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Random talk: Random walk and synchronizability in a moving neighborhood network[☆]

Maurizio Porfiri[a], Daniel J. Stilwell[b,*], Erik M. Bollt[c], Joseph D. Skufca[c]

[a] Department of Mechanical, Aerospace and Manufacturing Engineering, Polytechnic University, Brooklyn, NY 11201, USA
[b] The Bradley Department of Electrical and Computer Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA
[c] Department of Mathematics and Computer Science, Clarkson University, Postdam, NY 13699-5815, USA

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Abstract

We examine the synchronization problem for a group of dynamic agents that communicate via a moving neighborhood network. Each agent is modeled as a random walker in a finite lattice and is equipped with an oscillator. The communication network topology changes randomly and is dictated by the agents' locations in the lattice. Information sharing (talking) is possible only for geographically neighboring agents. This complex system is a time-varying jump nonlinear system. We introduce the concept of 'long-time expected communication network', defined as the ergodic limit of a stochastic time-varying network. We show that if the long-time expected network supports synchronization, then so does the stochastic network when the agents diffuse sufficiently quickly in the lattice.

Keywords: Synchronization; Random walk; Stochastic stability; Graph; Fast switching

1. Introduction

Over the past few years, synchronization in complex networks has attracted a massive research attention, see the excellent reviews [1–4]. Synchronization problems can be found in a wide variety of phenomena, ranging from epidemics [5], to biological systems [6], sociology [7], chemistry [8], nonlinear optics [9], and meteorology [10].

Despite the very large literature to be found, the great majority of research activities have been focused on static networks, whose connectivity and coupling strengths are constant in time. For example, static networks are assumed for the analysis of [11–18]. However, in many real-world complex networks, such as biological, epidemiological, and social networks, it is reasonable to assume that the coupling strengths and even the network topologies can evolve over time. Recent works such as [19–25] are among the few to consider time-dependent couplings.

To our knowledge, [24] is the first research attempt, in the synchronization literature, to naturally model the time evolution of communication networks. In particular, in [24] synchronization over a network of diffusing agents communicating within geographical neighborhoods is considered. Each agent carries an oscillator and moves ergodically in the environment.[1] A communication network is formed based on the agents' motion. When two diffusing agents are close, a communication link is established, and information sharing is possible: the agents talk.

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* Corresponding address: 369 Science Center, Department of Mathematics and Computer Science, Clarkson University, Postdam, NY 13699-5815, USA. Tel.: +1 315 268 2307; fax: +1 315 268 2395.
E-mail addresses: mporfiri@poly.edu (M. Porfiri), stilwell@vt.edu (D.J. Stilwell), bolltem@clarkson.edu (E.M. Bollt), jskufca@clarkson.edu (J.D. Skufca).

1 An ergodic measure, as it is used in dynamical systems [26], is an invariant measure such that any invariant set measures either zero or one. That means that "typical" orbits are likely to explore almost everywhere. We will use the phrase ergodic in an alternative setting in this paper, for Markov chains [27], which are similarly defined in terms of a stationary measure and are invariant event which measure either zero or one. This also implies that the system recurrently visits all states.

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When the same two agents move apart, the communication link is broken, and information sharing ceases. Through numerical experiments, it is shown that the oscillators' synchronization is possible even if the communication network is mostly disconnected at any frozen time instants. It is conjectured that the synchronization of the set of oscillators can be assessed by examining a time-averaged communication network that is computed from the underlying time-varying and sparsely connected network. The model proposed in [24] seems particularly promising for modeling social interactions and the spread of epidemics.

In [25] we made a first attempt to mathematically formalize our results of [24]. However, whereas in [24], our agents moved ergodically in an underlying state space, which automatically formed time varying networks, our work in [25] simply concerned deterministic switching between a finite collection of networks representing couplings between oscillators. Network switching was not due to the agents' motion. In this paper, for the first time we are able to put the results in [24] onto a much firmer footing, using a related system, where we assume the agents are randomly walking through an underlying graph. Our results here are similar in spirit to those in [24], in that fast switching allows even for instantaneously disconnected coupling networks to synchronize, but now we are able to state our results rigorously. Some of the methods of proof we use here are very closely related to those we developed in [25]. Here we show that suitably adapting our methods of proof in [25], and adding further substantial arguments, now allows us to rigorously address the problem where the fast switching is due to stochastically moving agents, similar to what we studied empirically in [24]. Furthermore, it is well known that for some classes of dynamical systems, the so-called "Markov maps" are dense in an appropriate space of ergodic transformations [28], such as skew tent maps [29]. Consequently, we believe the treatment via Markov chains in this paper provides a significant step toward the analysis of a broad class of moving neighborhood networks.

We consider a system of N identical agents that meander in a finite region. Each agent carries an oscillator and diffuses in the environment as a random walker. Each agent is described in terms of a spatial coordinate X, specifying its random walk, and a state variable x, characterizing the oscillator's state. We assume that the random walkers are independent and that diffusion takes place in a bounded lattice described by a finite, connected, and non-bipartite graph. In addition, we assume that the random walkers move only at prescribed instants in time, equally spaced with a period Δ. The period Δ is a measure of the time-scale of the diffusion process: the smaller Δ is, the faster the agents meander in the environment.

We associate with the random walks a time-varying communication network, represented by a graph on an N-dimensional vertex set that we name the *moving neighborhood graph*. The edges of the moving neighborhood graph are determined by the agents' locations in the lattice; that is, a link between two agents is present only when the corresponding random walkers occupy the same site of the lattice. Therefore, this graph is generally not connected at frozen instants in time. The time evolution of the moving neighborhood graph, due to motion of the agents in the lattice, is called the *network dynamics*, and is independent of the states of the oscillators. When a link is present between two agents in the moving neighborhood graph, the corresponding oscillator systems are dynamically coupled. The time evolution of the set of oscillators is called the *system dynamics*, and it is influenced by the network dynamics.

The set of oscillators coupled by the moving neighborhood graph are synchronized if all their states are equal. We determine sufficient conditions for asymptotic synchronization by combining results from Markov chains, stochastic stability, and fast switching theory. We define the long-time expected communication network as the ergodic limit of the moving neighborhood graph. In the long-time average, the network behaves like an all-to-all coupling scheme among the oscillators, and the related synchronization problem may be addressed by using the well-known master stability function, see e.g. [11,12,15]. We show that if the oscillators synchronize when coupled by the all-to-all network, then synchronization is possible if the period Δ is sufficiently small.

The paper is organized as follows. In Section 2, we review some basic concepts of Markov chains and graph theory that are used in the subsequent sections. In Section 3, we formally state the synchronization problem, describe the network and system dynamics, and derive their variational equations of motion. In Section 4, we present a few propositions on the stochastic stability of time-varying jump linear systems. In Section 5, we directly address the synchronization problem and elucidate our main contribution. In Section 6, we illustrate an application of the theoretical framework to a system of agents that move on a small-world planar network [30] while carrying Rössler oscillators. Conclusions appear in Section 7.

2. Review of relevant terms

2.1. Markov chains

A sequence of discrete valued random variables $X(k)$, $k \in \mathbb{Z}^+$ with sample space F is called a *Markov Chain* if it satisfies the *Markov Condition*:

$$\begin{aligned}&\mathsf{P}(X(k+1)=s|X(k)=x_k, X(k-1)=x_{k-1}, \dots X(1)=x_1)\\&\quad=\mathsf{P}(X(k+1)=s|X(k)=x_k)\end{aligned}$$

for all $k \in \mathbb{Z}^+$ and all $s, x_1, \dots, x_k \in F$. In this paper, we restrict ourselves to *homogeneous Markov chains,* which have the additional property that

$$\begin{aligned}&\mathsf{P}(X(k+1)=j|X(k)=i)=\mathsf{P}(X(2)=s|X(1)=x),\\&\quad\forall k\in\mathbb{Z}^+,\ \text{and}\ s,x\in F.\end{aligned}$$

Additionally, we assume that F is finite and we indicate its cardinality with $|F|$. Without loss of generality, we number the possible states of the chain using positive integers, so that $F = \{1, \dots |F|\}$. (For ease of description, when we say *Markov chain* in this paper, we are considering only the restriction to this smaller set of finite, homogeneous processes.)

The matrix $P = [p_{ij}]$, where $p_{ij} = \mathsf{P}(X(k+1) = j|X(k) = i)$, is called the transition matrix of the Markov chain and is a stochastic matrix, that is, it is nonnegative and all its rows sum to one. A matrix is nonnegative (positive) if all its entries are greater than or equal to zero. Since the rows of P sum to one, the $|F|$-vector $e_{|F|} = [1, \dots 1]^{\mathrm{T}}$ is always an eigenvector of P, corresponding to an eigenvalue equal to one. The random variable $X(0)$ is called the initial state of the Markov chain and its probability distribution $\pi(0) = [\pi_1(0), \dots \pi_{|F|}(0)]^{\mathrm{T}}$, defined as:

$$\pi_i(0) = \mathsf{P}(X(0) = i)$$

is called the initial distribution. The distribution of the chain at the kth time step $\pi(k) = [\pi_1(k), \dots \pi_{|F|}(k)]^{\mathrm{T}}$ is defined as:

$$\pi_i(k) = \mathsf{P}(X(k) = i)$$

and can be expressed in terms of the initial distribution and the transition matrix by using the Chapman–Kolmogorov equation, see e.g. Theorem 1.1 in Chapter 2 of [31], as:

$$\pi(k)^{\mathrm{T}} = \pi(0)^{\mathrm{T}} P^k. \tag{1}$$

Two states i and j are said to communicate if there exists $\bar{k} \in \mathbb{Z}^+$ such that the ij and ji entries of $P^{\bar{k}}$, say $p_{ij}^{(\bar{k})}$ and $p_{ji}^{(\bar{k})}$, are positive. If all the states in F communicate, the chain is called irreducible, and P is an irreducible matrix. The period d_i of a state i is defined by $d_i = \gcd\{n \in \mathbb{Z}^+ \setminus \{0\} : p_{ii}^{(n)} > 0\}$, and if $d_i = 1$ the state i is called aperiodic. For an irreducible Markov chain, all the states have the same period, see e.g. Section 2.4 in [31]. An irreducible aperiodic Markov chain is called ergodic, as explained in footnote 1, (when F is not finite additional assumptions are needed, since an irreducible chain is not necessarily positive recurrent, see e.g. Chapter 4.2 in [31]). The transition matrix of an ergodic Markov chain is a primitive stochastic matrix, see e.g. Chapter 6 of [31]. P being primitive means that there exists $\bar{k} \in \mathbb{Z}^+$ such that $P^{\bar{k}}$ is a positive matrix. A probability distribution π is called stationary if $\pi^{\mathrm{T}} = \pi^{\mathrm{T}} P$. If all the entries of π are positive and $\pi_i p_{ij} = \pi_j p_{ji}$ for all the states i and j, the Markov chain is called reversible. The spectrum of the transition matrix of an ergodic Markov chain is $\lambda_1, \dots \lambda_{|F|}$, with $\lambda_1 = 1$ having algebraic and geometric multiplicity equal to one and with $|\lambda_r| < 1$ for $r = 2, \dots |F|$. For a reversible Markov chain, all the eigenvalues of P are real. For an ergodic Markov chain, there exists a positive constant μ and a positive constant $\rho < 1$, such that:

$$|p_{ij}^{(k)} - \pi_j| \leq \mu\rho^k \tag{2}$$

where $p_{ij}^{(k)}$ is the ij entry of P^k, and π is the unique stationary distribution of the chain, see e.g. [32]. From (2), for any initial probability distribution and state $j \in F$, the distribution at the kth time step satisfies the ergodicity condition:

$$|\pi_j(k) - \pi_j| \leq \mu\rho^k. \tag{3}$$

Inequality (3) may be easily deduced from (2). Indeed, using (1) and considering that $\sum_{i=1}^{|F|} \pi_i(0) = 1$, the LHS of (3) is:

$$|\pi_j(k) - \pi_j| = \left| \sum_{i=1}^{|F|} p_{ij}^{(k)} \pi_i(0) - \pi_j \right| = \left| \sum_{i=1}^{|F|} (p_{ij}^{(k)} - \pi_j) \pi_i(0) \right|.$$

Applying the triangle inequality and using (2) we get:

$$|\pi_j(k) - \pi_j| \leq \sum_{i=1}^{|F|} |(p_{ij}^{(k)} - \pi_j)| \pi_i(0) \leq \mu\rho^k \sum_{i=1}^{|F|} \pi_i(0) = \mu\rho^k$$

which proves (3).

2.2. Graphs

A graph is a pair of sets $G = (E, V)$, where $V = \{1, \dots |V|\}$ and $E \subseteq V \times V$ are finite, see e.g. [33]. The elements of V are called nodes, vertices or sites, and the elements of E are *unordered pairs*, and are called edges or links. Two nodes $q, r \in V$ are neighbors if there is an edge connecting them, that is, if the *unordered pair* $(q, r) \in E$. A path from q to r is a sequence of distinct vertices, starting with q and ending with r, such that all the consecutive vertices are neighbors. The graph G is connected if there exists a path between every two vertices in V. If the graph is connected, all the components of d are nonzero. The graph is called bipartite if V can be partitioned into two subsets V_1 and V_2, such that every edge has one end in V_1 and the other in V_2. A graph is said to have a self-loop at node q if $(q, q) \in E$. The graph's topology can be algebraically represented by introducing the adjacency matrix $A = [a_{qr}]$, defined by:

$$a_{qr} = \begin{cases} 1 & \text{if } (q, r) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since edges are represented by unordered pairs, we may immediately infer that A is symmetric. The degree matrix $D = \mathrm{diag}(d)$ is a diagonal matrix, whose diagonal elements are $d_q = \sum_{r=1}^{n} a_{qr}$. A graph is said to be regular if all the nodes have the same degree. The Laplacian matrix $L = [l_{qr}]$ is defined as the difference between the adjacency matrix and the degree matrix, that is $L = D - A$. The graph's Laplacian is a symmetric positive semidefinite matrix. Spectral properties of graph Laplacians may be found, for example, in [34,35].

The Laplacian matrix is a zero row-sum matrix. Therefore, the null space of L contains the $|V|$-vector $e_{|V|} = [1, \dots 1]^{\mathrm{T}}$, corresponding to the zero eigenvalue. The multiplicity of the zero eigenvalue is one if and only if the graph is connected. The highest eigenvalue is less than or equal to $\max\{d_q + d_r : (q, r) \in E\}$ (see e.g. [35]), which is less than $2|V|$.

3. Problem statement

3.1. Moving neighborhood network

Let the network dynamics be described by a set of N independent random walkers $X_1, \dots X_N$ on a finite, non-bipartite connected graph: $G^{\mathrm{rw}} = (E^{\mathrm{rw}}, V^{\mathrm{rw}})$, see e.g. [36]. (We use the superscript $(\cdot)^{\mathrm{rw}}$ to denote that these objects are associated with the lattice graph that determines where the random walkers can move.) Each random walk represents the motion of an agent in the system. The initial probability distribution of each random walker X_q, $q = 1, \ldots, N$, is $\pi_q(0) = [\pi_{q1}(0), \ldots \pi_{q|V^{\mathrm{rw}}|}(0)]^{\mathrm{T}}$.

Consider a random walker X_q on G^{rw}. If at the kth time step, $k \in \mathbb{Z}^+$, X_q is located at the site i, that is $X_q(k) = i$, we allow the walker to move to any of its neighboring sites with equal probability, so that:

$$p_{ij} = \begin{cases} 1/d_i^{\mathrm{rw}} & (i, j) \in E^{\mathrm{rw}} \\ 0 & \text{otherwise.} \end{cases}$$

The sequence of positions $X_q(k)$ is a Markov chain on the set V^{rw}, with transition matrix $P = [p_{ij}]$. Since the graph is connected and not bipartite, the Markov chain is ergodic, see e.g. [36]. The stationary probability distribution is:

$$\pi_i = \frac{d_i^{\mathrm{rw}}}{\sum_{j=1}^{|V^{\mathrm{rw}}|} d_j^{\mathrm{rw}}} \tag{4}$$

and the chain is reversible. We remark that our analysis does not require that the transition probabilities of (4) be uniform, other than the need to recompute π_i in (4). Allowing arbitrary transition probabilities will allow us to discuss agents which move according to a Markov dynamical system [37] rather than random diffusion, which we focus on here.

The random walkers move independently from each other, and share the same transition matrix P, and stationary distribution π. From (3), for any initial probability distribution and site $j \in V^{\mathrm{rw}}$, the distribution at the kth time step satisfies (3). If the graph G^{rw} is regular, then from (4) $\pi_i = 1/|V^{\mathrm{rw}}|$.

Fig. 1 shows the dynamics of five random walkers ($N = 5$) meandering on a small-world planar graph G^{rw}, with ten sites ($|V^{\mathrm{rw}}| = 10$) for the first six time intervals ($k = 0, \ldots, 5$). The random walkers are indicated by gray balls, and the sites of the graph are shown as black dots. If multiple agents occupy the same site at a point in time, the condition is indicated by multiple stacked balls.

The overall state of the independent random walkers may be represented by a sole augmented Markov chain on a space of cardinality $|V^{\mathrm{rw}}|^N$, whose transition matrix is $P \otimes \cdots \otimes P$ (N times), where '$\otimes$' is the standard Kronecker product. The first state of the augmented chain corresponds to all the walkers in the first site, the second is obtained by changing the location of the Nth walker to the second site, and so on up to the Nth state of the augmented chain, which is determined by placing the Nth walker in the site $|V^{\mathrm{rw}}|$; the next set of $|V^{\mathrm{rw}}|$ states is obtained by following the same iterations on the Nth walker's site, but changing that of the $(N-1)$th walker to the second site. The procedure is then inductively iterated until all states of the augmented chain are determined. The augmented Markov chain is ergodic and reversible, and its stationary probability is $\pi \otimes \cdots \otimes \pi$.

Given that the static graph G^{rw} describes the lattice where the agents meander, we now introduce a second graph G^{mn} that describes the agents' talking. The moving neighborhood graph: $G^{\mathrm{mn}}(k) = (E^{\mathrm{mn}}(k), V^{\mathrm{mn}})$ is a sequence of random graphs, whose node set is $V^{\mathrm{mn}} = \{1, \ldots, N\}$, where we associate a node with each of the random walkers, and whose edges depend on the random walkers' locations in the graph G^{rw}. (Note that for the moving neighborhood graph, we use superscripts $(\cdot)^{\mathrm{mn}}$ to distinguish this communication graph from the lattice where the random walkers are moving.) In the kth time interval, the set of edges of $G^{\mathrm{mn}}(k)$ is defined by:

$$E^{\mathrm{mn}}(k) = \{(q, r) \in V^{\mathrm{mn}} \times V^{\mathrm{mn}},\ q \neq r : X_q(k) = X_r(k)\}$$

that is, the edge (q, r) is present at the kth time step if and only if the random walkers X_q and X_r occupy the same site in the graph G^{rw} during the kth step. Clearly, the moving neighborhood graph does not have self-loops.

Fig. 2 shows the dynamics of the moving neighborhood network generated by the random walks of the five agents shown in Fig. 1. For each of the time steps depicted in Fig. 1, we show the resultant $G^{\mathrm{mn}}(k)$. The nodes of $G^{\mathrm{mn}}(k)$ represent the agents, and links among them are present whenever the corresponding random walkers in Fig. 1 occupy the same site of G^{rw}. Note that each of these graphs has five nodes, one for each agent, and the associated graph Laplacian, $L^{\mathrm{mn}}(k)$ is 5×5.

For $q \neq r$, the qr entry of the expected value of the adjacency matrix of the moving neighborhood graph G^{mn} at the kth step is:

$$\mathsf{E}[a_{qr}^{\mathrm{mn}}(k)] = \sum_{i=1}^{|V^{\mathrm{rw}}|} \pi_{qi}(k)\pi_{ri}(k) \tag{5}$$

and it represents the probability that the qth and rth random walkers occupy the same site in the graph G^{rw} during the kth time interval. The expected value of the rth diagonal element of the degree matrix at the the kth time interval is:

$$\mathsf{E}[d_q^{\mathrm{mn}}(k)] = \sum_{r=1, r\neq q}^{N} \sum_{i=1}^{|V^{\mathrm{rw}}|} \pi_{qi}(k)\pi_{ri}(k) \tag{6}$$

and it represents the probability that the qth random walker occupies the same site of any other random walker in the graph G^{rw} during the kth time interval. Therefore, the expected value of the graph Laplacian in the kth time interval is: $\mathsf{E}[L^{\mathrm{mn}}(k)] = \mathsf{E}[D^{\mathrm{mn}}(k)] - \mathsf{E}[A^{\mathrm{mn}}(k)]$, and it is a zero-row sum matrix.

We have introduced the notion of the long-time expected graph, which describes the communication of the agents with respect to the stationary distribution π. We note that the sequence of random graphs $G^{\mathrm{mn}}(k)$ is used to generate several sequences of random variables, such as $A^{\mathrm{mn}}(k)$ and $L^{\mathrm{mn}}(k)$. For a sequence of random variables $Y(k)$, we introduce the ergodic limit, $\mathsf{E}^*[Y]$, defined by:

$$\mathsf{E}^*[Y] = \lim_{k\to\infty} \mathsf{E}[Y(k)]$$

if the limit exists. We note that for $q \neq r$, by using the ergodicity condition (3) and (5), we have:

$$\mathsf{E}^*[a_{qr}^{\mathrm{mn}}] = \sum_{i=1}^{|V^{\mathrm{rw}}|} \pi_i \pi_i$$

while $a_{qq}^{\mathrm{mn}}(k) = 0$. So:

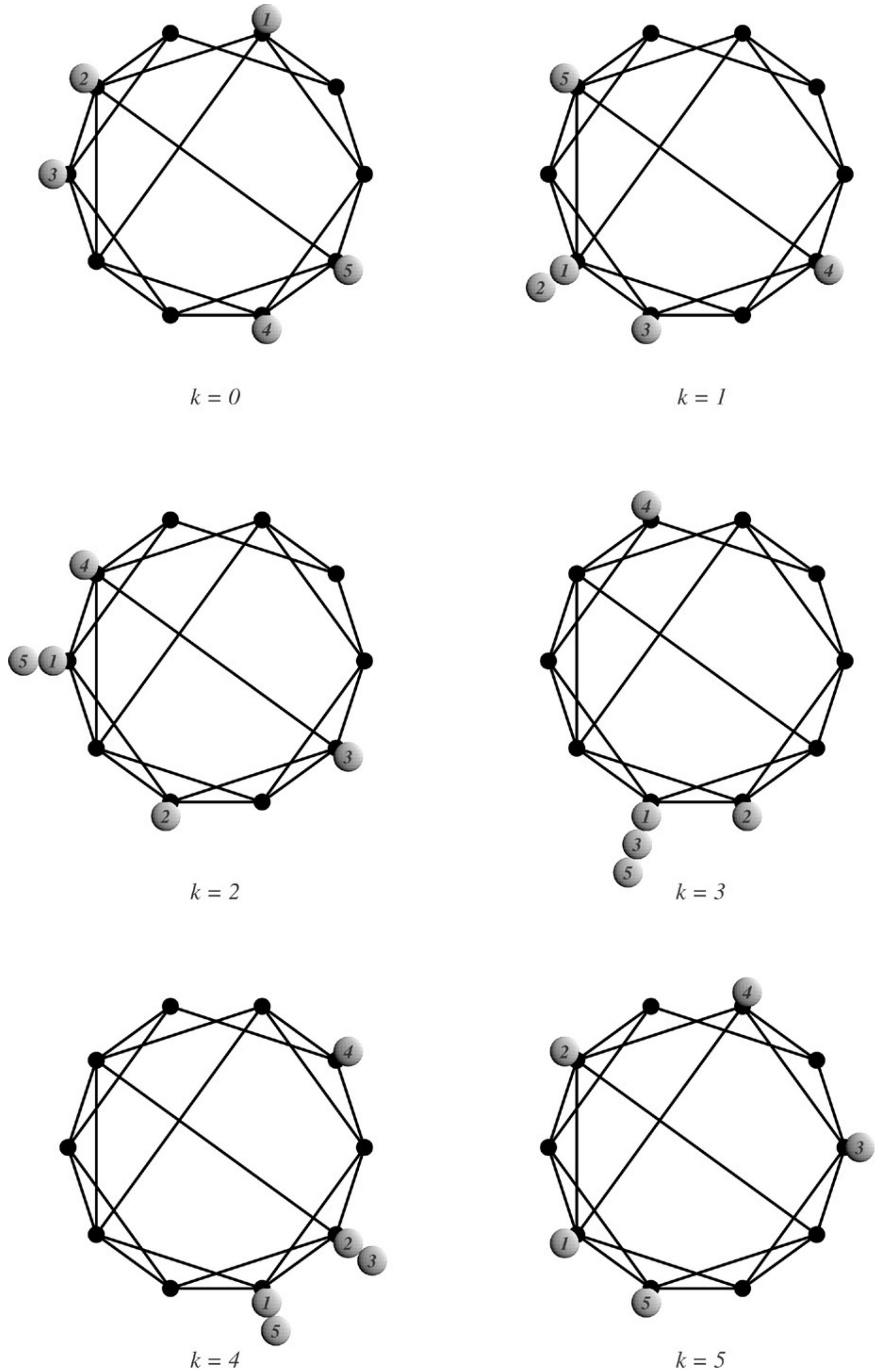

Fig. 1. Time evolution of five random walkers diffusing on a small-world planar network G^{rw}.

$$\mathsf{E}^*[A^{\mathrm{mn}}] = \pi^{\mathrm{T}}\pi[I_N - ee^{\mathrm{T}}]$$

where I_N is the $N \times N$ identity matrix. (This notation is adopted throughout.) We note that $\mathsf{E}^*[A^{\mathrm{mn}}]$ is not a binary matrix consisting of entries 0 and 1, and therefore cannot be described as an adjacency matrix. However, it does provide a description of the time-averaged connectivity within the network. From (6), we have:

$$\mathsf{E}^*[d_q^{\mathrm{mn}}] = \sum_{r=1, r\neq q}^{N} \sum_{i=1}^{|V^{\mathrm{rw}}|} \pi_i \pi_i$$

so that: $\mathsf{E}^*[D^{\mathrm{mn}}] = (N-1)\pi^{\mathrm{T}}\pi I_N$.

Therefore, we associate our notion of long-time expected graph with a weighted Laplacian matrix given by:

$$\mathsf{E}^*[L^{\mathrm{mn}}] = \pi^{\mathrm{T}}\pi[N I_N - e_N e_N^{\mathrm{T}}]. \tag{7}$$

The weighted Laplacian (7) represents a weighted all-to-all coupling among the random walkers, see e.g. [15]. The eigenvalues of $\mathsf{E}^*[L^{\mathrm{mn}}]$ are 0 (with multiplicity 1), and $\pi^{\mathrm{T}}\pi N$ (with multiplicity $N-1$).

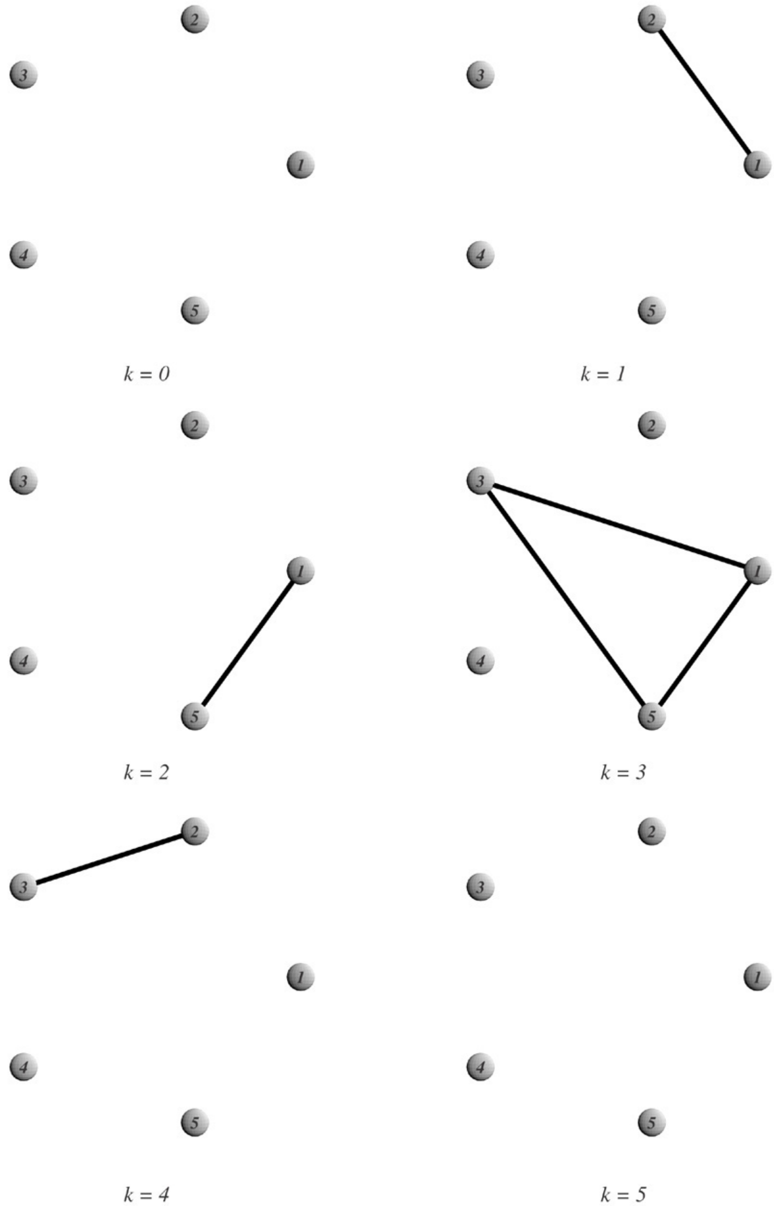

Fig. 2. Dynamics of the moving neighborhood network. For each of the time steps depicted in Fig. 1, we show the resultant $G^{\mathrm{mn}}(k)$.

3.2. *Synchronization problem*

Let each agent carry an oscillator characterized by an n dimensional autonomous dynamics. We study the time evolution of the complex dynamical system obtained by coupling the oscillators' dynamics according to the moving neighborhood network generated by their random walks. We assume that the random walkers do not move over time intervals of duration $\Delta > 0$. Jumps are allowed only at equally spaced transition instants $t_k = k\Delta, k \in \mathbb{Z}^+$. The resulting system dynamics is described by:

$$\dot{x}_q(t) = f(x_q(t)) + \sigma B \sum_{r=1}^{N} l_{qr}^{\mathrm{mn}}(t) x_r, \quad q = 1, \ldots, N, \; t \in \mathbb{R}^+ \tag{8}$$

where t is the time variable, $x_q \in \mathbb{R}^n$ is the random state vector of the qth agent, $f : \mathbb{R}^n \to \mathbb{R}^n$ describes the oscillators' individual dynamics, $B \in \mathbb{R}^{n\times n}$ describes coupling between oscillators, σ is the coupling strength and $L^{\mathrm{mn}}(t)$ is the graph Laplacian of the moving neighborhood network. In the time interval $T_k = [t_k, t_{k+1})$, the random process $L^{\mathrm{mn}}(t)$ corresponds to the random variable $L^{\mathrm{mn}}(k)$. Thus, it is a function of the random walks $X_1, \ldots, X_N$ at the kth time interval. We collect all the states of the system in the nN dimensional vector x. The initial conditions are prescribed at the initial time $t = 0$ as $x(0) = x_0$.

If for any $t \in \mathbb{R}^+$ all the oscillators have the same state $s(t)$, that is:

$$x_1(t) = \cdots = x_N(t) = s(t)$$

or equivalently:

$$x = e_N \otimes s$$

then we say that the agents are synchronized. Since e_N is in the null space of L^{mn}, a *synchronized solution s* is a solution of the individual agent equation, namely:

$$\dot{s} = f(s). \tag{9}$$

The manifold in $\mathbb{R}^{nN}$ consisting of all trajectories $e_N \otimes s(t)$, where $s(t)$ is a solution of (9) is called the synchronization manifold.

Synchronization can be assessed by examining the local stability of the oscillators with respect to the synchronization manifold. Linearizing each oscillator about the trajectory $s(t)$, which is assumed to be on the synchronization manifold, yields:

$$\dot{z}_q(t) = F(t)z_q(t) + \sigma B \sum_{r=1}^{N} l_{qr}^{\text{mn}}(t) z_r(t)$$

where:

$$z_q(t) = x_q(t) - s(t)$$

and $F(t)$ is the Jacobian of f evaluated at $s(t)$. Thus, the system of linearized coupled oscillators may be rewritten as:

$$\dot{z}(t) = (I_N \otimes F(t) + \sigma L^{\text{mn}}(t) \otimes B) z(t) \tag{10}$$

where $z(t) = [z_1^{\text{T}}(t), \ldots, z_N^{\text{T}}(t)]^{\text{T}}$. To assess the asymptotic stability of the set of oscillators, we partition the state of (10) into a component that evolves along the synchronization manifold, and a component that evolves transverse to the synchronization manifold. For analysis, it suffices to show that the component that evolves transverse to the synchronization manifold asymptotically approaches the synchronization manifold.

Let $W \in \mathbb{R}^{N\times(N-1)}$ satisfy $W^{\text{T}} e_N = 0$ and $W^{\text{T}} W = I_{N-1}$. Note that the state vector $z(t)$ in (10) can be decomposed as $z(t) = (W \otimes I_n)\zeta(t) + e_N \otimes z_s(t)$, where $\zeta = (W^{\text{T}} \otimes I_n) z$ is transverse to the synchronization manifold, and $z_s = \frac{1}{N}(e_N \otimes I_n)^{\text{T}} z$ is on the synchronization manifold. Note that:

$$((W \otimes I_n)\zeta)^{\text{T}} (e_N \otimes z_s) = 0.$$

The variational equation (10) in terms of ζ and z_s becomes:

$$\dot{z}_s(t) = F(t) z_s(t) + \sigma (e_N^{\text{T}} L^{\text{mn}}(t) W \otimes B) \zeta(t)$$

$$\dot{\zeta}(t) = (I_{N-1} \otimes F(t) + \sigma W^{\text{T}} L^{\text{mn}}(t) W \otimes B) \zeta(t). \tag{11}$$

We say that the set of oscillators (8) locally asymptotically synchronize almost surely at the synchronized solution $s(t)$ if (11) is almost surely asymptotically stable, see e.g. [38], that is, if ζ converges to zero almost surely for any $\zeta_0 \in \mathbb{R}^{(N-1)n}$ and any initial probability $\pi_q(0)$, $q = 1, \ldots, N$. The definition of almost sure convergence may be found, for example, in Chapter 5 of [39]. System (11) represents a jump linear time varying system, see e.g. [40].

We associate with the stochastic dynamic network (8) the deterministic dynamic network:

$$\dot{x}_q(t) = f(x_q(t)) + \sigma B \sum_{r=1}^{N} \mathsf{E}^*[l_{qr}^{\text{mn}}] x_r(t),$$
$$q = 1, \ldots, N, \ t \in \mathbb{R}^+ \tag{12}$$

where $\mathsf{E}^*[L^{\text{mn}}]$ is the long-time expected value of the graph Laplacian, as defined in (7).

Synchronization of the deterministic set of coupled oscillators (12) may be studied using the master stability function. As a representative parameter for the synchronizability of (12), we introduce the *friendliness* Φ of the graph G^{rw}, defined by:

$$\Phi = \|\pi\|_2^2 \tag{13}$$

where $\|\cdot\|_2$ is the Euclidean norm. The stability question reduces by linear perturbation analysis to a constraint upon the coupling parameter σ, the friendliness π, and the number of agents N of the form $\Phi N \in S$, where S is the stability region and is an interval of $\mathbb{R}^+$. For many oscillator dynamical systems (see e.g. [15]) the stability region is a bounded interval of the type $S = (\alpha_1, \alpha_2)$. The parameters α_1, α_2 are given by the master stability function, which is a property of the individual oscillator dynamic equation and of the coupling matrix B. Therefore, synchronization of (12) is generally expressed as a constraint on the control parameter σ, that is:

$$\frac{\alpha_1}{N\Phi} < \sigma < \frac{\alpha_2}{N\Phi}. \tag{14}$$

For large values of Φ (i.e. highly friendly networks) the set of oscillators (12) synchronizes for small values of the coupling parameter σ, while large coupling is required for achieving synchronization in unfriendly networks. In addition, we note that for a prescribed graph G^{rw}, synchronization for small coupling may also be possible by increasing the number of agents N. For a regular graph G^{rw}, and the product $N\Phi$ represents the average occupancy number of the graph G^{rw}.

Our main contribution is to show that if the static network in (12) supports synchronization, the stochastic network (8) also does so, if the random walkers are sufficiently fast (or equivalently, if the switching period, Δ, is sufficiently small).

4. A few propositions on stochastic stability

In this section we present two propositions on linear system stability that are used to determine sufficient conditions for almost sure asymptotic synchronization of (8). For clarity, we restate the well-known Borel–Cantelli lemma in the form presented in Lemma 1 of [41] (Chapter 8), but in the notation of this paper.

Lemma 1. *Consider the stochastic process $\xi(k)$ with $k \in \mathbb{Z}^+$, in $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ is a nonnegative function, and:*

$$\sum_{k=0}^{\infty} \mathsf{E}[f(\xi(k))] < \infty$$

then $f(\xi(k)) \xrightarrow{a.s.} 0$, that is $f(\xi(k))$ converges to zero almost surely.

The following proposition generalizes the claim in Theorem 1 of [41] (Chapter 8) to a class of time-varying jump linear systems.

Proposition 2. *Consider the stochastic system:*

$$\xi(k+1) = A(X(k), k)\xi(k) \tag{15}$$

where $X(k)$ is an ergodic Markov chain in F, $\xi(0) \in \mathbb{R}^m$, and A is a bounded matrix function. Suppose there is a sequence of symmetric bounded positive definite matrices $Q(k)$, such that for any $k \in \mathbb{Z}^+$:

$$\mathsf{E}^*[A(X(k), k)^{\mathrm{T}} Q(k+1) A(X(k), k)] - Q(k) = -C(k) \tag{16}$$

where E^* indicates expected value computed with respect to the stationary chain distribution π, and $C(k)$ is a sequence of symmetric positive definite matrices such that $C(k) > \nu$ for every $k \in \mathbb{Z}^+$ with $\nu > 0$. Then (15) is almost surely asymptotically stable, that is $\xi(k) \overset{a.s.}{\to} 0$ for any initial condition $\xi(0)$ and any initial distribution $\pi(0)$.

Proof. Define in $\mathbb{R}^m$ the quadratic Lyapunov function:

$$v(x, k) = \frac{1}{2} x^{\mathrm{T}} Q(k) x, \quad x \in \mathbb{R}^m$$

and the related random process:

$$V(k) = v(\xi(k), k). \tag{17}$$

Thus, using (15) and (17) iteratively, we obtain:

$$\mathsf{E}[V(1)] = \frac{1}{2}\xi(0)^{\mathrm{T}} \mathsf{E}[A(X(0), 0)^{\mathrm{T}} Q(1) A(X(0), 0)]\xi(0)$$

$$\mathsf{E}[V(2)|\xi(1)] = \frac{1}{2}\xi(1)^{\mathrm{T}} \mathsf{E}[A(X(1), 1)^{\mathrm{T}} Q(2) A(X(1), 1)]\xi(1)$$

$$\begin{aligned} &\mathsf{E}[V(k+1)|\xi(k)] \\ &\quad = \frac{1}{2}\xi(k)^{\mathrm{T}} \mathsf{E}[A(X(k), k)^{\mathrm{T}} Q(k+1) A(X(k), k)]\xi(k). \end{aligned} \tag{18}$$

The expected value in (18) may be algebraically manipulated to give:

$$\begin{aligned} &\mathsf{E}[A(X(k), k)^{\mathrm{T}} Q(k+1) A(X(k), k)] \\ &\quad = \mathsf{E}^*[A(X(k), k)^{\mathrm{T}} Q(k+1) A(X(k), k)] + R(k) \end{aligned} \tag{19}$$

where:

$$\begin{aligned} &\mathsf{E}[A(X(k), k)^{\mathrm{T}} Q(k+1) A(X(k), k)] \\ &\quad = \sum_{i=1}^{|F|} A(i, k)^{\mathrm{T}} Q(k+1) A(i, k) \pi_i(k) \\ &\mathsf{E}^*[A(X(k), k)^{\mathrm{T}} Q(k+1) A(X(k), k)] \\ &\quad = \sum_{i=1}^{|F|} A(i, k)^{\mathrm{T}} Q(k+1) A(i, k) \pi_i \end{aligned}$$

and $R(k)$ is defined by:

$$R(k) = \sum_{i=1}^{|F|} A(i, k)^{\mathrm{T}} Q(k+1) A(i, k)(\pi_i(k) - \pi_i).$$

Using the triangle inequality, Schwartz inequality, and (3), $R(k)$ may be bounded by:

$$\begin{aligned} \|R(k)\|_2 &\le \sum_{i=1}^{|F|} \|A(j, k)^{\mathrm{T}} Q(k+1) A(j, k)\|_2 \mu\rho^k \\ &\le |F| \mu\rho^k \alpha^2 \vartheta \end{aligned} \tag{20}$$

where for any $k \in \mathbb{R}^+$ and $j \in F$, $\|A(j, k)\|_2 \le \alpha$ and $\|Q(k)\|_2 \le \vartheta$. Substituting (16) and (19) into (18), we obtain:

$$\begin{aligned} \mathsf{E}[V(k+1)|\xi(k)] &= \frac{1}{2}\xi(k)^{\mathrm{T}}(Q(k) - C(k) + R(k))\xi(k) \\ &= v(\xi(k), k) - \frac{1}{2}\xi(k)^{\mathrm{T}}(C(k) - R(k))\xi(k). \end{aligned} \tag{21}$$

Since $C(k)$ is a positive definite matrix bounded away from zero and (20) holds with $\rho < 1$, it is possible to find an integer $\bar{k}$ and a positive constant κ, such that for $k \ge \bar{k}$ and for any $x \in \mathbb{R}^m$, $x^{\mathrm{T}}(C(k) - R(k))x \ge \kappa x^{\mathrm{T}} x$. Iterating (21), we obtain:

$$\mathsf{E}[V(k)|\xi(\bar{k})] = v(\xi(\bar{k}), \bar{k}) - \frac{1}{2}\sum_{i=\bar{k}}^{k-1} \mathsf{E}[\xi(i)^{\mathrm{T}}(C(i) - R(i))\xi(i)]$$

which gives:

$$\begin{aligned} v(\xi(\bar{k}), \bar{k}) &\ge v(\xi(\bar{k}), \bar{k}) - \mathsf{E}[V(k)|\xi(\bar{k})] \\ &= \frac{1}{2}\sum_{i=\bar{k}}^{k-1} \mathsf{E}[\xi(i)^{\mathrm{T}}(C(i) - R(i))\xi(i)] \\ &\ge \frac{\kappa}{2}\sum_{i=\bar{k}}^{k-1} \mathsf{E}[\xi(i)^{\mathrm{T}}\xi(i)]. \end{aligned}$$

Applying Lemma 1 to $V(k-\bar{k})$ with $f(x) = \kappa\frac{1}{2}x^{\mathrm{T}}x$, the claim follows. □

The following intuitive result is useful for assessing the stochastic asymptotic stability of a linear system in terms of its sampled discrete version.

Proposition 3. *Consider the stochastic linear system in $\mathbb{R}^{m_1 m_2}$:*

$$\dot{y} = (A(t) + B(t) \otimes H(t))y, \quad t \in \mathbb{R}^+$$

where $y(t) \in \mathbb{R}^{m_1 m_2}$, $A(t) \in \mathbb{R}^{m_1 m_2 \times m_1 m_2}$ and $B(t) \in \mathbb{R}^{m_1 \times m_1}$ are bounded and continuous functions for all $t \in \mathbb{R}^+$, and $H(t) \in \mathbb{R}^{m_2 \times m_2}$ is a bounded random process such that for some $\Delta > 0$ is constant for all $t \in [k\Delta, (k+1)\Delta)$ and switches at time instants $k\Delta$, for all $k \in \mathbb{Z}^+$. If $y(k\Delta) \overset{a.s.}{\to} 0$, then $y(t) \overset{a.s.}{\to} 0$.

Proof. For any $t \in [k\Delta, (k+1)\Delta)$, $y(t) = \Phi(t, k\Delta)y(k\Delta)$, where there exist positive constants α, β, and η such that, for any $t \in \mathbb{R}^+$:

$$\|A(t)\|_2 \le \alpha, \qquad \|B(t)\|_2 \le \beta, \qquad \|H(t)\|_2 \le \eta. \tag{22}$$

From Coppel's inequality, see e.g. [42], we have:

$$\|y(t)\|_2 \le \exp \int_{k\Delta}^{t} \mu_2(A(\tau) + B(\tau) \otimes H(\tau)) \mathrm{d}\tau \|y(k\Delta)\|_2 \tag{23}$$

where $\mu_2(\cdot)$ is the Euclidean measure. The Euclidean matrix measure for every $C \in \mathbb{R}^{m \times m}$ is defined by (see e.g. [43]):

$$\mu_2(C) = \lim_{h \to 0^+} \frac{\|I_m + hC\|_2 - 1}{h}. \tag{24}$$

Since, for any $C \in \mathbb{R}^{m\times m}$, $\mu_2(C) \le \|C\|_2$ from (23) we obtain:

$$\|y(t)\|_2 \le \exp\int_{k\Delta}^{(k+1)\Delta} \|A(\tau)+B(\tau)\otimes H(\tau)\|_2 \mathrm{d}\tau \|y(k\Delta)\|_2$$

which, by applying the triangle inequality, by using (22) and Fact 9.12.22 in [43], yields:

$$\|y(t)\|_2 \le \exp((\alpha+\beta\eta)\Delta)\|y(k\Delta)\|_2$$

for any $t \in [k\Delta, (k+1)\Delta)$. Thus, the claim follows. □

5. Synchronization through fast-switching

In this section, we apply the results of Section 4 to the stochastic variational system (11), and show that asymptotic synchronization is achieved almost surely if the deterministic network (12) asymptotically synchronizes and if the random walkers are moving sufficiently fast. By means of Theorem 4, the synchronization problem for the network of oscillators described by (8) reduces to the analysis of synchronization over a static network. Thus, we reduce the problem to one which has been extensively studied in the literature and may be addressed by using the well known method of master-stability function (MSF) analysis, see e.g. [15]. We remark that in the standard usage, the MSF identifies a parameter range based on *Lyapunov stability,* whereas our argument requires *uniform asymptotic stability* of the deterministic system. However, as indicated in [15]), an MSF can be developed based on any stability criteria, with the rest of the mathematical structure of the MSF argument remaining intact.

Theorem 4. *Consider the deterministic dynamic system:*

$$\dot{y}(t) = (I_{N-1}\otimes F(t) + \sigma W^{\mathrm{T}}\mathsf{E}^*[L^{\mathrm{mn}}]W\otimes B)y(t) \tag{25}$$

representing the linearized transverse dynamics of (12). Assume that $F(t)$ is bounded and continuous in $\mathbb{R}^+$. If (25) is uniformly asymptotically stable, there is a time-scale $\Delta^* > 0$ such that for any shorter time-scale $\Delta < \Delta^*$, the stochastic system (8) locally asymptotically synchronizes almost surely.

Proof. We define $M(t) \equiv I_{N-1}\otimes F(t)+\sigma W^{\mathrm{T}}L^{\mathrm{mn}}W\otimes B$, and rewrite the variational equation (11) as:

$$\dot{\zeta}(t) = M(t)\zeta(t). \tag{26}$$

Consequently, Eq. (25) may be compactly rewritten as:

$$\dot{y}(t) = \mathsf{E}^*[M(t)]y(t) \tag{27}$$

where $\mathsf{E}^*[M(t)] \equiv I_{N-1}\otimes F(t)+\sigma W^{\mathrm{T}}\mathsf{E}^*[L^{\mathrm{mn}}]W\otimes B$. With Proposition 3 in mind, we consider the sequences of sample data $\zeta(k\Delta)$ and $y(k\Delta)$. Let $\Gamma(k)$ and $\Theta(k)$ be the transition matrices of (26) and (27), respectively, over the time interval $T_k = [k\Delta, (k+1)\Delta)$. Thus:

$$\zeta((k+1)\Delta) = \Gamma(k)\zeta(k\Delta) \tag{28a}$$

$$y((k+1)\Delta) = \Theta(k)y(k\Delta). \tag{28b}$$

By hypothesis, $y(k\Delta)$ in (28b) asymptotically converges to zero for any initial condition $y(0)$. Our task is to show that $\zeta(k\Delta)$ in (28a) converges to zero almost surely for any initial condition and any initial probability distribution if Δ is sufficiently small.

Since F is bounded, there is a positive constant ϕ, such that $\|F(t)\|_2 \le \phi$ for all $t \in \mathbb{R}^+$. Define $\|B\|_2 = \beta$, then since $\|W\|_2 = 1$ and $\|L^{\mathrm{mn}}\|_2 \le 2N$:

$$\|M(t)\|_2 \le \alpha, \qquad \|\mathsf{E}^*[M(t)]\|_2 \le \alpha \tag{29}$$

for any $t \in \mathbb{R}^+$, where $\alpha = \phi + 2\sigma\beta N$.

Since (27) is uniformly asymptotically stable and $\mathsf{E}^*[M]$ is continuous, there exist positive scalars η, ρ, μ, and a symmetric continuously differentiable matrix $Q(t)$, such that (see e.g. Theorem 7.2 of [44]):

$$\eta I \le Q(t) \le \rho I, \tag{30}$$

where the inequality indicates sign-definitiveness. The related Lyapunov function:

$$V(t, y(t)) = \frac{1}{2}y(t)^{\mathrm{T}}Q(t)y(t)$$

satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}V((t, y(t))) \le -\mu\|y(t)\|_2^2.$$

For linear systems, uniform asymptotic stability is equivalent to uniform exponential stability, thus (see e.g. the proof of Theorem 7.4 of [44]):

$$\begin{aligned} &V((k+1)\Delta, y((k+1)\Delta)) - V(k\Delta, y(k\Delta)) \\ &\quad\le \frac{\rho}{2}(-1+\exp(-\mu\Delta/\rho))\|y(k\Delta)\|_2^2. \end{aligned} \tag{31}$$

From (31) the following condition arises:

$$\Theta(k)^{\mathrm{T}}Q((k+1)\Delta)\Theta(k) - Q(k\Delta) = -G(k) \tag{32}$$

where $G(k)$ is a sequence of symmetric positive definite matrices satisfying:

$$\|G(k)\|_2 \ge \rho(1-\exp(-\mu\Delta/\rho)). \tag{33}$$

In addition, since $\mathsf{E}^*[M]$ is bounded, by using Coppel's inequality, see e.g. [42], and by noting that the matrix measure of a matrix is less than or equal to its corresponding norm [43], we have:

$$\|\Theta(k)\|_2 \le \exp(\alpha\Delta). \tag{34}$$

Recalling the Peano–Baker expansion for $\Gamma(k)$ (see e.g. Chapter 3 of [44]), we have:

$$\begin{aligned} \Gamma(k) &= I_{(N-1)n} + \int_{k\Delta}^{(k+1)\Delta} M(\sigma_1)\mathrm{d}\sigma_1 \\ &\quad + \sum_{i=2}^{\infty}\int_{k\Delta}^{(k+1)\Delta} M(\sigma_1)\int_{k\Delta}^{\sigma_1}\cdots M(\sigma_{i-1})\int_{k\Delta}^{\sigma_{i-1}} M(\sigma_i)\mathrm{d}\sigma_i\cdots\mathrm{d}\sigma_1 \end{aligned}$$

we can express:

$$R(k) = \Gamma(k) - \Theta(k) \tag{35}$$

as:

$$R(k) = \int_{k\Delta}^{(k+1)\Delta} (M(\sigma) - \mathsf{E}^*[M(\sigma)])\mathrm{d}\sigma + \varepsilon(k) \tag{36}$$

where:

$$\varepsilon(k) = \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} M(\sigma_1) \int_{k\Delta}^{\sigma_1} \cdots \int_{k\Delta}^{\sigma_{i-1}} M(\sigma_i) \mathrm{d}\sigma_i \cdots \mathrm{d}\sigma_1$$
$$- \sum_{i=2}^{\infty} \int_{k\Delta}^{(k+1)\Delta} \mathsf{E}^*[M(\sigma_1)]$$
$$\times \int_{k\Delta}^{\sigma_1} \cdots \int_{k\Delta}^{\sigma_{i-1}} \mathsf{E}^*[M(\sigma_i)] \mathrm{d}\sigma_i \cdots \mathrm{d}\sigma_1. \quad (37)$$

By taking the ergodic limit, that is the long-run expected value of both sides of (36) we have:

$$\mathsf{E}^*[R(k)] = \mathsf{E}^*[\varepsilon(k)]. \quad (38)$$

From (29), the Euclidean norm of the first term on the RHS of (36) may be bounded by $2\Delta\alpha$. On the other hand, from (29) and (37), the Euclidean norm of $\varepsilon(k)$ is bounded by:

$$\|\varepsilon(k)\|_2 \leq \sum_{k=2}^{\infty} (2\Delta\alpha)^k = \sum_{k=0}^{\infty} (2\Delta\alpha)^k - (1 + 2\Delta\alpha)$$
$$= \exp(2\Delta\alpha) - 1 - 2\Delta\alpha \quad (39)$$

which implies:

$$\|\mathsf{E}^*[\varepsilon(k)]\|_2 \leq \exp(2\Delta\alpha) - 1 - 2\Delta\alpha. \quad (40)$$

Using (29) and (40), the Euclidean norm of $R(k)$ is bounded by:

$$\|R(k)\|_2 \leq \exp(2\Delta\alpha) - 1. \quad (41)$$

We emphasize that from (39) and (41), one obtains that:

$$\|R(k)\|_2 \leq O(\Delta), \qquad \|\varepsilon(k)\|_2 \leq O(\Delta^2).$$

Next, we show that for sufficiently small values of Δ, the matrix $Q(k\Delta)$ defines a quadratic Lyapunov function for the system (11) in the sense of Proposition 2. Indeed, by substituting (35) into (32), and by using (38), we obtain:

$$\mathsf{E}^*[\Gamma(k)^{\mathrm{T}} Q((k+1)\Delta)\Gamma(k)] - Q(k\Delta)$$
$$= \mathsf{E}^*[(R(k) + \Theta(k))^{\mathrm{T}} Q((k+1)(R(k) + \Theta(k)))]$$
$$- Q(k\Delta) = -G(k) + \Theta(k)^{\mathrm{T}} Q((k+1)\Delta)\mathsf{E}^*[\varepsilon(k)]$$
$$+ \mathsf{E}^*[\varepsilon(k)]^{\mathrm{T}} Q((k+1)\Delta)\Theta(k)$$
$$+ \mathsf{E}^*[R(k)^{\mathrm{T}} Q((k+1)\Delta)R(k)]. \quad (42)$$

Considering all the bounds presented above, that is (29), (30), (33), (34) and (40)–(42) yields:

$$\mathsf{E}^*[\Gamma(k)^{\mathrm{T}} Q((k+1)\Delta)\Gamma(k)] - Q(k) \leq -g(\Delta) I_{(N-1)n}$$

where the continuous function $g(\Delta)$ is defined by:

$$g(\Delta) = \rho(1 - \exp(-\mu\Delta/\rho))$$
$$- 2\exp(\alpha\Delta)\rho(\exp(2\Delta\alpha) - 1 - 2\Delta\alpha)$$
$$- \rho(\exp(2\Delta\alpha) - 1)^2.$$

It can be shown that $g(0) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}\Delta} g(0) = \mu > 0$. Thus, since $g(\Delta) \to -\infty$ as $\Delta \to \infty$, there exists Δ^* such that $g(\Delta) > 0$ for all $\Delta \in (0, \Delta^*)$, and by applying Proposition 2 the claim follows. □

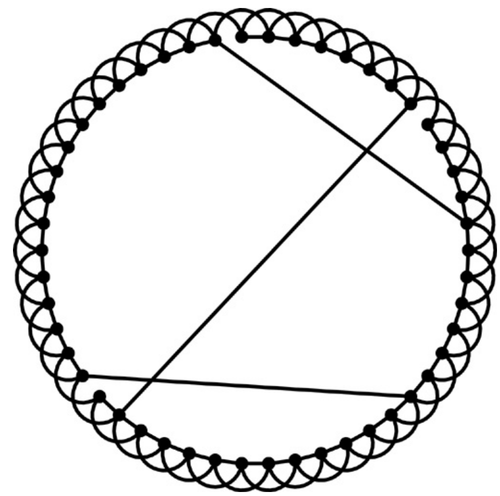

Fig. 3. Small-world graph G^{rw} used for the numerical illustration.

We remark that the stability claims in this paper are actually statements about the linearized system near a trajectory on the synchronization manifold of the full nonlinear system. Extending these results to the nonlinear system would depend upon the applicability of the *Hartman–Grobman* theorem [45]. However, for most chaotic systems, it is not possible to establish the hypothesis of that theorem. Additionally, to extend from the local nonlinear system to the full nonlinear system would require a proof of ergodicity for the system of coupled oscillators, which is an outstanding open problem for typical chaotic systems. Consequently, rather than stating theorems that the hypothesis cannot be established, we have limited our claims to properties of the linearized system.

6. Illustration by numerical simulation

For the purposes of illustration, we consider a set of $N = 20$ agents diffusing in the small-world planar graph G^{rw}, with $|V^{\mathrm{rw}}| = 50$ sites as depicted in Fig. 3. The graph G^{rw} is connected and non-bipartite. The friendliness of the network, as defined in (13), is $\Phi = 0.020$.

Each agent is equipped with a Rössler oscillator. When agents occupy the same site, their first state is coupled. Thus the system of oscillators is described by:

$$\begin{aligned} \dot{x}_{q1}(t) &= -x_{q2}(t) - x_{q3}(t) - \sigma \sum_{r=1}^{N} l_{qr}^{\mathrm{mn}} x_{r1}(t) \\ \dot{x}_{q2}(t) &= x_{q1}(t) + a x_{q2}(t) \\ \dot{x}_{q3}(t) &= b + x_{q3}(t)(x_{q1}(t) - c) \end{aligned} \quad (43)$$

where $q = 1, \ldots, N$, and a, b, c are constants.

By choosing the parameters $a = 0.2$, $b = 0.2$, and $c = 7$ from the stability region plot of Figure 2 of [15], we have that the stability regions α_1 and α_2 in (14) are $\alpha_1 = 0.2$ and $\alpha_1 = 2.3$. Therefore, from (14), the deterministic system (12) asymptotically synchronizes in the sense of the transverse Lyapunov exponents if $0.50 < \sigma < 5.7$. As previously remarked, this does not always mean the transverse dynamics are uniformly asymptotically stable. We choose $\sigma = 2$. Fig. 4

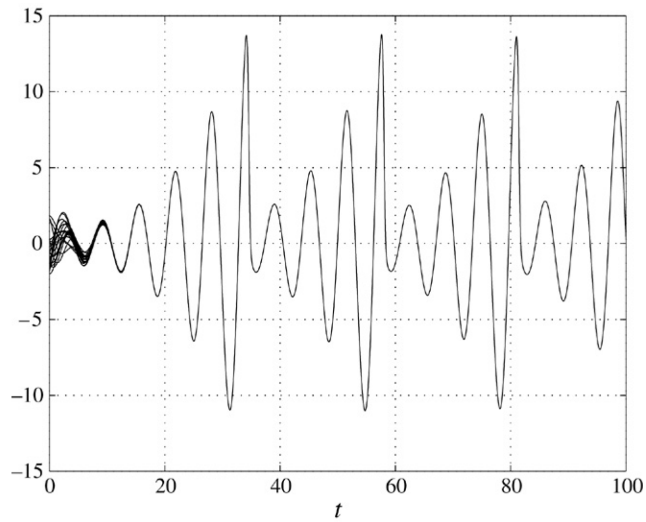

Fig. 4. Time evolution of the x_1 coordinate of the set of coupled Rössler oscillators using the long-time expected graph. Observe the asymptotic synchronization.

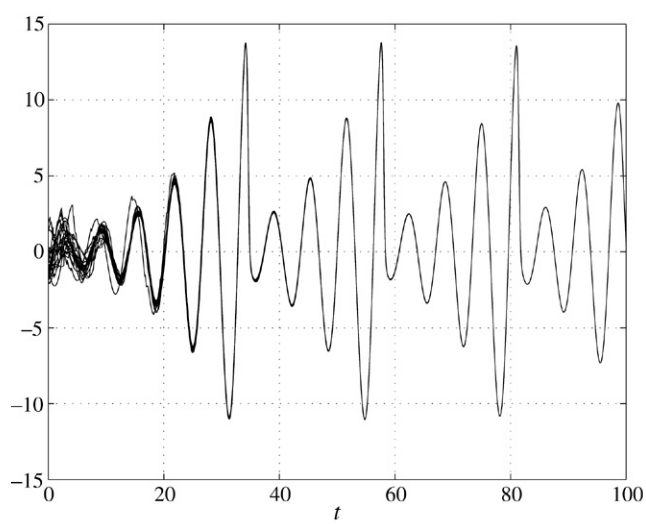

Fig. 5. Time evolution of the x_1 coordinate of the set of coupled Rössler oscillators using the moving neighborhood graph. Even though the neighborhood graph is instantaneously disconnected, fast switching allows for synchronization.

depicts the time evolution of the x_1 coordinate of the Rössler oscillators with static coupling given by the long-time expected graph.

For the stochastic network (45), we consider a switching period for the random walkers of $\Delta = 0.1$. Fig. 5 depicts the x_1 coordinate of the set of coupled Rössler oscillators coupled by the moving neighborhood graph, using the same initial conditions as in Fig. 4.

7. Conclusions

New generalizations on the synchronization of mutually coupled oscillators are presented. We pose the synchronization problem in a stochastic dynamic framework, where each agent diffuses in a finite lattice and carries an oscillator. The communication network topology evolves in time, and is determined by the agents' locations in the lattice. Communication takes place only within geographical neighborhoods. We introduce the concept of the long-time expected communication network, defined as the ergodic limit of the stochastic time-varying network. We utilize tools based on fast switching and stochastic stability, and show that synchronization is asymptotically achieved if the long-time expected network supports synchronization and if the agents are moving sufficiently fast in the lattice. A numerical simulation illustrates the theoretical achievements of the present paper. We expect the theoretical framework presented in this paper to provide a better understanding of synchronization problems in biological, epidemiological and social networks, where the dynamics of the agent cannot be ignored.

References

[1] A. Pikovsky, M. Rosemblum, J. Kurths, Synchronization, A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2001.

[2] J.M. Gonzalez-Miranda, Synchronization and Control of Chaos, Imperial College Press, London, 2004.

[3] G. Chen, X. Yu (Eds.), Chaos Control Theory and Applications, in: Lecture Notes in Control and Information Sciences, vol. 292, Springer, Berlin, 2003.

[4] S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares, C.S. Zhou, The synchronization of chaotic systems, Phys. Rep. 366 (2002) 1–101.

[5] D. He, L. Stone, Spatio-temporal synchronization of recurrent epidemics, Proc. R. Soc. Lond. Ser. B 270 (2003) 1519–1526.

[6] E. Mosekilde, Y. Maistrenko, D. Postnov, Chaotic Synchronization: Applications to Living systems, World Scientific, New Jersey, 2002.

[7] M.F. Laguna, G. Abramson, D.H. Zanette, Minorities in a model for opinion formation, Complexity 9 (4) (2004) 31–36.

[8] Y. Kuramoto, Chemical Oscillations, Waves and Turbulence, Springer, Berlin, 1984.

[9] J. Ohtsubop, Feedback induced instability and chaos in semiconductor lasers and their applications, Opt. Rev. 6 (1) (1999) 1–15.

[10] G.S. Duane, P.J. Webster, J.B. Weiss, Go-occurrence of northern and southern hemisphere blocks as partially synchronized chaos, J. Atmospheric Sci. 56 (24) (1999) 4183–4205.

[11] L.M. Pecora, T.L. Carroll, G.A. Johnson, D.J. Mar, Fundamentals of synchronization in chaotic systems, concepts, and applications, Chaos 7 (4) (1997) 520–543.

[12] M. Barahona, L.M. Pecora, Synchronization in small-world systems, Phys. Rev. Lett. 89 (5) (2002) 050101.

[13] M. Chen, D. Zhou, Synchronization in uncertain complex networks, Chaos 16 (2006) 013101.

[14] L. Kocarev, P. Amato, Synchronization in power-law networks, Chaos 15 (2) (2005) 024101.

[15] L. Pecora, T. Carroll, G. Johnson, G. Mar, K.S. Fink, Synchronization stability in coupled oscillator arrays: Solution for arbitrary configurations, Internat. J. Bifur. Chaos 10 (2000) 2.

[16] L.M. Pecora, T.L. Carroll, Master stability functions for synchronized coupled systems, Phys. Rev. Lett. 80 (10) (1998) 2109–2112.

[17] C.W. Wu, L.O. Chua, Synchronization in an array of linearly coupled dynamical systems, IEEE Trans. Circuits Syst. - I 42 (8) (1995) 430–447.

[18] C. Zhou, A.E. Motter, J. Kurths, Universality in the synchronization of weighted random networks, Phys. Rev. Lett. 96 (3) (2006) 034101.

[19] L. Kocarev, P. Janjic, U. Parlitz, T. Stojanovski, Controlling spatio-temporal chaos in coupled oscillatots by sporadic driving, Chaos Solitons Fractals 9 (1/2) (1998) 283–293.

[20] T. Shibata, K. Kaneko, Coupled mab gas: Structures formation and dynamics of interacting motile elements with internal dynamics, Physica D 181 (2003) 197–214.

[21] J. Lu, X. Yu, G. Chen, Chaos synchronization of general complex dynamical networks, Physica A 334 (2004) 281–302.

[22] J. Lu, G. Chen, A time-varying complex dynamical network model and its controlled synchronization criteria, IEEE Trans. Automat. Control 50 (6) (2005) 841–846.
[23] I.V. Belykh, V.N. Belykh, M. Hasler, Blinking model and synchronization in small-world networks with a time-varying coupling, Physica D 195 (1–2) (2004) 188–206.
[24] J.D. Skufca, E.M. Bollt, Communication and synchronization in disconnected networks with dynamic topology: Moving neighborhood networks, Math. Biosci. Eng. 1 (2) (2004) 347–359.
[25] D.J. Stilwell, E.M. Bollt, D.G. Roberson, Sufficient conditions for fast switching synchronization in time-varying network topologies, SIAM J. Appl. Dynam. Syst. 6 (1) (2006) 140–156.
[26] A. Lasota, M.C. Mackey, Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics, Springer, New York, 1998.
[27] J.R. Norris, R. Gill, Markov Chains, Cambridge University Press, Cambridge, UK, 1997.
[28] A. Boyarsky, P. Góra, Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimensions, Birkhauser, Boston, 1997.
[29] L. Billings, E. Bollt, Probability density functions of some skew tent maps, Chaos Solitons Fractals 12 (2) (2001) 365–376.
[30] D.J. Watts, S.H. Strogatz, Collective dynamics of small-world networks, Nature 393 (1998) 440–442.
[31] P. Bremaud, Markov Chains, Gibbs Fields, Monte Carlo Simulation, and Queues, 1999.
[32] P. Billingsley, Probability and Measure, Wiley, New York, 1986.
[33] R. Diestel, Graph Theory, Springer-Verlag, New York, NY, 1997.
[34] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, NY, 2001.
[35] W.N. Anderson, T.D. Morley, Eigenvalues of the laplacian of a graph, Linear Multilinear Algebra 18 (1985) 141–145.
[36] L. Lovasz, Combinatorics, in: Paul Erdos is Eighty, vol. 2, Janos Bolyai Mathematical Society, Budapest, 1996, p. 353398 (Ch. Random walks on graphs: A survey).
[37] E.M. Bollt, J.D. Skufca, Markov partitions, in: A. Scott (Ed.), Encyclopedia of Nonlinear Science, Routledge, New York, 2005.
[38] H.J. Kushner, Stochastic Stability and Control, Academic Press, New York, 1967.
[39] A. Gut, Probability: A Graduate Course, Springer, New York, NY, 2005.
[40] O.L.V. Costa, M.D. Fragoso, R.P. Marques, Discrete-Time Markov Jump Linear Systems, Springer, London, UK, 2005.
[41] H. Kushner, Introduction to Stochastic Control, Holt, Rinehart and Winston, Inc., New York, NY, 1971.
[42] C.J. Harris, J.F. Miles, Stability of Linear systems, Academic Press, New York, 1980.
[43] D.S. Bernstein, Matrix Mathematics, Princeton University Press, Princeton, NJ, 2005.
[44] W.J. Rugh, Linear System Theory, 2nd ed., Prentice Hall, London, 1996.
[45] L. Perko, Differential Equations and Dynamical Systems, Springer, New York, 2006.