau_i \geq \varphi_i)) = \mu]\\
&= \sum_{\mu=0}^{m} \binom{m}{\mu} \bar{q}^{\mu}(1-\bar{q})^{m-\mu}\text{Prob}(z \notin K(\tau_i \geq \varphi_i)_\mu)\\
&\leq \sum_{\mu=0}^{m} \binom{m}{\mu} \bar{q}^{\mu}(1-\bar{q})^{m-\mu} 2\sum_{j=0}^{d-1} \binom{\mu}{j} (\frac{\bar{u}(z)}{2\bar{q}})^j (1-\frac{\bar{u}(z)}{2\bar{q}})^{\mu-j}\\
&= 2\sum_{j=0}^{d-1} \binom{m}{j} \bar{q}^j (\frac{\bar{u}(z)}{2\bar{q}})^j \sum_{\mu=j}^{m} \binom{m-j}{m-\mu} (1-\bar{q})^{m-\mu}[\bar{q}(1-\frac{\bar{u}(z)}{2\bar{q}})]^{\mu-j}\\
&= 2\sum_{j=0}^{d-1} \binom{m}{j} (\frac{\bar{u}(z)}{2})^j \sum_{\mu=0}^{m-j} \binom{m-j}{m-j-\mu} (1-\bar{q})^{m-j-\mu}(\bar{q}-\frac{\bar{u}(z)}{2})^{\mu}\\
&= 2\sum_{j=0}^{d-1} \binom{m}{j} (\frac{\bar{u}(z)}{2})^j (1-\frac{\bar{u}(z)}{2})^{m-j}.
\end{aligned}$$

Then

$$\begin{aligned}
E_1 &:= m^2 \iint_{K_1^{(i)}} \text{Prob}(G1 \text{ and } G2)dydx\\
&\leq m^2 \iint_{K_1^{(i)}} 2\sum_{j=0}^{d-1} \binom{m}{j} (\frac{\bar{u}(z)}{2})^j (1-\frac{\bar{u}(z)}{2})^{m-j} dydx\\
&= 2m^2 \sum_{\lambda=1}^{m} \iint_{\substack{K_1^{(i)}\\ \frac{\lambda-1}{m} \leq \bar{u}(z) \leq \frac{\lambda}{m}}} \sum_{j=0}^{d-1} \binom{m}{j} (\frac{\bar{u}(z)}{2})^j (1-\frac{\bar{u}(z)}{2})^{m-j} dydx\\
&\leq \text{const}(d) m^2 \sum_{\lambda=1}^{m} \lambda^{d-1} e^{-\lambda/2} \text{meas}\{(x,y) \in K_1^{(i)} : \bar{u}(z) \leq \frac{\lambda}{m}\},
\end{aligned}$$

where the last inequality follows in the same way as (5.2) and (5.3). This time we set $\lambda_0 = \lfloor 8 \log m \rfloor$ and write

$$E_1 \le \text{const}(d) m^2 [\sum_{\lambda=1}^{\lambda_0} \lambda^{d-1} e^{-\lambda/2} \text{meas}\{(x,y) \in K_1^{(i)} : \bar{u}(z) \le \frac{\lambda}{m}\} + \sum_{\lambda=\lambda_0+1}^{m} \lambda^{d-1} e^{-\lambda/2}]. \tag{6.10}$$

The second sum is less than const(d)m^{-2}; cf. (5.4). For the first sum we need

Proposition 2. *$u(z) \le \dfrac{2d!}{(d-i)!}\bar{u}(z)$ if $\tau_i(z) \ge 2\varphi_i$.*

Proof. We use induction. For $i = 0$ the statement is

$$u_P(z) \le 2u_{P(\tau_0 \ge \varphi_0)}(z)$$

provided $\tau_0(z) \ge 2\varphi_0$. Observe that $M_{P(\tau_0\ge\varphi_0)}(z) = M_P(z) \cap H_0(\varphi_0, 2\tau_0(z) - \varphi_0)$ where $H_0(t_1, t_2)$ stands for the slab between $H_0(t_1)$ and $H_0(t_2)$. So by (3.2)

$$\begin{aligned} \frac{\bar{u}(z)}{u(z)} &= \frac{2\int_0^{\tau_0(z)} \text{vol}_{d-1}[M_{P(\tau_0\ge\varphi_0)}(z) \cap H_0(t)]dt}{2\int_0^{\tau_0(z)} \text{vol}_{d-1}[M_P(z) \cap H_0(t)]dt} \\ &= \frac{\int_{\varphi_0}^{\tau_0(z)} \text{vol}_{d-1}[M_P(z) \cap H_0(t)]dt}{\int_0^{\tau_0(z)} \text{vol}_{d-1}[M_P(z) \cap H_0(t)]dt} \ge \frac{1}{2} \end{aligned}$$

since $\tau_0(z) \ge 2\varphi_0$ and the integrand is a monotone function.

When $i \ge 1$, $t_0 := \max\{0, 2\tau_0(z) - \phi_0\}$ is the smallest t such that

$$M_{P(\phi_0,\ldots,\phi_{i-1})}(z) \cap H_0(t) \text{ and } M_{P(\phi_0,\ldots,\phi_{i-1},\tau_i\ge\varphi_i)}(z) \cap H_0(t)$$

are nonempty. Therefore (3.3), (3.4), and the induction hypothesis (also cf. (2.8)) imply

$$\begin{aligned} \frac{\bar{u}(z)}{u(z)} &= \frac{\text{vol}\, M_{P(\phi_0,\ldots,\phi_{i-1},\tau_i\ge\varphi_i)}(z)}{\text{vol}\, M_{P(\phi_0,\ldots,\phi_{i-1})}(z)} \\ &= \frac{2\int_{t_0}^{\tau_0(z)} \text{vol}_{d-1}[M_{P(\phi_0,\ldots,\phi_{i-1},\tau_i\ge\varphi_i)}(z) \cap H_0(t)]dt}{2\int_{t_0}^{\tau_0(z)} \text{vol}_{d-1}[M_{P(\phi_0,\ldots,\phi_{i-1})}(z) \cap H_0(t)]dt} \\ &\ge \frac{\frac{\tau_0(z)-t_0}{d} \text{vol}_{d-1}[M_{P(\phi_0,\ldots,\phi_{i-1},\tau_i\ge\varphi_i)}(z) \cap H_0(\tau_0(z))]}{(\tau_0(z) - t_0)\,\text{vol}_{d-1}[M_{P(\phi_0,\ldots,\phi_{i-1})}(z) \cap H_0(\tau_0(z))]} \\ &= \frac{1}{d}\frac{u_{Q(\tau_0(z))(\phi_1,\ldots,\phi_{i-1},\tau_i\ge\varphi_i)}(z)}{u_{Q(\tau_0(z))(\phi_1,\ldots,\phi_{i-1})}(z)} \ge \frac{1}{d}\frac{(d-i)!}{(d-1)!}\frac{1}{2}. \end{aligned}$$

□

Using Proposition 2 and Lemma 3 in the first sum of (6.10) we obtain

$$\sum_{\lambda=1}^{\lambda_0} \lambda^{d-1} e^{-\lambda/2} \text{meas}\{(x,y) \in K_1^{(i)} : \bar{u}(z) \le \frac{\lambda}{m}\}$$
$$\le \sum_{\lambda=1}^{\lambda_0} \lambda^{d-1} e^{-\lambda/2} \text{meas}\{(x,y) \in K_1^{(i)} : u(z) \le 2d!\frac{\lambda}{m}\}$$
$$\le \sum_{\lambda=1}^{\lambda_0} \lambda^{d-1} e^{-\lambda/2} \,\text{const}(P)(\frac{2d!\lambda}{m})^2 \log^{d-2} \frac{m}{2d!\lambda} \log\log \frac{m}{2d!\lambda}$$
$$\le \text{const}(P) m^{-2} \log^{d-2} m \log\log m.$$

This proves that

$$E_1 \le \text{const}(P) \log^{d-2} m \log\log m.$$

Case 2: $\tau_i(x) \le \tau_i(z) \le 2\varphi_i$. This time we set $\tilde{u} = u_{K(\tau_i \le \phi_i)}$ and $\tilde{q} = \text{vol}\, K(\tau_i \le \phi_i)$. In a similar way as in Case 1 we see that

$$\text{Prob}(G1 \text{ and } G2) \le \text{Prob}(G2)$$
$$\le \text{Prob}[z \notin \text{conv}(Y_m \cap K(\tau_i \le \phi_i))]$$
$$\le 2\sum_{j=0}^{d-1} \binom{m}{j} (\frac{\tilde{u}(z)}{2})^j (1 - \frac{\tilde{u}(z)}{2})^{m-j}.$$

Correspondingly,

$$E_2 := m^2 \iint_{K_2^{(i)}} \text{Prob}(G1 \text{ and } G2) dy dx$$
$$\le m^2 \iint_{K_2^{(i)}} 2\sum_{j=0}^{d-1} \binom{m}{j} (\frac{\tilde{u}(z)}{2})^j (1 - \frac{\tilde{u}(z)}{2})^{m-j}$$
$$\le \text{const}(d) m^2 \sum_{\lambda=1}^{m} \lambda^{d-1} e^{-\lambda/2} \text{meas}\{(x,y) \in K_2^{(i)} : \tilde{u}(z) \le \frac{\lambda}{m}\}$$
$$\le \text{const}(d) m^2 [\sum_{\lambda=1}^{\lambda_0} \lambda^{d-1} e^{-\lambda/2} \text{meas}\{(x,y) \in K_2^{(i)} : \tilde{u}(z) \le \frac{\lambda}{m}\}$$
$$+ \sum_{\lambda=\lambda_0+1}^{m} \lambda^{d-1} e^{-\lambda/2}]$$

where $\lambda_0 = \lfloor 8 \log m \rfloor$, again. Here we need

Proposition 3. $u(z) \le \frac{d!}{(d-i)!} \tilde{u}(z)$ *if* $\tau_i(z) \le \frac{\phi_i}{2}$.

Proof. By induction again. The case $i = 0$ is very simple, since $u_P(z) = u_{P(\phi_0)}(z)$ if $\tau_0(z) \le \frac{\phi_0}{2}$. When $i \ge 1$, the same reasoning as in the proof of Proposition 2 gives

$$\frac{\tilde{u}(z)}{u(z)} = \frac{\operatorname{vol} M_{P(\phi_0,\dots,\phi_i)}(z)}{\operatorname{vol} M_{P(\phi_0,\dots,\phi_{i-1})}(z)}$$

$$= \frac{2\int_{t_0}^{\tau_0(z)} \operatorname{vol}_{d-1}[M_{P(\phi_0,\dots,\phi_i)}(z) \cap H_0(t)]dt}{2\int_{t_0}^{\tau_0(z)} \operatorname{vol}_{d-1}[M_{P(\phi_0,\dots,\phi_{i-1})}(z) \cap H_0(t)]dt}$$

$$\geq \frac{\frac{1}{d}(\tau_0(z) - t_0) \operatorname{vol}_{d-1}[M_{P(\phi_0,\dots,\phi_i)}(z) \cap H_0(\tau_0(z))]}{(\tau_0(z) - t_0) \operatorname{vol}_{d-1}[M_{P(\phi_0,\dots,\phi_{i-1})}(z) \cap H_0(\tau_0(z))]}$$

$$\geq \frac{1}{d} \frac{u_{Q(\tau_0(z))(\phi_1,\dots,\phi_i)}(z)}{u_{Q(\tau_0(z))(\phi_1,\dots,\phi_{i-1})}(z)} \geq \frac{1}{d} \frac{(d-i)!}{(d-1)!}. \qquad \square$$

Observing (6.1) we see in the same way as in Case 1 that

$$E_2 \leq \operatorname{const}(P) \log^{d-2} m \log\log m.$$

Case 3: $\tau_i(z) \leq \tau_i(x)$. Of course, $\tau_i(x) \leq \varphi_i < \phi_i \leq \tau_i(y)$. Macbeath proved that the set $\{x \in K : u(x) \geq \varepsilon\}$ is convex (recall Section 2). This implies that u is maximal on ray(x, y) at x. Similarly as in Case 2 – but with x instead of z – we get

$$E_3 := m^2 \iint_{K_3^{(i)}} \operatorname{Prob}(G1 \text{ and } G2) dy dx$$

$$\leq m^2 \iint_{K_3^{(i)}} \operatorname{Prob}[x \notin \operatorname{conv}(Y_m \cap K(\tau_i \leq \phi_i))] dy dx$$

$$\leq \operatorname{const}(d) m^2 [\sum_{\lambda=0}^{\lambda_0} \lambda^{d-1} e^{-\lambda/2} \operatorname{meas}\{(x,y) \in K_3^{(i)} : \tilde{u}(x) \leq \frac{\lambda}{m}\}$$

$$+ \sum_{\lambda=\lambda_0+1}^{m} \lambda^{d-1} e^{-\lambda/2}]$$

with $\lambda_0 = \lfloor 8 \log m \rfloor$. Again $u(x) \leq d! \tilde{u}(x)$ by Proposition 3. Lemma 1 shows that $y \in M_K(z, \theta)$ with $\theta = (3d)^{d+2}$. As x lies on the segment connecting z and y we have $y \in M_K(x, \theta)$. Hence

$$\operatorname{meas}\{(x,y) \in K_3^{(i)} : \tilde{u}(x) \leq \frac{\lambda}{m}\}$$

$$\leq \operatorname{meas}\{(x,y) \in K_3^{(i)} : u(x) \leq d! \frac{\lambda}{m}\}$$

$$\leq \operatorname{meas}\{(x,y) \in P(\bar{\varphi}_i) \times K(\tau_i \geq \phi_i) : u(x) \leq d! \frac{\lambda}{m},\ y \in M_K(x, \theta)\}$$

$$= \int_{\substack{x \in P(\bar{\varphi}_i) \\ u(x) \leq d! \frac{\lambda}{m}}} \operatorname{vol}\{y \in K(\tau_i \geq \phi_i) : y \in M_K(x, \theta)\} dx.$$

Estimating the integrand by Lemma 2 and observing (6.1) we further see that

$$\begin{aligned}
&\text{meas}\{(x,y) \in K_3^{(i)} : \tilde{u}(x) \le \frac{\lambda}{m}\} \\
&\le \int\limits_{\substack{x\in P(\bar{\varphi}_i)\\ u(x)\le d!\frac{\lambda}{m}}} \text{const}(P)\tau_i(x)u(x)dx \\
&\le \text{const}(P)(\log m)^{-1}\frac{d!\lambda}{m} \int\limits_{\substack{x\in P(\bar{\varphi}_i)\\ u(x)\le d!\frac{\lambda}{m}}} 1dx.
\end{aligned}$$

By Theorem 6

$$\text{vol}\{x \in P(\bar{\varphi}_i) : u(x) \le d!\frac{\lambda}{m}\} \le \text{const}(d)\frac{d!\lambda}{m}\log^{d-1}\frac{d!\lambda}{m},$$

and therefore

$$\begin{aligned}
&\text{meas}\{(x,y) \in K_3^{(i)} : \tilde{u}(x) \le \frac{\lambda}{m}\} \\
&\le \text{const}(P)\frac{\lambda^2}{m^2}\log^{d-2} m.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\sum_{\lambda=0}^{\lambda_0} \lambda^{d-1}e^{-\lambda/2}\text{meas}\{(x,y) \in K_3^{(i)} : \tilde{u}(x) \le \frac{\lambda}{m}\} \\
&\le \text{const}(P)\frac{1}{m^2}\log^{d-2} m
\end{aligned}$$

and

$$E_3 \le \text{const}(P)\log^{d-2} m. \qquad \square$$

7. Proof of Theorem 1

Consider a convex body $K \in \mathscr{K}_1^d$. Define $N(\varepsilon)$ as the maximal number of pairwise disjoint caps of K, each of volume ε. (A cap of K is the intersection of K with a halfspace.) If K is a polytope, then $N(\varepsilon) \le \text{vert}\, K$ and $N(\varepsilon) = \text{vert}\, K$ for small enough ε. Conversely we have:

$$\text{If } N(\varepsilon) \text{ is bounded, then } K \text{ is a polytope.} \tag{7.1}$$

To prove this assume that $N(\varepsilon) \le N_0$, $N(\varepsilon_0) = N_0$, and take pairwise disjoint caps $C_1, \dots, C_{N_0}$, each of volume ε_0. Then $C_i = K \cap H_i$ with a halfspace H_i. Write H_i^ε for the halfspace contained in H_i such that $\text{vol}(K \cap H_i^\varepsilon) = \varepsilon$ for $0 \le \varepsilon \le \varepsilon_0$. By changing each H_i a little and decreasing ε_0 a little we may assume that $K \cap H_i^0$ is a single point z_i. We show now that $K = \text{conv}\{z_1, \dots, z_{N_0}\}$. Assume not, then there is a point z_0 on the boundary of K with $z_0 \notin \text{conv}\{z_1, \dots, z_{N_0}\}$. Then there is a halfspace H_0 with $z_0 \in \text{int}\, H_0$ and $z_i \notin H_0$ ($i = 1, \dots, N_0$). Then the cap $H_0^\varepsilon \cap K$

is disjoint from all the other caps $H_i^\varepsilon \cap K$ for sufficiently small ε, a contradiction proving (7.1).

Now we prove (1.2). Let first K be a polytope. If it is not a simplex, it has at least $d+2$ vertices, each vertex belongs to at least d edges, and, generally, each k-face belongs to at least $d-k$ faces of dimension $k+1$. Hence $T(K) \geq (d+2)d!$, and Theorem 2 gives

$$\liminf \frac{E(K,n)}{E(\Delta,n)} = \frac{T(K)}{T(\Delta)} \geq \frac{(d+2)d!}{(d+1)!} = 1 + \frac{1}{d+1}$$

unless K is a simplex. So assume K is not a polytope. For $\varepsilon > 0$ small, find $N(\varepsilon)$ and pairwise disjoint caps $C_1, \ldots, C_{N(\varepsilon)}$ of volume ε. Let $C_i = K \cap H_i$ and $C_i^* = K \cap H_i^*$ where the halfspace H_i^* is contained in H_i with its boundary hyperplane halving the width of C_i in direction orthogonal to H_i. Clearly, for $\eta > 0$ small enough

$$\{x \in C_i^* : u_{C_i}(x) \leq \eta\} = \{x \in C_i^* : u_K(x) \leq \eta\}.$$

The proof of Theorem 2 of [BáLa], applied to C_i (cf. (2.2)), yields

$$\operatorname{vol}\{x \in C_i^* : u_{C_i}(x) \leq \eta\} \geq \operatorname{const}(d)\eta \log^{d-1} \frac{\varepsilon}{\eta}.$$

Choosing $\varepsilon = \sqrt{\eta}$ we obtain

$$\begin{aligned}
\operatorname{vol} K(u_K \leq \eta) &\geq \sum_{i=1}^{N(\sqrt{\eta})} \operatorname{vol}\{x \in C_i^* : u_K(x) \leq \eta\} \\
&= \sum_{i=1}^{N(\sqrt{\eta})} \operatorname{vol}\{x \in C_i^* : u_{C_i}(x) \leq \eta\} \\
&\geq \operatorname{const}(d) N(\sqrt{\eta})\eta \log^{d-1} \frac{1}{\eta}
\end{aligned}$$

and consequently, by (2.1),

$$E(K,n) \geq \operatorname{const}(d) N(\frac{1}{\sqrt{n}}) \frac{\log^{d-1} n}{n}.$$

Since $N(\frac{1}{\sqrt{n}})$ is unbounded by (7.1), this shows that

$$\liminf \frac{E(K,n)}{E(\Delta,n)} \geq \liminf \operatorname{const}(d) N(\frac{1}{\sqrt{n}}) = \infty. \qquad \square$$

8. Proof of the lemmata

Proof of Lemma 1. The set $K(v \geq \varepsilon)$ is convex as it is the intersection of closed halfspaces. By Lemma F of [Bá] it does not contain any line segment on its boundary provided $\varepsilon > 0$. Therefore the maximal v-value on aff(a,b) is attained at a unique point c^*, and there is a hyperplane H^* containing aff(a,b) such that $K(v \geq v(c^*)) \cap H^* = \{c^*\}$. From Lemma G of [Bá] we know that if C is a cap with $K(v \geq \varepsilon) \cap C = \{x\}$, a single point, then $C \subset M(x, 3d)$ provided ε is sufficiently small. Hence the cap C^* cut off from K by H^* is contained in $M(c^*, 3d)$, and consequently

$$\frac{\|a - c^*\|}{\|b - c^*\|} \leq 3d.$$

Now if c^* is on the line segment connecting c and b, clearly

$$\frac{\|a - c\|}{\|b - c\|} \leq \frac{\|a - c^*\|}{\|b - c^*\|} \leq 3d,$$

and we are done. So assume c^* is on the line segment connecting c and a. Since u is maximal at c, $u(c) \geq u(c^*)$. Write $Q^* = K \cap H^*$. Let the width of C^* be h in the direction orthogonal to H^*. As $C^* \subset M(c^*, 3d)$, the width of $M(c^*)$ in the same direction is at least $\frac{2}{3d}h$. Considering (3.2), (3.3), and (3.4) we see that

$$u(c) \leq 2hu_{Q^*}(c),$$
$$u(c^*) \geq \frac{1}{d}\frac{2}{3d}hu_{Q^*}(c^*).$$

Let L be the $(d-2)$-dimensional plane in H^* through b orthogonal to $\mathrm{aff}(a, b)$, and let σ be the maximal $(d-2)$-dimensional volume of a section of Q^* with a plane that is parallel to L. Then

$$u_{Q^*}(c) \leq 2\|b - c\|\sigma.$$

On the other hand, $C^* \subset M(c^*, 3d)$ implies $Q^* \subset M_{Q^*}(c^*, 3d)$ and thus $\mathrm{vol}_{d-1} Q^* \leq (3d)^{d-1} \mathrm{vol}_{d-1} M_{Q^*}(c^*)$, i.e.

$$u_{Q^*}(c^*) \geq \frac{1}{(3d)^{d-1}} \mathrm{vol}_{d-1} Q^*.$$

As $\mathrm{vol}_{d-1} Q^* \geq \frac{1}{d-1}\|a - b\|\sigma$,

$$u_{Q^*}(c^*) \geq \frac{1}{(d-1)(3d)^{d-1}}\|a - b\|\sigma.$$

Hence

$$1 \leq \frac{u(c)}{u(c^*)} \leq 3d^2 \frac{u_{Q^*}(c)}{u_{Q^*}(c^*)} \leq 6d^2(d-1)(3d)^{d-1}\frac{\|b - c\|}{\|a - b\|}$$
$$\leq (3d)^{d+2}\frac{\|b - c\|}{\|a - b\|},$$

and $\frac{\|a-b\|}{\|b-c\|} \leq (3d)^{d+2}$ gives $\frac{\|a-c\|}{\|b-c\|} \leq (3d)^{d+2}$. □

Proof of Lemma 2. Set, as in the proof of Theorem 4, $K = P(\bar{\phi}_{i-1})$ and $K(\tau_i \geq \phi_i) = P(\bar{\phi}_{i-1}, \tau_i \geq \phi_i)$. We may assume $\tau_0(x) \leq \frac{\phi_0}{\theta+1}$ which implies that $K(\tau_0 \geq \phi_0) \cap M_K(x, \theta)$ is empty, proving the lemma when $i = 0$.

For $i \geq 1$ we first consider the case $\theta = 1$. Recall the definition of Q in (1.8), set $q = \mathrm{cone}\, F_1 \cap H_0(t_0)$ and define

$$x^* = x + (1 - \tau_0(x)t_0^{-1})q.$$

Assume now $i > 1$. It is not difficult to see that for $0 \le t \le 2\tau_0(x)$

$$M_K(x) \cap H_0(t) \subseteq (-1 + tt_0^{-1})q + M_{Q(\phi_1,\dots,\phi_{i-1})}(x^*). \tag{8.1}$$

$(M_K(x) \cap H_0(t)$ is empty if $t > 2\tau_0(x)$.) From

$$K(\tau_i \ge \phi_i) = \operatorname{cone} Q(\phi_1, \dots, \phi_{i-1}, \tau_{i-1}^{(Q)} \ge \phi_i) \cap H_0(0, \phi_0)$$

(cf. (2.8)) it follows that for $0 \le t \le t_0$

$$K(\tau_i \ge \phi_i) \cap H_0(t) \subseteq (-1 + tt_0^{-1})q + Q(\phi_1, \dots, \phi_{i-1}; \tau_{i-1}^{(Q)} \ge \phi_i). \tag{8.2}$$

(8.1), (8.2), and the induction hypothesis yield

$$\begin{aligned}
&\mathrm{vol}_{d-1}[K(\tau_i \ge \phi_i) \cap M_K(x) \cap H_0(t)] \\
&\quad = \mathrm{vol}_{d-1}[Q(\phi_1, \dots, \phi_{i-1}; \tau_{i-1}^{(Q)} \ge \phi_i) \cap M_{Q(\phi_1,\dots,\phi_{i-1})}(x^*)] \\
&\quad \le \mathrm{const}(Q)\tau_{i-1}^{(Q)}(x^*)\,\mathrm{vol}_{d-1} M_{Q(\phi_1,\dots,\phi_{i-1})}(x^*) \\
&\quad = \mathrm{const}(Q)\tau_i(x)\,\mathrm{vol}_{d-1} M_{Q(\tau_0(x))(\phi_1,\dots,\phi_{i-1})}(x),
\end{aligned}$$

since $\tau_{i-1}^{(Q)}(x^*) = \tau_i(x)$ as $i > 1$ (cf. (2.5)) and $M_{Q(\phi_1,\dots,\phi_{i-1})}(x^*)$ is congruent to $M_{Q(\tau_0(x))(\phi_1,\dots,\phi_{i-1})}(x)$. Then

$$\begin{aligned}
&\mathrm{vol}[K(\tau_i \ge \phi_i) \cap M_K(x)] \\
&\quad = \int_0^{2\tau_0(x)} \mathrm{vol}_{d-1}[K(\tau_i \ge \phi_i) \cap M_K(x) \cap H_0(t)]dt \\
&\quad \le 2\tau_0(x)\,\mathrm{const}(Q)\tau_i(x)\,\mathrm{vol}_{d-1} M_{Q(\tau_0(x))(\phi_1,\dots,\phi_{i-1})}(x) \\
&\quad \le \mathrm{const}(P)\tau_i(x)\,\mathrm{vol}\, M_K(x),
\end{aligned}$$

where the last step follows from (3.4).

Special care is needed when $i = 1$. Then the hyperplane $H(F_1)$ supports K and so $M_K(x)$ lies between the hyperplanes $H(F_1)$ and $2x - H(F_1)$ which is the reflection of $H(F_1)$ through x. The slab between these hyperplanes intersects Q in $Q(\tau_0^{(Q)} \le 2\tau_0(x)\tau_1(x)t_0^{-1})$. So we have instead of (8.1)

$$M_K(x) \cap H_0(t) \subseteq (-1 + tt_0^{-1})q + Q(\tau_0^{(Q)} \le 2\tau_0(x)\tau_1(x)t_0^{-1}).$$

On the other hand, using (2.4) we get

$$K(\tau_1 \ge \phi_1) \cap H_0(t) \subseteq (-1 + tt_0^{-1})q + Q(\tau_0^{(Q)} \ge t\phi_1 t_0^{-1}).$$

Hence $K(\tau_1 \ge \phi_1) \cap M_K(x) \cap H_0(t)$ is empty unless $t\phi_1 t_0^{-1} \le 2\tau_0(x)\tau_1(x)t_0^{-1}$. Thus

$$\begin{aligned}
&\mathrm{vol}[K(\tau_1 \geq \phi_1) \cap M_K(x)] \\
&= \int_0^{2\tau_0(x)} \mathrm{vol}_{d-1}[K(\tau_1 \geq \phi_1) \cap M_K(x) \cap H_0(t)]dt \\
&\leq \int_0^{2\tau_0(x)\tau_1(x)\phi_1^{-1}} \mathrm{vol}_{d-1}[M_K(x) \cap H_0(t)]dt \\
&\leq 2\tau_0(x)\tau_1(x)\phi_1^{-1}\,\mathrm{vol}_{d-1}[M_K(x) \cap H_0(\tau_0(x))] \\
&\leq d\phi_1^{-1}\tau_1(x)\,\mathrm{vol}\,M_K(x).
\end{aligned}$$

If $\theta > 1$, $x + \theta(K - x) \supset K \supset K(\tau_i \geq \phi_i)$ implies

$$\begin{aligned}
&K(\tau_i \geq \phi_i) \cap M_K(x, \theta) \\
&= K(\tau_i \geq \phi_i) \cap \{x + \theta[(K - x) \cap (x - K)]\} \\
&= K(\tau_i \geq \phi_i) \cap [x + \theta(K - x)] \cap [x + \theta(x - K)] \\
&= K(\tau_i \geq \phi_i) \cap [x + (K - x)] \cap [x + \theta(x - K)],
\end{aligned}$$

and as $K = \mathrm{cone}\,Q(\phi_1, \ldots, \phi_{i-1}) \cap H_0(0, \phi_0)$, it follows from $\tau_0(x) \leq \frac{\phi_0}{\theta+1}$ that

$$\begin{aligned}
&[x + (K - x)] \cap [x + \theta(x - K)] \\
&= K \cap [(\theta + 1)x - \theta K] \\
&= \frac{\theta+1}{2}x + [(K - \frac{\theta+1}{2}x) \cap (\frac{\theta+1}{2}x - K)] \\
&= M_K(\frac{\theta+1}{2}x, 1).
\end{aligned}$$

Consequently

$$K(\tau_i \geq \phi_i) \cap M_K(x, \theta) = K(\tau_i \geq \phi_i) \cap M_K(\frac{\theta+1}{2}x, 1).$$

On the other hand, $\tau_i(\frac{\theta+1}{2}x) = \tau_i(x)$ and

$$\begin{aligned}
M_K(\frac{\theta+1}{2}x, 1) &= [x + (K - x)] \cap [x + \theta(x - K)] \\
&\subset [x + \theta(K - x)] \cap [x + \theta(x - K)] \\
&= M_K(x, \theta).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathrm{vol}[K(\tau_i \geq \phi_i) \cap M_K(x, \theta)] \\
&= \mathrm{vol}[K(\tau_i \geq \phi_i) \cap M_K(\frac{\theta+1}{2}x, 1)] \\
&\leq \mathrm{const}(P)\tau_i(\frac{\theta+1}{2}x)\,\mathrm{vol}\,M_K(\frac{\theta+1}{2}x, 1) \\
&\leq \mathrm{const}(P)\tau_i(x)\,\mathrm{vol}\,M_K(x, \theta). \qquad \square
\end{aligned}$$

Proof of Lemma 3. We are going to use Theorem 6 of [BáLa] and Theorems 7 and 8 of [Bá]. They – or rather their proofs – say the following:

For a convex body $K \subset \mathscr{K}_1^d$ and $\varepsilon \le \varepsilon_0(d)$ assume that $z_1, \dots, z_N$ is a system of points maximal with respect to the following two properties: $u(z_j) = \varepsilon$ for every $j = 1, \dots, N$ and $M(z_j, \frac{1}{2}) \cap M(z_k, \frac{1}{2}) = \emptyset$ for every $j, k = 1, \dots, N$, $j \neq k$. According to Macbeath, the set $K(u \ge \varepsilon)$ is convex (recall Section 2) and does not contain any line segment on its boundary (recall Section 3), so for every z_j there is a halfspace H_j^+ with $K(u \ge \varepsilon) \cap H_j^+ = \{z_j\}$. Now, by Theorem 6 of [BáLa]

$$\bigcup_{j=1}^{N} [M(z_j, \frac{1}{2}) \cap H_j^+] \subset K(u \le \varepsilon) \subset \bigcup_{j=1}^{N} M(z_j, 5), \tag{8.3}$$

and by Theorems 7 and 8 of [Bá]

$$\{(x, y) \in K \times K : u(x, y) \le \varepsilon\} \subset \bigcup_{j=1}^{N} M(z_j, 15d) \times M(z_j, 15d). \tag{8.4}$$

Again set $K = P(\bar{\phi}_{i-1})$ and $K(\tau_i \ge \phi_i) = P(\bar{\phi}_{i-1}, \tau_i \ge \phi_i)$. As K is a polytope, by Theorem 6, $\operatorname{vol} K(u \le \varepsilon) \le \text{const}(P) \varepsilon \log^{d-1} \frac{1}{\varepsilon}$. On the other hand, $\operatorname{vol}[M(z_j, \frac{1}{2}) \cap H_j^+] = 2^{-(d+1)} \varepsilon$. Hence

$$N \le \text{const}(P) \log^{d-1} \frac{1}{\varepsilon}. \tag{8.5}$$

Claim. *If $z \notin S(T, 2\eta)$ and H^+ is any halfspace containing z in its bounding hyperplane, then*

$$\operatorname{vol}[M(z, \frac{1}{2}) \cap H^+ \setminus S(T, \eta)] \ge \frac{1}{d! 2^d} \operatorname{vol} M(z, \frac{1}{2}).$$

Proof. By induction on d. The case $d = 1$ is trivial. Since

$$M(z, \frac{1}{2}) \subset H_0(\frac{1}{2}\tau_0(z), \frac{3}{2}\tau_0(z))$$

and the last set is disjoint from $S(T, \eta)$ whenever $\eta < \frac{1}{2}\tau_0(z)$, only the case $\tau_0(z) \le 2\eta$ has to be considered.

As $z \in H_0(0, 2\eta)$ and $z \notin S(T, 2\eta) = \operatorname{cone} S_Q(T_Q, 2\eta) \cap H_0(0, 2\eta)$ (cf. (1.11)), clearly $z_Q \notin S_Q(T_Q, 2\eta)$ (cf. (2.3)). Then, by the induction hypothesis, for any halfspace H_Q^+ in $H_0(t_0)$ containing z_Q on its boundary

$$\operatorname{vol}_{d-1}[M_Q(z_Q, \frac{1}{2}) \cap H_Q^+ \setminus S_Q(T_Q, \eta)] \ge \frac{1}{(d-1)! 2^{d-1}} \operatorname{vol}_{d-1} M_Q(z_Q, \frac{1}{2}).$$

Choosing $H_Q^+ := \operatorname{cone}[H^+ \cap H_0(\tau_0(z))] \cap H_0(t_0)$ and replacing $H_0(t_0)$ by $H_0(\tau_0(z))$ we obtain

$$\operatorname{vol}_{d-1}[M(z, \frac{1}{2}) \cap H_0(\tau_0(z)) \cap H^+ \setminus \operatorname{cone} S_Q(T_Q, \eta)]$$
$$\ge \frac{1}{(d-1)! 2^{d-1}} \operatorname{vol}_{d-1}[M(z, \frac{1}{2}) \cap H_0(\tau_0(z))].$$

(The set $H^+ \cap H_0(\tau_0(z))$ may, exceptionally, coincide with the whole $H_0(\tau_0(z))$. In this case one has to perturb H_0.) The point $\frac{1}{2}z$ has distance $\frac{1}{2}\tau_0(z)$ from the $(d-1)$-dimensional set $M(z, \frac{1}{2}) \cap H_0(\tau_0(z)) \cap H^+ \backslash \operatorname{cone} S_Q(T_Q, \eta)$. Both the point and the set lie in $M(z, \frac{1}{2}) \cap H^+ \backslash S(T, \eta)$. Thus

$$
\begin{aligned}
&\operatorname{vol}[M(z, \frac{1}{2}) \cap H^+ \backslash S(T, \eta)] \\
&\geq \frac{1}{d} \frac{\tau_0(z)}{2} \operatorname{vol}_{d-1}[M(z, \frac{1}{2}) \cap H_0(\tau_0(z)) \cap H^+ \backslash S(T, \eta)] \\
&\geq \frac{1}{d} \frac{\tau_0(z)}{2} \frac{1}{(d-1)!2^{d-1}} \operatorname{vol}[M(z, \frac{1}{2}) \cap H_0(\tau_0(z))] \\
&\geq \frac{1}{d!2^d} \operatorname{vol} M(z, \frac{1}{2}),
\end{aligned}
$$

where the last step follows from (3.3). □

The Claim shows that for $z_j \notin S(T, 2\eta)$

$$\operatorname{vol}[M(z, \frac{1}{2}) \cap H_j^+ \backslash S(T, \eta)] \geq \frac{1}{d!4^d} \varepsilon.$$

On the other hand, by (4.4)

$$\operatorname{vol}[P(u \leq \varepsilon) \backslash \bigcup_T S(T, 2\eta)] \leq \operatorname{const}(P) \varepsilon \log^{d-2} \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}$$

if we choose $\eta = (\log \frac{1}{\varepsilon})^{-1}$. Then (8.3) shows that the number of points z_j outside $\cup_T S(T, 2\eta)$ is at most

$$\operatorname{const}(P) \log^{d-2} \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}. \tag{8.6}$$

Further, (8.4) implies

$$
\begin{aligned}
&\{(x, y) \in P(\bar{\varphi}_i) \times K(\tau_i \geq \phi_i) : u_K(x, y) \leq \varepsilon\} \\
&\subseteq \bigcup_{j=1}^{N} [M(z_j, 15d) \cap P(\bar{\varphi}_i)] \times [M(z_j, 15d) \cap K(\tau_i \geq \phi_i)].
\end{aligned} \tag{8.7}
$$

Consider now a point $z_j \in S(T, 2\eta)$ for some tower T. It follows from Lemma 2 that if the tower T does not start with the chain of faces $F_0 \subset F_1 \subset \ldots \subset F_i$, then

$$\operatorname{vol}[M(z_j, 15d) \cap P(\bar{\varphi}_i)] \leq \operatorname{const}(P) \varepsilon (\log \frac{1}{\varepsilon})^{-1}. \tag{8.8}$$

When T starts with this chain of faces, then, again by Lemma 2,

$$\operatorname{vol}[M(z_j, 15d) \cap K(\tau_i \geq \phi_i)] \leq \operatorname{const}(P) \varepsilon (\log \frac{1}{\varepsilon})^{-1}. \tag{8.9}$$

Taking the measure of the sets in (8.7) we get

$$\operatorname{meas}\{(x,y) \in P(\bar{\varphi}_i) \times K(\tau_i \geq \phi_i) : u_K(x,y) \leq \varepsilon\}$$
$$\leq \sum_{j=1}^{N} \operatorname{vol}[M(z_j, 15d) \cap P(\bar{\varphi}_i)] \operatorname{vol}[M(z_j, 15d) \cap K(\tau_i \geq \phi_i)].$$

By (8.6) there are at most const(P) $\log^{d-2} \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}$ terms with $z_j \notin \cup_T S(T, 2\eta)$, and as both factors in each term are less than const(d)ε, the sum of these terms is at most

$$\text{const}(P)\varepsilon^2 \log^{d-2} \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}.$$

By (8.8) the terms with $z_j \in S(T, 2\eta)$ are less than const(P)$\varepsilon(\log \frac{1}{\varepsilon})^{-1}$ times const(d)ε if T does not start with $F_0 \subset F_1 \subset \ldots \subset F_i$, and by (8.9) less than const(d)ε times const(P)$\varepsilon(\log \frac{1}{\varepsilon})^{-1}$ if T starts with $F_0 \subset F_1 \subset \ldots \subset F_i$. As by (8.5) there are at most const(P) $\log^{d-1} \frac{1}{\varepsilon}$ terms, the sum of terms with $z_j \in \cup_T S(T, 2\eta)$ is at most

$$\text{const}(P)\varepsilon^2 \log^{d-2} \frac{1}{\varepsilon}.$$

Therefore

$$\operatorname{meas}\{(x,y) \in P(\bar{\varphi}_i) \times K(\tau_i \geq \phi_i) : u_K(x,y) \leq \varepsilon\}$$
$$\leq \text{const}(P)\varepsilon^2 \log^{d-2} \frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}. \qquad \square$$

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