Random test examples with known minimum for convex semi-infinite programming problems

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Abstract

A significant research activity has occurred in the area of convex semi-infinite optimization in the recent years. Many new theoretical, algorithm and computational contribution has been obtained . Despite these numerous contributions, there still exits a lack of representative convex semi-infinite test problems. Test problems are of major importance for researchers interested in the algorithmic development. This article is motivated by the scarcity of convex semi-infinite test problems and describes a procedure for generating convex semi-infinite families of test problems with optimal solution and optimal value known.

Keywords: Convex Semi-infinite optimization

Mathematical Subject Classification (2000):

1 Introduction

The main objective of this paper is to give a collection of problems of semi-infinite convex programming.

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$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to:} & g(t,x) \leq 0, x \in Q \\ & t \in T \end{array} \tag{1}$$

where T is the compact metric space of the parameters and usually Q will be $\mathbb{R}^n$. We confine ourselves to ordinary convex semi-infinite programming problems of the form: $f : \mathbb{R}^n \to \mathbb{R}$ convex on $\mathbb{R}^n$ and level bounded on the feasible set $C := \{x \in \mathbb{R}^n : G(x) \leq 0\}$, where $G(x) := \max\{g_t(x) : t \in T\}$, and $g : T \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and the constraint functions g_t are convex on $\mathbb{R}^n$ for all $t \in T$. Moreover, the involved functions, f and g_t, $t \in T$, are assumed to be $\mathcal{C}^1$.

The interest of these examples leans on the practical difficulty to find explicit examples with some random initial parameters and with an arbitrary dimension with known solutions that can be used to explore the efficiency of the computational tools. In the paper [4] one can find a set of examples written with AMPL tool (the so-called SIPAMPL). These functions can be run with some difficulties through a NEOS server. Additionally we can use the AMPL student version that can be downloaded for free.

The examples are constructed via charts of a differentiable manifolds and using curves as parameter spaces T. When the analytical properties of these charts are "good enough", we obtain families of examples composing these charts with examples that work in $\mathbb{R}^n$. These examples can be produced randomly (by modifying some parameters of the smooth manifold considered). In this paper we concentrate on surface examples and we recall well-known charts for them. The ace of the hole is that the solutions of this semi-infinity programming examples that we consider can be determined in advance geometrically and thus we can test the accuracy of the results obtained and the precision of the procedure.

The main known references in this realm so far are contained in [4] and references therein. The implementations presented at the end of the paper have been inspired by the computational programming in [1] and are accurately described and implemented in [2].

2 First family of random examples with $\dim T = 1$

Let us recall the general use of spherical coordinates for the n-dimensional sphere S^n. In general, the relation $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$ between the parameters $x_1, x_2, \ldots, x_n$ can be described by using spherical coordinates. Indeed, consider the usual base $\{e_1, e_2, \ldots, e_n\}$ of the vectors of $\mathbb{R}^n$. Let $\vec{OP}$ be the vector position of the point $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and define $\omega_{n-1} = \text{ang}(\vec{OP}, e_n)$ (angle between the vectors $\vec{OP}$ and e_n), $\omega_{n-2} = \text{ang}(\vec{OP}_{n-1}, e_{n-1})$ where $\vec{OP}_{n-1}$ is the projection of the vector $\vec{OP}$ on $0+ < e_1, e_2, \ldots, e_{n-1} >$ parallel to the direction $< e_n >$, $\ldots$, $\omega_i = \text{ang}(\vec{OP}_{i+1}, e_{i+1})$

where $\vec{OP}_{i+1}$ is the projection of the vector $\vec{OP}$ on $0+ < e_1, e_2, \ldots, e_{i+1} >$ parallel to the direction $< e_{i+2}, \ldots, e_n >$, $\ldots$, $\omega_1 = \text{ang}(\vec{OP}_2, e_2)$ where $\vec{OP}_2$ is the projection of the vector $\vec{OP}$ on $0+ < e_1, e_2 >$ parallel to the direction $< e_3, \ldots, e_n >$. Then we can write

$$\left\{\begin{array}{lcr} x_1(t) & = & r\sin\omega_1\sin\omega_2\ldots\sin\omega_{n-2}\sin\omega_{n-1}, \\ x_2(t) & = & r\cos\omega_1\sin\omega_2\ldots\sin\omega_{n-2}\sin\omega_{n-1}, \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \qquad , \\ x_{n-2}(t) & = & r\cos\omega_{n-3}\sin\omega_{n-2}\sin\omega_{n-1}, \\ x_{n-1}(t) & = & r\cos\omega_{n-2}\sin\omega_{n-1}, \\ x_n(t) & = & r\cos\omega_{n-1}, \end{array}\right. \tag{2}$$

where $\omega_1 \in [0, 2\pi[$ and $\omega_i \in [0, \pi], i = 2,\ n-1$.

2.1 Curve for the test problem 1

From (2), we can consider the following parametrization: $\omega_1 = t$, $\omega_2 = \cdots = \omega_{n-1} = t/2$. Then we obtain,

$$\left\{\begin{array}{lcl} x_1(t) & = & r\sin(t)\sin(t/2)(n-1), \\ x_2(t) & = & r\cos(t)\sin(t/2)(n-1), \\ x_3(t) & = & r\cos(t/2)\sin(t/2)(n-2), \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ x_k(t) & = & r\cos(t/2)\sin(t/2)(n-k+1), \\ \ldots & \ldots & \ldots\ldots\ldots\ldots \\ x_{n-1}(t) & = & r\cos(t/2)\sin(t/2), \\ x_n(t) & = & r\cos(t/2), \end{array}\right.$$

where $t \in [0, 2\pi]$.

This curve is included in the sphere of radius R, with

$$R^2 = r^2(1 + n(n-2) + \frac{1}{6}(n-1)(n-2)(2n-3)).$$

Then we can reduce the curve to the sphere of radius r dividing for the norm of a point of the curve.

$$\|x(t)\| = \sqrt{r^2(1 + n(n-2)sin^2(t/2) + \frac{1}{6}(n-1)(n-2)(2n-3))sin^2(t/2)cos^2(t/2)}.$$

Then, the following curve for the test problems described in Section 2.3 is obtained,

$$\begin{cases} x_1(t) & = & \frac{r\sin(t)\sin(t/2)(n-1)}{\|x(t)\|}, \\ x_2(t) & = & \frac{r\cos(t)\sin(t/2)(n-1)}{\|x(t)\|}, \\ x_3(t) & = & \frac{r\cos(t/2)\sin(t/2)(n-2)}{\|x(t)\|}, \\ \dots & \dots & \dots\dots\dots \\ x_k(t) & = & \frac{r\cos(t/2)\sin(t/2)(n-k+1)}{\|x(t)\|}, \\ \dots & \dots & \dots\dots\dots \\ x_{n-1}(t) & = & \frac{r\cos(t/2)\sin(t/2)}{\|x(t)\|}, \\ x_n(t) & = & \frac{r\cos(t/2)}{\|x(t)\|}, \end{cases} \tag{3}$$

where $t \in [0, 2\pi]$.

2.2 Curve for the test problem 2

From (2), we can consider the above-mentionated parametrization: $\omega_1 = t$, $\omega_2 = \cdots = \omega_{n-1} = t/2$. Then we can modify the curve given in (3) in the form,

$$\begin{cases} x_1(t) & = & r\sin(t)\sin(t/2), \\ x_2(t) & = & r\cos(t)\sin(t/2), \\ x_3(t) & = & r\cos(t/2)\sin(t/2), \\ \dots & \dots & \dots\dots\dots \\ x_k(t) & = & r\cos(t/2)\sin(t/2), \\ \dots & \dots & \dots\dots\dots \\ x_{n-1}(t) & = & r\cos(t/2)\sin(t/2), \\ x_n(t) & = & r\cos(t/2), \end{cases}$$

where $t \in [0, 2\pi]$.

This curve is included in the sphere of radius R, with $R^2 = r^2(n-1)$ .As in section 2.1, we can reduce this curve to the sphere of radius r.

$$\|x(t)\| = r\sqrt{1 + (n-3)\cos^2(t/2)\sin^2(t/2)}.$$

Then, the following curve for the test problems described in Section 2.3 is obtained,

$$\begin{cases} x_1(t) & = & r\sin(t)\sin(t/2)/\|x(t)\|, \\ x_2(t) & = & r\cos(t)\sin(t/2)/\|x(t)\|, \\ x_3(t) & = & r\cos(t/2)\sin(t/2)/\|x(t)\|, \\ \dots & \dots & \dots\dots\dots \\ x_k(t) & = & r\cos(t/2)\sin(t/2)/\|x(t)\|, \\ \dots & \dots & \dots\dots\dots \\ x_{n-1}(t) & = & r\cos(t/2)\sin(t/2)/\|x(t)\|, \\ x_n(t) & = & r\cos(t/2)/\|x(t)\|, \end{cases} \tag{4}$$

where $t \in [0, 2\pi]$.

2.3 Generating optimization problems

From the curves 3 and 4 above-mentioned we can generate a convex semi-infinite test problem with known optimal solution and known optimal value in the following way,

i) Get $r > 0$ (in a randomized way)

ii) Given $t_0 \in [0, 2\pi]$ (in a randomized way), we can consider the point $(x_1^0, \ldots, x_n^0)$ with $x_i^0 = x_i(t_0)$, $i = 1, \ldots, n$. The tangent plane in this point is $x_1^0 x_1 + \cdots + x_n^0 x_n = r^2$.

iii) Consider the closed and bounded (thus compact) region.

$$x_1(t)x_1 + \cdots + x_n(t)x_n \leq r^2,\ t \in [0, 2\pi].$$

iv) Consider the point $(c_1, \ldots, c_n)$ with $c_i = kx_i^0$, $i = 1, \ldots, n$, with $(k > 0)$, obtained in a randomized way.

The convex semi-infinite optimization programming problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n (x_i - c_i)^2 \\ \text{subject to:} & x_1(t)x_1 + \cdots + x_n(t)x_n - r^2 \leq 0, \\ & t \in [0, 2\pi], \end{array} \tag{5}$$

has $(x_1^0, \ldots, x_n^0)$ as optimal solution and $k^2 r^2$ as objective value.

Observe also that the next possibility can also be considered in this set of problems,

The convex semi-infinite optimization programming problem

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n x_i^0 x_i \\ \text{subject to:} & x_1(t)x_1 + \cdots + x_n(t)x_n - r^2 \leq 0, \\ & t \in [0, 2\pi], \end{array} \tag{6}$$

has $(x_1^0, \ldots, x_n^0)$ as optimal solution and $-r^2$ as objective value.

In the first table below, we can see the practical implementation for $k = 1/r$ (for which the objective function is 1) this hold whenever $r < 1$.

If $r \geq 1$ then the distance is zero and the solution is zero.

In the second table, we are computing the maximum function of the tangent plane to a point of the sphere and for this one we obtain r^2.

2.4 Second family of random examples with $\dim T = 2$

From (2), we consider the parametrization: $\omega_1 = t_1$, $\omega_2 = \cdots = \omega_{n-1} = t_2$ to obtain the surface:

$$\begin{cases} x_1(t_1,t_2) & = & r\sin(t_1)\sin(t_2)/\|x(t_1,t_2)\|, \\ x_2(t_1,t_2) & = & r\cos(t_1)\sin(t_2)/\|x(t_1,t_2)\|, \\ x_3(t_1,t_2) & = & r\cos(t_2)\sin(t_2)/\|x(t_1,t_2)\|, \\ \dots & \dots & .... \\ x_k(t_1,t_2) & = & r\cos(t_2)\sin(t_2)/\|x(t_1,t_2)\|, \\ \dots & \dots & .... \\ x_{n-1}(t_1,t_2) & = & r\cos(t_2)\sin(t_2)/\|x(t_1,t_2)\|, \\ x_n(t_1,t_2) & = & r\cos(t_2)/\|x(t_1,t_2)\|, \end{cases}$$

where $(t_1,t_2) \in [0,2\pi] \times [0,\pi]$ and $\|x(t_1,t_2)\| = r\sqrt{1 + (n-3)\cos^2(t_2)\sin^2(t_2)}$.

Table 1: Results for quadratic objective function (the one in5) executed with a solver of convex semi-infinity programming up to dimension 1000. with $\epsilon = 0.001$ with 1-dimensional space of parameters, T.

Ndim	*Iter*	$f*$	*f-eval*	*g-eval*	*PrItGrad*	*CPU-time*	*stop-crt*
3	12	0.9993	8636	6494	5	0.296	7.285e-004
4	12	0.9993	11275	9032	5	0.406	7.362e-004
5	12	0.9992	14823	12362	5	0.546	7.513e-004
6	12	0.9992	18005	15440	6	0.671	7.639e-004
7	12	0.9992	21902	19170	6	0.797	7.738e-004
8	12	0.9992	23880	21230	6	0.953	7.816e-004
9	12	0.9992	26759	24085	6	1.156	7.868e-004
10	12	0.9992	29001	26365	6	1.281	7.907e-004
15	12	0.9992	44718	41920	7	2.124	8.076e-004
20	12	0.9992	64793	61703	8	3.733	8.331e-004
30	12	0.9992	91756	88789	7	6.156	8.116e-004
40	12	0.9992	130870	127670	8	11.062	7.870e-004
50	12	0.9992	149173	146239	7	13.983	7.762e-004
75	12	0.9992	240928	237748	8	30.451	7.756e-004
100	12	0.9992	243641	241218	6	36.748	7.759e-004
150	12	0.9992	420680	417883	7	90.933	7.517e-004
200	12	0.9993	583250	580337	7	154.496	7.373e-004
250	12	0.9993	672318	669628	7	208.106	7.263e-004
500	11	0.9990	1321775	1319126	7	751.427	9.926e-004
1000	11	0.9991	2437563	2435117	6	2623.110	9.437e-004

Table 2: Results for linear objective function (the one in6) executed with a solver of convex semi-infinity programming up to dimension 1000. with $\epsilon = 0.001$ with 1-dimensional space of parameters, T.

Ndim	*Iter*	$f*$	*f-eval*	*g-eval*	*PrItGrad*	*CPU-time*	*stop-crt*
3	9	-0.6407	9640	7255	10	0.296	6.666e-004
4	9	-0.6407	14212	11393	13	0.437	6.520e-004
5	8	-0.6409	23717	19793	17	0.750	9.065e-004
6	8	-0.6409	29508	25319	17	0.937	9.224e-004
7	8	-0.6409	24761	21680	15	0.796	8.510e-004
8	8	-0.6409	29420	26164	15	1.062	9.053e-004
9	8	-0.6409	44203	39800	20	1.688	8.961e-004
10	8	-0.6409	38294	34823	17	1.531	8.562e-004
15	9	-0.6406	64560	60530	16	2.890	6.207e-004
20	9	-0.6407	74922	71355	14	4.110	6.630e-004
30	8	-0.6410	136654	132246	18	8.719	9.763e-004
40	9	-0.6406	142392	138915	13	11.547	6.178e-004
50	8	-0.6409	204996	200973	19	18.345	9.188e-004
75	9	-0.6409	271980	268395	14	33.346	8.620e-004
100	9	-0.6406	370370	366696	16	53.847	5.675e-004
150	9	-0.6406	631567	627377	17	128.847	5.748e-004
200	9	-0.6406	888875	884445	16	225.520	6.008e-004
250	9	-0.6406	1147834	1143253	18	342.421	5.874e-004
500	8	-0.6409	2208663	2204247	19	1207.312	9.467e-004
1000	8	-0.6408	3489689	3486195	14	3761.142	7.707e-004

3 Other families of random examples

In this section we present an effective method to generate infinite families of examples which are given from constructions in the euclidean space $\mathbb{R}^k$ with associated compact regions. These constructions are exported to hypersurfaces on $\mathbb{R}^n$ using local diffeomorphisms. The main point of the construction consists in considering a smooth mapping between two fiber bundles: The tangent bundles of $\mathbb{R}^k$ and the tangent bundle to the hypersurface.

We then apply this method to construct families of examples on spheres of arbitrary dimension $\mathbb{S}^n$.

3.1 A smooth mapping between fiber bundles

An hypersurface H of $\mathbb{R}^n$ is a smooth submanifold of $\mathbb{R}^n$ of dimension $n-1$. In particular, it is a smooth manifold of dimension $n-1$ and therefore it is locally diffeomorphic to the euclidean space $\mathbb{R}^{n-1}$. This local diffeomorphism is classically established via a collection of local charts (U,ϕ) with U an open set of the manifold.

We require the existence of a collection of charts (U_i,ϕ_i) in such a way that $\cup_i U_i = H$ and $\phi_i : U_i \longrightarrow \mathbb{R}^{n-1}$ is an homeomorphism [1]. An additional compatibility condition is imposed on the collection (U_i,ϕ_i) that ensures the transition from topological manifold to a smooth manifold: the mappings $\phi_j \circ \phi_i^{-1}$ have to be a smooth mapping of $\mathbb{R}^{n-1}$. We may restrict the level of differentiability of our manifolds by requiring these mappings to be of $\mathcal{C}^k$ type.

These charts allow to transport smooth structures from $\mathbb{R}^{n-1}$ to the given manifold. In particular we may try to play the following game.

Consider a submanifold C of dimension k of $\mathbb{R}^{n-1}$. We may use the local charts to transport this submanifold onto the hypersurface $H \in \mathbb{R}^n$.

Consider now the following diagram:

$$C \xrightarrow{i} R^k \xrightarrow{\phi} H \subset R^n$$

where i stands for inclusion and ϕ for the local chart (we are assuming that the image of C is contained in an open subset U). These diagrams are local ones but are often sufficient to produce families of examples that take compact regions formed by tangent lines to the submanifold C to compact regions formed by tangent lines to the image by ϕ. The way to do it, is to visualize these tangent lines as elements of the tangent bundles of the corresponding manifolds. Recall that the tangent bundle of a manifold M, is defined by $T(M) = \{(p,v), p \in M, v \in T_p(M)\}$. Given a smooth map between manifolds M_1 and M_2, $f : M_1 \longrightarrow M_2$, the differential map d_pf at each point defines a smooth mapping of tangent spaces at the points: $d_pf : T_p(M_1) \longrightarrow T_{f(p)}(M_2)$ (these differentials can be easily computed via local charts).

By varying the point $p \in M_1$ we obtain a bundle map $df : T(M_1) \longrightarrow T(M_2)$ which is, indeed, a smooth map.

We may then consider the following diagram:

$$T(C) \xrightarrow{di} T(R^k) \xrightarrow{d\phi} T(H) \subset T(R^n)$$

where di is the bundle isomorphism given by the differential of the inclusion map and $d\phi$ is the bundle isomorphism given by the differential of a chart map.

[1] In the case the hypersurface $H \in \mathbb{R}^n$ is compact we can even assume this collection of charts is finite.

The composition $d\phi \circ di$ gives a smooth map $S : T(C) \longrightarrow T(H)$. In particular, since this mapping is continuous, the image of a compact set of $T(C)$ by S is a compact set in $T(H)$.

This proves the following,

Proposition 3.1 *A closed and bounded region on $T(C)$ is transformed on a compact and bounded region of $T(H)$ by diffeomorphisms.*

This proposition will be used in our *test-problems generating machinery.*

3.2 Example: The stereographic projection to the sphere

Consider $S^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + ... + x_{n+1}^2 = 1\}$ and consider the open sets

$$U_1 = S^n \setminus \{(0, ..., 0, 1)\} \quad \text{and} \quad U_2 = S^n \setminus \{(0, ..., 0, -1)\}$$

together with the stereographic mapping on the hyperplane $x_{n+1} = 0$

given by

$$\phi_1 : U_1 \longrightarrow \mathbb{R}^n \quad , \quad (x_1, ..., x_{n+1}) \mapsto (\frac{1}{1 - x_{n+1}})(x_1, ..., x_n),$$

$$\phi_2 : U_2 \longrightarrow \mathbb{R}^n \quad , \quad (x_1, ..., x_{n+1}) \mapsto (\frac{1}{1 + x_{n+1}})(x_1, ..., x_n).$$

These two charts cover the sphere and the mappings are convex thus we can compose any problem (for instance the ones considered in previous sections) that works in $\mathbb{R}^2$ to obtain examples on the sphere.

Remark 3.1 *Observe that the convexity is preserved for composition of convex functions.*

References

[1] A. Auslender, M.A. Goberna and M.A. López, *Penalty and smoothing methods for convex semi-infinite programming.* Math. Oper. Res. 34 (2009) 303-319.

[2] A. Auslender, A. Ferrer, M.A. Goberna and M.A. López, *Comparative Study of RPSALG algorith for convex semi-infinite programming*, work in progress, 2013.

[3] V. Guillemin and A. Pollack, *Differential topology* Prentice-Hall, New Jersey, 1974.

[4] A. Vaz, E. Fernandes and M. Gomes, SIPAMPL: Semi-infinite programming with AMPL. ACM Transactions on Mathematical Software 30 (2004) 47-61.