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Random Sampling of Random Processes: Stationary Point Processes

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This is the first of a series of papers treating randomly sampled random processes. Spectral analysis of the resulting samples presupposes knowledge of the statistics of $\{t_n\}$, the random point process whose variates represent the sampling times. We introduce a class of stationary point processes, whose stationarity (as characterized by any of several equivalent criteria) leads to wide-sense stationary sampling trains when applied to wide-sense stationary processes. Of greatest importance are the nth forward [backward] recurrence times (distances from t to the nth point thereafter [preceding]), whose distribution functions prove more useful to the computation of covariances than interval statistics, and which possess remarkable properties that facilitate the analysis.

The moments of the number of points in an interval are evaluated by weighted sums of recurrence time distribution functions, the moments being finite if and only if the associated sum converges. If the first moment is finite, these distribution functions are absolutely continuous, and obey some convexity relations. Certain formulas relate recurrence statistics to interval length statistics, and conversely; further, the latter are also suitable for a direct evaluation of moments of points in intervals.

Our point process requires neither independent nor identically distributed interval lengths. It embraces most of the common sampling schemes (e.g., periodic, Poisson, jittered), as well as some new models. Of particular interest are point processes obtained from others by a random deletion of points (skip processes), as for instance a jittered cyclically periodic process with (random or systematic) skipping. Computation of the statistics for several point processes yields new results of interest not only for their own sake, but also of use for spectral analyses appearing in other papers of this series.

LIST OF SYMBOLS

t_n	Random point on the real line
$L_n(t)$	Length of time required for the nth point after time t to occur
$L_{-n}(t)$	Length of time between the nth point before the time t and t itself
$N(t, x)$	Number of points in the interval $(t, t + x]$
β	Average number of points per unit time
$E_n(t, x)$	Represents the event that "there are at least n points in the interval $(t, t + x]$"
$A_n(t, x)$	Represents the event that "there are exactly n points in the interval $(t, t + x]$"
$G_n(x)$	Distribution function of $L_n(t)$ (and of $L_{-n}(t)$)
$p(n, x)$	Probability that there are exactly n points in $(t, t + x]$
$g_n^*(s)$	$= E[e^{-sL_n(t)}] = \int_{0-}^{\infty} e^{-s\sigma}\, dG_n(\sigma)$
$F_n(x)$	Distribution (in a rough sense) of n successive intervals (a precise definition is given in Section VI)
$f_n^*(s)$	$= \int_{0-}^{\infty} e^{-s\sigma}\, dF_n(\sigma)$
q	Probability of skipping a point
$\{u_n\}$	Time-jitter error process
$\gamma(s)$	$= E[e^{-su_n}]$

I. INTRODUCTION

This is the first of a series of papers treating stochastic sampling of wide-sense stationary random processes from a unified viewpoint. The sampling occurs at random times t_n, and may take the form of a pulse train (Leneman, 1966a), stepwise sample-and-hold (Leneman, 1966b, 1966c), linear interpolation between sample values (Leneman and Lewis, 1966a), or one of various other modulation schemes. Under the basic assumptions on $\{t_n\}$, the new process created by the sampling procedure is again a wide-sense stationary process. By using techniques introduced in the later papers of this series, and based on the theory presented here, we may then compute spectra of the randomly modulated process (Leneman 1966a, 1966b, 1966c, 1966d), discuss the mean square error associated with the reconstruction of the sampled process from the (randomly timed) samples (Beutler, 1966; Leneman and Lewis, 1966a, 1966b, 1966c), and study minimum mean square error recovery (Leneman, 1966d). We are also able to study pulse-modulated control systems (Leneman, 1966e).

The key to the spectral analysis of stochastic modulations lies in a suitable description of $\{t_n\}$. Our definition yields remarkable properties that facilitate the spectral computations of the above cited references. At the same time, the definition is sufficiently inclusive to admit as special cases jittered, periodic, Poisson, and skip sampling, in addition to some new sampling schemes of practical importance. Analysis of the spectral properties of stochastically modulated signals is undertaken in the referenced publications; our present concern is the underlying theory, on which the spectral analysis is necessarily based.

It is convenient to think of the t_n as points on the line (representing time), and to speak of them as such. In this language, we require the following of $\{t_n\}$ as our basic assumption: the joint statistics of the respective numbers of points in any set of intervals are invariant under a translation of these intervals. We also demand that $t_k \leqq t_{k+1}$ for all k. A process $\{t_n\}$ meeting these conditions has been called a stationary point process (hereafter abbreviated s.p.p.) (Wold, 1949; McFadden, 1962; Beutler and Leneman, 1966) and we shall use the same nomenclature.

The requirement that the point process $\{t_n\}$ be an s.p.p. assures the wide-sense stationarity of samples (of whatever kind) from a wide-sense stationary random process, while placing a minimum of constraint on the sampling sequence. For instance, that $\{t_n\}$ is an s.p.p. does *not* imply either that the interval lengths $\tau_k = t_k - t_{k-1}$ between successive points are independent, or that the τ_k are identically distributed.

A random sampling theory could have been based on other (existent) point processes, but each of those known has major deficiencies that severely circumscribe its applicability to random sampling. For instance, one might consider an equilibrium renewal process (Cox, 1962), only to find that the requirement of identically distributed mutually independent intervals precludes jittered sampling and sampling in (random) bursts. There are also mathematical difficulties connected with limits and indexing of the sample points. A more promising point process has been proposed by McFadden (1962) whose definition of stationarity (in terms of interval statistics) we have borrowed.[1] Indeed, McFadden's work has provided much of the underlying motivation for our investiga-

[1] The same definition was independently introduced by Wold (1949) who failed to develop its consequences.

tions. Unfortunately, McFadden's description of stationary point processes proved difficult to use in our work. In the first place, his point process suffers from "floating indices," so that one cannot connect expressions such as "there are n points in $(t, t + x]$" with set-theoretic expressions for events in a probability space. Secondly, there are some defective statements of which the most damaging is an assertion that stationarity of the point process (in the sense mentioned) implies that the intervals τ_k between points constitute a discrete parameter stationary stochastic process.[2] The latter is not a serious defect if one is willing to insert interval stationarity as an additional hypothesis. However, some interesting sampling processes would be eliminated thereby, so that it is preferable to proceed in a different direction making no use of the hypothesis. Finally, our analysis demands that we obtain a number of s.p.p. properties that McFadden fails to adduce in his paper.

Whereas renewal theory regards intervals between points as basic, we find it more convenient to work principally with the statistics of forward recurrences, i.e., the lengths of times $L_n(t)$ required for n points to occur after time t. The distribution functions of the $L_n(t)$ provide a direct description of the statistics of sample locations relative to arbitrary t, which is precisely the information needed for the determination of autocorrelations of sampled signals. Moreover, the distributions of $L_n(t)$ possess convexity and absolute continuity properties that facilitate computations, and lead to a comprehensive theory.

In what follows, we shall define point processes t_n, from which we obtain such notions as forward and backward recurrence times, and numbers of points $N(t, x)$ in intervals $(t, t + x]$. Stationarity is then introduced; this concept can be expressed in any of several equivalent forms, employing either interval or forward [backward] recurrence statistics. The distribution functions of the forward recurrence times are found to possess convexity and absolute continuity properties. Moments of $N(t, x)$ can be expressed in terms of these distributions as series whose convergences are necessary and sufficient conditions for finiteness of the moments. The mean number of points in an interval is always a linear function of interval length. Further, if some order moment of number of points in an interval is finite over any interval of positive (small) length, it is finite for any (finite) interval, and is given by an absolutely continuous function of the interval length.

Several examples of s.p.p. are provided. These include the familiar

[2] This was pointed out by Professor William L. Root, who provided a simple example representative of a class of sampling schemes that may be met in practice.

periodic process with random phase, and the jitter process discussed elsewhere by Balakrishnan (1960) and Brown (1963). Also well-known is the Poisson process; strangely, the mention of Poisson sampling in the literature (see e.g., Black, 1953) is not accompanied by suitable analysis. There are also other processes which have not been analyzed, such as the skip process. This process, which is generated by deleting in random fashion the points of another s.p.p., is here combined with possible jitter, and treated in some detail.

For the sake of brevity, we have often omitted details and proofs, most of which may be found in the much lengthier exposition (Beutler and Leneman, 1966). In most cases, the missing proofs relate to facts easily accepted intuitively but extremely tedious to verify rigorously. To cite just one example, it is clear that the sum of the numbers of points in two adjacent intervals is equal to the number of points in the entire interval (i.e., $N(t, x + y) = N(t, x) + N(t + x, y)$); yet, the proof based on the set-theoretic definition of $N(t, x)$ (rather than its intuitive interpretation in terms of points in an interval) requires several pages of set manipulations.

II. RANDOM POINT PROCESSES—DEFINITIONS AND PRELIMINARIES

A random point process is a statistical description for any physical phenomenon characterized by an enumerable, ordered sequence of specific occurrences. Electrons emitted in a vacuum tube, customers entering a store, pulse-type interference in a communication channel, random times of sampling a random process—all these can be defined as random point processes. To be more precise, a random point process is described by

DEFINITION 2.1. *Let* $\{\tau_n\}$, $n = 0, \pm 1, \pm 2, \cdots$ *be a discrete parameter random process such that (with probability one)*

$$\tau_n \geqq 0 \quad \textit{for all} \quad n \tag{2.1}$$

and all the τ_n *are finite-valued. If we take*

$$t_n = \begin{cases} \tau_0 + \sum_1^n \tau_k & \textit{for} \quad n \geqq 1 \\ \tau_0 & \textit{for } n = 0 \\ \tau_0 - \sum_n^{-1} \tau_k & \textit{for } n \leqq -1 \end{cases} \tag{2.2}$$

then $\{t_n\}$ *is called a random point process.*

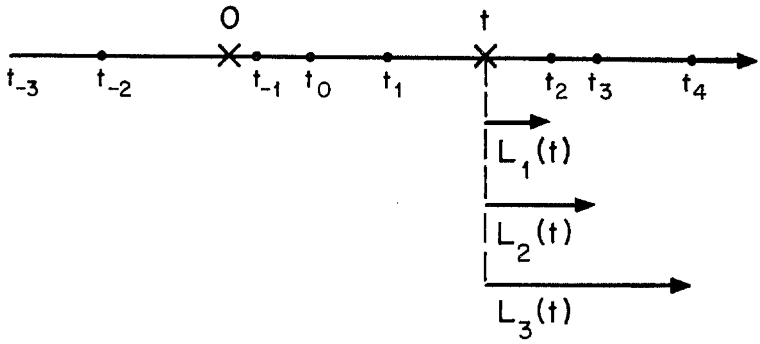

FIG. 1. Forward recurrence times

It is clear from the definition that $\{t_n\}$ is an ordered nondecreasing sequence each of whose members is finite-valued. One could regard τ_0 as an initial (random) phase, and the other τ_n as the interval lengths of the sequence.

All pertinent sets and random variables may be expressed in terms of countable set operations on the "basic building block" sets $B_n(t)$, which are defined for any integer n and time t by

$$B_n(t) = \{\omega : t_n(\omega) \leqq t\}. \tag{2.3}$$

Here ω is an element of the probability space Ω on which the τ_n are defined, and the measurability of the τ_n implies that $B_n(t)$ is also measurable. We shall often suppress the ω variable, writing $[t_n \leqq t]$ when we mean $B_n(t)$. We observe for future reference that $B_{n+1}(t) \subset B_n(t)$, and that, when $s \leqq t$, $B_n(s) \subset B_n(t)$.

For each t, we define a new discrete parameter process $\{L_k(t)\}$, $k = 1, 2, \cdots$, where $L_k(t)$ is the kth forward recurrence time, that is, the length of time required for the kth point after t to occur (compare Cox, 1962, p. 27). The $L_k(t)$ are illustrated in Fig. 1. For a more formal approach, let

$$E_n(t, x) = \bigcup_m [B_{m+1}^*(t) \cap B_{m+n}(t + x)], \qquad x \geqq 0, n \geqq 1, \tag{2.4}$$

where the union is taken over all integers, B_n is as in (2.3), and * denotes the complement of a set. Evidently, $E_n(t, x)$ carries the intuitive meaning of "at least n points fall in the interval $(t, t + x]$." An appropriate definition for $L_n(t)$ is then

DEFINITION 2.2. *$L_n(t)$ is the random variable satisfying*

$$[L_n(t) \leqq x] = E_n(t, x). \tag{2.5}$$

Another important concept in our considerations is $N(t, x)$, the number of points in $(t, t + x]$. If we write

$$A_n(t, x) = E_n(t, x) \cap E^*_{n+1}(t, x) \tag{2.6}$$

it is obvious that $A_n(t, x)$ represents the event "exactly n points in $(t, t + x]$." Accordingly, $N(t, x)$ is specified by

Definition 2.3. *$N(t, x)$ is the random variable satisfying*

$$[N(t, x) = n] = A_n(t, x), \qquad n = 0, 1, \cdots . \tag{2.7}$$

For $n = 0$, consistency requires that we let $E_0(t, x) = \Omega$.

Corresponding to the forward recurrence time notion, and equally useful, is the idea of backward recurrence times. For a positive integer n, the backward recurrence time $L_{-n}(t)$ is to be the time interval between the nth point before t and t itself, as shown in Fig. 2. This is accomplished by the extension of $E_n(t, x)$ to negative integers; for positive n, let

$$E_{-n}(t, x) = \bigcup_m [B^*_{m+1}(t - x) \cap B_{m+n}(t)]. \tag{2.8}$$

Definition 2.2 may now be taken to hold also for negative integers, on which it defines backward rather than forward recurrence times. For the sake of completeness, we note that $L_0(t) = 0$ follows in (2.5) from the definition of $E_0(t, x)$.

From a comparison of (2.4) and (2.8), we see that

$$[L_{-n}(t) \leqq x] = [L_n(t - x) \leqq x]. \tag{2.9}$$

As we shall soon see, the stationarity of $\{t_n\}$ implies a certain symmetry of distribution functions of recurrence times in the sense that the proba-

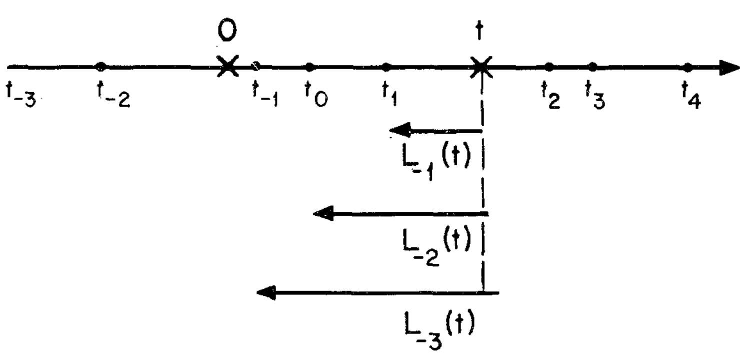

Fig. 2. Backward recurrence times

bilities $P[L_n(t) \leqq x]$ and $P[L_{-n}(t) \leqq x]$ are equal. This property does not extend to higher-order (multivariate) distribution functions of the recurrence times.

For simplicity, we will henceforth consider only forward recurrence times, the properties of backward recurrence times being entirely similar. The distribution function of $L_n(t)$ will be denoted by

$$G_n(x; t) = P[L_n(t) \leqq x], \tag{2.10}$$

where P indicates "probability of." The right side of (2.10) could equally well have been written as $P[E_n(t, x)]$, as we can see from the definition of $L_n(t)$. We further introduce the probability

$$p(n, x; t) = P[N(t, x) = n] \tag{2.11}$$

which is also $P[A_n(t, x)]$. The probabilities given by (2.10) and (2.11) are related to each other through (2.6) and the inclusions on the B_n. Thus it can be seen that

$$p(n, x; t) = G_n(x; t) - G_{n+1}(x; t). \tag{2.12}$$

In general, $N(t, x)$ and $N(t + y, x)$ need not have the same probability distribution. Relatively little can then be said regarding properties of arbitrary point processes. For this reason, such nonstationary processes are of little interest to us, and will not be analyzed further here. On the other hand, many physical situations suggest that the entire probability structure of the occurrence pattern does not vary with time. This seems to be the case for emission of electrons in a vacuum tube, assuming that the tube is in steady-state operation. The same is true of certain random sampling situations, as in jittered sampling, skip sampling due to loss of samples associated with jamming of signals, or deliberately random encoded signals; in each case, it is assumed that conditions of operations remain essentially unchanged for a period that is long relative to that for which the receiver is active. A stationarity assumption is therefore appropriate, especially since it renders the random point process amenable to the computation of sampling spectra and other results.

III. STATIONARITY OF RANDOM POINT PROCESSES

Although several possible definitions of stationarity for a random point process may occur to the reader, these definitions in fact turn out to be equivalent. Perhaps the most intuitively satisfying one is that of McFadden (1962) who required that the number of points in a set of fixed time intervals be invariant under any time shift which preserves the lengths and spacing of those intervals, i.e.,

DEFINITION 3.1. *$\{t_n\}$ is a stationary point process (s.p.p.) if, for each $x_1, x_2, \cdots, x_n$, each $t_1, t_2, \cdots, t_n$, each integer set $k_1, k_2, \cdots, k_n$, and any h*

$$P\left[\bigcap_{j=1}^{n} A_{k_j}(t_j, x_j)\right] = P\left[\bigcap_{j=1}^{n} A_{k_j}(t_j + h, x_j)\right]. \tag{3.1}$$

One of the consequences of Definition 3.1 is that $N(t, x)$, the number of points in $(t, t + x]$, is a finite-valued random variable. Indeed, the basic definition of the point process implies that no realization of t_n (except on a fixed zero probability set) can have more than two limit points. An argument based on the additivity of the probability measure, together with (3.1) applied for $n = 1$, leads to the desired conclusion. It follows that (the A_k being disjoint)

$$\sum_{k=0}^{\infty} p(k, x; t) = 1 \tag{3.2}$$

for any x and t. Further, we deduce that

$$\lim_{n\to\infty} G_n(x; t) = 0, \tag{3.3}$$

which is equivalent to (3.2), as we may verify by summing of (2.12). We remark that, in accordance with the preceding discussion, an s.p.p. cannot have finite limit points, except possibly on a fixed set of zero probability. Specifically excluded are point processes such as those generated by the zero crossings of a Brownian motion of Ornstein-Uhlenbeck process.

An apparently weaker requirement than (3.1) is that this equation hold for numbers of points in successive adjacent intervals, i.e., whenever $t_{j+1} = t_j + x_j$. But in fact, (3.1) is then actually valid for arbitrary choices of t_j and x_j. This knowledge is useful, for it provides a criterion of stationarity that is easier to verify than that demanded by Definition 3.1.

Stationarity may also be described by forward (backward) recurrence statistics. A necessary and sufficient condition that a point process be stationary is that

$$P\left[\bigcap_{j=1}^{n} E_{k_j}(t, x_j)\right] = P\left[\bigcap_{j=1}^{n} E_{k_j}(t + h, x_j)\right] \tag{3.4}$$

for each set $x_1, x_2, \cdots, x_n$, each set of positive (negative) integers

$k_1, k_2, \cdots, k_n$, any h, and some t. If (3.4) is satisfied for positive (negative) integer k_j, then for any integer combination (with variable signs), and any set of times $t_1, t_2, \cdots, t_n$, we have

$$P\left[\bigcap_{j=1}^{n} E_{k_j}(t_j, x_j)\right] = P\left[\bigcap_{j=1}^{n} E_{k_j}(t_j + h, x_j)\right]. \tag{3.5}$$

Thus, (3.1), (3.4), (3.5), and the condition on successive adjacent intervals (the apparent weakening of (3.1)) all imply one another, and are therefore equivalent in defining or verifying stationarity.

In view of these properties, $p(n, x; t)$ is the same for all t, so that we may write $p(n, x)$ whenever this probability refers to an s.p.p. The equivalent conditions on recurrence times likewise renders $G_n(x; t)$ independent of t, so that we are also justified in suppressing t there. Since in the remainder of the paper we restrict our considerations to s.p.p., the subscript t will not appear hereafter.

IV. CONVEXITY AND ABSOLUTE CONTINUITY OF DISTRIBUTION FUNCTIONS

As we have noted in the introduction, our work tends to utilize forward recurrence statistics rather than interval statistics. One of the reasons for our preference lies in the nature of the $G_n(x)$, which possess properties that are of considerable value in later calculations. Most of these depend ultimately on convexity properties which we now proceed to develop.

Combining (2.4) and (2.6) leads to

$$[x < L_1(t) \leqq x + h] = A_0(t, x) \cap E_1(t + x, h) \tag{4.1}$$

whence

$$G_1(x + h) - G_1(x) = P[A_0(t, x) \cap E_1(t + x, h)]. \tag{4.2}$$

Now the right side of (4.1) is subject to the containment relation

$$\begin{aligned}[A_0(t, x_2) \cap E_1(t + x_2, h)] \\ \subset [A_0(t + x_2 - x_1, x_1) \cap E_1(t + x_2, h)]\end{aligned} \tag{4.3}$$

valid whenever $x_1 \leqq x_2$. If (4.2) is applied to both sides of (4.3), we have the fundamental inequality

$$G_1(x_2 + h) - G_1(x_2) \leqq G_1(x_1 + h) - G_1(x_1). \tag{4.4}$$

Hence G_1 is concave (Hardy, Littlewood, and Polya, 1951) and since

G_1 must have a point of continuity in every interval

$$G_1[\alpha x + (1 - \alpha)y] \geqq \alpha G_1(x) + (1 - \alpha)G_1(y) \tag{4.5}$$

for $x, y \geqq 0$ and $0 \leqq \alpha \leqq 1$. In particular, $G_1(\alpha x) \geqq \alpha G_1(x)$.

The convexity of G_1 implies not only that G_1 is continuous (apply (4.4)), but even that G_1 is absolutely continuous on any interval $[\delta, \infty)$, $\delta > 0$. We now prove this result. Without loss of generality, we may assume that G_1 is continuous at δ, since G_1 (being a distribution function) has a continuity point in every interval. Next, pick h so small that $v = \delta - h > 0$. Then for every set of h_j such that $h = \sum h_j$, and every set $\{x_j\}$ such that $\delta \leqq x_1 < x_2 < x_3 < \cdots$, we have

$$G_1(x_k + h_k) - G_1(x_k) \leqq G_1\left(v + \sum_{j=0}^{k} h_j\right) - G_1\left(v + \sum_{j=0}^{k-1} h_j\right) \tag{4.6}$$

where $h_0 = 0$, and (4.6) is true by virtue of (4.4). Summing over $k = 1, 2, \cdots$ yields

$$\sum_k [G_1(x_k + h_k) - G_1(x_k)] \leqq G_1(\delta) - G_1(\delta - h). \tag{4.7}$$

Since the right side of (4.7) becomes small with h (independent of the choice of x_k or h_k), our assertion is proved.

The absolute continuity of G_1 permits us to write

$$G_1(x) = G_1(\delta) + \int_\delta^x g_1(u)\, du \quad \text{for} \quad x \geqq \delta > 0. \tag{4.8}$$

In (4.8), g_1 is a derivative of G_1; because G_1 is concave, we may take g_1 to be a monotone nonincreasing function. By taking the limit $\delta \to 0$ in (4.8), we find that this equation reduces to

$$G_1(x) = G_1(0+) + \int_0^x g_1(u)\, du. \tag{4.9}$$

We shall find that if the mean of $N(t, x)$ is finite (and under even weaker conditions), $G_1(0+) = 0$. It is then possible to deal with a density function g_1 (which is monotone for positive argument) rather than a distribution function.

Although (4.9) generalizes to G_n, $n = 1, 2, 3, \cdots$, the relation (4.4) need not hold for $n > 1$. However, if we define

$$S_n(x) = \sum_{k=1}^{n} G_k(x) \tag{4.10}$$

we shall find that for $x_1 \leqq x_2$ and any $n = 1, 2, \cdots$

$$S_n(x_2 + h) - S_n(x_2) \leqq S_n(x_1 + h) - S_n(x_1), \qquad h \geqq 0. \quad (4.11)$$

To prove (4.11), we observe that

$$P[x < L_m(t) \leqq x + h] = P\left[\bigcup_{k=0}^{m-1} \{A_k(t, x) \cap E_{m-k}(t + x, h)\}\right]. \quad (4.12)$$

The A_k are disjoint, so that (4.12) implies

$$G_m(x + h) - G_m(x) = \sum_{k=0}^{m-1} P[A_k(t, x) \cap E_{m-k}(t + x, h)]. \quad (4.13)$$

Both sides of (4.13) are summed on m over the first n integers, and the (finite) summations interchanged; there results

$$S_n(x + h) - S_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} P[A_{j-k}(t, x) \cap E_k(t + x, h)]. \quad (4.14)$$

On the other hand, it may be seen that for any $x_1 \leqq x_2$ and $h \geqq 0$

$$\begin{aligned} &\bigcup_{j=k}^{n} [A_{j-k}(t, x_2) \cap E_k(t + x_2, h)] \\ &\qquad \subset \bigcup_{j=k}^{n} [A_{j-k}(t + x_2 - x_1, x_1) \cap E_k(t + x_2, h)] \end{aligned} \quad (4.15)$$

so that, applying (4.14) with x replaced respectively by x_1 and x_2, we have (4.11).

Since (4.11) corresponds to (4.4) in every respect, the reasoning leading to (4.9) is equally applicable to the S_n. Thus for each $n = 1, 2, \cdots$ and positive x

$$S_n(x) = S_n(0+) + \int_0^x s_n(u)\, du \quad (4.16)$$

where s_n is a derivative of S_n, and s_n may be taken to be monotone non-increasing. We remark that if the mean of $N(t, x)$ is finite, $S_n(0+) = 0$ for each n, so that S_n is everywhere differentiable.

Further, we may conclude from $G_n = S_n - S_{n-1}$ in combination with (4.16) that each G_n is absolutely continuous (except perhaps at the origin) with

$$G_n(x) = G_n(0+) + \int_0^x g_n(u)\, du \quad (4.17)$$

for positive x. Again, if $N(t, x)$ has finite mean for any positive x, $G_n(0+) = 0$ for each n; furthermore, each s_n is bounded, so that $g_n = s_n - s_{n-1}$ is both bounded and of bounded variation. This means that the forward recurrence times are described by a rather simple probability density function.

V. MOMENTS OF THE NUMBER OF POINTS IN A TIME INTERVAL

The number $N(t, x)$ of points in a given time interval $(t, t + x]$ is of equal interest with recurrence times, and deserves particular attention. As will be seen in the following, the distribution functions G_n turn out to provide ideal tools for the study of the moments of $N(t, x)$. We shall state some of the results more pertinent to applications; since detailed proofs are often delicate and/or tedious (see Beutler and Leneman, 1966), we shall normally provide only outlines of proofs.

Since $E\{[N(t, x)]^k\} = \sum_{n=1}^{\infty} n^k p(n, x)$, we obtain by a substitution from (2.12) (compare Takacs, 1960, Eq. 3.1(3) for renewal processes):

THEOREM 5.1.

$$E\{[N(t, x)]^k\} = \sum_{n=1}^{\infty} [n^k - (n - 1)^k] G_n(x), \tag{5.1}$$

the two sides of (5.1) *being finite or infinite together.*

In the statement of Theorem 5.1, as elsewhere, we accept $+\infty$ as the limit of a sum of positive terms divergent in the usual sense. By using the Minkowski inequality, we prove

THEOREM 5.2. *If* $E\{[N(t, y)]^k\} < \infty$ *for some* $y > 0$, $E\{[N(t, x)]^k\} < \infty$ *for every positive finite* x.

COROLLARY 5.2.1. *If* $\sum_{n=1}^{\infty} [n^k - (n - 1)^k] G_n(x)$ *converges to a finite limit for some* $x > 0$, *it converges for every* x, *uniformly in each interval* $[0, x_0]$.

The corollary combines Theorems 5.1 and 5.2, and the uniformity of convergence follows from the nondecreasing character of the nonnegative summands. The same two theorems also yield

COROLLARY 5.2.2. $N(t, x)$ *has finite moments of all orders, i.e.,* $E\{[N(t, x)]^k\} < \infty$ *for all* $k = 1, 2, \cdots$ *if and only if*

$$\lim_{n\to\infty} n^j G_n(y) = 0 \tag{5.2}$$

for some $y > 0$ *and each* $j = 1, 2, \cdots$.

For the first moment of $N(t, x)$, there are certain special relationships; indeed, we can specify the precise form of the mean as a function of x.

THEOREM 5.3. *$N(t, x)$ has a finite first moment if and only if*

$$\sum_{n=1}^{\infty} \lim_{h \to 0+} [G_n(h)/h] < \infty; \tag{5.3}$$

if either is finite, we have

$$E[N(t, x)] = \beta x \tag{5.4}$$

where β is a finite constant. Moreover,

$$G_n(0+) = 0, \qquad n = 1, 2, \cdots, \tag{5.5}$$

$$\sum_{n=1}^{\infty} g_n(x) = \beta, \tag{5.6}$$

and

$$\lim_{h \to 0+} \sum_{n=1}^{\infty} [G_n(h)/h] = \sum_{n=1}^{\infty} \lim_{h \to 0+} [G_n(h)/h] = \beta. \tag{5.7}$$

The key to the proof of the theorem lies in the equality $N(t, x + y) = N(t, x) + N(t + x, y)$; if expectations are taken of both sides, a functional equation of the form $f(x + y) = f(x) + f(y)$ is obtained, with $f(x) = E[N(t, x)]$. Thus (5.4) is proved. The other results stem from Theorem 5.1 with $k = 1$, and interchanges of limits and summations validated by the nondecreasing character of each of the positive summands as $h \to 0+$. Theorem 5.3 is also intuitively meaningful in that β is the average number of points per unit time.

VI. INTERVAL STATISTICS, RECURRENCE TIMES, AND MOMENTS

In renewal theory, most results are expressed as formulas involving interval statistics, in particular F_n, $n = 1, 2, \cdots$, where F_n is the distribution function for the length of n successive intervals. Since the intervals of an s.p.p. need not be identically distributed, it is not possible to find distribution functions to which we can impute the same meaning, nor can we take advantage of mutual independence of intervals, such as is assumed for renewal processes. However, by formulas similar to those applicable to renewal processes (cf. Takacs, 1960, p. 50), we are able to find distribution functions F_n that refer in a rough sense to the average of n successive interval lengths. In case the s.p.p. has mutually independent identically distributed intervals (like a renewal process), the F_n have the same interpretation as they do for a renewal process.

We shall assume throughout this section that the first moment of $N(t, x)$ exists with parameter β. Then we take $F_n(x) = 0$ for $x < 0$, and for $x \geqq 0$

$$F_n(x) = 1 - \beta^{-1} s_n(x), \tag{6.1}$$

where the s_n have been defined in Section IV. It is easy to show (since the s_n are nonincreasing and tend toward zero) that each F_n is a distribution function. Moreover, we obtain from an integration by parts

THEOREM 6.1.

$$\int_0^\infty x\, dF_n(x) = n/\beta. \tag{6.2}$$

Note that (6.2) reflects our intuitive interpretation of the F_n, in the sense that the average length of n successive intervals should be n times the average length of an interval. In turn, the average length of an interval should be β^{-1}, the reciprocal of the average number of points per unit time. In renewal theory $\sum_{n=1}^{\infty} F_n(x) = H(x)$ is called the renewal function, and is related to the moments of the number of points in $(0, x]$ (see Cox, 1962, Secs. 4.1 and 4.6). Similar relations are found in the study of s.p.p. Without proof, we state

THEOREM 6.1. *$E\{[N(t, x)]^2\} < \infty$ if and only if $H(x) < \infty$, and then*

$$E\{[N(t, x)]^2\} = \beta x + 2\beta \int_0^x H(u)\, du. \tag{6.3}$$

This theorem is stated and proved (Beutler and Leneman, 1966) in more general form applicable to moments of arbitrary order. One is able to conclude from the integral form of the result that $E\{[N(t, x)]^k\}$ is an absolutely continuous function of x, with nondecreasing derivative, and that $E\{[N(t, x)]^k\} = 0(x)$ as $x \to 0$. For s.p.p. with finite second moments, there are additional results useful in computing spectra of randomly sampled signals. We have

THEOREM 6.2. *Let $E\{[N(t, x)]^2\} < \infty$, and take $s \leqq s + x \leqq t \leqq t + y$. Then*

$$E[N(s, x)N(t, y)] = \beta \int_0^y [H(u + t - s) - H(u + t - s - x)]\, du. \tag{6.4}$$

Formulas such as (6.4) are of use in obtaining the second moment properties of signals sampled by a train of delta functions or very narrow pulses.

VII. EXAMPLES AND CLASSES OF STATIONARY POINT PROCESSES

Several examples of s.p.p. are presented in this section. Proofs of their stationarity is given elsewhere (Beutler and Leneman, 1966), using whichever of the equivalent stationarity criteria of Section III is most easily applied to each particular process. In this paper, our principal concern is with the computation of the F_n and G_n .

It would be expected that there is a Poisson s.p.p. corresponding to the renewal process of the same type. The simplest and most appealing construction—all τ_k exponentially distributed and mutually independent —unfortunately fails to satisfy the stationarity criteria.[3] Indeed, it can be shown that if the τ_k are mutually independent, and all τ_k , $k \neq 0$, are exponentially distributed, there is no distribution of τ_0 leading to an s.p.p. However, the process we shall describe has all the properties demanded of a Poisson process, at least insofar as required for random sampling. Let the τ_k , $k \neq -1$ be mutually independent and exponentially distributed, i.e., their probability densities are given by

$$f_1(x) = \beta e^{-\beta x}, \tag{7.1}$$

and let τ_{-1} be specified by

$$\tau_{-1} = \tau_0 + \tau \tag{7.2}$$

where τ is independent of the τ_k , $k \neq -1$, and has the same probability density (7.1). It is easily shown that this point process satisfies (3.1) over disjoint intervals and is therefore an s.p.p.; furthermore, $N(t, x)$ has a Poisson distribution with parameter βx, and the number of points on disjoint intervals are mutually independent random variables. From the above, one obtains

$$g_n(x) = \beta(\beta x)^{n-1} e^{-\beta x}/(n - 1)! \tag{7.3}$$

and thus by (5.6) and (5.4) of Theorem 5.3

$$E[N(t, x)] = \beta x. \tag{7.4}$$

By a slightly more involved calculation, based on Theorem 5.1, we obtain another result classical for the Poisson renewal process, namely,

[3] The notation and the relation between the τ_n and the random point process $\{t_n\}$ are those of Definition 2.1.

$$E\{[N(t, x)]^2\} = \beta x + (\beta x)^2. \tag{7.5}$$

Moments of all orders could be computed from (5.1), but we shall not do this, remarking only that it is immediately apparent from (5.2) and the exponential character of G_n that all moments are finite.

In spectral calculation on randomly sampled stochastic processes, the generating function for $L_n(t)$ is often used. This function is defined as

$$g_n^*(s) = E[e^{-sL_n(t)}]. \tag{7.6}$$

In the case of the Poisson s.p.p. just discussed, we find that $g_n^*(s) = [\beta/(\beta + s)]^n$.

The periodic s.p.p. arises in situations which manifest a periodically repeated pattern of points, the intervals between successive points being determinate. The simplest case (simple periodicity) has $\tau_k = T, k \neq 0$, in which T is the period, with τ_0 being uniformly distributed over $(0, T]$. This s.p.p. is easily generalized to the following: let $\tau_1, \tau_2, \cdots, \tau_N$ be positive numbers such that

$$\sum_1^N \tau_k = T, \tag{7.7}$$

and for general positive $j = mN + i$, $\tau_j = \tau_i$. For negative indices, the τ_j are defined to continue the spacing pattern. Finally, τ_0 is (again) uniformly distributed on $(0, T]$.

The process just described can be shown to be an s.p.p. by verifying (3.4), a rather tedious procedure carried out elsewhere (Beutler and Leneman, 1966). An easier and more intuitive calculation, based on the fact that the probability of an arbitrary t falling in an interval of length τ_k is τ_k/T, shows that g_1 is a stepwise-constant function with $g_1(0+) = NT^{-1}$, and $g_1(x) = 0$ for $x \geqq \max_{1 \leqq k \leqq N} \tau_k$. We shall not pursue the general computation of these densities further, and only note that quite precise upper and lower bounds can be obtained. Also, $E[N(t, x)]$ is most easily computed from an ergodic theorem (see Beutler and Leneman, 1966) which yields the (not surprising) result $E[N(t, x)] = Nx/T$.

The case $N = 1$, $\tau_1 = T$, is of course, well-known. Here $g_1(x) = 1/T$ on $[0, T]$, and $g_1(x) = 0$ otherwise. For higher indices, we have $g_n(x) = g_1(x - [n - 1]T)$ from which for all $n = 1, 2, \cdots$

$$g_n^*(s) = [(1 - e^{sT})/sT]e^{-s(n-1)T}. \tag{7.8}$$

The zero crossings of an Ornstein-Uhlenbeck (symmetric stationary gaussian Markov) process do not constitute (in our definition) an s.p.p. because these crossings are not a denumerable ordered sequence. Nonetheless, our theory yields a result that is generally derived by rather lengthy procedures. It has been found by several authors (see e.g., Slepian, 1963) that

$$G_1(x) = 1 - (2/\pi) \sin^{-1} (e^{-x}). \tag{7.9}$$

Since $G_1(x)/x$ tends toward infinity as x tends toward zero, we conclude (assuming Theorem 5.3 to be applicable) immediately that $E[N(t, x)] = \infty$, replacing a much more difficult computation used elsewhere (see, e.g., Siegert, 1950).

A point process not previously analyzed thoroughly in the literature will be treated next. We call this s.p.p. the skip process, in recognition of its creation from the remaining points after points of an existing s.p.p. have been deleted at random. This corresponds to a physical model of a pulse amplitude modulated communication system, in which faulty transmission, reception, or perhaps jamming has expunged some of the sample pulses. Missing return pulses in a pulse radar system are also covered by this model. Finally, one can imagine queueing models in which some of the customers do not, after all, require service, or where defections occur from the queue.

Suppose that $\{t_n'\}$ is an s.p.p., and that $\{x_n\}$ is a discrete parameter stationary process, independent of $\{t_n'\}$, consisting of zeros and ones. Now form the new point process having a point at each time t_n' whenever $x_n = 1$, and no point at t_n' whenever $x_n = 0$. The new point process $\{t_n\}$, obtained from the remaining points, can then be shown to be an s.p.p. (Beutler and Leneman, 1966).

The simplest type of skip process is generated by mutually independent deletions; each point has probability $q < 1$ of being expunged. Then

$$G_n(x) = (1 - q)^n \sum_{m=0}^{\infty} q^m C_{n+m-1}^{n-1} G'_{n+m}(x), \tag{7.10}$$

where C_k^j is the number of ways of taking k objects j at a time, that is, $C_k^j = k!/j!(k - j)!$, and the primed quantities refer to the original s.p.p. $\{t_n'\}$. Moreover, if $\{t_n'\}$ is an s.p.p. with independent, identically distributed intervals, then $\{t_n\}$ is likewise an s.p.p. with these same properties. For such a process, it can be seen that

$$g_1^*(s) = (1 - q)\,\frac{g_1'^*(s)}{1 - qf_1'^*(s)} \tag{7.11}$$

where $f_1'^*(s)$ is the generating function for the interval τ_1' (in the process $\{t_n'\}$), that is

$$f_1'^*(s) = E[e^{-s\tau_1'}] = \int_{0-}^{\infty} e^{-su}\, dF_1'(u). \tag{7.12}$$

We also have

$$f_1^*(s) = (1 - q)\,\frac{f_1'^*(s)}{1 - qf_1'^*(s)} \tag{7.13}$$

where $f_1^*(s)$ denotes the generating function for the interval τ_1 of the skip process $\{t_n\}$. From there, $g_n^*(s)$ is readily computed, in view of the relation

$$g_n^*(s) = g_1^*(s)[f_1^*(s)]^{n-1}. \tag{7.14}$$

More generally, if we define (whether or not the intervals are independent and/or identically distributed) $f_1^* = \int_{0-}^{\infty} e^{-su}\, dF_1(u)$, and if $f_n^* = (f_1^*)^n$, then (7.14) holds. Conversely, (7.14) implies that $f_n^* = (f_1^*)^n$. For instance, these relationships hold for the Poisson s.p.p., even though the intervals (e.g., τ_{-1}) are not identically distributed.

We specialize the above to a simple periodic process $\{t_n'\}$ of period T with independent probabilities q of skipping any one point. Because $f_1'^*(s) = e^{-sT}$ and (correspondingly) $g_1'^*(s) = (1 - e^{-sT})/sT$, it follows that

$$g_1^*(s) = (1 - q)(1 - e^{-sT})/[sT(1 - qe^{-sT})] \tag{7.15}$$

and

$$f_1^*(s) = [(1 - q)e^{-sT}]/(1 - qe^{-sT}). \tag{7.16}$$

A similar problem is that of a Poisson s.p.p. $\{t_n'\}$ with parameter β' and independent probabilities q of skipping. The skip process is then another Poisson s.p.p., but with parameter $\beta = (1 - q)\beta'$; this is similar to a classical result (e.g., Parzen, 1962).

As we have already stated, the jitter process has been studied (Balakrishnan, 1962; Brown, 1963), not only because it is amenable to simple analysis, but also because it represents the time base for nominally equally spaced repetitive signals (e.g., sampling pulse trains), perturbed by the small timing errors one might find in actual sampling systems. Accordingly, we shall suppose that $\{t_n'\}$ is a periodic s.p.p. whose period is T, and that $\{u_n\}$ is a discrete parameter stationary process with $0 \leqq u_n < T$ for each n.[4] Then u_n are of course the perturbations on the periodic points t_n'. The jittered (or nearly-periodic) process $\{t_n\}$ is then specified by $t_n = t_n' + u_n$, from which the jittered interval lengths are $\tau_n = T + (u_n - u_{n-1})$ for $n \neq 0$. That the new process $\{t_n\}$ is also an s.p.p. is proved by verifying that (3.4) holds (see Beutler and Leneman (1966) for this proof).

In this paper we shall obtain explicit results only for jitters u_n that are pairwise independent. We shall call

$$\gamma(s) = E[\exp(-su_k)], \tag{7.17}$$

which is the same for any k. Because of the independence assumed for the u_n, $E\{\exp[-s(u_k - u_j)]\} = \gamma(s)\gamma(-s)$ whenever $j \neq k$. If now $f_n^*(s)$ denotes the generating function for $\sum_{k=1}^{n} \tau_k$, $n \geqq 1$, that is

$$f_n^*(s) = E\left[\exp\left(-s\sum_{k=1}^{n}\tau_k\right)\right],$$

we find that

$$f_n^*(s) = e^{-nsT}\gamma(s)\gamma(-s). \tag{7.18}$$

For this s.p.p., $\beta = T^{-1}$ by a direct calculation. Using $\beta = T^{-1}$ in (6.1), and subtracting successive equations to obtain g_n, one has from (7.18)

$$g_1^*(s) = [1 - \gamma(s)\gamma(-s)e^{-sT}]/sT$$

and

$$g_n^*(s) = \gamma(s)\gamma(-s)e^{-nsT}(e^{sT} - 1)/sT, \qquad n \geqq 2. \tag{7.19}$$

As another example, we combine the jittered process with skip sampling; we recall here that the application of skip sampling to any s.p.p. generates a new s.p.p. The jitter process just discussed will be considered here. There is an independent probability q of the elimination of each point of the jittered process. The skip-jittered process (or nearly-periodic process with skips) will then have identically distributed intervals, and the generating function for the sum of intervals $\sum_{k=1}^{n} \tau_k$ becomes

[4] Since the purpose of the restriction on the u_n is to maintain the ordering of the t_n the same as that of the t_n', the u_n may be restricted to any desired interval of length T.

$$f_n^*(s) = \gamma(s)\gamma(-s)[(1 - q)e^{-sT}]^n/[1 - qe^{-sT}]^n. \tag{7.20}$$

This result is most readily computed by observing that (7.10) holds with G_n replaced by F_n, and G'_{n+m} replaced by F'_{n+m}. If we now take the Laplace transform of both sides, and substitute from (7.19), the infinite sum can be evaluated in closed form, thus leading to (7.20). By means of (6.1) and the knowledge that $\beta = (1 - q)/T$, we can obtain an expression for each g_n^* in terms of f_n^* and f_{n+1}^*. Hence, (7.20) permits us to calculate each of the g_n as another new result.

For our final example, we consider the scheduled skip process. In its simplest form, every alternate point is skipped by our choice of $x_n = 0.5[1 + (-1)^n X]$, where X is independent of $\{t_n'\}$, and takes on values of $+1$ or -1, each with probability one-half. If this form of skipping is applied to a Poisson point process, or one with identically distributed independent intervals, we have $f_n^* = f_{2n}'^* = (f_1'^*)^{2n}$, so that again $g_n^* = g_1^*(f_1^*)^{n-1}$ for the new process. In particular, the Poisson process with scheduled skips (sometimes called the alternate Poisson point process) has

$$f_n^*(s) = [\beta'/(s + \beta')]^{2n}. \tag{7.21}$$

Now $g_n^* = g_1^*(f_{n-1}^*)$, where we may compute g_1^* from (6.1) and (7.21), with $\beta = 0.5\beta'$. The result is then

$$g_n^*(s) = 0.5(\beta')^{2n-1}(s + 2\beta')/(s + \beta')^{2n}. \tag{7.22}$$

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