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Random scaling and sampling of Brownian motion

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Dedicated to the memory of Professor Kiyosi Itô

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Abstract. In this paper, we provide a survey of recent distributional results obtained for Brownian type processes observed up to some random times. We focus on the case of hitting times and inverse local times and consider the situation where the processes are randomly sampled through a uniform random variable. We present various explicit formulas, some of them being quite remarkable.

1. Introduction.

Let $(B_t,\ t \geq 0)$ denote a one-dimensional standard Brownian motion, starting from 0. The scaling property of B states that for all $c > 0$,

$$\left(\frac{1}{\sqrt{c}} B_{ct},\ t \geq 0\right) \underset{\mathcal{L}}{=} (B_t,\ t \geq 0),$$

where we write $\underset{\mathcal{L}}{=}$ for equality in law. This result invites to consider more generally transformations of B by random scaling, that is processes $(\widetilde{B}_t,\ t \leq 1)$ of the form

$$(\widetilde{B}_t,\ t \leq 1) = \left(\frac{1}{\sqrt{T}} B_{tT},\ t \leq 1\right),$$

for some random times T, and to compare the law of $\widetilde{B}$ with that of $(B_t,\ t \leq 1)$, or some related processes. In Section 2, we identify the law of $\widetilde{B}$ when

$$T = T_a = \inf\{t :\ B_t = a\}$$

and

$$T = \tau_l = \inf\{t :\ L_t > l\},$$

where (L_t) denotes the local time process at 0 of B.

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In general, it is not obvious to obtain such an identification result in a reasonable form. Thus, as a first "estimation" or "approximation", one may look for the law of

$$\frac{1}{\sqrt{T}} B_{UT},$$

for some random variable U, independent of B and uniform on $[0,1]$. We present those laws in Section 3 for $T = T_a$ and $T = \tau_l$. They are in fact rather remarkable.

In Section 4, we show that the results from Section 2 together with those from Section 3 enable to derive several unexpected and simple formulas for various quantities related to some very classical Brownian type processes, namely the Brownian bridge, the Brownian meander and the three-dimensional Bessel process. More precisely, we investigate the laws of these processes when sampled with an independent uniform random variable. The results of this section are somehow in the spirit of those given by Pitman in [**8**], where some distributional properties of these processes sampled with (several) independent uniform random variables are studied. Actually, some alternative proofs can be obtained using the results in that reference.

In Section 5, we consider some natural random variables associated to $\widetilde{B}_U = (1/\sqrt{T})B_{UT}$, namely

$$X = \frac{1}{T^{3/2}} \int_0^T ds B_s \quad \text{and} \quad Y = \frac{1}{T^{3/2}} \int_0^T ds |B_s|.$$

Indeed, these two variables are the expectations of $\widetilde{B}_U$ and of its absolute value conditional on B. Of course the law of $\widetilde{B}_U$ enables to compute the expectations of X and Y. However, obtaining higher moments turns out to be quite intricate in general. We derive the moments of order two of X and Y in the case $T = \tau_l$ in Section 5. To do so, we use a bang-bang process version of the second Ray–Knight theorem.

Apart from the new computations in Section 5, the nature of this paper is essentially that of a survey, mainly based on the papers [**5**], [**10**], [**11**], to which we refer for more information.

2. On the law of $(\widetilde{B}_t,\ t \le 1)$ in two cases.

We recall in Theorem 2.1 some absolute continuity results enabling to describe the law of $(\widetilde{B}_t,\ t \le 1)$ in the cases $T = T_a$ and $T = \tau_l$.

THEOREM 2.1. *Let $F : \mathbb{C}([0,1],\mathbb{R}) \to \mathbb{R}_+$ be a measurable functional. Then the following identities hold*:

$$\mathbb{E}\left[F\left(\frac{B_{uT_a}}{\sqrt{T_a}},\ u \le 1\right)\right] = \mathbb{E}\left[F(R_1 - R_{1-u},\ u \le 1)\frac{1}{R_1^2}\right], \tag{1}$$

where R denotes a three-dimensional Bessel process starting from 0;

$$\mathbb{E}\left[F\left(\frac{B_{u\tau_l}}{\sqrt{\tau_l}},\ u \le 1\right)\right] = \sqrt{\frac{2}{\pi}}\mathbb{E}\left[F\big(b(u),\ u \le 1\big)\frac{1}{\lambda_1^0}\right], \tag{2}$$

where $(\lambda_u^0,\ u \le 1)$ *denotes the local time at* 0 *of the standard Brownian bridge* $(b(u),\ u \le 1)$.

We note that (in fact simply by Brownian scaling), the quantities on the left hand side of (1), resp. (2), do not depend on a, resp. l. Therefore, to make the discussion simpler, we shall take $a = 1$ and $l = 1$ from now on. Both results (1) and (2) are presented, somewhat independently from each other, in Biane, Le Gall and Yor, see [**3**], where the emphasis is rather put on (2), and the process

$$\left(\frac{B_{u\tau_1}}{\sqrt{\tau_1}},\ u \le 1\right)$$

is called pseudo-Brownian bridge.

Here we intend to connect (1) and (2), or more precisely to show how (1) may be obtained from (2). Thanks to Lévy's construction of reflected Brownian motion from Brownian motion, there is the identity:

$$\mathbb{E}\left[F\left(\frac{L_{u\tau_1} - |B_{u\tau_1}|}{\sqrt{\tau_1}},\ u \le 1\right)\right] = \mathbb{E}\left[F\left(\frac{B_{uT_1}}{\sqrt{T_1}},\ u \le 1\right)\right]. \tag{3}$$

From (2), the left hand side of (3) is equal to

$$\sqrt{\frac{2}{\pi}}\mathbb{E}\left[F\big(\lambda_u^0 - |b(u)|,\ u \le 1\big)\frac{1}{\lambda_1^0}\right].$$

Since the law of the standard Brownian bridge is invariant by time reversal, this can be written

$$\sqrt{\frac{2}{\pi}}\mathbb{E}\left[F\big(\lambda_1^0 - (\lambda_{1-u}^0 - |b(1-u)|),\ u \le 1\big)\frac{1}{\lambda_1^0}\right].$$

Now, Biane and Yor have shown in [**4**] that

$$(\lambda_v^0 + |b(v)|,\ v \le 1)$$

is distributed as a standard Brownian meander $(m(v),\ v \le 1)$. Thus, the left hand side of (3) is equal to

$$\sqrt{\frac{2}{\pi}}\mathbb{E}\left[F\big(m(1) - m(1-u),\ u \le 1\big)\frac{1}{m(1)}\right].$$

We also recall Imhof's relation, see [**3**], [**7**]:

$$\mathbb{E}\big[F\big(m(u),\ u\leq 1\big)\big]=\sqrt{\frac{2}{\pi}}\mathbb{E}\Big[F\big(R_u,\ u\leq 1\big)\frac{1}{R_1}\Big]. \tag{4}$$

Therefore, the left hand side of (3) is also equal to

$$\mathbb{E}\Big[F(R_1-R_{1-u},\ u\leq 1)\frac{1}{R_1^2}\Big],$$

thus obtaining (1).

Note that, in the above discussion, we have shown the deep links which exist between (1), (2), (3) and (4). We will furthermore explore these connections in Section 4, in the random sampling context.

3. On the law of $\widetilde{B}_U = B_{UT}/\sqrt{T}$ for $T = T_1$ and $T = \tau_1$.

We now give some results related to a uniform and independent sampling of the randomly scaled Brownian motion.

3.1. A result on the pseudo-Brownian bridge.

Our main result in [**11**] is the following identity in law for the pseudo-Brownian bridge.

Theorem 3.1. *We have*

$$\Big(\frac{B_{U\tau_1}}{\sqrt{\tau_1}},\frac{1}{\sqrt{\tau_1}},L_{U\tau_1}\Big)\underset{\mathcal{L}}{=}\Big(\frac{1}{2}B_1,L_1,\Lambda\Big),$$

with Λ *a uniform random variable on* $[0,1]$*, independent of* (B_1,L_1)*.*

It is well-known that the law of (B_1,L_1) admits a density on $\mathbb{R}\times\mathbb{R}^+$ with value at point (x,l) given by

$$\frac{1}{\sqrt{2\pi}}(|x|+l)\exp\Big(-\frac{(l+|x|)^2}{2}\Big).$$

From this expression, we deduce for fixed time $s\geq 0$ the useful factorization:

$$(|B_s|,L_s)\underset{\mathcal{L}}{=}R_s(1-U,U).$$

Using elementary computations, this last expression enables to derive from Theorem 3.1 some interesting and simple formulas related to the joint law of $(B_{U\tau_1},\tau_1)$. For example, we have the following corollary.

Corollary 3.1. *The triplet*

$$\Big(|B_{U\tau_1}|,L_{U\tau_1},\frac{1+2|B_{U\tau_1}|}{\sqrt{\tau_1}}\Big)$$

has the law of the triplet of independent variables

$$\left(\frac{1}{2}\left(\frac{1}{U}-1\right),\Lambda,R_1\right).$$

Note that some further interesting variables can be added in the joint law presented in Theorem 3.1 using results in [**9**], see also [**2**].

3.2. The case $T = T_1$.

Again with the help of Lévy's equivalence theorem, we may obtain a version of Theorem 3.1 in which we replace τ_1 by T_1. Let

$$M_u = \sup_{s\le u} B_s.$$

There is the following identity in law.

THEOREM 3.2. *We have*

$$\left(\frac{B_{UT_1}}{\sqrt{T_1}},\frac{1}{\sqrt{T_1}},M_{UT_1}\right)\underset{\mathcal{L}}{=}\left(\Lambda L_1-\frac{1}{2}|B_1|,L_1,\Lambda\right),$$

with Λ *a uniform random variable on* $[0,1]$*, independent of* (B_1, L_1).

We note in particular that the random variable α defined by

$$\alpha = \frac{B_{UT_1}}{\sqrt{T_1}}$$

satisfies

$$\mathbb{E}[\alpha] = 0.$$

Indeed,

$$\mathbb{E}\left[\Lambda L_1-\frac{1}{2}|B_1|\right]=\left(\frac{1}{2}-\frac{1}{2}\right)\mathbb{E}[|B_1|]=0.$$

This centering property of α intrigued us and led us to look for the law itself of α. This law, established in [**5**], is quite remarkable, as the following statement shows.

THEOREM 3.3. *The law of* α *admits a density* h *which satisfies for* $y \ge 0$

$$h(y)=\sqrt{\frac{2}{\pi}}\int_0^2\frac{dw}{1+w}\exp\left(-\frac{2y^2}{w^2}\right)$$

and for $y \le 0$

$$h(y) = \sqrt{\frac{2}{\pi}}\log(3)\exp(-2y^2).$$

Hence, conditional on $(\alpha > 0)$, the law of α^+ is a mixture of absolute Gaussian laws, whereas conditional on $(\alpha < 0)$, α^- is distributed as the absolute value of a Gaussian random variable. Note in particular the continuity of h.

3.3. Some elements of proof.

The proof of Theorem 3.1 was obtained in [**11**] by computing for every $a > 0$, $c > 0$ and $l \leq 1$

$$\mathbb{E}\Big[\Big|\frac{B_{U\tau_1}}{\sqrt{\tau_1}}\Big|^a\Big|\frac{1}{\sqrt{\tau_1}}\Big|^c 1_{\{L_{U\tau_1}\leq l\}}\Big],$$

and showing that it is equal to the desired quantity obtained from the right hand side of the expression in Theorem 3.1. Rather than reproducing precisely this proof, let us simply present the kind of computations which is needed to complete such proof. Let ϕ be a bounded measurable function and $\lambda > 0$. One of the key points in the proof of Theorem 3.1 in [**11**] is to be able to obtain semi-explicit expressions for quantities such as

$$I = \mathbb{E}\Big[\int_0^{\tau_1} ds\phi(|B_s|)\exp\Big(-\frac{\lambda^2}{2}\tau_1\Big)\Big].$$

This can be done as follows. The process

$$\exp\Big(\lambda(L_s - |B_s|) - \frac{\lambda^2}{2}s\Big)$$

is a martingale. Thus, we get

$$\begin{aligned}
I &= \exp(-\lambda)\mathbb{E}\Big[\int_0^\infty ds\phi(|B_s|)1_{\{L_s<1\}}\exp\Big(\lambda(L_s - |B_s|) - \frac{\lambda^2 s}{2}\Big)\Big]\\
&= \exp(-\lambda)\int_0^\infty ds\sqrt{\frac{2}{\pi s^3}}\int_0^\infty dx\phi(x)\int_0^1 dl(x+l)\mathrm{e}^{-(x+l)^2/2s}\mathrm{e}^{\lambda(l-x)}\mathrm{e}^{-\lambda^2 s/2}\\
&= 2\exp(-\lambda)\int_0^\infty dx\phi(x)\int_0^1 dl\mathrm{e}^{\lambda(l-x)}\mathrm{e}^{-\lambda(x+l)}\\
&= 2\int_0^\infty dx\phi(x)\exp\big(-\lambda(2x+1)\big).
\end{aligned}$$

Going back to the definition of I, and considering the symmetry of the Brownian distribution, we derive in particular from the previous computations that for $\lambda \geq 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}\left[L^x_{\tau_1} \exp\left(-\frac{\lambda^2}{2}\tau_1\right)\right] = \exp\big(-\lambda(2|x|+1)\big),$$

with $L^x_{\tau_1}$ the local time at point x at time τ_1. This might also be obtained from the Ray–Knight theorem stating that $(L^x_{\tau_1},\ x \geq 0)$ and $(L^{-x}_{\tau_1},\ x \geq 0)$ are two independent BESQ(0) processes starting from 1.

4. The Brownian bridge, Brownian meander and Bessel process under uniform sampling.

Theorem 3.1 and its corollaries can be used together with the absolute continuity relations mentioned in Section 2 in order to obtain some quite explicit formulas for the distributions of the Brownian bridge, Brownian meander and Bessel process under uniform sampling. We give some examples below, see [**10**] for more details. The following result holds for the Brownian bridge.

THEOREM 4.1. *For any non negative measurable functions f and g, we have*

$$\mathbb{E}\left[f\big(b(U), \lambda^0_1\big) g\left(\frac{\lambda^0_U}{\lambda^0_1}\right)\right] = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[f\left(\frac{1}{2}B_1, L_1\right)L_1\right]\mathbb{E}[g(\Lambda)], \tag{5}$$

with Λ a uniform random variable on $[0,1]$, independent of (B_1, L_1).

Thus, λ^0_U/λ^0_1 is a uniform random variable on $[0,1]$, independent of the pair $\big(b(U), \lambda^0_1\big)$ which is distributed according to (5) with $g = 1$, see also [**1**].

Surprisingly simple expressions for some densities and (conditional) expectations of quantities related to the Brownian bridge can be obtained from Theorem 4.1. For example, we easily retrieve that λ^0_1 has the same law as $\sqrt{2\mathcal{E}}$, with $\mathcal{E}$ an exponential random variable, that is λ^0_1 is Rayleigh distributed (which is in fact a classical result, see [**3**], [**4**], [**7**]). Further results can be found in [**10**].

Let

$$i_u = \inf_{u \leq t \leq 1} m(t).$$

Theorem 4.1 can be reinterpreted as follows for the Brownian meander.

THEOREM 4.2. *For any non negative measurable functions f and g, we have*

$$\mathbb{E}\left[f\big(m(U), m(1)\big) g\left(\frac{i_U}{m(1)}\right)\right] = \sqrt{\frac{\pi}{2}}\mathbb{E}\left[f\left(\frac{1}{2}|B_1| + \Lambda L_1, L_1\right)L_1 g(\Lambda)\right],$$

with Λ a uniform random variable on $[0,1]$, independent of (B_1, L_1).

Finally, we have the following result for the three-dimensional Bessel process, where

$$J_u = \inf_{u \leq t \leq 1} R_t.$$

THEOREM 4.3. *For any non negative measurable functions f and g, we have*

$$\mathbb{E}\left[f(R_U, R_1) g\left(\frac{J_U}{R_1}\right)\right] = \mathbb{E}\left[f\left(\frac{1}{2}|B_1| + \Lambda L_1, L_1\right) L_1^2 g(\Lambda)\right],$$

with Λ a uniform random variable on $[0,1]$, independent of (B_1, L_1).

5. Computation of two second moments.

We now turn to some random variables closely related to $B_{U\tau_1}/\sqrt{\tau_1}$. Let

$$X = \frac{1}{\tau_1^{3/2}} \int_0^{\tau_1} ds B_s = \mathbb{E}\left[\frac{B_{U\tau_1}}{\sqrt{\tau_1}}\middle| B\right]$$

and

$$Y = \frac{1}{\tau_1^{3/2}} \int_0^{\tau_1} ds |B_s| = \mathbb{E}\left[\frac{|B_{U\tau_1}|}{\sqrt{\tau_1}}\middle| B\right].$$

Although we are able to compute the law of $B_{U\tau_1}/\sqrt{\tau_1}$, namely

$$\frac{B_{U\tau_1}}{\sqrt{\tau_1}} \underset{\mathcal{L}}{=} \frac{1}{2} B_1,$$

we do not know the laws of X and Y. There are the immediate inequalities:

$$\mathbb{E}[X^2] \le \mathbb{E}[Y^2] \le \mathbb{E}\left[\left(\frac{B_{U\tau_1}}{\sqrt{\tau_1}}\right)^2\right] = \frac{1}{4}$$

and

$$(\mathbb{E}[|X|])^2 \le (\mathbb{E}[|Y|])^2 = \frac{1}{4}\left(\frac{2}{\pi}\right) \le \mathbb{E}[Y^2].$$

The exact values of $\mathbb{E}[X^2]$ and $\mathbb{E}[Y^2]$ are given in the following theorem.

THEOREM 5.1. *There are the formulas*:

$$\mathbb{E}[X^2] = 2(a - b), \quad \mathbb{E}[Y^2] = 2(a + b),$$

with $2a = (1/4)(11/16)$ and $2b = (1/4)(1/16)$. Hence

$$\mathbb{E}[X^2] = \frac{1}{4}\left(\frac{5}{8}\right), \quad \mathbb{E}[Y^2] = \frac{1}{4}\left(\frac{3}{4}\right).$$

Before giving the proof of Theorem 5.1, we provide a Ray–Knight type theorem for the bang-bang process. This result will be the key point of the proof of Theorem 5.1. The bang-bang process with parameter $\lambda \in \mathbb{R}^+$ is defined as the solution of the equation:

$$X_t = \beta_t - \lambda \int_0^t \mathrm{sign}(X_s) ds, \tag{6}$$

where (β_t) is a Brownian motion, see for example [**6**] for details. We write $\tilde{L}_t^a$ the local time at point a at time t of the bang-bang process and set

$$\tilde{\tau}_l = \inf \big\{ t : \tilde{L}_t^0 > l \big\}.$$

The bang-bang process version of the second Ray–Knight theorem goes as follow.

THEOREM 5.2. *The process $(\tilde{L}_{\tilde{\tau}_l}^a,\ a \geq 0)$ is distributed as Z_a which satisfies*

$$Z_a = l + 2 \int_0^a \sqrt{Z_x} d\gamma_x - 2\lambda \int_0^a dx Z_x,$$

with γ a Brownian motion.

PROOF. To fix ideas, we take (X_t) as the coordinate process on the canonical path space. Furthermore, we denote by $\mathbb{P}$ the Wiener measure and $\mathbb{P}^\lambda$ the law of the bang-bang process. Under $\mathbb{P}^\lambda$,

$$X_t + \lambda \int_0^t \mathrm{sign}(X_s) ds$$

is a Brownian motion. Moreover, by Girsanov's theorem, we get

$$\mathbb{P}_{\mathcal{F}_t}^\lambda = \exp\bigg(- \lambda \int_0^t \mathrm{sign}(X_s) dX_s - \frac{\lambda^2}{2} t \bigg) \mathbb{P}_{\mathcal{F}_t}.$$

Then, applying Tanaka's formula, we deduce

$$\mathbb{P}_{\mathcal{F}_t}^\lambda = \exp\bigg(- \lambda(|X_t| - L_t) - \frac{\lambda^2}{2} t \bigg) \mathbb{P}_{\mathcal{F}_t},$$

and therefore

$$\mathbb{P}_{\mathcal{F}_{\tau_l}}^\lambda = \exp\bigg(\lambda l - \frac{\lambda^2}{2} \tau_l \bigg) \mathbb{P}_{\mathcal{F}_{\tau_l}}. \tag{7}$$

Also, from Ito's excursions theory, we get the independence of the positive and negative parts of the local times up to τ_l. This, together with a symmetry argument and the occupation formula give

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^{+\infty} da L_{\tau_l}^{-a}\Big)\Big] = \Big(\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\tau_l\Big)\Big]\Big)^{1/2} = \exp\Big(-\frac{\lambda l}{2}\Big).$$

Thus, using again the independence property of the local times together with (7), we get

$$\begin{aligned}
&\mathbb{E}\big[F\big(\tilde{L}_{\tilde{\tau}_l}^a,\ 0 \le a \le b\big)\big] \\
&\quad = \mathbb{E}_{\mathbb{P}^\lambda}\big[F\big(L_{\tau_l}^a,\ 0 \le a \le b\big)\big] \\
&\quad = \mathbb{E}\Big[F\big(L_{\tau_l}^a,\ 0 \le a \le b\big)\exp\Big(\lambda l - \frac{\lambda^2}{2}\int_0^{+\infty} da L_{\tau_l}^a - \frac{\lambda l}{2}\Big)\Big] \\
&\quad = \mathbb{E}\Big[F\big(L_{\tau_l}^a,\ 0 \le a \le b\big)\exp\Big(\frac{\lambda l}{2} - \frac{\lambda^2}{2}\Big(\int_0^b da L_{\tau_l}^a + \int_b^{+\infty} da L_{\tau_l}^a\Big)\Big)\Big].
\end{aligned}$$

From the usual Ray–Knight theorem, we obtain that this last quantity is equal to

$$\mathbb{E}\Big[F\big(Z_a^l,\ 0 \le a \le b\big)\exp\Big(\frac{\lambda l}{2} - \frac{\lambda^2}{2}\Big(\int_0^b da Z_a^l + \int_b^{+\infty} da Z_a^l\Big)\Big)\Big], \tag{8}$$

where (Z_a^z) is a 0-dimensional squared Bessel process, that is the unique strong solution of

$$Z_a^z = z + 2\int_0^a \sqrt{Z_x^z}\, dX_x.$$

To compute (8), we need to consider

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_b^{+\infty} da Z_a^l\Big)\Big| Z_a^l,\ 0 \le a \le b\Big],$$

which is equal to

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^{+\infty} dx Z_x^z\Big)\Big]\Big|_{z=Z_b^l},$$

by the Markov property of (Z_a^z). Since

$$Z_a^z = \beta^z_{(4\int_0^a dx Z_x^z)},$$

with (β^z) a Brownian motion starting from z, using that 0 is a trap for (Z_a^z), we have that $4\int_0^{+\infty} dx Z_x^z$ is equal to the first hitting time of 0 by β^z. Therefore we deduce

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda^2}{2}\int_0^{+\infty} dx Z_x^z\Big)\Big] = \exp\Big(-\frac{\lambda z}{2}\Big).$$

Consequently,

$$\mathbb{E}\big[F\big(\tilde{L}^a_{\tilde{\tau}_l},\ 0 \le a \le b\big)\big] = \mathbb{E}\bigg[F\big(Z^l_a,\ 0 \le a \le b\big)\exp\bigg(-\lambda\int_0^b \sqrt{Z^l_a}dX_a - \frac{\lambda^2}{2}\int_0^b daZ^l_a\bigg)\bigg].$$

Now set $\mathbb{Q}$ the probability with density

$$\exp\bigg(-\lambda\int_0^b \sqrt{Z^l_a}dX_a - \frac{\lambda^2}{2}\int_0^b daZ^l_a\bigg)$$

with respect to $\mathbb{P}$. Applying again Girsanov's theorem, we have that under $\mathbb{Q}$, the process

$$X_a + \lambda\int_0^a \sqrt{Z^l_x}dx$$

is a Brownian motion. Thus, we finally obtain

$$\mathbb{E}\big[F\big(\tilde{L}^a_{\tilde{\tau}_l},\ 0 \le a \le b\big)\big] = \mathbb{E}_{\mathbb{Q}}\big[F\big(Z^l_a,\ 0 \le a \le b\big)\big] = \mathbb{E}\big[F\big(Z_a,\ 0 \le a \le b\big)\big],$$

where

$$Z_a = l + 2\int_0^a \sqrt{Z_x}d\gamma_x - 2\lambda\int_0^a dxZ_x,$$

with γ a Brownian motion. □

We now give the proof of Theorem 5.1.

PROOF. We write

$$X = \frac{1}{\tau_1^{3/2}}\bigg(\int_0^{\tau_1} dsB^+_s - \int_0^{\tau_1} dsB^-_s\bigg),$$

and

$$Y = \frac{1}{\tau_1^{3/2}}\bigg(\int_0^{\tau_1} dsB^+_s + \int_0^{\tau_1} dsB^-_s\bigg).$$

Thanks to the symmetry properties of the Brownian motion, we obtain

$$\mathbb{E}[X^2] = 2(a-b), \quad \mathbb{E}[Y^2] = 2(a+b),$$

with

$$a = \mathbb{E}\bigg[\frac{1}{\tau_1^3}\bigg(\int_0^{\tau_1} dsB^+_s\bigg)^2\bigg], \quad b = \mathbb{E}\bigg[\frac{1}{\tau_1^3}\bigg(\int_0^{\tau_1} dsB^+_s\bigg)\bigg(\int_0^{\tau_1} dsB^-_s\bigg)\bigg].$$

We have

$$\int_0^{\tau_1} dsB_s^+ = \int_0^\infty dxxL_{\tau_1}^x, \quad \int_0^{\tau_1} dsB_s^- = \int_0^\infty dyyL_{\tau_1}^{-y}$$

and

$$\frac{1}{\tau_1^3} = \frac{1}{\Gamma(3)}\int_0^\infty dtt^2\mathrm{e}^{-t\tau_1} = \frac{1}{2}\int_0^\infty d\lambda\frac{\lambda^5}{4}\mathrm{e}^{-(\lambda^2/2)\tau_1}.$$

Consequently, with the same notation as in Theorem 5.2, using (7), we get

$$\begin{aligned} a &= \frac{1}{4}\int_0^\infty d\lambda\lambda^5\int_0^\infty dyy\int_y^\infty dxx\mathbb{E}\big[L_{\tau_1}^y L_{\tau_1}^x\mathrm{e}^{-(\lambda^2/2)\tau_1}\big] \\ &= \frac{1}{4}\int_0^\infty d\lambda\lambda^5\mathrm{e}^{-\lambda}\int_0^\infty dyy\int_y^\infty dxx\mathbb{E}[\tilde{L}_{\tilde{\tau}_1}^y\tilde{L}_{\tilde{\tau}_1}^x], \end{aligned} \tag{9}$$

and by independence and equidistribution,

$$\begin{aligned} b &= \frac{1}{8}\int_0^\infty d\lambda\lambda^5\mathbb{E}\bigg[\bigg(\int_0^\infty dyyL_{\tau_1}^y\bigg)\bigg(\int_0^\infty dxxL_{\tau_1}^{-x}\bigg)\mathrm{e}^{-(\lambda^2/2)\tau_1}\bigg] \\ &= \frac{1}{8}\int_0^\infty d\lambda\lambda^5\mathrm{e}^{-\lambda}\mathbb{E}\bigg[\bigg(\int_0^\infty dyy\tilde{L}_{\tilde{\tau}_1}^y\bigg)\bigg(\int_0^\infty dxx\tilde{L}_{\tilde{\tau}_1}^{-x}\bigg)\bigg] \\ &= \frac{1}{8}\int_0^\infty d\lambda\lambda^5\mathrm{e}^{-\lambda}\bigg(\mathbb{E}\bigg[\int_0^\infty dyy\tilde{L}_{\tilde{\tau}_1}^y\bigg]\bigg)^2. \end{aligned}$$

Solving the differential equation deduced from Theorem 5.2, we get

$$\mathbb{E}\big[\tilde{L}_{\tilde{\tau}_1}^y\big] = \exp(-2\lambda y),$$

which yields

$$b = \frac{1}{8}\int_0^\infty d\lambda\frac{\lambda^5\mathrm{e}^{-\lambda}}{(2\lambda)^4} = \frac{1}{8\times 16}.$$

It remains to compute a. Let $Z_x = \tilde{L}_{\tilde{\tau}_1}^x$. Using again Theorem 5.2, we have for $0 \le y \le x$:

$$\mathbb{E}[Z_x|Z_y] = Z_y - 2\lambda\int_y^x dz\mathbb{E}[Z_z|Z_y].$$

Hence

$$\mathbb{E}[Z_x|Z_y] = \exp\big(-2\lambda(x-y)\big)Z_y$$

and

$$\mathbb{E}[Z_y Z_x] = \mathbb{E}[Z_y^2] \exp\big(- 2\lambda(x-y)\big).$$

We now compute $\mathbb{E}[Z_y^2]$. From Theorem 5.2, we get

$$Z_y^2 = 1 + 2\int_0^y Z_x dZ_x + 4\int_0^y dx Z_x.$$

Thus,

$$\mathbb{E}[Z_y^2] = 1 - 4\lambda \int_0^y dx \mathbb{E}[Z_x^2] + 4\int_0^y dx \mathbb{E}[Z_x].$$

Now, the method of the variation of constants yields

$$\begin{aligned}\mathbb{E}[Z_y^2] &= \exp(-4\lambda y)\left(1 + 4\int_0^y dx \exp(2\lambda x)\right)\\ &= \left(1 - \frac{2}{\lambda}\right)\exp(-4\lambda y) + \frac{2}{\lambda}\exp(-2\lambda y).\end{aligned}$$

The $(dydx)$ integral in (9) is equal to

$$\begin{aligned}&\int_0^\infty dy y \mathbb{E}[Z_y^2] \int_y^\infty dx x \exp\big(- 2\lambda(x-y)\big)\\ &= \int_0^\infty dy y \mathbb{E}[Z_y^2]\left(\frac{1}{4\lambda^2} + \frac{y}{2\lambda}\right).\end{aligned}$$

Thus, it is also equal to

$$\begin{aligned}&\int_0^\infty dy y \left(\frac{1}{4\lambda^2} + \frac{y}{2\lambda}\right)\left[\left(1 - \frac{2}{\lambda}\right)\mathrm{e}^{-4\lambda y} + \frac{2}{\lambda}\mathrm{e}^{-2\lambda y}\right]\\ &= \frac{1}{4\lambda^2}\left[\left(1 - \frac{2}{\lambda}\right)\frac{1}{(4\lambda)^2} + \frac{2}{\lambda}\frac{1}{(2\lambda)^2}\right] + \frac{1}{2\lambda}\int_0^\infty dy y^2 \left[\left(1 - \frac{2}{\lambda}\right)\mathrm{e}^{-4\lambda y} + \frac{2}{\lambda}\mathrm{e}^{-2\lambda y}\right]\\ &= \left(1 - \frac{2}{\lambda}\right)\frac{1}{\lambda^4}\frac{1}{32} + \frac{3}{8\lambda^5}\\ &= \frac{1}{32\lambda^4} + \frac{5}{16}\frac{1}{\lambda^5}.\end{aligned}$$

Finally, going back to (9), we have obtained

$$
\begin{aligned}
a &= \frac{1}{4}\int_0^{\infty} d\lambda \lambda^5 \mathrm{e}^{-\lambda}\left(\frac{1}{32\lambda^4} + \frac{5}{16}\frac{1}{\lambda^5}\right) \\
&= \frac{1}{4}\left(\frac{1}{32} + \frac{5}{16}\right) \\
&= \frac{11}{8\times 16}.
\end{aligned}
$$

□

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