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# Random Polytopes in the $d$-Dimensional Cube

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**Abstract.** Let $C^d$ be the set of vertices of a $d$-dimensional cube, $C^d = \{(x_1, \ldots, x_d): x_i = \pm 1\}$. Let us choose a random $n$-element subset $A(n)$ of $C^d$. Here we prove that Prob(the origin belongs to the conv $A(2d + x\sqrt{2d})) = \Phi(x) + o(1)$ if $x$ is fixed and $d \to \infty$. That is, for an arbitrary $\varepsilon > 0$ the convex hull of more than $(2+\varepsilon)d$ vertices almost always contains 0 while the convex hull of less than $(2-\varepsilon)d$ points almost always avoids it.

## 1. Convex Hull of Subsets of Vertices

Let $C^d$ denote the set of vertices of a $d$-dimensional cube, $C^d \subset \mathbb{R}^d$, $C^d = \{(x_1, \ldots, x_d): x_i = \pm 1\}$. Let $A(n)$ be a random $n$-element subset of $C^d$. We have $\binom{2^d}{n}$ possibilities for $A(n)$, hence Prob($A(n)$ has property $\pi$) = (# of $n$-tuples of $C^d$ with property $\pi$)$\Big/\binom{2^d}{n}$. The threshold function of the property $\pi$ is $n_d$ if for every $\varepsilon > 0$ we have

$$\text{Prob}(A((1+\varepsilon)n_d) \text{ has property } \pi) \to 1 \qquad \text{whenever} \quad d \to \infty$$

and

$$\text{Prob}(A((1-\varepsilon)n_d) \text{ has property } \pi) \to 0 \qquad \text{whenever} \quad d \to \infty.$$

J. Mycielski posed the following problem: How large should we choose $n$ so that conv $A(n)$, the convex hull of $A(n)$, contains almost surely the origin 0 [2]? P. Erdös conjectured that the threshold function of this property is $O(d)$ [2]. This was proved by Komlós [5] in 1980.

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**Theorem 1.1** [5]. $1.3d < n_d < 4.4d$, *i.e.*,

$$\lim_{d\to\infty} \operatorname{Prob}(0 \in \operatorname{conv} A(1.3d)) = 0$$

*and*

$$\lim_{d\to\infty} \operatorname{Prob}(0 \in \operatorname{conv} A(4.4d)) = 1.$$

Here we improve this result. Denote $1/\sqrt{2\pi}\int_{-\infty}^{c} e^{-x^2/2}\,dx$ by $\Phi(c)$, as usual.

**Theorem 1.2.** *Let c be a real number. Then*

$$\lim_{d\to\infty} \operatorname{Prob}(0 \in \operatorname{conv} A(2d + c\sqrt{2d})) = \Phi(c).$$

**Corollary 1.3.** $n_d = 2d$.

Our results are strongly related to an old theorem of Wendel [7] (see Theorem 4.1). More results and an extensive literature about random polytopes can be found in a recent paper of Buchta and Müller [1]. See also Mycielski [6].

## 2. Lemmas

P. Erdös conjectured that a random $\pm 1$ matrix is almost always regular. This was proved by Komlós [4] in 1967:

**Lemma 2.1** [4]. *Let M be a $d \times d$ random $\pm 1$ matrix (i.e., every entry $a_{ij}$ is chosen independently and with probabilities* $\operatorname{Prob}(a_{ij} = 1) = \operatorname{Prob}(a_{ij} = -1) = \frac{1}{2}$). *Then*

$$\operatorname{Prob}(M \textit{ is regular}) > 1 - O(1/\sqrt{d}). \tag{1}$$

Komlós conjectures that one can replace $O(1/\sqrt{d})$ by $O(1/(1+\varepsilon)^d)$ for some positive $\varepsilon$. The following lemma is a simple generalization of a result due to Harding [3]. Let $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^d$ be a point-set ($p_i = p_j$ is possible). The partition $P = U \cup V$ is *induced by a hyperplane* if there exists a hyperplane $H \subset \mathbb{R}^d$ such that $P \cap H = \varnothing$ and $H$ splits every segment $[u, v]$ for $u \in U$, $v \in V$. Denote by $h(P)$ the number of such partitions of $P$, $h(d, n) = \max\{h(P)\colon P \subset \mathbb{R}^d, |P| = n\}$. Harding proved that

$$h(d, n) = \binom{n-1}{d} + \cdots + \binom{n-1}{0}.$$

Denote by $a(k, P)$ the number of affine dependent $k$-tuples of $P$. The lower bound in the following lemma is an easy consequence of a theorem of Winder [8] (see also Zaslavsky [9]).

**Lemma 2.2.** *For every $P \subset \mathbb{R}^d$, $|P| = n$ we have*

$$h(d, n) - \sum_{2 \le k \le d+1} a(k, P) \leqslant h(P) \leqslant h(d, n) = \sum_{k \le d} \binom{n-1}{k}. \qquad (2)$$

This lower bound is not the best possible, but it is sufficient for our purposes.

## 3. Proof of Theorem 1.2

The first observation is that we can neglect those $A(n)$'s which contain a pair of opposite vertices because the probability of this event tends to 0 when $d \to \infty$. In fact, we have

$$\operatorname{Prob}(A(n) \cap (-A(n)) \neq \emptyset) \le \frac{2^d \binom{2^d}{n-2}}{\binom{2^d}{n}} < \frac{n^2}{2^d} = o(1) \qquad \text{if} \quad n = O(d) \text{ and } d \to \infty. \qquad (3)$$

The main idea in our argument is that we obtain a random $n$-set $A$ in two steps. First we choose $n$ pairs from the $2^{d-1}$ pairs of the form $\{x, -x\}$, $x \in C^d$. Then we choose an element from each pair. Let $H$ be a hyperplane of $\mathbb{R}^d$ in general position with respect to $C^d$, $0 \notin H$. For $x \in C^d$ we denote the point $H \cap (0, x)$ by $\pi(x)$.

**Lemma 3.1.** *Let $x_1, \ldots, x_n \in C^d$, $\Pi = \{\pi(x_i): 1 \leqslant i \leqslant n\}$ and*

$$Y = \{\{y_1, \ldots, y_n\}: y_i = x_i \text{ or } -x_i\}.$$

*Then # (members of $Y$ whose convex hull avoids 0) $= 2h(\Pi)$.*

*Proof.* If the convex hull of $\{y_1, \ldots, y_n\}$ avoids 0 then there exists a hyperplane $H_0$ through 0 such that it separates $\{y_1, \ldots, y_n\}$ from $\{-y_1, \ldots, -y_n\}$. Hence $H_0 \cap H$ induces a partition of $\Pi$. Moreover, the converse is also true, every induced partition yields two members of $Y$. □

Using Lemma 2.2 for $n = 2d + c\sqrt{2d}$ we get

$$\operatorname{Prob}(0 \notin \operatorname{conv} A(n)) = \frac{1}{2^n \binom{2^{d-1}}{n}} \sum 2h(\Pi) \leqslant \frac{h(d, n)}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}} \sum_{i < n/2 - (c\sqrt{n})/2} \binom{n-1}{i} = \Phi(-c) + O\left(\frac{1}{n}\right).$$

(Actually we have calculated $\operatorname{Prob}(0 \notin \operatorname{conv} A(n) | A \cap (-A) = \emptyset)$ and used (3).)

Moreover, (3) implies

$$\operatorname{Prob}(0 \notin \operatorname{conv} A(n)) \geqslant \frac{h(d, n)}{2^{n-1}} - \frac{1}{2^{n-1}\binom{2^{d-1}}{n}} \sum_{\Pi} \sum_{1 \leqslant k \leqslant d} a(k, \Pi). \qquad (4)$$

Here Komlós' theorem (i.e., Lemma 2.1) implies

$$\frac{1}{\binom{2^{d-1}}{n}}\sum a(k,\Pi)\leqslant O\left(\frac{1}{\sqrt{d}}\right)\binom{n}{d}.$$

Similarly

$$\frac{1}{2^{d-1}}\sum_{\Pi} a(k,\Pi)$$

$$=\binom{n}{k}\text{Prob}(k\text{ random }\pm 1\text{ sequences of length }d\text{ are linearly dependent})$$

$$\leq\binom{n}{k}O\left(\frac{1}{\sqrt{d}}\right).$$

Hence (4) gives

$$\text{Prob}(0\notin\text{conv }A(n))\geqslant\frac{h(d,n)}{2^{n-1}}-\frac{\sum_{i\leqslant d}\binom{n}{i}}{2^{n-1}}O\left(\frac{1}{\sqrt{d}}\right)=\Phi(-c)+O\left(\frac{1}{n}\right)-O\left(\frac{1}{\sqrt{d}}\right).$$

## 4. Final Remarks

If Komlós' conjecture is true, then the method given above yields, for $n=O(d)$,

$$\text{Prob}(0\notin\text{conv }A(n))=\frac{h(n,d)}{2^{n-1}}+O(d^2/(1+\varepsilon)^d).$$

Of course, this method can be used in all cases when the underlying set of the points $T$ is symmetric, and $\text{Prob}(x_1,\ldots,x_d, T\text{ are linearly dependent})=1-o(1)$, for example,

**Theorem 4.1** (Wendel [7]). *Let $B^d$ denote the d-dimensional ball. Let us choose an n-element set P randomly. Then*

$$\text{Prob}(0\notin\text{conv }P)=h(d,n)/2^{n-1}.$$

Actually, Wendel used a similar inductional method (but he did not need our lemmas except the equality $h(P)=h(d,n)$ for affine independent $n$-sets $P\subset\mathbb{R}^d$). Finally, it is easy to see that in our case ($n\sim 2d$, $d\to\infty$) we have

$$\text{Prob}(0\in\text{int conv }P)-\text{Prob}(0\in\text{conv }P)\to 0.$$

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