



Random Walk in Balls and an Extension of the Banach Integral in Abstract Spaces

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Abstract

We describe the construction of a random walk in a Banach space \mathbb{B} with a quasi-orthogonal Schauder basis and show that it is a martingale. Next we prove that under certain additional assumptions the described random walk converges a.s. and in $L^p(\mathbb{B})$, $1 \leq p < \infty$, to a random element ξ , which generates a probability measure with support contained in the unit ball $B \subset \mathbb{B}$. Moreover, we define the Banach integral with respect to the distribution of ξ for a class of bounded, Borel measurable real-valued functions on B . Next some examples of nonstandard Banach spaces with quasi-orthogonal Schauder bases are presented; furthermore, examples which demonstrate the possibility of applications of all the obtained results in spaces ℓ^p , $1 \leq p < \infty$ and $L^p[0, 1]$, $1 < p < \infty$ are given.

Keywords Banach random walk · Martingale · Radon–Nikodym property · Quasi-orthogonal Schauder basis

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1 Introduction

Let $\{e_i, i \geq 1\}$ be a complete orthonormal system (CONS) in a separable Hilbert space $(\mathbb{H}, |\cdot|)$ and let $\pi_n(x) = \sum_{i=1}^n x_i e_i$ for $x = \sum_{i=1}^\infty x_i e_i \in \mathbb{H}$. Denote by K_n and B the unit balls in \mathbb{R}^n and \mathbb{H} , respectively.

In a paper published as addendum to the Saks monograph *Theory of the Integral*, Banach [1] described the most general form of a nonnegative linear functional F (satisfying additional conditions analogous to some properties of the Lebesgue integral, thus called by Banach \mathcal{L} -integral), defined on the linear set \mathcal{L} of bounded, Borel measurable functions $\Phi: B \rightarrow \mathbb{R}$, namely

$$F(\Phi) = \lim_{n \rightarrow \infty} F_n(\Phi),$$

where

$$\begin{aligned}
 F_n(\Phi) &= \int_{K_n} \Phi(\pi_n(x)) g_n(x_1, \dots, x_n) dx_1 \dots dx_n, \\
 g_n(x_1, \dots, x_n) &= \mathbb{1}_{K_n}(x_1, \dots, x_n) \frac{g(x_1) g\left(x_2/\sqrt{1-x_1^2}\right) \dots g\left(x_n/\sqrt{1-(x_1^2+\dots+x_{n-1}^2)}\right)}{\sqrt{(1-x_1^2) \cdot [1-(x_1^2+x_2^2)] \cdot \dots \cdot [1-(x_1^2+\dots+x_{n-1}^2)]}},
 \end{aligned}$$

$g: [-1, 1] \rightarrow [0, \infty)$ is Borel measurable and integrable with $\int_{-1}^1 g(t) dt = 1$, and $\mathbb{1}_A$ denotes the indicator of the set A .

In fact, Banach [1] considered only the case when g is the density of the uniform distribution on $[-1, 1]$, and a more general case was treated by Banek [2]. Furthermore, Banek [2] observed that

$$F_n(\Phi) = E\Phi(Z_n),$$

where $\{Z_n\}$ is the so-called Banach random walk (BRW) in $B \subset \mathbb{H}$ given by the random linear combination $Z_n = \sum_{i=1}^n X_i e_i$, $n \geq 1$, of elements of CONS $\{e_i\}$ in \mathbb{H} , with coefficients X_i that are dependent r.v.'s defined recursively as follows: X_1 is a r.v. having density g concentrated on the interval $[-1, 1]$, and if the r.v.'s X_1, \dots, X_{n-1} are already defined, then X_n is defined as a r.v. with probability density $g\left(x_n/\sqrt{1-(X_1^2+\dots+X_{n-1}^2)}\right)$ in the random interval

$$\left[-\sqrt{1-(X_1^2+\dots+X_{n-1}^2)}, \sqrt{1-(X_1^2+\dots+X_{n-1}^2)}\right].$$

The last observation forms a probabilistic background to the purely deterministic Banach construction of the \mathcal{L} -integral for a class of bounded, Borel measurable functions defined in $B \subset \mathbb{H}$.

It is worth mentioning that Banach [1] considered two various special cases: (1) the mapping Φ is defined on a compact metric space and (2) Φ is defined in the unit ball of a separable Hilbert space.

In this paper, we describe a generalized BRW $\{Z_n, n \geq 1\}$ with values in the unit ball of a Banach space. Moreover, we give a criterion for the existence of the Banach–Lebesgue integral

$$F(\Phi) = \lim_{n \rightarrow \infty} F_n(\Phi) \quad (1)$$

in terms of the constructed BRW $\{Z_n, n \geq 1\}$, where Φ is a bounded, Borel measurable real-valued function defined in the unit ball of a Banach space.

2 Banach Random Walk in a Banach space

Let $(\mathbb{B}, \|\cdot\|)$ be an infinite-dimensional Banach space with a Schauder basis $\{b_n, n \geq 1\}$. Then each vector $x \in \mathbb{B}$ possesses a unique series expansion $x = \sum_{k=1}^{\infty} x_k b_k$, and thus, for all $n \geq 1$, the projections $\pi_n: \mathbb{B} \rightarrow \mathbb{B}$, given by $\pi_n(x) = \sum_{k=1}^n x_k b_k$, are well defined. Denote

$$B_n = \{\pi_n(x) \in \mathbb{B}: \|\pi_n(x)\| \leq 1\}, \quad B = \{x \in \mathbb{B}: \|x\| \leq 1\},$$

and put

$$\alpha_1 = \inf \{t \in \mathbb{R}: \|tb_1\| \leq 1\}, \quad \beta_1 = \sup \{t \in \mathbb{R}: \|tb_1\| \leq 1\} (= -\alpha_1).$$

Furthermore, given any point $\pi_{n-1}(x) \in B_{n-1}$, $n \geq 2$, define inductively

$$\begin{aligned} \alpha_n &= \alpha_n(\pi_{n-1}(x)) = \inf \{t \in \mathbb{R}: \|\pi_{n-1}(x) + tb_n\| \leq 1\}, \\ \beta_n &= \beta_n(\pi_{n-1}(x)) = \sup \{t \in \mathbb{R}: \|\pi_{n-1}(x) + tb_n\| \leq 1\}. \end{aligned}$$

It is clear that $\alpha_n \leq 0 \leq \beta_n$, and $[\alpha_n, \beta_n]$, $n \geq 1$, are bounded intervals in \mathbb{R} , for

$$\|\pi_{n-1}(x) + tb_n\| \leq 1 \Rightarrow |t| \cdot \|b_n\| \leq \|\pi_{n-1}(x)\| + 1, \quad n \geq 1,$$

where $\pi_{n-1}(x) = \pi_0(x) = 0$ for $n = 1$. Obviously, α_n and β_n depend on $\pi_{n-1}(x) \in B_{n-1}$ and b_n , and in general the intervals $[\alpha_n, \beta_n]$, $n \geq 2$, need not be symmetric about 0. In addition, it may happen that for some $n \geq 2$ the interval $[\alpha_n, \beta_n]$ reduces to the single point $[0, 0] = \{0\}$. To fix a standard length of the first interval, without loss of generality we may and do assume that $\|b_1\| = 1$ (but we do not require that $\|b_n\| = 1$ for all $n \geq 2$). In such a situation, $\alpha_1 = -1$ and $\beta_1 = 1$.

Let G_n , $n \geq 1$, be arbitrary probability distributions concentrated on $[-1, 1] \subset \mathbb{R}$, i.e., $G_n([-1, 1]) = 1$ for all $n \geq 1$. Define inductively on a probability space (Ω, \mathcal{F}, P) a sequence of (dependent) real r.v.'s $\{X_n, n \geq 1\}$ and, associated with it, a sequence of \mathbb{B} -valued random elements (r.e.) $\{Z_n, n \geq 1\}$. Namely, let X_1 be a r.v. with distribution G_1 and let $Z_1 = X_1 b_1$. Then $X_1(\omega) \in [\alpha_1, \beta_1] = [-1, 1]$ a.s., and thus, we may define X_2 as a r.v. distributed according to G_2 , scaled linearly in such a

way that it is concentrated on $[\alpha_2, \beta_2] = [\alpha_2(Z_1(\omega)), \beta_2(Z_1(\omega))]$. In other words, X_2 is a r.v. with distribution function

$$G_2 \left(\frac{2t - [\beta_2(Z_1(\omega)) + \alpha_2(Z_1(\omega))]}{\beta_2(Z_1(\omega)) - \alpha_2(Z_1(\omega))} \right), \quad t \in \mathbb{R},$$

whenever $\beta_2 - \alpha_2 > 0$, and then we put $Z_2 = X_1b_1 + X_2b_2$. Next, given any value $X_2(\omega)$, and *a fortiori* $Z_2(\omega) \in B_2$ a.s., we define X_3 as a r.v. with distribution G_3 scaled linearly in such a way that it is concentrated on the interval $[\alpha_3, \beta_3] = [\alpha_3(Z_2(\omega)), \beta_3(Z_2(\omega))]$, and then put $Z_3 = X_1b_1 + X_2b_2 + X_3b_3$, etc. More generally, if X_1, \dots, X_{n-1} and Z_1, \dots, Z_{n-1} are already defined in such a manner that $Z_{n-1}(\omega) \in B_{n-1}$ a.s., then X_n is a r.v. with distribution function

$$G_n \left(\frac{2t - [\beta_n(Z_{n-1}(\omega)) + \alpha_n(Z_{n-1}(\omega))]}{\beta_n(Z_{n-1}(\omega)) - \alpha_n(Z_{n-1}(\omega))} \right), \quad t \in \mathbb{R},$$

provided $\beta_n - \alpha_n > 0$ and $Z_n = X_1b_1 + X_2b_2 + \dots + X_nb_n = Z_{n-1} + X_nb_n$.

As was already mentioned, it may happen that for some $n \geq 1$ and $Z_n(\omega) \in B_n$ the interval $[\alpha_{n+1}, \beta_{n+1}] = [\alpha_{n+1}(Z_n(\omega)), \beta_{n+1}(Z_n(\omega))]$ is reduced to the single point $\{0\}$; in this situation, we assume that the distribution G_{n+1} is transformed in such a way that it assigns the unit mass to the one point set $\{0\}$. Although in such a case $Z_{n+1}(\omega) = Z_n(\omega)$, the next random interval $[\alpha_{n+2}, \beta_{n+2}] = [\alpha_{n+2}(Z_{n+1}(\omega)), \beta_{n+2}(Z_{n+1}(\omega))]$, defined by means of the successive basic vector b_{n+2} , need not be equal to $\{0\}$; thus, the process is still continued.

Definition 1 The sequence of \mathbb{B} -valued r.e.’s $\{Z_n, n \geq 1\}$ obtained in the way described above is called *Banach random walk* (BRW) in a Banach space \mathbb{B} .

It seems that the main idea of Banach’s [1] construction of \mathfrak{L} -integral was the symmetry of mappings corresponding to the symmetry of Lebesgue measures in $\mathbb{R}^n, n \geq 1$, which led to convergence of the integral functional in (1). Therefore, we introduce in addition the following notion:

Definition 2 The Schauder basis $\{b_n, n \geq 1\}$ is called *quasi-orthogonal*, if

$$\beta_{n+1}(\pi_n(x)) = -\alpha_{n+1}(\pi_n(x)) \tag{2}$$

for all $n \geq 1$ and $x \in \mathbb{B}$ such that $\pi_n(x) \in B_n$.

Recall that the basis $\{b_n, n \geq 1\}$ in a Banach space $(\mathbb{B}, \|\cdot\|)$ is said to be unconditional, if for all $n \geq 1, x_k \in \mathbb{R}$ and $\epsilon_k = \pm 1, 1 \leq k \leq n$, we have

$$\left\| \sum_{k=1}^n \epsilon_k x_k b_k \right\| = \left\| \sum_{k=1}^n x_k b_k \right\|. \tag{3}$$

The definition of a quasi-orthogonal Schauder basis in a Banach space seems to be similar to the condition defining an unconditional basis, but in spite of this, these two

notions are not equivalent. If the basis $\{b_n, n \geq 1\}$ is unconditional, i.e., (3) holds, then it is obviously quasi-orthogonal, but the converse need not be true. To explain the notion of quasi-orthogonality, below we present the construction of relevant examples of Banach spaces with quasi-orthogonal Schauder bases which are not unconditional. It should be pointed out that many familiar Banach spaces possess unconditional Schauder bases (thus in fact quasi-orthogonal) consisting of unit vectors, but the class of Banach spaces with quasi-orthogonal bases is substantially larger than the class of spaces with unconditional bases.

3 Banach Spaces with Quasi-Orthogonal Bases

The quasi-orthogonal basis is constructed sequentially, step by step: given any basic vectors b_1, b_2, \dots, b_n , the element b_{n+1} of the basis is chosen in such a way that for arbitrary $x_1, x_2, \dots, x_n \in \mathbb{R}$ satisfying condition $x_1b_1 + \dots + x_nb_n = \pi_n(x) \in B_n$, Eq. (2) is satisfied. As will be seen condition (3) for this property is not necessary.

1. Spaces of bounded sums and conditionally convergent series

Let $S = \mathbb{R}^{\mathbb{N}} = \{x = (x_1, x_2, \dots) : x_k \in \mathbb{R} \text{ for all } k \geq 1\}$ be the set of all infinite sequences of real numbers. Define a function $\|\cdot\| : S \rightarrow [0, \infty]$ by

$$\begin{aligned} \|x\| &= \sup \{|x_1 + x_2|, |x_1 - x_2|, |x_1 + x_2 + x_3|, |x_1 + x_2 - x_3|, \\ &\quad \dots, |x_1 + \dots + x_{n-1} + x_n|, |x_1 + \dots + x_{n-1} - x_n|, \dots\} \\ &= \sup \{|x_1| + |x_2|, |x_1 + x_2| + |x_3|, \dots, |x_1 + \dots + x_n| + |x_{n+1}|, \dots\}, \end{aligned}$$

and next put

$$S_b = \{x = (x_1, x_2, \dots) \in S : \|x\| < \infty\}.$$

Then $\|\cdot\|$ is a norm in S_b , and $(S_b, \|\cdot\|)$ is a Banach space. The space S_b consists of all bounded sequences $(x_1, x_2, \dots) \in S$ of real numbers with bounded partial sums $s_n = x_1 + \dots + x_n$; namely, if $x \in S_b$ and $\|x\| = M < \infty$, then $|x_n| \leq M$ and $|s_n| \leq M$ for all $n \geq 1$. Conversely, if there exists a constant $0 \leq M < \infty$ such that $|x_n| \leq M$ and $|s_n| \leq M$ for all $n \geq 1$, then $|s_n \pm x_{n+1}| \leq |s_n| + |x_{n+1}| \leq 2M$, and thus, $\|x\| \leq 2M$, i.e., $x = (x_1, x_2, \dots) \in S_b$. Therefore, $(S_b, \|\cdot\|)$ may be called the *space of bounded sums*.

However, the space S_b of sequences of real numbers is not separable. To show this, consider the family $2^{\mathbb{N}}$ of all the subsets of the set $\mathbb{N} = \{1, 2, \dots\}$, and for $\emptyset \neq A = \{k_1, k_2, \dots\} \subseteq \mathbb{N}$, where $k_1 < k_2 < \dots$, define x_A as the sequence with terms $x_{k_{2j-1}} = 1, x_{k_{2j}} = -1, j \geq 1$, and $x_i = 0$ otherwise, along with $x_\emptyset = (0, 0, \dots)$. Then $\|x_A\| \leq 2$, while $\|x_A - x_B\| \geq 1$ whenever $A, B \subseteq \mathbb{N}, A \neq B$. Since the set $2^{\mathbb{N}}$ is uncountable, the space S_b is nonseparable. Hence in the context of our requirements, the space S_b is inadequate.

Consider the set

$$S_c = \left\{ x = (x_1, x_2, \dots) \in S_b : \text{the series } \sum_k x_k \text{ converges} \right\}.$$

In other words, S_c is the set of all sequences $x = (x_1, x_2, \dots) \in S_b$ for which a finite limit $\lim_n s_n = s \in \mathbb{R}$ exists. It can be easily verified that $(S_c, \|\cdot\|)$ is also a Banach space. Moreover, the space S_c is separable. Indeed, the set of elements

$$\{e_n = (\delta_{kn}, k \geq 1), n = 1, 2, \dots\},$$

where $\delta_{kn} = 0$ for $k \neq n$ and $\delta_{nn} = 1$, is a basis of the space S_c , and finite linear combinations of vectors e_n with rational coefficients from a countable dense subset in S_c . Moreover, from the definition of the norm $\|\cdot\|$ it follows that the basis $\{e_n, n \geq 1\}$ is quasi-orthogonal, but it is not unconditional, because sums of the form $\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_n x_n, x = (x_1, x_2, \dots) \in S_c$, need not be convergent for all combinations of signs $\epsilon_k = \pm 1$. Taking into account the above properties, $(S_c, \|\cdot\|)$ may be called the Banach space of conditionally convergent series.

Since the existence of $\lim_n s_n = s \in \mathbb{R}$ implies that $\lim_n x_n = 0$, we conclude that $x \in S_c \Rightarrow x \in c_0$. The Banach space c_0 of sequences of real numbers convergent to 0 is usually considered with the supremum norm $|x|_\infty = \sup\{|x_1|, |x_2|, \dots\}$, but the two norms $\|\cdot\|$ and $|\cdot|_\infty$ restricted to S_c are not equivalent. To see this, consider the sequence of points $\{x^{(n)}, n \geq 1\}$,

$$\begin{aligned} x^{(1)} &= (1, -1/2, 1/3, -1/4, 1/5, -1/6, 1/7, -1/8, \dots), \\ x^{(2)} &= (1, 1/2, 1/3, -1/4, 1/5, -1/6, 1/7, -1/8, \dots), \\ x^{(3)} &= (1, 1/2, 1/3, 1/4, 1/5, -1/6, 1/7, -1/8, \dots), \\ &\vdots \end{aligned}$$

and put $x^{(\infty)} = (1, 1/2, 1/3, 1/4, 1/5, 1/6, \dots, 1/n, 1/(n+1), \dots)$.

Then $\|x^{(n)}\| < \infty$ for all $n \geq 1$, so that $\{x^{(n)}, n \geq 1\} \subset S_c$. Moreover, $|x^{(n)} - x^{(\infty)}|_\infty = 1/n \rightarrow 0$, while $\|x^{(n)} - x^{(\infty)}\| = \infty, n = 1, 2, \dots$ which is a consequence of the fact that $\sum_n 1/n = \infty$. Therefore, the inclusion $S_c \subset c_0$ is valid only for sets, but it is not true for Banach spaces, $(S_c, \|\cdot\|) \not\subset (c_0, |\cdot|_\infty)$.

A similar effect as in the case of the space S_c can be obtained for every fixed system of signs $\epsilon = (\epsilon_1, \epsilon_2, \dots) \in \{-1, 1\}^{\mathbb{N}}$ and the norm given by

$$\|x\|_\epsilon = \|(\epsilon_1 x_1, \epsilon_2 x_2, \epsilon_3 x_3, \dots)\| \text{ for } x = (x_1, x_2, \dots).$$

In this way, we obtain a Banach space $(S_{c,\epsilon}, \|\cdot\|_\epsilon)$, where

$$S_{c,\epsilon} = \left\{ x = (x_1, x_2, \dots) \in S : \|x\|_\epsilon < \infty \text{ and the series } \sum_k \epsilon_k x_k \text{ converges} \right\}.$$

The basis $\{e_n, n \geq 1\}$ in $S_{c,\epsilon}$ is quasi-orthogonal, but it is not unconditional. Note now that $\ell^1 \subseteq S_{c,\epsilon}$ for each $\epsilon \in \{-1, 1\}^{\mathbb{N}}$, therefore $\ell^1 \subseteq \bigcap_{\epsilon \in \{-1, 1\}^{\mathbb{N}}} S_{c,\epsilon}$. On the other hand, assuming that $x = (x_1, x_2, \dots) \in \bigcap_{\epsilon \in \{-1, 1\}^{\mathbb{N}}} S_{c,\epsilon}$ and taking $\epsilon^{(x)} = (\text{sign } x_1, \text{sign } x_2, \dots, \text{sign } x_n, \dots)$ we have

$$\epsilon_1^{(x)}x_1 + \epsilon_2^{(x)}x_2 + \dots + \epsilon_n^{(x)}x_n = |x_1| + |x_2| + \dots + |x_n|, \quad n \geq 1.$$

Hence, we infer that

$$\sum_{n=1}^{\infty} |x_n| = \sup_n \left\{ \left| \epsilon_1^{(x)}x_1 + \epsilon_2^{(x)}x_2 + \dots + \epsilon_n^{(x)}x_n \right| \right\} \leq \|x\|_{\epsilon^{(x)}},$$

where $\epsilon^{(x)} \in \{-1, 1\}^{\mathbb{N}}$, thus $x \in \ell^1$. Consequently, $\ell^1 = \bigcap_{\epsilon \in \{-1, 1\}^{\mathbb{N}}} S_{c,\epsilon}$.

Define next a function $\|\cdot\|_{\cap} : \bigcap_{\epsilon \in \{-1, 1\}^{\mathbb{N}}} S_{c,\epsilon} \rightarrow [0, \infty]$ by the formula: $\|x\|_{\cap} = \sup \{ \|x\|_{\epsilon}, \epsilon \in \{-1, 1\}^{\mathbb{N}} \}$. Since

$$|\epsilon_1x_1 + \epsilon_2x_2 + \dots + \epsilon_{n-1}x_{n-1} \pm \epsilon_nx_n| \leq |x_1| + |x_2| + \dots + |x_n|$$

for each $\epsilon \in \{-1, 1\}^{\mathbb{N}}$, where the inequality \leq may be replaced by the equality whenever $\epsilon_k = \text{sign } x_k, 1 \leq k \leq n - 1$ and $\pm \epsilon_n = \text{sign } x_n$, the function $\|\cdot\|_{\cap}$ assumes only finite values and in fact $\|x\|_{\cap} = \sum_n |x_n|$, i.e., $\|\cdot\|_{\cap}$ is the norm equal precisely to the norm $|x|_1 = \sum_n |x_n|$ of the space ℓ^1 . Therefore, one can write

$$\left(\ell^1, |\cdot|_1 \right) = \bigcap_{\epsilon \in \{-1, 1\}^{\mathbb{N}}} (S_{c,\epsilon}, \|\cdot\|_{\epsilon}).$$

In this sense, the Banach space $(\ell^1, |\cdot|_1)$ in comparison with the space $(S_c, \|\cdot\|)$ is “relatively small”. It is also worth mentioning that the basis $\{e_n, n \geq 1\}$ in S_c (as well as in $S_{c,\epsilon}$) is monotone, i.e., for every choice of scalars $\{x_n, n \geq 1\} \subseteq \mathbb{R}$, the sequence of numbers $\{ \|\sum_{k=1}^n x_k e_k\|, n \geq 1 \}$ is nondecreasing.

Recall now that a basis $\{b_n, n \geq 1\}$ of a Banach space $(\mathbb{B}, \|\cdot\|)$ is called boundedly complete if, for every sequence of scalars $\{x_n, n \geq 1\} \subseteq \mathbb{R}$ such that $\sup_{n \geq 1} \|\sum_{k=1}^n x_k b_k\| < \infty$, the series $\sum_{n=1}^{\infty} x_n b_n$ converges in norm of \mathbb{B} . Unfortunately, the basis $\{e_n, n \geq 1\}$ in S_c or $S_{c,\epsilon}$ is not boundedly complete.

2. Spaces of conditionally convergent series with rates of convergence

The space S_c described here may be the prototype for a wide class of various spaces with quasi-orthogonal Schauder bases that are not unconditional. For instance, consider the spaces S_c^p of (conditionally) summable sequences of real numbers spanned on vectors of the basis $\{e_n, n \geq 1\}$, with norms likewise in $\ell^p, 1 \leq p < \infty$, (that describe rates of convergence)

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \|R_n(x)\|^p \right)^{1/p},$$

where $R_n(x) = \sum_{k=n}^{\infty} x_k e_k$ for $x = (x_1, x_2, \dots) \in S$; spaces $S_{c,w}$ with norms determined by some positive weights $w = (w_1, w_2, \dots)$, $w_i > 0$,

$$\|x\|_{1,w} = \sum_{n=1}^{\infty} w_n \|R_n(x)\|,$$

say geometrical weights $w = (w_1, w_2, \dots) = (q^1, q^2, \dots)$, $q > 0$, or more generally, spaces $S_{c,w}^p$ equipped with norms

$$\|x\|_{p,w} = \left(\sum_{n=1}^{\infty} w_n \|R_n(x)\|^p \right)^{1/p}, \quad 1 \leq p < \infty, \text{ etc.}$$

Since convergence of the series on the right-hand side of the definition of $\|x\|_p$ implies that partial sums of the series $\sum_n x_n$ satisfy Cauchy’s criterion, the basis $\{e_n, n \geq 1\}$ in $(S_c^p, \|\cdot\|_p)$ is evidently monotone and boundedly complete. Thus in view of Dunford’s theorem [4, Ch. III, §1, Th. 6, p. 64], these spaces possess the Radon–Nikodym property (RNP)—see [4, Ch. III, §1, p. 61] for the definition of this notion. Moreover, the equality

$$\left\| \sum_{j=k}^n x_j e_j \right\| = \left\| \sum_{j=k}^{n-1} x_j e_j - x_n e_n \right\|$$

valid for all $(x_1, x_2, \dots) \in S$ and $1 \leq k < n < \infty$ implies that the basis $\{e_n, n \geq 1\}$ in $(S_c^p, \|\cdot\|_p)$ is again quasi-orthogonal, but it is not unconditional. A similar conclusion can be also derived for spaces equipped with norms $\|\cdot\|_{p,w}$, provided that w is a suitable sequence of weights.

3. Spaces of bounded sums of functions and conditionally convergent function series

Let $\{q_1 = 0, q_2 = 1, q_3, q_4, \dots\} \subset [0, 1] \subset \mathbb{R}$ be a countable set of numbers dense in $[0, 1]$ (arranged in any order), and let $\{e_n(t), n \geq 1\}$ be the system of Schauder hat functions in $[0, 1]$ defined as follows: $e_1(t) = 1 - t$ and $e_2(t) = t$, $0 \leq t \leq 1$; for $n > 2$ the points q_1, \dots, q_{n-1} divide the interval $[0, 1]$ into $n - 2$ subintervals, and if $[q_i, q_j]$ is the subinterval which contains the point q_n , then $e_n(t) = 0$ for $t \in [0, q_i] \cup [q_j, 1]$, $e_n(q_n) = 1$, and e_n is a linear function in the interval $[q_i, q_n]$ as well as in $[q_n, q_j]$. It is known that the described system of Schauder hat functions forms a basis for the space $C[0, 1]$ of real-valued continuous functions in $[0, 1]$ with the supremum norm

$$\|f\|_{\infty} = \sup_{0 \leq t \leq 1} |f(t)|,$$

see, e.g., [10, Prop. 2.3.5, p. 29].

Now with every sequence $x = (x_1, x_2, \dots) \subset \mathbb{R}^{\mathbb{N}}$, we associate an element g having coefficients (x_1, x_2, \dots) in the basis $\{e_n(t), n \geq 1\}$, formally written as $g := (x_1 e_1 + x_2 e_2 + \dots)$, and define

$$\begin{aligned} \|g\| &= \left\| \sup \{ |x_1 e_1 + x_2 e_2|, |x_1 e_1 - x_2 e_2|, \dots, |x_1 e_1 + \dots + x_{n-1} e_{n-1} + x_n e_n|, \right. \\ &\quad \left. |x_1 e_1 + \dots + x_{n-1} e_{n-1} - x_n e_n|, \dots \} \right\|_\infty \\ &= \left\| \sup_{n \geq 1} \{ |x_1 e_1 + \dots + x_n e_n| + |x_{n+1} e_{n+1}| \} \right\|_\infty, \end{aligned}$$

along with

$$S_b [0, 1] = \{g = (x_1 e_1 + x_2 e_2 + \dots) : \|g\| < \infty\}.$$

Then $(S_b [0, 1], \|\cdot\|)$ is a (nonseparable) Banach space. The space $S_b [0, 1]$ consists of elements $g = (x_1 e_1 + x_2 e_2 + \dots)$ with finite sums of functions $x_1 e_1 + x_2 e_2 + \dots + x_n e_n$ bounded uniformly in $0 \leq t \leq 1$ and $n \geq 1$, but not necessarily convergent function series $x_1 e_1 + x_2 e_2 + \dots$.

Furthermore, denote

$$S_c [0, 1] = \left\{ g = (x_1 e_1 + x_2 e_2 + \dots) \in S_b [0, 1] : \begin{array}{l} \text{the series } \sum_n x_n e_n \\ \text{converges in norm } \|\cdot\| \end{array} \right\}.$$

It can be shown that $S_c [0, 1]$ as a set of functions is identically equal to $C[0, 1]$, and on account of the well-known open mapping theorem, the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$; therefore, $(S_c [0, 1], \|\cdot\|)$ treated as a function Banach space with its norm topology is the same as $(C[0, 1], \|\cdot\|_\infty)$. Since finite linear combinations $\sum_{k=1}^n w_k e_k$ with rational coefficients form a countable dense set in $C[0, 1]$, the space $S_c [0, 1]$ is separable. The Schauder basis $\{e_n(t), n \geq 1\}$ in $(S_c [0, 1], \|\cdot\|)$ is monotone, because finite linear combinations of hat functions $e_n(t), n \geq 1$, are piecewise linear with an increasing number of nodes. However, the basis $\{e_n(t), n \geq 1\}$ is not boundedly complete, for bounded finite linear combinations of basic functions need not define a conditionally uniformly convergent series of functions. Moreover, the basis $\{e_n(t), n \geq 1\}$ in $S_c [0, 1]$ is quasi-orthogonal with respect to $\|\cdot\|$, but it is not quasi-orthogonal in $(C[0, 1], \|\cdot\|_\infty)$, and it is not unconditional. More precisely, properties of a given Schauder basis in $C[0, 1]$ depend on the shape of the unit sphere, and from our considerations it follows that for each Schauder basis $\{e_n(t), n \geq 1\}$ there can be defined a norm $\|\cdot\|$ equivalent to the original supremum norm $\|\cdot\|_\infty$ in $C[0, 1]$, such that the basis $\{e_n(t), n \geq 1\}$ becomes quasi-orthogonal with respect to $\|\cdot\|$, although the same basis need not be quasi-orthogonal with respect to $\|\cdot\|_\infty$.

By analogy to $S_{c,\epsilon}$ one can now define the spaces $S_{c,\epsilon} [0, 1]$ consisting of function series $\sum_n \epsilon_n x_n e_n$ convergent (conditionally) uniformly in $0 \leq t \leq 1$ with norms

$$\|(x_1 e_1 + x_2 e_2 + \dots)\|_\epsilon = \|(\epsilon_1 x_1 e_1 + \epsilon_2 x_2 e_2 + \dots)\|,$$

for all sequences of signs $\epsilon = (\epsilon_1, \epsilon_2, \dots) \in \{-1, 1\}^{\mathbb{N}}$. The intersection

$$\bigcap_{\epsilon \in \{-1, 1\}^{\mathbb{N}}} S_{c, \epsilon} [0, 1] := \ell^1 [0, 1]$$

is then the Banach space of function series convergent absolutely uniformly in $0 \leq t \leq 1$, with the norm

$$\sup \left\{ \|(x_1 e_1 + x_2 e_2 + \dots)\|_{\epsilon} : \epsilon \in \{-1, 1\}^{\mathbb{N}} \right\} = \sup_{0 \leq t \leq 1} \sum_{n=1}^{\infty} |x_n e_n(t)|.$$

The last formula follows from the fact that Schauder hat functions $e_n(t)$, $n \geq 1$, are nonnegative, and in a more general case this is a consequence of a theorem by Sierpiński, cf. [10, Prop. 1.5.7, p. 19]. Clearly, the basis $\{e_n(t), n \geq 1\}$ in $\ell^1 [0, 1]$ is quasi-orthogonal, monotone, unconditional and boundedly complete; thus, $\ell^1 [0, 1]$ possesses the RNP, cf. [4, Ch. III, §1, Th. 6, p. 64 and Ch. III, §3, Corollary 8, p. 83].

To estimate the rate of (uniform) convergence of function series of the form $g(t) = \sum_{n=1}^{\infty} x_n e_n(t)$, $t \in [0, 1]$, we may introduce various norms similar as in ℓ^p , $1 \leq p < \infty$; namely, let $(R_n g)(t) = \sum_{k \geq n} x_k e_k(t)$, and let

$$S_c^p [0, 1] = \left\{ g = \sum_{n=1}^{\infty} x_n e_n(t) \in S_c [0, 1] : \|g\|_p = \left(\sum_{n=1}^{\infty} \|(R_n g)\|^p \right)^{1/p} < \infty \right\}.$$

Arguing as in example 2, we conclude that the basis $\{e_n(t), n \geq 1\}$ in the space $(S_c^p [0, 1], \|\cdot\|_p)$ is quasi-orthogonal, but it is not unconditional. Moreover, the basis $\{e_n(t), n \geq 1\}$ in $(S_c^p [0, 1], \|\cdot\|_p)$ is monotone and boundedly complete; thus, by Dunford’s theorem the spaces $(S_c^p [0, 1], \|\cdot\|_p)$ possess the RNP, see [4, Ch. III, §1, Th. 6, p. 64].

Remark 1 The same idea as above leads to other examples of Banach spaces of a similar kind. For example, let $I = \{i_n, n \geq 1\}$ be a sequence of positive integers such that $1 \leq i_n \leq n$, $i_n \nearrow \infty$ and $n - i_n \nearrow \infty$ as $n \rightarrow \infty$, and for $x = \{x_n\} \in \mathbb{R}^{\infty}$, let

$$\|x\|_{I,p} = \left\{ \sum_{n=1}^{\infty} (|x_{i_n} + \dots + x_n| + |x_{n+1}|)^p \right\}^{1/p},$$

where $1 \leq p < \infty$, and

$$\|x\|_{I,\infty} = \sup_{n \geq 1} \{|x_{i_n} + \dots + x_n| + |x_{n+1}|\}.$$

Define

$$\mathbb{B}_{I,p} = \left\{ x = \{x_n\} \in \mathbb{R}^{\infty} : \text{the series } \sum_n x_n e_n \text{ converges in norm } \|\cdot\|_{I,p} \right\},$$

$1 \leq p \leq \infty$. Then $\mathbb{B}_{I,p}$, $1 \leq p \leq \infty$, are Banach spaces such that $\{e_n, n \geq 1\}$ is a quasi-orthogonal basis with respect to $\|\cdot\|_{I,p}$, but in general not unconditional. Moreover, for $1 \leq p < \infty$ the basis $\{e_n, n \geq 1\}$ in $\mathbb{B}_{I,p}$ is boundedly complete, and thus, these spaces possess the RNP, cf. [4, Ch. III, §1, Th. 6, p. 64]. Next, if $I^{(1)} = \{i_n^{(1)}, n \geq 1\}$, $I^{(2)} = \{i_n^{(2)}, n \geq 1\}$, \dots , $I^{(k)} = \{i_n^{(k)}, n \geq 1\}$ are some sequences of positive integers satisfying conditions: $1 \leq i_n^{(1)} < \dots < i_n^{(k)} \leq n$, $i_n^{(j)} \nearrow \infty$, $1 \leq j \leq k$, $i_n^{(j+1)} - i_n^{(j)} \nearrow \infty$, $1 \leq j < k$, and $n - i_n^{(k)} \nearrow \infty$ as $n \rightarrow \infty$, then for $n - i_n \geq k \geq 2$ (replacing i_n by $i_n^{(1)}$), the sums $|x_{i_n} + \dots + x_n| + |x_{n+1}|$ can be divided into blocks of the form

$$\left| x_{i_n^{(1)}} + \dots + x_{i_n^{(2)}-1} \right| + \left| x_{i_n^{(2)}} + \dots + x_{i_n^{(3)}-1} \right| + \dots + \left| x_{i_n^{(k)}} + \dots + x_n \right| + |x_{n+1}|;$$

in addition, for $i \geq 1$, instead of $x_i e_i$, the terms $\epsilon_i x_i e_i$ can be used, where $\{\epsilon_i, i \geq 1\}$ are various sequences of signs ± 1 . By means of these expressions, in an analogous way as above, new norms and new Banach spaces with quasi-orthogonal bases can be defined.

4 Convergence of the Banach Random Walk

According to the standard terminology, the measure G on the Borel σ -field $\mathcal{B}(\mathbb{R})$ is called here symmetric, if $G(-A) = G(A)$ for all $A \in \mathcal{B}(\mathbb{R})$.

Lemma 1 *Let $\{b_n, n \geq 1\}$ be a quasi-orthogonal Schauder basis in a real Banach space $(\mathbb{B}, \|\cdot\|)$, and let $\{G_n, n \geq 1\}$ be a sequence of symmetric probability distributions concentrated on the interval $[-1, 1] \subset \mathbb{R}$. Then the BRW $\{Z_n, n \geq 1\}$ is a \mathbb{B} -valued martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n) = \sigma(Z_1, Z_2, \dots, Z_n)$, $n \geq 1$.*

Proof From the construction of BRW in a Banach space, it follows that $Z_n \in B$ for all $n \geq 1$, i.e., the r.e.'s Z_n are bounded and therefore Bochner integrable. Moreover, each Z_n is \mathcal{F}_n -measurable; thus,

$$E[Z_{n+1} | \mathcal{F}_n] = E[Z_n + X_{n+1} b_{n+1} | \mathcal{F}_n] = Z_n + E[X_{n+1} | \mathcal{F}_n] b_{n+1} \text{ a.s.}$$

Hence it suffices to show the equality $E[X_{n+1} | \mathcal{F}_n] = 0$ a.s. But the last statement is obvious, since it is known that X_{n+1} possesses a symmetric distribution in the a.s. symmetric interval $[\alpha_{n+1}(Z_n(\omega)), \beta_{n+1}(Z_n(\omega))] = [-\beta_{n+1}(Z_n(\omega)), \beta_{n+1}(Z_n(\omega))]$. □

Lemma 2 *Let $\{Z_n, n \geq 1\}$ be the BRW in a Banach space \mathbb{B} with a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$ and let $\{\mathcal{F}_n, n \geq 1\}$ be the defined above filtration associated with $\{Z_n, n \geq 1\}$. Denote by \mathcal{F}_∞ the σ -field generated by the field $\bigcup_{n=1}^\infty \mathcal{F}_n$, i.e., $\mathcal{F}_\infty = \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$. Then the following statements are true:*

(a) *There exists a vector measure $\nu: \mathcal{F}_\infty \rightarrow \mathbb{B}$ with bounded variation, absolutely continuous with respect to P , such that*

$$\langle Z_n, x^* \rangle \rightarrow \frac{d \langle \nu, x^* \rangle}{dP} \text{ a.s. for all } x^* \in \mathbb{B}^*,$$

(b) *If there is a r.e. $\xi \in L^1(\mathcal{F}_\infty; \mathbb{B})$ such that*

$$\langle Z_n, x^* \rangle \rightarrow \langle \xi, x^* \rangle \text{ a.s.}$$

for each $x^* \in \mathbb{B}^*$, then

$$\|Z_n - \xi\| \rightarrow 0 \text{ a.s.}$$

Proof Since the BRW $\{Z_n, n \geq 1\}$ satisfies the condition $Z_n \in B, n \geq 1$, we have $\sup_{n \geq 1} E \|Z_n\| \leq 1 < \infty$. Therefore, our result is a direct consequence of a theorem given by Stegall, which can be found in [11, Ch. II, §4.3, Prop. 4.3, p. 132]. \square

Lemma 3 *For each set $A \in \mathcal{F}_\infty$ there exists*

$$\lim_{n \rightarrow \infty} \int_A Z_n dP = V(A)$$

in the strong topology of \mathbb{B} , and the mapping $V: \mathcal{F}_\infty \rightarrow \mathbb{B}$ is a countably additive vector measure.

Proof Observe first that in view of Jensen’s inequality for conditional expectations in a Banach space, the sequence $\{\|Z_n\|, \mathcal{F}_n, n \geq 1\}$ is a real-valued submartingale, cf. [11, Ch. II, §4.1, (g), p. 127], or [12]. Furthermore,

$$\sup_{n \geq 1} E \|Z_n\|^p \leq 1 < \infty \text{ for each } 1 \leq p < \infty,$$

so that r.v.’s $\{\|Z_n\|, n \geq 1\}$ are uniformly integrable, which implies a.s. convergence $\|Z_n\| \rightarrow Z_\infty$ (and in L^1), where $Z_\infty \in L^p = L^p(\mathbb{R})$ for every fixed $1 \leq p < \infty$, see, e.g., [8, Ch. IV, Th. IV-1-2, p. 62, and Prop. IV-5-24, p. 91]. In particular,

$$\int_A \|Z_n\| dP \rightarrow \int_A Z_\infty dP$$

for each measurable set $A \in \mathcal{F}$. Next, if $B \in \bigcup_n \mathcal{F}_n$, then by the martingale property of $\{Z_n, n \geq 1\}$,

$$\int_B Z_n dP \rightarrow V(B)$$

strongly in \mathbb{B} . Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be chosen in such a way that $\int_C Z_\infty dP < \varepsilon/3$ whenever $C \in \mathcal{F}_\infty$ and $P[C] < \delta$. Given any set $A \in \mathcal{F}_\infty$, select $B \in \bigcup_n \mathcal{F}_n$ satisfying condition $P[A \div B] < \delta$. Obviously,

$$\begin{aligned} \left\| \int_A Z_n dP - \int_A Z_m dP \right\| &\leq \left\| \int_A Z_n dP - \int_B Z_n dP \right\| \\ &+ \left\| \int_B Z_n dP - \int_B Z_m dP \right\| + \left\| \int_B Z_m dP - \int_A Z_m dP \right\|. \end{aligned}$$

Moreover,

$$\left\| \int_A Z_n dP - \int_B Z_n dP \right\| \leq \int_{A \setminus B} \|Z_n\| dP + \int_{B \setminus A} \|Z_n\| dP = \int_{A \div B} \|Z_n\| dP.$$

Passing to the limit as $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} \int_{A \div B} \|Z_n\| dP = \int_{A \div B} Z_\infty dP < \varepsilon/3,$$

thus

$$\left\| \int_A Z_n dP - \int_B Z_n dP \right\| < \varepsilon/3$$

for sufficiently large $n \geq n_0$. Since the sequence $\{\int_B Z_n dP, n \geq 1\}$ is Cauchy in \mathbb{B} , we also conclude that

$$\left\| \int_B Z_n dP - \int_B Z_m dP \right\| < \varepsilon/3$$

for all large enough $m > n \geq n_1$. Consequently,

$$\left\| \int_A Z_n dP - \int_A Z_m dP \right\| < \varepsilon,$$

whenever $m > n \geq \max\{n_0, n_1\}$. In other words, the sequence of integrals $\{\int_A Z_n dP, n \geq 1\}$ is Cauchy in $(\mathbb{B}, \|\cdot\|)$, and therefore, there exists

$$\lim_{n \rightarrow \infty} \int_A Z_n dP = V(A)$$

in the strong topology of \mathbb{B} for each set $A \in \mathcal{F}_\infty$. It can be easily seen that $V: \mathcal{F}_\infty \rightarrow \mathbb{B}$ is finitely additive. Let $A_1, A_2, \dots \in \mathcal{F}_\infty$ be an arbitrary sequence of pairwise disjoint sets. Notice that

$$\left\| \lim_{n \rightarrow \infty} \int_{\bigcup_{k=1}^\infty A_k} Z_n dP - \lim_{n \rightarrow \infty} \int_{\bigcup_{k=1}^m A_k} Z_n dP \right\| = \lim_{n \rightarrow \infty} \left\| \int_{\bigcup_{k=m+1}^\infty A_k} Z_n dP \right\|,$$

and thus, to prove countable additivity of V it is enough to show that

$$\lim_{n \rightarrow \infty} \left\| \int_{\bigcup_{k=m+1}^{\infty} A_k} Z_n dP \right\| \rightarrow 0$$

as $m \rightarrow \infty$. Taking m_0 so large that $P \left[\bigcup_{k=m+1}^{\infty} A_k \right] < \delta$ for $m \geq m_0$, we obtain

$$\lim_{n \rightarrow \infty} \left\| \int_{\bigcup_{k=m+1}^{\infty} A_k} Z_n dP \right\| \leq \lim_{n \rightarrow \infty} \int_{\bigcup_{k=m+1}^{\infty} A_k} \|Z_n\| dP = \int_{\bigcup_{k=m+1}^{\infty} A_k} Z_{\infty} dP < \varepsilon/3$$

provided $m \geq m_0$, which terminates the proof. □

The above Lemma 3 enables us to apply the Lebesgue decomposition theorem for the vector measure V , see [4, Ch. I, §5, Th. 9, p. 31].

Lemma 4 *Let*

$$V(A) = \lim_{n \rightarrow \infty} \int_A Z_n dP, \quad A \in \mathcal{F}_{\infty},$$

and let $V = H + J$, $|H| \ll P$, $|J| \perp P$, be the Lebesgue decomposition of V with respect to P , where $|H|$, $|J|$ are variations of H and J , respectively. Then $\lim_{n \rightarrow \infty} Z_n$ exists a.s. if and only if H has a Radon–Nikodym derivative $h \in L^1(\mathcal{F}; \mathbb{B})$. Moreover, in this case $\lim_{n \rightarrow \infty} Z_n = E(h|\mathcal{F}_{\infty})$ a.s.

Proof Arguing as above, we easily note that $\{Z_n, \mathcal{F}_n, n \geq 1\}$ is an $L^1(\mathbb{B})$ -bounded martingale (here, and in the sequel $L^p(\mathbb{B}) = L^p(\Omega, \mathcal{F}, P; \mathbb{B})$, $1 \leq p < \infty$); thus, the conclusion follows from the martingale pointwise convergence theorem given in [4, Ch. V, §2, Th. 9, p. 130]. □

Lemma 5 *Let \mathbb{B} be a Banach space with the RNP and a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$. Moreover, let $\{G_n, n \geq 1\}$ be a sequence of symmetric probability distributions concentrated on $[-1, 1] \subset \mathbb{R}$. Then the BRW martingale $\{Z_n, n \geq 1\}$ converges strongly a.s. and in $L^p(\mathbb{B})$ for each fixed $1 \leq p < \infty$.*

Proof The limit $\lim_{n \rightarrow \infty} Z_n$ of the martingale $\{Z_n, n \geq 1\}$ exists in $L^p(\mathbb{B})$ -norm, if and only if $\sup_{n \geq 1} \|Z_n\|_p^p = \sup_{n \geq 1} E \|Z_n\|^p < \infty$, where $1 < p < \infty$, which is evident as $Z_n \in B$ for $n \geq 1$. The last observation implies uniform integrability of random elements $\{Z_n, n \geq 1\}$, and we have obviously $\sup_{n \geq 1} \|Z_n\|_1 = \sup_{n \geq 1} E \|Z_n\| \leq 1 < \infty$. Thus $\lim_{n \rightarrow \infty} Z_n$ exists as well in $L^1(\mathbb{B})$ -norm in view of the martingale mean convergence theorem, cf. [4, Ch. V, §2, Corollary 4, p. 126]. It is also well known that an $L^1(\mathbb{B})$ convergent martingale converges a.s. to its $L^1(\mathbb{B})$ -limit, see [4, Ch. V, §2, Th. 8, p. 129], or [11, Ch. II, §4.3, Th. 4.2, p. 131 and Th. 4.3, p. 136]. □

Corollary 1 *If \mathbb{B} is a Banach space with a quasi-orthogonal boundedly complete Schauder basis $\{b_n, n \geq 1\}$, then the BRW $\{Z_n, n \geq 1\}$ in \mathbb{B} converges strongly a.s. and in $L^p(\mathbb{B})$ for each fixed $1 \leq p < \infty$.*

Proof By a theorem of Dunford, if a Banach space \mathbb{B} possesses a boundedly complete Schauder basis, then \mathbb{B} has the RNP, cf. [4, Ch. III, §1, Th. 6, p. 64]. Hence and from Lemma 5, the assertion of Corollary 1 follows. \square

Corollary 2 *Let \mathbb{B} be a reflexive Banach space which has a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$. Then the \mathbb{B} -valued BRW $\{Z_n, n \geq 1\}$ converges strongly a.s. and in $L^p(\mathbb{B})$ for each fixed $1 \leq p < \infty$. In particular, if $\mathbb{B} = \mathbb{H}$ is a Hilbert space with a basis $\{b_n, n \geq 1\}$ which forms a CONS in \mathbb{H} , then the last statement remains valid.*

Proof It is fairly well known from a theorem of Phillips that reflexive Banach spaces have the RNP, see [4, Ch. III, §3, Corollary 4, p. 82]. Since each Hilbert space is reflexive, we conclude that $\mathbb{B} = \mathbb{H}$ has the RNP. Thus, an application of Lemma 5 concludes the proof. \square

Theorem 1 *Let $\phi: \mathbb{B} \rightarrow \mathbb{R}$ be a bounded and continuous mapping in a Banach space \mathbb{B} which has the RNP and a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$. If $\{Z_n, n \geq 1\}$ is the BRW in \mathbb{B} , then*

$$\phi(Z_n) \rightarrow \phi(\xi) \text{ a.s. and in } L^p = L^p(\mathbb{R}), \quad 1 \leq p < \infty,$$

where $\xi = \lim_{n \rightarrow \infty} Z_n$ a.s. and in $L^p(\mathbb{B})$ -norm for all $1 \leq p < \infty$. In particular, there exists

$$\lim_{n \rightarrow \infty} E\phi(Z_n) = E\phi(\xi).$$

Proof Since ϕ is continuous and the assumptions of Lemma 5 are fulfilled, we conclude that $\phi(Z_n) \rightarrow \phi(\xi)$ a.s. But in addition ϕ is assumed to be bounded, thus using the Lebesgue-dominated convergence theorem we obtain also convergence $\phi(Z_n) \rightarrow \phi(\xi)$ in L^p , $1 \leq p < \infty$. The last statement of the theorem follows from the estimate

$$|E\phi(Z_n) - E\phi(\xi)| \leq E|\phi(Z_n) - \phi(\xi)| \rightarrow 0.$$

\square

Corollary 3 *The assertion of Theorem 1 remains valid for a Banach space \mathbb{B} with a boundedly complete quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$, as well as for a Hilbert space $\mathbb{B} = \mathbb{H}$ with the Schauder basis $\{b_n, n \geq 1\}$ that forms a CONS in \mathbb{H} .*

5 The Banach Functional Integral in a Class of Banach Spaces

Let $\mathcal{C}_b = \{\phi: B \rightarrow \mathbb{R}; \phi\text{-bounded and continuous}\}$. Observe that \mathcal{C}_b has the following properties:

- (i₁) the set \mathcal{C}_b is a real linear space,
- (i₂) if $\phi \in \mathcal{C}_b$, then $|\phi| \in \mathcal{C}_b$.

Define a functional $f: \mathcal{C}_b \rightarrow \mathbb{R}$ by the formula:

$$f(\phi) = \lim_{n \rightarrow \infty} E\phi(Z_n) = E\phi(\xi), \tag{4}$$

where $\{Z_n, n \geq 1\}$ is a BRW in the Banach space \mathbb{B} , and $\xi = \lim_{n \rightarrow \infty} Z_n$ a.s. and in $L^p(\mathbb{B})$, $1 \leq p < \infty$.

It can be easily seen that the mapping f satisfies the following conditions:

- (ii₁) $f: \mathcal{C}_b \rightarrow \mathbb{R}$ is a linear functional,
- (ii₂) the functional f is nonnegative, i.e., $f(\phi) \geq 0$ whenever $\phi \in \mathcal{C}_b$ and $\phi \geq 0$,
- (ii₃) if 1^o $\{\phi_n\} \subset \mathcal{C}_b$, $\psi \in \mathcal{C}_b$, 2^o $|\phi_n| \leq \psi$ for $n \geq 1$, and 3^o $\lim_{n \rightarrow \infty} \phi_n(x) = 0$ for all $x \in B$, then $\lim_{n \rightarrow \infty} f(\phi_n) = 0$.

Notice that 3^o implies P -a.s. convergence $\phi_n(\xi) \rightarrow 0$; thus, the last condition follows from the classical Lebesgue-dominated convergence theorem applied to integrals $E\phi_n(\xi)$, $n = 1, 2, \dots$ (Actually, in our approach we can even replace condition 3^o by a weaker assumption $\phi_n \rightarrow 0$ in $P \circ \xi^{-1}$ -measure.) Consequently, the functional f satisfies all the conditions given in §2 of the Banach paper [1]. Therefore, for our functional f the Banach Th. 1, §3, p. 322, [1] is valid. In this way, we obtain the following result.

Theorem 2 *Let $\{Z_n, n \geq 1\}$ be a BRW in a Banach space \mathbb{B} with the RNP and a quasi-orthogonal Schauder basis $\{b_n, n \geq 1\}$, in particular in a Banach space \mathbb{B} with a boundedly complete quasi-orthogonal Schauder basis. Then, the functional f given by (4) has an extension to the additive functional F on the linear set $\mathcal{L} \supset \mathcal{C}_b$ of all bounded, Borel measurable functions $\Phi: B \rightarrow \mathbb{R}$. Moreover, the extended functional F on \mathcal{L} possesses all the properties (A)–(E) and (R) specified in §1 of the Banach paper [1], analogous to the Lebesgue integral.*

Remark 2 The approach presented above is a generalization of the method proposed by Banach [1] for the construction of the so-called \mathcal{L} -integral—an analogue of the Lebesgue integral in abstract spaces.

Theorem 3 *Let \mathbb{B} be a Banach space with the RNP and a quasi-orthogonal Schauder basis, in particular—a Banach space with a boundedly complete quasi-orthogonal Schauder basis. Then each sequence of symmetric probability distributions $\{G_n, n \geq 1\}$ concentrated on the interval $[-1, 1] \subset \mathbb{R}$ generates a probability measure Γ on the Borel σ -field \mathcal{B} in \mathbb{B} , given by*

$$\Gamma(A) = E\mathbb{1}_A(\xi), \quad A \in \mathcal{B}.$$

The measure Γ is equal to the limit distribution of the described above BRW $\{Z_n, n \geq 1\}$ in \mathbb{B} , thus $\text{supp } \Gamma \subseteq B$.

Proof Obviously, Γ is nonnegative and normalized so that $\Gamma(\mathbb{B}) = \Gamma(B) = 1$. It suffices to verify countable additivity of Γ , but it follows immediately from the properties of the integral $E(\cdot)$. The last conclusion can also be easily shown in a direct way. To this end, let $A_1, A_2, \dots \in \mathcal{B}$ be arbitrary disjoint sets. Since $\text{supp } \Gamma \subseteq B$, we have $\Gamma(A) = \Gamma(A \cap B) = E\mathbb{1}_{A \cap B}(\xi)$, $A \in \mathcal{B}$. Observe next that

$$E \mathbb{1}_{\bigcup_{j=1}^n (A_j \cap B)} (\xi) = E \left(\sum_{j=1}^n \mathbb{1}_{A_j \cap B} (\xi) \right) = \sum_{j=1}^n E \mathbb{1}_{A_j \cap B} (\xi) = \sum_{j=1}^n \Gamma (A_j),$$

and

$$0 \leq \sum_{j=1}^n \mathbb{1}_{A_j \cap B} (x) = \mathbb{1}_{\bigcup_{j=1}^n (A_j \cap B)} (x) \nearrow \mathbb{1}_{\bigcup_{n=1}^\infty (A_n \cap B)} (x) \leq \mathbb{1}_B (x), \quad x \in B.$$

Hence, on account of the Lebesgue monotone convergence theorem,

$$\begin{aligned} \sum_{n=1}^\infty \Gamma (A_n) &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \Gamma (A_j) = \lim_{n \rightarrow \infty} E \mathbb{1}_{\bigcup_{j=1}^n (A_j \cap B)} (\xi) \\ &= E \left(\lim_{n \rightarrow \infty} \mathbb{1}_{\bigcup_{j=1}^n (A_j \cap B)} (\xi) \right) = E \mathbb{1}_{\left(\bigcup_{n=1}^\infty A_n\right) \cap B} (\xi) = \Gamma \left(\bigcup_{n=1}^\infty A_n \right). \end{aligned}$$

□

From the construction of the BRW in a Banach space, it follows immediately that the limit distribution $\Gamma = P \circ \xi^{-1}$ of the BRW is sign-invariant with respect to the Schauder basis $\{b_n, n \geq 1\}$, in the sense that for each set $A \in \mathcal{B}$ and every sequence $\epsilon = \{\epsilon_1, \epsilon_2, \dots\}$ of signs $\epsilon_k \in \{-1, 1\}$, $k \geq 1$, we have

$$\Gamma (A) = \Gamma (\epsilon A),$$

where $\epsilon A = \{\sum_{k=1}^\infty \epsilon_k x_k b_k \in \mathbb{B} : \sum_{k=1}^\infty x_k b_k \in A\}$. It is also clear that each sign-invariant measure is symmetric, thus

$$\Gamma (A) = \Gamma (-A) \text{ for all } A \in \mathcal{B}.$$

By analogy to the notion of the Wiener measure, we propose to call Γ the Banach measure in a Banach space. One may expect that the Banach measure will play a similarly important role in Banach spaces as is played by the Gaussian measure constructed by Gross [5], cf. Bogachev [3], or Kuo [6].

6 Examples

1. Let $\mathbb{B} = \ell^p$, $1 \leq p < \infty$, and let $b_n = e_n = (0, \dots, 0, 1, 0, \dots)$, $n \geq 1$, where 1 is the n th term of the sequence $(0, \dots, 0, 1, 0, \dots)$. Then $\pi_n(x) = \sum_{k=1}^n x_k b_k = (x_1, \dots, x_n, 0, \dots)$ for $x = \sum_{n=1}^\infty x_n b_n = (x_1, x_2, \dots) \in \ell^p$, and thus

$$|\pi_n(x) + t b_{n+1}|_p^p = \sum_{k=1}^n |x_k|^p + |t|^p \leq 1 \Leftrightarrow |t| \leq \left(1 - \sum_{k=1}^n |x_k|^p \right)^{1/p}.$$

Hence it follows that $\{b_n, n \geq 1\}$ is a quasi-orthogonal Schauder basis in ℓ^p . (In fact, the considered basis is unconditional.) Moreover, if

$$\sup_{n \geq 1} |\pi_n(x)|_p = \sup_{n \geq 1} \left| \sum_{k=1}^n x_k b_k \right|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \leq M < \infty,$$

then the series $\sum_{n=1}^{\infty} x_n b_n$ converges in ℓ^p . Thus, the basis $\{b_n, n \geq 1\}$ is boundedly complete and in consequence each space $\ell^p, 1 \leq p < \infty$, has the RNP. It is also well known that for $p > 1$ the spaces ℓ^p are reflexive, which implies as well that they have the RNP. Therefore, all the above results are valid for Banach spaces $\mathbb{B} = \ell^p, 1 \leq p < \infty$.

2. Let $\mathbb{B} = L^p [0, 1]$, where $1 \leq p < \infty$. Consider the system of Haar functions: $h_1^0(s) = 1, s \in [0, 1]$, and

$$h_k^n(s) = 2^n \cdot \mathbb{1}_{[(2k-2)/2^{n+1}, (2k-1)/2^{n+1})}(s) - 2^n \cdot \mathbb{1}_{[(2k-1)/2^{n+1}, (2k)/2^{n+1})}(s)$$

for $k = 1, 2, \dots, 2^n, n = 1, 2, \dots, s \in [0, 1]$. It is known that the system of Haar functions forms a Schauder basis in $L^p [0, 1]$, see, e.g., [9, Th. 24.17, pp. 290–295], or [7, Part II, Prop. 2.c.1, p. 150]. For convenience of the reader, we sketch here the proof that the Haar basis in $L^p [0, 1]$ is quasi-orthogonal.

Proposition 1 *The system of Haar functions is a quasi-orthogonal basis in $L^p [0, 1], 1 \leq p < \infty$.*

Proof Let the Haar functions be arranged in a sequence that is divided into blocks, each of 2^n members, numbered by upper indices $n = 0, 1, 2, \dots$,

$$\Lambda = \left\{ \left(h_1^0 \right), \left(h_1^1, h_2^1 \right), \left(h_1^2, h_2^2, h_3^2, h_4^2 \right), \left(h_1^3, h_2^3, h_3^3, h_4^3, h_5^3, h_6^3, h_7^3, h_8^3 \right), \dots \right\}.$$

We make now two crucial observations: 1^0 for a fixed $n \geq 0$ within the same n th block the Haar functions $h_k^n, 1 \leq k \leq 2^n$, have nonoverlapping supports, 2^0 the function of the form $c_1^0 h_1^0 + c_1^1 h_1^1 + c_2^1 h_2^1 + \dots + c_1^{n-1} h_1^{n-1} + \dots + c_{2^{n-1}}^{n-1} h_{2^{n-1}}^{n-1}$, where $c_j^i \in \mathbb{R}$ are arbitrarily fixed coefficients, is constant in each interval of the form $[(k-1)/2^n, k/2^n), k = 1, 2, \dots, 2^n$.

Suppose now that $\pi_{k-1}^n(x) = x_1^0 h_1^0 + x_1^1 h_1^1 + x_2^1 h_2^1 + \dots + x_{k-1}^n h_{k-1}^n$ and the next element of Λ is h_k^n . Then for any parameter $t \in \mathbb{R}$,

$$\begin{aligned} \|\pi_{k-1}^n(x) + t h_k^n\|_p^p &= \int_0^1 |\pi_{k-1}^n(x)(s) + t h_k^n(s)|^p ds = \int_0^{(k-1)/2^n} |\pi_{k-1}^n(x)(s)|^p ds \\ &+ \int_{(k-1)/2^n}^{k/2^n} |\pi_{2^{n-1}}^{n-1}(x)(s) + t h_k^n(s)|^p ds \\ &+ \int_{k/2^n}^1 |\pi_{2^{n-1}}^{n-1}(x)(s)|^p ds = I_1 + I_2 + I_3, \end{aligned}$$

where $\pi_{2^{n-1}}^{n-1}(x)(s) = \pi_{2^{n-1}}^{n-1}(x)((2k-1)/2^{n+1}) = c, s \in [(k-1)/2^n, k/2^n)$. The first and third integrals on the right-hand side do not depend on the parameter t , and the middle term is equal to

$$\begin{aligned} I_2 &= \int_{(k-1)/2^n}^{(2k-1)/2^{n+1}} |c + t2^n|^p ds + \int_{(2k-1)/2^{n+1}}^{k/2^n} |c - t2^n|^p ds \\ &= |c + t2^n|^p \cdot \frac{1}{2^{n+1}} + |c - t2^n|^p \cdot \frac{1}{2^{n+1}} := r(t). \end{aligned}$$

Since $r(t) = r(-t)$, and

$$\begin{aligned} \inf \{t \in \mathbb{R} : r(t) \leq 1 - I_1 - I_3\} &= -\sup \{-t \in \mathbb{R} : r(-t) \leq 1 - I_1 - I_3\} \\ &= -\sup \{t' \in \mathbb{R} : r(t') \leq 1 - I_1 - I_3\}, \end{aligned}$$

we conclude that $\alpha_k^n = -\beta_k^n$. The same argument remains valid when π_{k-1}^n is replaced by $\pi_{2^{n-1}}^{n-1}$, and π_k^n is replaced by π_1^n , and thus, the system of Haar functions forms a quasi-orthogonal basis in $L^p[0, 1]$. □

Remark 3 It is clear that the Haar functions for $n \geq 1$ can be modified as follows:

$$h_k^n(s) = 2^{n/p} \cdot \mathbb{1}_{[(2k-2)/2^{n+1}, (2k-1)/2^{n+1})}(s) - 2^{n/p} \cdot \mathbb{1}_{[(2k-1)/2^{n+1}, (2k)/2^{n+1})}(s),$$

$k = 1, 2, \dots, 2^n, s \in [0, 1]$. Then the above Proposition 1 for modified Haar functions remains true, and in addition, we have $\|h_k^n\|_p = 1$ for all k, n .

Evidently, all the spaces $L^p[0, 1], 1 < p < \infty$, are reflexive Banach spaces, and thus, they possess the RNP, which is a straightforward consequence of Phillips’ theorem, cf. [4, Ch. III, §3, Corollary 6, p. 82]. Unfortunately, the space $L^1[0, 1]$ does not have the RNP, see [4, Ch. VII, p. 219].

Hence, it follows that all the results presented in previous sections are valid for Banach spaces $L^p[0, 1], 1 < p < \infty$.

Remark 4 Proposition 1 together with observation that the space $L^1[0, 1]$ does not possess the RNP implies the following conclusion: the existence of a quasi-orthogonal Schauder basis in a Banach space is not a sufficient condition for the RNP. The same conclusion follows from the fact that the space $(S_c[0, 1], \|\cdot\|)$ (isometrically isomorphic to $(C[0, 1], \|\cdot\|_\infty)$) does not have the RNP.

3. To illustrate the technique of computations of Banach \mathfrak{L} -integrals based on the method described here, we calculate, for instance, two “rarefied” absolute p th moments of $\xi = \sum_{n=1}^\infty X_n b_n$ in $\ell^p, 1 \leq p < \infty$, determined by lacunary series $\sum_{n=1}^\infty X_{2n-1} b_{2n-1}$ and $\sum_{n=1}^\infty X_{2n} b_{2n}$, namely $E(\sum_{n=1}^\infty |X_{2n-1}|^p)$ and $E(\sum_{n=1}^\infty |X_{2n}|^p)$, where ξ is the limit random element of the BRW generated by the sequence of identical uniform distributions on $[-1, 1] \subset \mathbb{R}$. Since the basis $\{b_n, n \geq 1\}$ in ℓ^p is unconditional, the lacunary series considered here are strongly convergent, see, e.g., [7, Prop. 1.c.1, p. 15, and p. 19].

Observe first that if $\pi_{\text{odd}}(x) = \sum_{n=1}^{\infty} x_{2n-1} b_{2n-1}$ for $x = \sum_{n=1}^{\infty} x_n b_n \in \ell^p$, then $|\pi_{\text{odd}}(x)|_p = (\sum_{n=1}^{\infty} |x_{2n-1}|^p)^{1/p}$, thus

$$\begin{aligned} & \left| \left(\sum_{n=1}^{\infty} |x_{2n-1}|^p \right)^{1/p} - \left(\sum_{n=1}^{\infty} |x'_{2n-1}|^p \right)^{1/p} \right| = \left| |\pi_{\text{odd}}(x)|_p - |\pi_{\text{odd}}(x')|_p \right| \\ & \leq |\pi_{\text{odd}}(x - x')|_p = \left(\sum_{n=1}^{\infty} |x_{2n-1} - x'_{2n-1}|^p \right)^{1/p} \leq |x - x'|_p \rightarrow 0 \end{aligned}$$

as $x \rightarrow x'$ in ℓ^p -norm $|\cdot|_p$. It follows that $\ell^p \ni x \mapsto (\sum_{n=1}^{\infty} |x_{2n-1}|^p)^{1/p}$ is a continuous function. Therefore, $\ell^p \ni x \mapsto \sum_{n=1}^{\infty} |x_{2n-1}|^p$, as well as $\ell^p \ni x \mapsto \sum_{n=1}^{\infty} |x_{2n}|^p$ are continuous, and both these maps are bounded in the unit ball $B \subset \ell^p$.

According to the construction of the BRW in ℓ^p , the density of X_1 is equal to $f_1(x_1) = \mathbb{1}_{[-1,1]}(x_1)/2$, and the density of (X_1, X_2, \dots, X_n) for $n > 1$ is given by

$$\begin{aligned} & f_n(x_1, x_2, \dots, x_n) \\ & = \frac{\mathbb{1}_{K_n}(x_1, x_2, \dots, x_n)}{2^n \cdot ([1 - |x_1|^p] \cdot [1 - (|x_1|^p + |x_2|^p)] \cdots [1 - (|x_1|^p + \dots + |x_{n-1}|^p)])^{1/p}}, \end{aligned}$$

where $K_n = K_n(0, 1)$ is the unit ball with center zero and radius 1 in \mathbb{R}^n , equipped with the ℓ^p -norm $|(x_1, \dots, x_n)|_{n,p} = (\sum_{k=1}^n |x_k|^p)^{1/p}$. Hence,

$$E |X_1|^p = \int_{-1}^1 \frac{|x_1|^p}{2} dx_1 = 2 \int_0^1 \frac{x_1^p}{2} dx_1 = \frac{x_1^{p+1}}{p+1} \Big|_0^1 = \frac{1}{p+1}.$$

Moreover, for $n > 1$,

$$\begin{aligned} E |X_n|^p & = \int_{K_n} |x_n|^p \cdot f_n(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ & = \int_{K_{n-1}} \left(2 \int_0^{[1 - (|x_1|^p + \dots + |x_{n-1}|^p)]^{1/p}} x_n^p \cdot f_n(x_1, x_2, \dots, x_n) dx_n \right) \\ & \quad \times dx_1 dx_2 \dots dx_{n-1} \\ & = \int_{K_{n-1}} \frac{x_n^{p+1}}{p+1} \Big|_0^{[1 - (|x_1|^p + \dots + |x_{n-1}|^p)]^{1/p}} \\ & \quad \times \frac{f_{n-1}(x_1, x_2, \dots, x_{n-1})}{[1 - (|x_1|^p + \dots + |x_{n-1}|^p)]^{1/p}} dx_1 dx_2 \dots dx_{n-1} \\ & = \int_{K_{n-1}} \frac{[1 - (|x_1|^p + \dots + |x_{n-1}|^p)]}{p+1} \cdot f_{n-1}(x_1, x_2, \dots, x_{n-1}) \end{aligned}$$

$$\begin{aligned} & \times dx_1 dx_2 \dots dx_{n-1} \\ & = \frac{1}{p+1} \left\{ 1 - (E |X_1|^p + E |X_2|^p + \dots + |X_{n-1}|^p) \right\}. \end{aligned}$$

Thus,

$$E |X_2|^p = \frac{1}{p+1} \left\{ 1 - E |X_1|^p \right\} = \frac{1}{p+1} \left(1 - \frac{1}{p+1} \right) = \frac{1}{p+1} \frac{p}{p+1},$$

and, by induction,

$$\begin{aligned} E |X_n|^p & = \frac{1}{p+1} \left\{ 1 - \left(\frac{1}{p+1} + \frac{p}{(p+1)^2} + \dots + \frac{p^{n-2}}{(p+1)^{n-1}} \right) \right\} \\ & = \frac{1}{(p+1)} \left\{ 1 - \frac{1}{p+1} \cdot \frac{1 - p^{n-1}/(p+1)^{n-1}}{1 - p/(p+1)} \right\} = \frac{p^{n-1}}{(p+1)^n}. \end{aligned}$$

Therefore,

$$E \left(\sum_{n=1}^{\infty} |X_{2n-1}|^p \right) = \sum_{n=1}^{\infty} \frac{p^{2n-1-1}}{(p+1)^{2n-1}} = \frac{1}{p+1} \cdot \frac{1}{1 - p^2/(p+1)^2} = \frac{p+1}{2p+1},$$

and

$$E \left(\sum_{n=1}^{\infty} |X_{2n}|^p \right) = \sum_{n=1}^{\infty} \frac{p^{2n-1}}{(p+1)^{2n}} = \frac{p}{(p+1)^2} \cdot \frac{1}{1 - p^2/(p+1)^2} = \frac{p}{2p+1}.$$

In consequence,

$$E |\pi_{\text{odd}}(\xi)|_p^p = \frac{p+1}{2p+1}, \quad E |\xi - \pi_{\text{odd}}(\xi)|_p^p = \frac{p}{2p+1},$$

so that

$$E |\xi|_p^p = E \left(\sum_{n=1}^{\infty} |X_n|^p \right) = \frac{p+1}{2p+1} + \frac{p}{2p+1} = 1.$$

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References

1. Banach, S.: The Lebesgue integral in abstract spaces. In: Saks, S. (ed.) *Theory of the Integral*. Dover Publication, New York (1964)
2. Banek, T.: Banach random walk in the unit ball $S \subset \ell^2$ and chaotic decomposition of $\ell^2(S, P)$. *J. Theor. Probab.* **29**(4), 1728–1735 (2016)
3. Bogachev, V.I.: *Gaussian Measures*. American Mathematical Society, Providence (1998)
4. Diestel, J., Uhl, J.J.: *Vector Measures*. American Mathematical Society, Providence (1977)
5. Gross, L.: Abstract Wiener spaces. In: *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability*, vol. 2, pp. 31–42 (1965)
6. Kuo, H.H.: *Gaussian Measures in Banach Spaces*. Lecture Notes in Mathematics, vol. 463. Springer, Berlin (1975)
7. Lindenstrauss, J., Tzafriri, L.: *Classical Banach Spaces I, II*. Springer, Berlin (1977, 1979)
8. Neveu, J.: *Discrete Parameter Martingales*. North Holland, Amsterdam (1975)
9. Schilling, R.L.: *Measures, Integrals and Martingales*. Cambridge University Press, Cambridge (2005)
10. Semadeni, Z.: *Schauder Bases in Banach Spaces of Continuous Functions*. Lecture Notes in Mathematics, vol. 918. Springer, Berlin (1982)
11. Vakhania, N.N., Tarieladze, V.I., Chobanyan, S.A.: *Probability Distributions on Banach Spaces*. D. Reidel, Kluwer, Dordrecht (1987)
12. Zapala, A.M.: Jensen's inequality for conditional expectations in Banach spaces. *Real Anal. Exch.* **26**(2), 541–552 (2000/2001)

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