



Random Walk in deterministically changing environment

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Abstract. We consider a random walk with transition probabilities weakly dependent on an environment with a deterministic, but strongly chaotic, evolution. We prove that for almost all initial conditions of the environment the walk satisfies the CLT.

1. Introduction

The continuing interest in the limit properties of random walks has generated a remarkable amount of literature (see Sznitman, 2004; Zeitouni, 2004 for a review of the field). In particular, many papers have addressed the case of random walks in dynamical environments. Apart from few papers in which special hypotheses are imposed on the form of the transition probabilities (implying that the process is reversible with respect to the stationary measure of the environment, see Kipnis and Varadhan, 1986) the authors have usually investigated the case in which the evolution of the environment is described by a Markov process with positive transition probabilities and the transition probabilities of the walk are close to constant (see, e.g., Boldrighini et al., 2000; Bandyopadhyay and Zeitouni, 2006 and references therein). While such a situation has recently been settled in great generality (Dolgopyat et al., 2007) the case of more complex local dynamics and/or far from constant transition probabilities is still wide open.

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In this paper we address the first issue establishing conditions under which the CLT holds for a deterministic local evolution with strong chaotic properties. Such a CLT is established for almost all the initial conditions of the environment with respect to a (natural) stationary measure (this is commonly called a *quenched CLT*).

The deterministic dynamics is taken to be independent at each site (although our method can be easily extended to weakly interacting cases, cf. Keller and Liverani, 2005; Dolgopyat et al., 2007). The single site dynamics is a piecewise expanding one dimensional map. While multidimensional expanding dynamics could be treated similarly the case of Anosov map poses a real problem. Indeed the technique used to control the environment dynamics is borrowed from the study of coupled map lattices (more precisely from Keller and Liverani, 2006) and the extension of such a technique to coupled Anosov systems is still missing. In general, the extension to more general dynamics (with substantially weaker ergodic properties) would be of interest, but to obtain results in this direction new ideas seem to be needed.

Note that the present strategy differs, in its probabilistic part, from the one used in Dolgopyat et al. (2007). In particular, it is not necessary to prove absolutely continuity of the invariant measure of the environment as seen from the particle with respect to the invariant measure of the environment in a fixed reference frame. We hope that this simplification may be helpful in treating more general cases.

The plan of the paper is the following. In section 2 the system under investigation is explained in detail and the main result of the paper is precisely stated. This main result (Theorem 2.7) follows after estimating the asymptotic independence of two random walks in the same environment (Lemma 2.8). In Section 3 we study ergodic properties of the environment. Section 4 contains the proof of the annealed (averaged) invariance principle. In section 5 the proof of Lemma 2.8 is reduced to an estimate on the number of close encounters (Lemma 5.2). Lemma 5.2 is proven in Section 6.

Convention 1.1. *In this paper we will use C to designate a generic constant depending only on the quantities appearing in the Assumptions 1, 2, 3, 4 below. We will use instead $C_{a,b,c,\dots}$ for constants depending also on the parameters a, b, c, \dots . Consequently the actual numerical value of such constants may vary from one occurrence to the next. On the contrary we will use C_1, C_2, \dots , to designate constants whose value is fixed through the paper.*

2. Model and Results

Let $I = [0, 1]$ and $T : I \rightarrow I$ be a piecewise \mathcal{C}^2 topologically mixing map such that $|D_x T| \geq \lambda > 2$ for each $x \in I$ for which the derivative is well defined. Then, $I^{\mathbb{Z}^d} =: \Theta$ is the space of environments (it is a measurable space with the product (Borel) σ -algebra \mathcal{T}) and $\theta \in \Theta$ is an environment on \mathbb{Z}^d . This environment evolves deterministically according to a map $F : \Theta \rightarrow \Theta$.

Assumption 1 (*Environment Dynamics*). *For each $\theta \in \Theta$*

$$(F(\theta))_q := T(\theta_q).$$

*That is, the evolution is independent at each site.*¹

¹The case of weakly coupled maps can be treated similarly by using the techniques introduced in Keller and Liverani (2006) and used here to study the present, simpler, case.

The evolution of the environment can be thus seen as a deterministic Markov process on the space $\Theta^{\mathbb{N}} =: \Omega$ such that, for all $(\theta^n)_{n \in \mathbb{N}} := \boldsymbol{\theta} \in \Omega$, $\theta^n := F^n(\theta^0)$. If μ_0 is the unique absolutely continuous invariant measure of T ,² then $\mu^e := \otimes_{p \in \mathbb{Z}^d} \mu_0$ is the natural invariant measure for F we are interested in. In fact, it is possible to show (Keller and Liverani, 2006) that it is the only invariant measure in a reasonably large class of measures; see the precise statement below.

We consider a bounded increment random walk X_n in such environment. More precisely, let $\Lambda := \{z \in \mathbb{Z}^d : \|z\| \leq C_0\}$ and $\Delta_n = X_{n+1} - X_n$, then the process is defined by the transition probabilities

$$\mathbb{P}(\{\Delta_n = z\} \mid X_n, \theta^0) = \pi_z(\tau^{X_n} \theta^n) \quad (2.1)$$

where $\pi_z \equiv 0$ for all $z \notin \Lambda$, $\pi_z(\theta)$ depends only on $\{\theta_q\}_{q \in \Lambda}$, and, for each $z \in \mathbb{Z}^d$, $(\tau^z \theta)_i := \theta_{i+z}$. We will be interested in the measure \mathbf{P}_ν on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ determined by the above process when the environment is started with the measure ν and the walk starts from zero. We will use the notation \mathbf{P}^e for \mathbf{P}_{μ^e} . Finally, we will use \mathbb{E} for the expectation with respect to the latter measure and \mathbb{E}_ν for the expectations with respect to the process \mathbf{P}_ν .

Assumption 2 (Regularity). *The functions $\{\pi_z\}_{z \in \Lambda}$ belong to \mathcal{C}^1 .*

The next assumption depends on a parameter $\varepsilon > 0$.

Assumption 3 (Perturbative regime). *There exists $\{a_z\}_{z \in \Lambda} \subset \mathbb{R}_+$, $\sum_{z \in \Lambda} a_z = 1$, such that*

$$\|\pi_z - a_z\|_{\mathcal{C}^1} \leq a_z \varepsilon.$$

In the following, when we will say ‘‘assumption 3 holds for ε_i ’’ we will mean that it holds with $\varepsilon = \varepsilon_i$. The values ε_i will be taken small enough for Theorem 2.2, Proposition 2.4, Lemma 2.5 and Lemma 3.5 to hold.

Assumption 4 (Ellipticity). *For each $l \in \mathbb{Z}^d \setminus \{0\}$, the function $\left| \sum_{z \in \Lambda} \pi_z e^{i \langle l, z \rangle} \right| \in \mathcal{C}^0(I^\Lambda, \mathbb{R}_+)$ is not identically equal to 1.*

It is well known that to study the properties of X_n it is convenient to study the process of the environment as seen from the particle. In fact, such a process can be considered in several fashions of which the following will be relevant in the sequel.

2.1. Process of the environment as seen from the particle. Consider the process $\boldsymbol{\omega} =: (\omega^n)_{n \in \mathbb{N}} \in \Omega$ described by the action of the Markov operator $S : L^\infty(\Theta) \rightarrow L^\infty(\Theta)$ defined by

$$Sf(\boldsymbol{\omega}) := \sum_{z \in \Lambda} \pi_z(\boldsymbol{\omega}) f \circ F(\tau^z \boldsymbol{\omega}) =: \sum_{z \in \Lambda} S_z f. \quad (2.2)$$

Remark 2.1. It is easy to verify that the process $\boldsymbol{\omega}$, $\omega^0 = \theta$, has the same distribution as the process $(\tau^{X_n} \theta^n)_{n \in \mathbb{N}}$, $\theta^0 = \theta$.

We can then consider the measure \mathbb{P}_ν on Ω of the associated Markov process started with a measure ν .

²For the existence and uniqueness (among measures absolutely continuous w.r.t. Lebesgue) see Baladi (2000).

In analogy with the techniques used in the study of coupled map lattices, see Keller and Liverani (2006), it is then natural to restrict the space of measures on which S' acts.³ To this end we start by defining the following norms

$$\begin{aligned} |\mu| &:= \sup_{|\varphi|_{\mathcal{C}^0(\Theta, \mathbb{R})} \leq 1} \mu(\varphi) \\ \|\mu\| &:= \sup_{i \in \mathbb{Z}^d} \sup_{|\varphi|_{\mathcal{C}^0(\Theta, \mathbb{R})} \leq 1} \mu(\partial_{\theta_i} \varphi) \end{aligned} \quad (2.3)$$

We then consider the Banach space of complex valued measures⁴

$$\mathcal{B} := \{\mu \in \mathcal{M}(\Theta) : \|\mu\| < \infty\}. \quad (2.4)$$

It is easy to check that such measures have finite dimensional marginals absolutely continuous w.r.t. Lebesgue and the densities are functions of bounded variations with variations bounded by the norm of the measure. Moreover μ^e is the unique invariant measure for F belonging to \mathcal{B} , Keller and Liverani (2006).

Theorem 2.2. *For each dynamics F satisfying assumption 1 and transition probabilities satisfying assumption 2, the operator S' , is a bounded operator on \mathcal{B} . In addition, there exists $\varepsilon_0 > 0$, depending on F , such that if assumption 3 holds for ε_0 , then there exists a unique invariant probability measure $\mu^w \in \mathcal{B}$ ($S'\mu^w = \mu^w$). This measure enjoys the following properties: There exists $\eta \in (0, 1)$ such that for each $\nu \in \mathcal{B}$ and local functions $\varphi, \phi \in \mathcal{C}^0$ each depending only on L variables with the two sets of dependency having distance at least M*

- (1) $|\nu(S^n \phi) - \mu^w(\phi)\nu(1)| \leq CL\eta^n |\phi|_\infty \|\nu\|$
- (2) $|\mu^w(\varphi\phi) - \mu^w(\varphi)\mu^w(\phi)| \leq CL\eta^{\frac{M}{2C_0}} |\varphi|_\infty |\phi|_\infty$.

The proof of the above theorem can be found in Section 3.2.

Remark 2.3. Theorem 2.2 implies that the process \mathbb{P}_{μ^w} is a stationary (and ergodic) process.

2.2. Annealed statistical properties.

Proposition 2.4. *For each dynamics F satisfying assumption 1 and transition probabilities satisfying assumption 2, if assumption 3 is satisfied for ε_0 (where $\varepsilon_0 > 0$ is as in Theorem 2.2), then there exists a vector $v \in \mathbb{R}^d$ and a matrix $\Sigma^2 \geq 0$ such that, for each probability measure $\nu \in \mathcal{B}$ we have*

$$\begin{aligned} \frac{1}{N} \mathbb{E}_\nu(X_N) &\rightarrow v \\ \frac{X_N - vN}{\sqrt{N}} &\Rightarrow \mathcal{N}(0, \Sigma^2) \quad \text{under } \mathbf{P}_\nu. \end{aligned}$$

Moreover, there exists $C_1 > 0$ such that, setting $\tilde{X}_N := X_N - vN$, the following inequality holds for all $N \in \mathbb{N}$ and $t \in \mathbb{R}^d$:

$$\left| \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_N \rangle} \right) - e^{-\frac{1}{2} \langle t, \Sigma^2 t \rangle} \right| \leq C_1 (1 + \|t\|^3) N^{-\frac{1}{2}} \|\nu\|.$$

Finally, if assumption 4 is also satisfied, then $\Sigma^2 > 0$.

³As usual the dual operator S' is defined as $S'\nu(f) = \nu(Sf)$ for all $f \in \mathcal{C}^0$.

⁴By $\mathcal{M}(\Theta)$ we designate the set of complex valued finite Borel measures on Θ .

Proof. Let us start noticing that

$$\frac{1}{N}\mathbb{E}_\nu(X_N) = \frac{1}{N}\sum_{k=0}^{N-1}\mathbb{E}_\nu(\Delta_k) = \frac{1}{N}\sum_{k=0}^{N-1}\mathbb{E}_\nu(\mathbb{E}_\nu(\Delta_k | \mathcal{F}_k)),$$

where $\mathcal{F}_k := \sigma\{\theta^0, X_1, \dots, X_k\}$. The relevance of the process as seen from the particle is due to the following fact:

$$\mathbb{E}_\nu(\Delta_k | \mathcal{F}_k) = \sum_{z \in \Lambda} z \pi_z(\tau^{X_k} \theta^k) = \sum_{z \in \Lambda} z \pi_z(\omega^k) =: g(\omega^k). \quad (2.5)$$

Thus,

$$\frac{1}{N}\mathbb{E}_\nu(X_N) = \frac{1}{N}\sum_{k=0}^{N-1}\nu(g(\omega^k)) = \frac{1}{N}\sum_{k=0}^{N-1}[(S')^k \nu](g).$$

Accordingly, Theorem 2.2 implies

$$\lim_{N \rightarrow \infty} \frac{1}{N}\mathbb{E}_\nu(X_N) = \mu^w(g) =: v. \quad (2.6)$$

To prove the CLT let $\tilde{\Delta}_n = \tilde{X}_{n+1} - \tilde{X}_n$, then

$$\mathbb{E}_\nu\left(e^{\frac{i}{\sqrt{N}}\langle t, \tilde{X}_N \rangle}\right) = \mathbb{E}_\nu\left(e^{\frac{i}{\sqrt{N}}\langle t, \tilde{X}_{N-1} \rangle} \mathbb{E}_\nu\left(e^{\frac{i}{\sqrt{N}}\langle t, \Delta_{N-1-v} \rangle} | \mathcal{F}_{N-1}\right)\right).$$

Since

$$\mathbb{E}_\nu\left(e^{\frac{i}{\sqrt{N}}\langle t, \Delta_k - v \rangle} | \mathcal{F}_k\right) = \sum_{z \in \Lambda} \pi_z(\tau^{X_k} \theta^k) e^{\frac{i}{\sqrt{N}}\langle t, z - v \rangle},$$

it is natural to introduce the operators, for all $t \in \mathbb{C}^d$,

$$\mathcal{M}_t h(\theta) := \sum_{z \in \Lambda} \pi_z(\theta) e^{\langle t, z - v \rangle} h(\tau^z F(\theta)) = \sum_{z \in \Lambda} e^{\langle t, z - v \rangle} S_z h. \quad (2.7)$$

Then,

$$\mathbb{E}_\nu\left(e^{\frac{i}{\sqrt{N}}\langle t, \Delta_k - v \rangle} | \mathcal{F}_k\right) = (\mathcal{M}_{it/\sqrt{N}} 1)(\tau^{X_k} \theta^k),$$

and the reader can then check, by induction, the formula

$$\mathbb{E}_\nu\left(e^{\frac{i}{\sqrt{N}}\langle t, \tilde{X}_N \rangle}\right) = \nu(\mathcal{M}_{it/\sqrt{N}}^N 1). \quad (2.8)$$

The operator \mathcal{M}'_t acting on the space \mathcal{B} is an analytic perturbation of the operator $S' = \mathcal{M}'_0$. Unfortunately, S' does not have a nice spectrum on \mathcal{B} , so in order to apply usual perturbation theory, it is necessary to lift the dynamics to an appropriate space in the spirit of Bardet et al. (2007). We do so in section 3.3 where we prove the following result.

Lemma 2.5. *Under the assumptions of Theorem 2.2 there exists $C_2 > 0$ and a function α_t analytic in $\{\|t\| < C_2\}$ such that for each $n \in \mathbb{N}$, probability measure $\nu \in \mathcal{B}$ and local function f depending on L variables we have*

$$\begin{aligned} |\nu(\mathcal{M}_t^n f)| &\leq CL |\alpha_t^n| \|f\|_\infty \|\nu\| \\ \nu(\mathcal{M}_t^n 1) &= \alpha_t^n (1 + \mathcal{O}(t\|\nu\|)) + \mathcal{O}(\eta^n \|\nu\|). \end{aligned}$$

Moreover, $\alpha_0 = 1$, $\dot{\alpha}_0 = 0$ and $\ddot{\alpha}_0 \geq 0$ (the ‘‘dot’’ stands for the derivatives with respect to t). Finally, if assumption 4 is also satisfied, then $\ddot{\alpha}_0 > 0$.

Using Lemma 2.5 and setting $\Sigma^2 := \tilde{\alpha}_0$, we have

$$\nu(\mathcal{M}_{it/\sqrt{N}}^N 1) = \alpha_{it/\sqrt{N}}^N (1 + \mathcal{O}(tN^{-\frac{1}{2}} \|\nu\|)) + \mathcal{O}(\eta^N \|\nu\|).$$

We can finally compute, for $\|t\| \leq CN^{\frac{1}{6}}$ and N large enough,⁵

$$\begin{aligned} \mathbb{E}_\nu \left(e^{\frac{i}{\sqrt{N}} \langle t, \tilde{X}_N \rangle} \right) &= \alpha_{it/\sqrt{N}}^N + \mathcal{O} \left(\frac{1 + \|t\|}{\sqrt{N}} \|\nu\| \right) \\ &= \left(1 - \frac{1}{2N} \langle t, \Sigma^2 t \rangle + \mathcal{O}(\|t\|^3 N^{-\frac{3}{2}}) \right)^N + \mathcal{O} \left(\frac{1 + \|t\|}{\sqrt{N}} \|\nu\| \right) \\ &= e^{-\frac{1}{2} \langle t, \Sigma^2 t \rangle + \mathcal{O}(\|t\|^3 N^{-\frac{1}{2}})} + \mathcal{O} \left(\frac{1 + \|t\|}{\sqrt{N}} \|\nu\| \right), \end{aligned}$$

from which Proposition 2.4 follows. \square

Next, we need a large deviations estimate.

Lemma 2.6. *Under the assumptions of Theorem 2.2 and assumption 4 there exists $a_0 > 0$ such that for each $\nu \in \mathcal{B}$, $n, m \in \mathbb{N}$ and $a \in (0, a_0)$ the following holds true*

$$\mathbf{P}_\nu \left(\left\{ \left| \frac{1}{m} (\tilde{X}_{n+m} - \tilde{X}_n) \right| \geq a \right\} \right) \leq C e^{-Ca^2 m} (\|\nu\| + 1).$$

Proof. Again this large deviation result can be obtained by perturbation theory of the operator S' . Indeed, for each $w \in \mathbb{R}^d$, $\|w\| = 1$ and $t \in \mathbb{R}$,

$$\begin{aligned} \mathbf{P}_\nu \left(\left\{ \frac{1}{m} \langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle \geq a \right\} \right) &\leq \mathbb{E}_\nu \left(e^{t(\langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle - am)} \right) \\ &= e^{-tam} [(S')^n \nu](\mathcal{M}_{tw}^m 1). \end{aligned}$$

Since by Theorem 2.2 $\sup_{n \in \mathbb{N}} \|(S')^n \nu\| \leq C \|\nu\|$, we can apply Lemma 2.5 and obtain, for $t \leq C_2$,

$$\begin{aligned} \mathbf{P}_\nu \left(\left\{ \frac{1}{m} \langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle \geq a \right\} \right) &\leq C e^{-tam} \alpha_{tw}^m (1 + C|t| \|\nu\|) + C \eta^m \|\nu\| \\ &\leq C e^{-tam} \left(1 + \frac{t^2}{2} \langle w, \Sigma^2 w \rangle + \mathcal{O}(|t|^3) \right)^m (1 + \|\nu\|) + C \eta^m \|\nu\| \\ &\leq C e^{-tam + \frac{mt^2}{2} \langle w, \Sigma^2 w \rangle + \mathcal{O}(m|t|^3)} (1 + \|\nu\|) + C \eta^m \|\nu\|. \end{aligned}$$

Finally, choosing $t = \frac{a}{\langle w, \Sigma^2 w \rangle}$ and a_0 so small that the term $\mathcal{O}(t^3)$ is small with respect to $\frac{a_0^2}{2\langle w, \Sigma^2 w \rangle}$, $\frac{a_0}{\langle w, \Sigma^2 w \rangle} < C_2$ and $e^{-\frac{a_0^2}{2\langle w, \Sigma^2 w \rangle}} \geq \eta$,

$$\mathbf{P}_\nu \left(\left\{ \frac{1}{m} \langle w, \tilde{X}_{n+m} - \tilde{X}_n \rangle \geq a \right\} \right) \leq C e^{-Ca^2 m} (1 + \|\nu\|).$$

We then conclude by noticing that the above estimate for all the w in the set $\{\pm e_i\}_{i=1}^d$, where $\{e_i\}_{i=1}^d$ is the standard base of \mathbb{R}^d , implies the Lemma. \square

⁵If $\|t\| \leq CN^{\frac{1}{6}}$ the last statement of Proposition 2.4 is obvious: the left hand side is ≤ 2 .

2.3. *Main result: Quenched C.L.T.* Let \mathbf{P}_θ be the measure \mathbf{P}^e conditioned to starting the environment in the configuration θ . We will use \mathbb{E}_θ for the expectation with respect to \mathbf{P}_θ .

Theorem 2.7. *For each dynamics F satisfying assumption 1 and transition probabilities satisfying assumptions 2 and 4, if assumption 3 is satisfied for ε_1 (where $\varepsilon_0 \geq \varepsilon_1 > 0$ is as in Lemma 3.5), then (using the same notation as in Proposition 2.4), for μ^e almost all $\theta \in \Theta$ the following holds*

(a) $\frac{1}{N}X_N \rightarrow v \quad \mathbf{P}_\theta \text{ a.s.};$

(b) $\frac{X_N - vN}{\sqrt{N}} \Rightarrow \mathcal{N}(0, \Sigma^2) \text{ under } \mathbf{P}_\theta.$

Proof. Lemma 2.6 implies the bound $\mathbf{P}^e(\{|N^{-1}X_N - v| \geq \varepsilon\}) \leq Ce^{-C\varepsilon^2 N}$; (a) follows then by applying Borel-Cantelli.

To prove (b) let $\alpha \in (0, 1)$ be a number to be specified later. Combining Lemma 2.6 and Borel-Cantelli Lemma we see that for any $\delta > 0$, \mathbf{P}^e -almost surely for any k and $0 \leq j \leq 2^{(1-\alpha)k}$ we have⁶

$$\max_{m \in [2^k + j2^{2\alpha k}, 2^k + (j+1)2^{2\alpha k}]} |\tilde{X}_m - \tilde{X}_{2^k + j2^{2\alpha k}}| \leq C_{\theta, X, \delta} 2^{(\frac{\alpha}{2} + \delta)k} \quad (2.9)$$

By Fubini Theorem for almost every θ (2.9) holds \mathbf{P}_θ almost surely. Therefore it is enough to prove the convergence along the subsequence $n_{jk} = 2^k + j2^{2\alpha k}$ where $0 \leq j \leq 2^{(1-\alpha)k}$.

To conclude it suffices to prove that there exists $\beta > 0$ and $b \in \mathbb{N}$ such that for each smooth function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ compactly supported in a box of size L the following inequality holds

$$\mathbb{E} \left(\left| \mathbb{E}_\theta(\varphi(N^{-\frac{1}{2}}\tilde{X}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) \right|^2 \right) \leq C_L |\varphi|_{C^b} N^{-\beta}, \quad (2.10)$$

where $\mathbb{E}_{\mathcal{N}(0, \Sigma^2)}$ is the expectation with respect to the Gaussian measure $\mathcal{N}(0, \Sigma^2)$. Indeed, denote

$$\xi_{jk} = \mathbb{E}_\theta \left(\varphi \left(\frac{\tilde{X}_{n_{jk}}}{\sqrt{n_{jk}}} \right) \right).$$

Then (2.10) and Chebyshev inequality imply

$$\mathbf{P}^e \left(\left\{ |\xi_{jk} - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)| \geq \varepsilon \right\} \right) \leq C_L |\varphi|_{C^b} \varepsilon^{-2} n_{jk}^{-\beta}. \quad (2.11)$$

Hence, by finally choosing α such that $\alpha + \beta > 1$, $\xi_{jk} \rightarrow \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)$ almost surely. Next, choose a family φ_m which is dense in $\mathcal{C}_0^0(\mathbb{R}^d)$. Then, for almost every θ , we have

$$\mathbb{E}_\theta \left(\varphi_m \left(\frac{\tilde{X}_{n_{jk}}}{\sqrt{n_{jk}}} \right) \right) \rightarrow \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi_m)$$

for all m . Then, for any such θ

$$\mathbb{E}_\theta \left(\varphi \left(\frac{\tilde{X}_{n_{jk}}}{\sqrt{n_{jk}}} \right) \right) \rightarrow \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)$$

for any continuous compactly supported φ proving (b).

⁶The X in $C_{\theta, X, \delta}$ stands for the dependence on the random walk realization $\{X_n\}_{n \in \mathbb{N}}$.

The result is then proved provided (2.10) is true. It turns out that (2.10) can be conveniently interpreted in terms of two independent walks X_N, Y_N in the same environment. In fact, calling \mathbb{E}^2 the expectation with respect to such a process it follows

$$\begin{aligned} \mathbb{E} \left(\left| \mathbb{E}_\theta(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) \right|^2 \right) &= \mathbb{E}^2(\varphi(N^{-\frac{1}{2}} \tilde{X}_N) \varphi(N^{-\frac{1}{2}} \tilde{Y}_N)) \\ &\quad - 2\mathbb{E}(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) + \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)^2 \\ &= \mathbb{E}^2(\varphi(N^{-\frac{1}{2}} \tilde{X}_N) \varphi(N^{-\frac{1}{2}} \tilde{Y}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)^2 + \mathcal{O}(L^d \|\varphi\|_{\mathcal{C}^{d+1}} N^{-\beta}). \end{aligned}$$

where we have used the quantitative estimate in the Proposition 2.4.⁷

We have thus reduced the proof of the theorem to proving the following.

Lemma 2.8.

$$\left| \mathbb{E}^2(\varphi(N^{-\frac{1}{2}} \tilde{X}_N) \varphi(N^{-\frac{1}{2}} \tilde{Y}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi)^2 \right| \leq C \|\varphi\|_{\mathcal{C}^b} N^{-\beta}.$$

Lemma 2.8 is proved in Section 5. □

3. Proofs: Environment

In this section we establish all the needed properties of the environment dynamics. The basic idea is to prove that the environment enjoys very strong mixing properties. Our first aim is to prove Theorem 2.2, that is exponential decay of space-time correlations.

Note that $S1 = 1$, hence

$$|S'\mu| \leq |\mu|, \tag{3.1}$$

that is S is a contraction in the $|\cdot|$ norm. In addition, it is possible to prove that for each $\sigma \in (2\lambda^{-1}, 1)$ there exists $A, B, \varepsilon_0 > 0$ and such that, if assumption 3 is satisfied for ε_0 , then for each $n \in \mathbb{N}$ and $\mu \in \mathcal{B}$,⁸

$$\|(S')^n \mu\| \leq A\sigma^n \|\mu\| + B|\mu|. \tag{3.2}$$

For finitely many sites the above estimate would suffice to prove that the operator S' is quasi-compact and this, together with the topologically mixing assumption, would imply the existence of a spectral gap. Unfortunately, such a proof is based on the compactness of the unit ball $\{\mu \in \mathcal{B} : \|\mu\| \leq 1\}$ in the topology of the $|\cdot|$ norm which fails when one considers infinitely many sites.

The obvious idea is to use explicitly the fact that the dynamics in different sites are independent, hence the system has a product structure, yet this is a subtle issue. To understand better the situation, let us recall few facts about the single

⁷If $\hat{\varphi}$ is the Fourier transform of φ , then Proposition 2.4 yields, for each $\rho < \frac{1}{6}$ and $b \in \mathbb{N}$,

$$\begin{aligned} \left| \mathbb{E}(\varphi(N^{-\frac{1}{2}} \tilde{X}_N)) - \mathbb{E}_{\mathcal{N}(0, \Sigma^2)}(\varphi) \right| &\leq C \int_{\|t\| \geq N^\rho} |\hat{\varphi}(t)| dt + C \int_{\|t\| \leq N^\rho} (1 + \|t\|^3) N^{-\frac{1}{2}} |\hat{\varphi}(t)| dt \\ &\leq C_d L^d \|\varphi\|_{\mathcal{C}^b} \int_{N^\rho}^\infty x^{-b+d-1} dx + C_d L^d \|\varphi\|_{\mathcal{C}^0} N^{-\frac{1}{2}} \int_0^{N^\rho} (1+x^3) x^{d-1} dx, \end{aligned}$$

which gives the advertised result provided $b > d$ and ρ is chosen small enough.

⁸See (2.4) for the definition of \mathcal{B} . Inequality (3.2) follows directly from (3.7) and Lemma 3.4.

site systems. At each site we have the dynamical system $(I = [0, 1], T)$. Let us consider the norm in $\mathcal{M}(I)$ given by

$$\|\nu\|_0 = \sup_{|\varphi|_{\mathcal{C}^0} \leq 1} \mu(\varphi'),$$

where φ' is the derivative of φ . The Banach space $B = \{\nu \in \mathcal{M} : \|\nu\|_0 < \infty\}$ consists of measures absolutely continuous with respect to the Lebesgue measure $m_{\mathcal{L}}$. In addition, if $d\nu = h dm_{\mathcal{L}}$, then the density h is a function of bounded variation and $|h|_{BV} = \|\nu\|$.⁹ By a change of variable one can compute that, if $d\nu = h dm_{\mathcal{L}}$, then $d(T'\nu) = (\mathcal{L}h)dm_{\mathcal{L}}$, where the operator \mathcal{L} is defined as

$$\mathcal{L}h(x) = \sum_{y \in T^{-1}(x)} |D_y T|^{-1} h(y).$$

The operator \mathcal{L} is often called the Ruelle-Perron-Frobenius transfer operator. It is well known that, if T is topologically mixing, then the operator \mathcal{L} , acting on BV has 1 as a simple eigenvalue (corresponding to the unique invariant measure absolutely continuous with respect to Lebesgue) and enjoys a spectral gap, that is there exists $\eta_0 \in (0, 1)$ such that the rest of the spectrum is strictly contained in a disk of radius η_0 (see, e.g., Baladi, 2000 for details). Clearly the above implies that T' has a spectral gap when acting on B . Unfortunately, it turns out that the tensor products of B is a too small a space to be really useful for our purposes.¹⁰ This is the reason why we have introduced the spaces \mathcal{B} which is a generalization of measures with density of bounded variations to the infinite dimensional setting. Yet on such a space S' does not behave very well and we will use an abstract covering space on which the dynamics will exhibit a spectral gap.

More precisely, we would like to introduce a Banach space $\overline{\mathcal{B}}$ and two (possibly only partially defined) maps $\Psi : \mathcal{B} \rightarrow \overline{\mathcal{B}}$ and $\text{Pr} : \overline{\mathcal{B}} \rightarrow \mathcal{B}$ and an operator $\mathbf{S} : \overline{\mathcal{B}} \rightarrow \overline{\mathcal{B}}$ such that the dynamics of the latter covers the dynamics of S' as illustrated by the following commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{B}} & \xrightarrow{\mathbf{S}^n} & \overline{\mathcal{B}} \\ \Psi \uparrow & & \downarrow \text{Pr} \\ \mathcal{B} & \xrightarrow{(S')^n} & \mathcal{B} \end{array} \quad (3.3)$$

We will first define the space $\overline{\mathcal{B}}$ and the map \mathbf{S} . Then we will prove Theorem 2.2 by proving that \mathbf{S} has a spectral gap on $\overline{\mathcal{B}}$. Next we will obtain other, more refined, results by using the same strategy (albeit applied to different operators).

3.1. Covering dynamics. First we define the above mentioned abstract space. Let $\overline{\mathcal{B}} := \mathbb{C} \times [\times_{p \in \mathbb{Z}^d} \mathcal{B}_p]$ where¹¹

$$\mathcal{B}_p := \{\mu \in \mathcal{B} : \mu(\varphi) = 0 \forall \varphi \in \mathcal{C}^0(\Theta) \text{ that do not depend on } \theta_p\}.$$

⁹The equality of the norms follows from the usual weak definition of BV , the fact that the measures must be absolutely continuous can be easily proved by approximating a measure with finite norm by one with a smooth density (just use a mollifier) and remembering that the unit ball of BV is compact in L^1 . See Keller and Liverani (2005) for more details.

¹⁰The problem is already present for two sites since $BV(I) \otimes BV(I) \neq BV(I^2)$.

¹¹For example, if ν_p, ν'_p ($p \in \mathbb{Z}^d$) are probability measures on I such that $\nu_p = \nu'_p$ for all $p \neq q$, and we set $\nu := \otimes_{p \in \mathbb{Z}^d} \nu_p$, $\nu' := \otimes_{p \in \mathbb{Z}^d} \nu'_p$, then $\nu - \nu' \in \mathcal{B}_q$.

The vector space $\overline{\mathcal{B}}$ is a Banach space when equipped with the norm

$$\|\boldsymbol{\mu}\| := \sup\{|c_\mu|, \|\mu_p\| : p \in \mathbb{Z}^d\}.$$

Here we use the notational convention that an element $\boldsymbol{\mu} \in \overline{\mathcal{B}}$ has components $c_\mu \in \mathbb{C}$ and $\bar{\mu} := (\mu_p)_p$ with $\mu_p \in \mathcal{B}_p$.

Next we define a projection $\text{Pr} : D \subset \overline{\mathcal{B}} \rightarrow \mathcal{B}$ and a map $\Psi : \mathcal{B} \rightarrow \overline{\mathcal{B}}$ allowing to transfer objects between the two spaces.

Let $\mu_* \in \mathcal{B}$ be a fixed probability measure on I then $m = \otimes_{\mathbb{Z}^d} \mu_*$ is a product probability measure on Θ . For each $\boldsymbol{\mu} = (c_\mu, (\mu_p)_p) \in \overline{\mathcal{B}}$ and local function f we define

$$\text{Pr } \boldsymbol{\mu}(f) := c_\mu m(f) + \sum_{p \in \mathbb{Z}^d} \mu_p(f), \quad (3.4)$$

which makes clear in which sense $\overline{\mathcal{B}}$ ‘‘covers’’ \mathcal{B} (or $\mathcal{M}(\Theta)$).

Remark 3.1. Note that, although $\text{Pr } \boldsymbol{\mu}(f)$ is well defined on each local function, $\text{Pr } \boldsymbol{\mu}$ is not necessarily a measure. Let $\overline{\mathcal{B}}_M \subset \overline{\mathcal{B}}$ be such that the elements of $\text{Pr } \overline{\mathcal{B}}_M$ give rise to bounded linear functionals on the space of local functions, and hence identify uniquely a measure.¹² We will call such a measure $\text{Pr } \boldsymbol{\mu}$.

The choice of the map Ψ is quite arbitrary, we will fix a convenient one. Consider a strict total ordering \prec of \mathbb{Z}^d such that $0 \prec p$ for each $p \in \mathbb{Z}^d \setminus \{0\}$ and the sets $\{q : q \prec p\}$ are finite for each $p \in \mathbb{Z}^d$.¹³

Let q_+ be the successor of q (that is, $q \prec q_+$ and there are no $q' \in \mathbb{Z}^d$ such that $q \prec q' \prec q_+$). For each $q \in \mathbb{Z}^d$ we can then consider the σ -algebra \mathcal{F}_q^0 determined by all the variables $\theta_{q'}$ with $q \preceq q'$, hence \mathcal{F}_0^0 is the complete σ -algebra. Next, for each $f \in C^0(\Theta)$ and $q \in \mathbb{Z}^d$, define the operator $J_q f = m(f | \mathcal{F}_q^0) - m(f | \mathcal{F}_{q_+}^0)$. For each local function f we can write¹⁴

$$f = m(f) + \sum_{q \in \mathbb{Z}^d} J_q(f)$$

Accordingly, for each $\mu \in \mathcal{B}$ we define $\mu_q(f) := J'_q \mu(f) \in \mathcal{B}_q$, and the lift

$$\Psi(\mu) := (\mu(1), (J'_q \mu)_q).$$

Note that Ψ is a bounded operator. Indeed if $q \prec p$, then

$$|J'_q \mu(\partial_{\theta_p} \varphi)| \leq |\mu| |J_q(\partial_{\theta_p} \varphi)|_\infty \leq 2|\mu| \|\mu_*\| |\varphi|_\infty.$$

If $q \succ p$, then

$$|J'_q \mu(\partial_{\theta_p} \varphi)| \leq |\mu(\partial_{\theta_p} J_q \varphi)| \leq \|\mu\| |J_q(\varphi)|_\infty \leq 2\|\mu\| |\varphi|_\infty.$$

Finally, for $q = p$, we have $|J'_q \mu(\partial_{\theta_p} \varphi)| \leq (|\mu| \|\mu_*\| + \|\mu\|) |\varphi|_\infty$. In other words there exists $C_3 > 0$, depending on the choice of μ_* , such that

$$\|\Psi(\mu)\| \leq C_3 \|\mu\|. \quad (3.5)$$

Clearly, $\Psi(\mathcal{B}) \subset D$ and for each $\mu \in \mathcal{B}$ it holds true

$$\text{Pr}(\Psi(\mu)) = \mu.$$

Now that we know how to lift measures, we can address the dynamics.

¹²Since the local functions are dense in the continuous ones by the Stone-Weierstrass theorem.

¹³For example, one can start from zero and spiral out over larger and larger cubical shells.

¹⁴As f is local there exists a box $\Lambda_* \subset \Theta$ such that f depends only on the variables $\{\theta_q : q \in \Lambda_*\}$, but this means that the sum consists only of finitely many terms.

For all $z \in \mathbb{Z}^d$, $\tau^z \mu_q(\varphi) := \mu_q(\varphi \circ \tau^z) = 0$ if φ does not depend on θ_{q-z} . Thus we can define the decomposition for $\tau^z \mu$ via the decomposition $\mu_q = J'_q \mu$ of μ :

$$\tau^z \mu = \sum_{q \in \mathbb{Z}^d} (\tau^z \mu)_q := \sum_{q \in \mathbb{Z}^d} \tau^z \mu_{q-z}.$$

Setting $\Lambda_1 = \cup_{z \in \Lambda} \tau^{-z} \Lambda$, we can define the covering dynamics $\mathbf{S}\mu$ by $(c_\mu, \bar{S}\bar{\mu} + \bar{\zeta}c_\mu)$ where $\bar{\zeta} := (\zeta_q)$ with $\zeta_q := J'_q S' m =: J'_q \zeta$ and

$$(\bar{S}\bar{\mu})_p = \begin{cases} \sum_{z \in \Lambda} S'_z \mu_{p-z} + \sum_{q \in \Lambda_1} \sum_{z \in \Lambda} J'_p \hat{S}'_z \mu_{q-z} & \text{for each } p \notin \Lambda_1, \\ \sum_{z \in \Lambda} A'_z \mu_{p-z} + \sum_{q \in \Lambda_1} \sum_{z \in \Lambda} J'_p \hat{S}'_z \mu_{q-z} & \text{for each } p \in \Lambda_1, \end{cases} \quad (3.6)$$

where $S_z f(\omega) := \pi_z(\omega) f \circ F \circ \tau^z(\omega)$, $A_z f(\omega) := a_z f \circ F \circ \tau^z(\omega)$ and $\hat{S}_z := S_z - A_z$. It is easy to check the following.

Lemma 3.2. *The operator \mathbf{S} is well defined as a bounded operator from $\bar{\mathcal{B}}$ to $\bar{\mathcal{B}}$. For each $\mu \in \bar{\mathcal{B}}$ and continuous local function f we have $\Pr(\mathbf{S}\mu)(f) = \Pr(\mu)(Sf)$ which implies that for each $n \in \mathbb{N}$ and $\mu \in \mathcal{B}$ we have $\Pr \mathbf{S}^n \Psi \mu = S^{n'} \mu$*

We have thus established a setting in which the commutative diagram (3.3) holds true.

3.2. Mixing properties of the environment. We have now the necessary machinery to deal with the statistical properties of the environment.

Proof of Theorem 2.2 . Let us first discuss the environment dynamics F . Its basic properties are described by the so called Lasota-Yorke inequalities asserting that there exists $B > 0$ such that, for all $n \in \mathbb{N}$,¹⁵

$$\begin{aligned} |F' \mu| &\leq |\mu| \\ \|(F')^n \mu\| &\leq (2\lambda^{-1})^n \|\mu\| + B|\mu|. \end{aligned} \quad (3.7)$$

Note that the above implies that $\{\|(F')^n \mu\|\}_{n \in \mathbb{N}}$ is bounded.

Lemma 3.3. *There exists $\eta_* \in (\eta_0, 1)$ such that, for each $q \in \mathbb{Z}^d$, $\mu_q \in \mathcal{B}_q$,*

$$\|(F')^n \mu_q\| \leq C\eta_*^n \|\mu_q\|.$$

¹⁵The first is trivial since $|F' \mu(\varphi)| = |\mu(\varphi \circ F)| \leq |\mu| |\varphi|_\infty$. For the second, given any smooth local function φ , let $\varphi_{\theta_{\neq q}}(\xi) := \varphi(\theta^\xi)$, where $\theta_p^\xi = T\theta_p$ for each $p \neq q$ while $\theta_q^\xi = \xi$. Next introduce a function ϕ , piecewise linear in the variable θ_q such that $\varphi_{\theta_{\neq q}}(\theta_p) - \phi(\theta) = 0$ for each θ_p on the discontinuity values of T . By construction $\varphi_{\theta_{\neq q}}(T\theta_p) - \phi(F\theta)$ is then a Lipschitz function in θ_p , thus

$$\begin{aligned} |F' \mu(\partial_{\theta_q} \varphi)| &= |\mu((\partial_{\theta_q}(\varphi - \phi)) \circ F)| + |\mu| |\partial_{\theta_q} \phi|_\infty \leq |\mu(\partial_{\theta_q}(|D_{\theta_q} T|^{-1}(\varphi - \phi) \circ F))| + C|\mu| |\varphi|_\infty \\ &\leq \|\mu\| |D_{\theta_q} T|^{-1}(\varphi - \phi) \circ F|_\infty + C|\mu| |\varphi|_\infty \leq [2\lambda^{-1} \|\mu\| + C|\mu|] |\varphi|_\infty. \end{aligned}$$

The above yields $\|F' \mu\| \leq 2\lambda^{-1} \|\mu\| + C|\mu|$ which iterated yields the wanted result with $B = (1 - 2\lambda^{-1})^{-1}C$. See Baladi (2000); Keller and Liverani (2005) if more details are needed.

Proof. For each local function $\varphi \in C^0$, we can define $\varphi_{\theta \neq q}^n(\xi) := \varphi(\theta^{n,\xi})$, where $\theta_p^{n,\xi} = T^n \theta_p$ for each $p \neq q$ while $\theta_q^{n,\xi} = \xi$,

$$\begin{aligned} |(F')^n \mu_q(\varphi)| &= \left| \mu_q \left(\partial_{\theta_q} \int_0^1 [\chi_{[0,\theta_q]}(\xi) - \theta_q] \varphi_{\theta \neq q}^n(T^n \xi) d\xi \right) \right| \\ &\leq \|\mu_q\| \cdot \left| \int_0^1 \mathcal{L}^n [\chi_{[0,\theta_q]}(\xi) - \theta_q] \cdot \varphi_{\theta \neq q}^n(\xi) d\xi \right|_{\infty} \\ &\leq C \|\mu_q\| \|\mathcal{L}^n [\chi_{[0,\theta_q]} - \theta_q]\|_{BV} |\varphi|_{\infty} \leq C \|\mu_q\| \eta_0^n |\varphi|_{\infty}, \end{aligned}$$

by the spectral gap of \mathcal{L} and the fact that $m_{\mathcal{L}}(\chi_{[0,\theta_q]} - \theta_q) = 0$, i.e. it is a zero average function. Then, by the Lasota-Yorke inequality,

$$\begin{aligned} \|(F')^{j+k} \mu_q\| &\leq (2\lambda^{-1})^j \|(F')^k \mu_q\| + B|(F')^k \mu_q| \\ &\leq (2\lambda^{-1})^j [(2\lambda^{-1})^k \|\mu_q\| + B|\mu|] + BC\eta_0^k \|\mu_q\| \\ &\leq [(2\lambda^{-1})^{j+k} + (2\lambda^{-1})^j B + BC\eta_0^k] \|\mu_q\|. \end{aligned}$$

The result follows by optimizing the choice of $j + k = n$. \square

Lemma 3.4. *Multiplication by a C^1 local function is a bounded operator on \mathcal{B} .*

Proof. For any smooth local functions ψ, ϕ , $|\psi|_{\infty} \leq 1$ we have

$$|\nu(\phi \cdot \partial_{\theta_i} \psi)| = \left| \nu \left(\partial_{\theta_i} \int_0^{\theta_i} (\phi \partial_{\theta_i} \psi) \right) \right| \leq \|\nu\| \left| \int_0^{\theta_i} \phi \partial_{\theta_i} \psi \right|_{\infty} \leq 3\|\nu\| |\phi|_{C^1}. \quad (3.8)$$

\square

To use the above facts, it is convenient to introduce a more compact notation for the pieces that make up the operator \bar{S} . Let $\mathbb{1}_A : \mathbb{Z}^d \rightarrow \{0, 1\}$ be the characteristic function of the set $A \subset \mathbb{Z}^d$. Then define the operators $K_{z,p,q,\sigma} : \mathcal{B} \rightarrow \mathcal{B}$ by $K_{z,p,q,0} := \mathbb{1}_{\{p\}}(q+z)A'_z$ and $K_{z,p,q,1} := \mathbb{1}_{\Lambda_1^c}(p)\mathbb{1}_{\{p\}}(q+z)\hat{S}'_z + \mathbb{1}_{\Lambda_1}(q+z)J'_p\hat{S}'_z$. With this notation (3.6) can be rewritten as

$$(\bar{S}\bar{\mu})_p = \sum_{z \in \Lambda} \sum_{\sigma \in \{0,1\}} \sum_{q \in \mathbb{Z}^d} K_{z,p,q,\sigma} \mu_q.$$

Hence, iterating,

$$(\bar{S}^n \bar{\mu})_{q_0} = \sum_{z_1, \dots, z_n \in \Lambda} \sum_{\sigma_1, \dots, \sigma_n \in \{0,1\}} \sum_{q_1, \dots, q_n \in \mathbb{Z}^d} K_{z_1, q_0, q_1, \sigma_1} \cdots K_{z_n, q_{n-1}, q_n, \sigma_n} \mu_{q_n}. \quad (3.9)$$

By assumption 3, Lemma 3.3, Lemma 3.4 and the inequalities (3.7) it follows that there exists a constant $C_4 > 0$, depending only on F and π_z , such that $\sum_q \|K_{z,p,q,1} \mu_q\| \leq C_4 \varepsilon a_z \|\bar{\mu}\|$ and

$$\sum_{q_1, \dots, q_n \in \mathbb{Z}^d} \|K_{z_1, q_0, q_1, 0} \cdots K_{z_n, q_{n-1}, q_n, 0} \mu_{q_n}\| \leq C_4 a_{z_1} \cdots a_{z_n} \eta_*^n \|\bar{\mu}\|$$

Accordingly, if $C_4^2 \varepsilon + \eta_* < 1$, then there exists $n_* \in \mathbb{N}$ and $\eta \in (\eta_*, 1)$ such that $C_4(\eta_* + C_4^2 \varepsilon)^{n_*} \leq \eta^{n_*} < 1$. This means that every z_1, \dots, z_{n_*} term in (3.9) will be smaller than $\eta^{n_*} a_{z_1} \cdots a_{z_{n_*}} \|\bar{\mu}\|$, hence for all $n \in \mathbb{N}$,

$$\|(\bar{S}^n \bar{\mu})\| \leq C_4 \eta^{n-n_*} \|\bar{\mu}\|. \quad (3.10)$$

Since $\mathbf{S}^n \boldsymbol{\mu} = (c_\mu, \bar{S}^n \bar{\mu} + c_\mu \sum_{k=1}^{n-1} \bar{S}^k \bar{\zeta})$ and the series $\bar{\zeta}_* = \sum_{k=1}^{\infty} \bar{S}^k \bar{\zeta}$ converges by (3.10), it follows that $\boldsymbol{\mu}^w := (1, \bar{\zeta}_*)$ is an invariant vector for \mathbf{S} . In addition

$$\|\mathbf{S}^n \boldsymbol{\mu} - c_\mu \boldsymbol{\mu}^w\| \leq C \eta^n \|\boldsymbol{\mu}\|, \quad (3.11)$$

That is the operator \mathbf{S} on $\bar{\mathcal{B}}$ has 1 as a simple maximal eigenvalue and a spectral gap. From this result we can obtain the decay of temporal correlation simply by projecting down to \mathcal{B} . Indeed, let μ be a probability measure and ϕ be a smooth local function depending only on the sites $A \subset \mathbb{Z}^d$ and let L be the cardinality of A , then, by Lemma 3.2 and (3.11), (3.5),

$$\begin{aligned} |\mu(\phi \circ S^n) - \Pr(\boldsymbol{\mu}^w)(\phi)| &= |\Pr(\mathbf{S}^n(\Psi(\mu) - \boldsymbol{\mu}^w))(\phi)| \\ &= \left| \sum_{q \in A} (\mathbf{S}^n(\Psi(\mu) - \boldsymbol{\mu}^w))_q(\phi) \right| \\ &\leq \sum_{q \in A} \|(\mathbf{S}^n(\Psi(\mu) - \boldsymbol{\mu}^w))_q\| |\phi|_\infty \leq CL \eta^n (\|\mu\| + C) |\phi|_\infty. \end{aligned} \quad (3.12)$$

Thus, remembering (3.1), $\boldsymbol{\mu}^w \in \bar{\mathcal{B}}_M$, that is it gives rise to a bounded linear functional on local functions. Accordingly, we can define the measure $\mu^w = \Pr(\boldsymbol{\mu}^w)$ which will be invariant by S' . Equation (3.12) gives then the temporal correlation decay for such a measure and, together with (3.2), implies that $\mu^w \in \mathcal{B}$.

To have the spatial decay of correlations note that if φ and ϕ are supported at a distance M , then their support, under the dynamics, grows at most linearly in time, thus it will take a time $\frac{M}{2C_0}$ before the supports have a common variable. Accordingly, since $\varphi\phi$ depends on $2L$ variables, (3.12) (applied repeatedly to the product measure m) implies

$$\begin{aligned} \mu^w(\varphi\phi) &= (S')^{M/2C_0} m(\varphi\phi) + \mathcal{O}(L \eta^{M/2C_0} |\varphi\phi|_\infty) \\ &= m(S^{M/2C_0} \varphi) m(S^{M/2C_0} \phi) + \mathcal{O}(L \eta^{M/2C_0} |\varphi\phi|_\infty) \\ &= \mu^w(g) \mu^w(\phi) + \mathcal{O}(L \eta^{M/2C_0} |\varphi\phi|_\infty). \end{aligned}$$

□

3.3. Perturbation Theory. In this section we prove Lemma 2.5.

We deal with operators of the type $\mathcal{M}_t f := \sum_{z \in \Lambda} S_z(e^{(t, z-v)} f)$ where $t \in \mathbb{C}^d$. The problem is to study the spectrum for small t .¹⁶

Proof of Lemma 2.5. First of all we need to lift the operator to our covering space. The obvious solution is to define $\mathbf{M}_t \boldsymbol{\mu}$ by

$$\left(c_\mu m(\mathcal{M}_t 1) + \sum_{p \in \Lambda_1} \sum_{z \in \Lambda} e^{(t, z-v)} \mu_{p-z}(\hat{S}_z 1), \bar{S}_t \bar{\mu} + \bar{\zeta} c_\mu \right)$$

where $\bar{\zeta} = (\zeta_q) := (J'_q \mathcal{M}'_t m)$ and

$$(\bar{S}_t \bar{\mu})_q = \begin{cases} \sum_{z \in \Lambda} e^{(t, z-v)} S'_z \mu_{q-z} + \sum_{\substack{p \in \Lambda_1 \\ z \in \Lambda}} e^{(t, z-v)} J'_q \hat{S}'_z \mu_{p-z} & \forall q \notin \Lambda_1, \\ \sum_{z \in \Lambda} e^{(t, z-v)} A'_z \mu_{q-z} + \sum_{\substack{p \in \Lambda_1 \\ z \in \Lambda}} e^{(t, z-v)} J'_q \hat{S}'_z \mu_{p-z} & \forall q \in \Lambda_1. \end{cases} \quad (3.13)$$

¹⁶This problem is already well investigated, see in particular Bardet et al. (2007), here we treat it in detail only because we need some explicit estimates not readily available in the literature.

A direct computation shows that, for each smooth local function φ , $\Pr(\mathbf{M}_t \boldsymbol{\mu})(\varphi) = \Pr(\boldsymbol{\mu})(\mathcal{M}_t \varphi)$, thus the lift covers the dynamics. In addition, one can easily check that $\mathbf{M}_0 = \mathbf{S}$ and that \mathbf{M}_t is analytic in t .¹⁷ Accordingly, standard perturbation theory implies that there exists $\alpha_t, \boldsymbol{\mu}_t$, analytic in t , such that $\mathbf{M}_t \boldsymbol{\mu}_t = \alpha_t \boldsymbol{\mu}_t$ with $\alpha_0 = 1, \boldsymbol{\mu}_0 = \boldsymbol{\mu}^w$.

We will normalize $\boldsymbol{\mu}_t$ so that $\boldsymbol{\mu}_t = (1, \bar{\mu}_t)$. Setting $\mu_t := \Pr(\boldsymbol{\mu}_t)$, for each fixed local function f , $\mu_t(f) = \Pr \boldsymbol{\mu}_t(f)$ is analytic in t since the sum implicit in the right hand side is just a finite sum.¹⁸ However Lemma 2.5 requires a more quantitative information.

By the arguments of section 3.2 (see (3.10)) it follows that $\mathbf{M}_0 = \mathbf{S} = \Pi + R$ where $\Pi^2 = \Pi$, $\Pi R = R \Pi = 0$ and $\|R^n\| \leq C \eta^n$, for all $n \in \mathbb{N}$. Thus by standard perturbation theory (see Kato, 1966), $\mathbf{M}_t = \alpha_t \Pi_t + R_t$ where $|\alpha_0 - \alpha_t| \leq C \|t\|$, $\|\Pi_t - \Pi\| \leq C \|t\|$, $\|R_t^n\| \leq C \eta_t^n$, with $\eta_t \leq \eta + C \|t\|$. Hence, $\|\mathbf{M}_t^n\| \leq C |\alpha_t|^n + C \eta_t^n$ and, for each local function f depending only on L variables

$$|\mathcal{M}_t^n \nu(f)| = |\Pr(\mathbf{M}_t^n \Psi(\nu))(f)| \leq L |f|_\infty \|\mathbf{M}_t^n \Psi(\nu)\| \leq C |\alpha_t|^n \|\nu\| L |f|_\infty$$

provided that $|\alpha_t| \geq \eta_t$ which holds for all $\|t\| < B$ for some $B > 0$. This proves the first inequality of Lemma 2.5.

To prove the second note that $\Pi_t \boldsymbol{\nu} = \ell_t(\boldsymbol{\nu}) \boldsymbol{\mu}_t$ with $\ell_0(\boldsymbol{\nu}) = [\boldsymbol{\nu}]_0$ hence¹⁹

$$\begin{aligned} \mathcal{M}_t^n \boldsymbol{\nu}(1) &= [\mathbf{M}_t^n \Psi(\boldsymbol{\nu})]_0 = [\alpha_t^n \Pi_t \Psi(\boldsymbol{\nu}) + \mathcal{O}(\eta_t^n \|\boldsymbol{\nu}\|)]_0 \\ &= [\alpha_t^n \Pi_0 \Psi(\boldsymbol{\nu})]_0 + \mathcal{O}((\eta_t^n + \alpha_t^n C t) \|\boldsymbol{\nu}\|) = \alpha_t^n (1 + \mathcal{O}(t \|\boldsymbol{\nu}\|)) + \mathcal{O}(\eta_t^n \|\boldsymbol{\nu}\|). \end{aligned}$$

Finally, to study the derivatives of α we use the relation $\mu_t(\mathcal{M}_t \varphi) = \alpha_t \mu_t(\varphi)$ for any local smooth function φ . Differentiating with respect to t yields

$$\begin{aligned} \dot{\mu}_t(\mathcal{M}_t \varphi) + \mu_t(\dot{\mathcal{M}}_t \varphi) &= \dot{\alpha}_t \mu_t(\varphi) + \alpha_t \dot{\mu}_t(\varphi) \\ \dot{\mu}_t(\mathcal{M}_t \varphi) + 2\dot{\mu}_t(\dot{\mathcal{M}}_t \varphi) + \mu_t(\ddot{\mathcal{M}}_t \varphi) &= \ddot{\alpha}_t \mu_t(\varphi) + 2\dot{\alpha}_t \dot{\mu}_t(\varphi) + \alpha_t \ddot{\mu}_t(\varphi) \\ \dot{\mu}_t(1) &= \dot{\mu}_t(1) = 0, \end{aligned} \quad (3.14)$$

Since $\dot{\mathcal{M}}_t = \sum_{z \in \Lambda} (z - v) e^{(t, z - v)} S_z$ and $\ddot{\mathcal{M}}_t = \sum_{z \in \Lambda} (z - v) \otimes (z - v) e^{(t, z - v)} S_z$ the above equations, for $t = 0$ imply (substituting $\varphi = 1$)

$$\dot{\alpha}_0 = \mu^w(\dot{\mathcal{M}}_0 1) = \mu^w(g - v) = 0,$$

where g is defined in (2.5) and we have used (2.6). Next, substituting in the first of the (3.14), $\varphi = \sum_{k=0}^{n-1} S^k \phi$, for some local function ϕ , we have

$$\sum_{k=0}^{n-1} \dot{\mathcal{M}}_0 \mu^w(S^k \phi) = \dot{\mu}_0((\mathbb{1} - S) \sum_{k=0}^{n-1} S^k \phi) = \dot{\mu}_0(\phi) - \Pr(\mathbf{S}^{n-1} \dot{\boldsymbol{\mu}}_0)(\phi).$$

¹⁷For the first assertion note that

$$\sum_{p \in \Lambda_1} \sum_{z \in \Lambda} \mu_{p-z}(\hat{S}_z 1) = \sum_{p \in \mathbb{Z}^d} \sum_{z \in \Lambda} \mu_{p-z}(\hat{S}_z 1) = \sum_{p \in \mathbb{Z}^d} \sum_{z \in \Lambda} \mu_p(S_z 1) = \sum_{p \in \mathbb{Z}^d} \mu_p(S 1) = 0.$$

For the latter just write it as power series of t .

¹⁸Note that μ_t is not necessarily a measure and gives rise to an analytic object only when applied to a local function. We will abuse notations by writing $\dot{\mu}_t$ to mean the functional on local functions defined by $\Pr(\frac{d}{dt} \boldsymbol{\mu}_t)(\varphi)$.

¹⁹Here we use the notation $[\boldsymbol{\nu}]_0$ to designate the components c_ν of the vector $\boldsymbol{\nu} = (c_\nu, \bar{\nu})$.

By (3.11), taking the limit for n to infinity, we have

$$\dot{\mu}_0(\phi) = \sum_{k=0}^{\infty} \dot{\mathcal{M}}_0' \mu^w(S^k \phi). \quad (3.15)$$

Finally, the second of the (3.14), setting $\varphi = 1$ and $t = 0$, yields²⁰

$$\begin{aligned} \ddot{\alpha}_0 &= 2 \sum_{n=0}^{\infty} \mu^w(\dot{\mathcal{M}}_0 S^n \dot{\mathcal{M}}_0 1) + \mu^w(\ddot{\mathcal{M}}_0 1) \\ &= 2 \sum_{n=0}^{\infty} \mathbb{E}_{\mu^w} \left(\tilde{\Delta}_n \otimes \tilde{\Delta}_0 \right) + \mathbb{E}_{\mu^w} \left(\tilde{\Delta}_0 \otimes \tilde{\Delta}_0 \right). \end{aligned} \quad (3.16)$$

Since $\dot{\mathcal{M}}1$ is a local function the sum is convergent. Hence

$$\begin{aligned} \ddot{\alpha}_0 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[2 \sum_{k,m=0}^n \mathbb{E}_{\mu^w} \left(\tilde{\Delta}_{m+k} \otimes \tilde{\Delta}_k \right) + \mathbb{E}_{\mu^w} \left(\tilde{\Delta}_k \otimes \tilde{\Delta}_k \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{\mu^w} \left(\tilde{X}_n \otimes \tilde{X}_n \right) \geq 0. \end{aligned}$$

Finally, if there exists $w \in \mathbb{R}^d$ such that $\ddot{\alpha}_0 w = 0$, it means (from (3.16) and Theorem 2.2) that there exists a constant $C_5 > 0$ such that, for all $n \in \mathbb{N}$,²¹

$$\mathbb{E}_{\mu^w} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) \leq C_5; \quad \mathbb{E} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) \leq C_5$$

We can thus extract a subsequence $\{n_j\}$ such that $\langle w, \tilde{X}_{n_j} \rangle$ converges weakly almost surely to a random variable Z . Let $\psi = \mathbb{E}_{\mu^w}(Z | \mathcal{F}_0)$ and $\tilde{g}_w = \langle w, g - v \rangle$, then, for each \mathcal{F}_0 measurable smooth local function φ ,

$$\begin{aligned} \mathbb{E}_{\mu^w}(\varphi(\psi - S\psi)) &= \lim_{j \rightarrow \infty} \mathbb{E}_{\mu^w}(\varphi(\tilde{X}_{n_j} - \tilde{X}_{n_j+1})) = \lim_{j \rightarrow \infty} \mu^w(\varphi(\tilde{g}_w - S^{n_j+1} \tilde{g}_w)) \\ &= \mu^w(\varphi \tilde{g}_w), \end{aligned}$$

²⁰Remember that $\ddot{\alpha}_t = (\partial_{t_i} \partial_{t_j} \alpha_t)$ is a $d \times d$ matrix.

²¹The latter follows by Theorem 2.2. Let $\beta = \sum_z \langle w, z - v \rangle^2 \pi_z$, and $G_j = \sum_z \langle w, z - v \rangle S_z(S^j \langle w, g - v \rangle)$, then

$$\mathbb{E} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) = \sum_{k=0}^{n-1} \mu^e(S^k \beta) + \sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} \mu^e(S^k G_j)$$

and $|\mu^e(S^k \beta) - \mu^w(\beta)| \leq C\eta^k$, $|\mu^e(S^k G_j) - \mu^w(G_j)| \leq Cj^d \eta^k$. Moreover, setting $\nu_{k,z}(\varphi) := \langle w, z - v \rangle \mu^e(S^k S_z \varphi)$, for ε in assumption 3 such that $(1 + 3\varepsilon)2\lambda^{-1} < 1$ we have (by equation (3.7) and (3.8)) $\|\nu_{k,z}\| \leq C$. Hence Theorem 2.2 yields

$$|\mu^e(S^k G_j)| \leq \left| \sum_z \nu_{k,z}(S^j(g - v)) \right| \leq C\eta^j$$

Thus we can write

$$\left| \mathbb{E} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) - \mathbb{E}_{\mu^w} \left(|\langle w, \tilde{X}_n \rangle|^2 \right) \right| \leq C \sum_{k=0}^n \eta^k + C \sum_{j=1}^{n-1} \left[\sum_{k=0}^j \eta^k + \sum_{k=j+1}^{n-j} j^d \eta^k \right] \leq C.$$

where we have used Theorem 2.2. Thus $\tilde{g}_w = \psi - S\psi$, μ^w -a.s.; a similar argument proves the same identity μ^e -a.s.. This implies that, setting $M_0 = 0$, and

$$\begin{aligned} M_{n+1} - M_n &= \langle w, \tilde{\Delta}_n \rangle - \mathbb{E}(\langle w, \tilde{\Delta}_n \rangle \mid \mathcal{F}_n) + \psi(\omega^{n+1}) - S\psi(\omega^n) \\ &= \langle w, \tilde{\Delta}_n \rangle + \psi(\omega^{n+1}) - \psi(\omega^n), \end{aligned}$$

M_n is a \mathbb{P}_{μ^w} stationary martingale. Moreover,

$$\langle w, \tilde{X}_n \rangle = M_n - \psi(\omega^n) + \psi(\omega^0).$$

From this it follows that

$$\begin{aligned} C &\geq \mathbb{E}_{\mu^w}(|M_n|^2) = \sum_{k=1}^{n-1} \mathbb{E}_{\mu^w}(|\langle w, \tilde{\Delta}_k \rangle + \psi(\omega^{k+1}) - \psi(\omega^k)|^2) \\ &= \sum_{k=1}^{n-1} \mathbb{E}_{\mu^w}(|\langle w, \tilde{\Delta}_k \rangle + \psi(\omega^{k+1})|^2 - |\psi(\omega^k)|^2) \\ &= (n-1) \left[\mathbb{E}_{\mu^w}(|\langle w, \tilde{\Delta}_1 \rangle + \psi(\omega^1)|^2 - |\tilde{g}_w(\omega^0) + S\psi(\omega^0)|^2) \right]. \end{aligned}$$

Thus $\sum_z \pi_z |\langle w, z - v \rangle + \psi \circ F \circ \tau^z|^2 = |\sum_z \pi_z (\langle w, z - v \rangle + \psi \circ F \circ \tau^z)|^2$, that is $\langle w, z - v \rangle + \psi \circ F \circ \tau^z = \tilde{g}_w + S\psi = \psi$, μ^w -a.s..

Next, let $\alpha_z = \mu^w(\pi_z)$, then, $\sum_z \alpha_z = 1$ and $\sum_z \langle w, z - v \rangle \alpha_z = \mu^w(\tilde{g}_w) = 0$. Hence,

$$\sum_z \alpha_z \psi \circ F \circ \tau^z = \psi \quad \mu^w\text{-a.s..}$$

Note that the operator $S_\alpha \varphi := \sum_z \alpha_z \varphi \circ F \circ \tau^z$ defines a Markov process with invariant measure μ^e and satisfies the hypothesis of Theorem 2.2. Since $\psi = \sum_{k=0}^{n-1} S^k \tilde{g}_w + S^n \psi$, for each $\phi \in L^2(\mu^w)$,

$$\lim_{j \rightarrow \infty} \mu^w \left(\phi \sum_{k=0}^{n_j-1} S^k \tilde{g}_w \right) = \lim_{j \rightarrow \infty} \mathbb{E}_{\mu^w}(\phi(\omega^0) \langle w, X_{n_j} \rangle) = \mu^w(\phi \psi).$$

In addition, assumption 3 implies that setting,²² for each smooth local function ϕ , $\nu_{n,\phi}(\varphi) = \mu^w(\phi S_\alpha^n \varphi)$,

$$|\nu_{n,\phi}(\varphi)| \leq \frac{(1+\varepsilon)^n}{(1-\varepsilon)^n} |\phi|_\infty \mu^w(S^n |\varphi|) = \frac{(1+\varepsilon)^n}{(1-\varepsilon)^n} |\phi|_\infty \mu^w(|\varphi|).$$

Thus $\nu_{n,\phi}$ is absolutely continuous with respect to μ^w with density $\rho_{n,\phi} \in L^\infty(\mu^w)$. Accordingly,²³

$$\begin{aligned} \mu^w(\phi \psi) &= \mu^w(\phi S_\alpha^n \psi) = \lim_{j \rightarrow \infty} \sum_{k=0}^{n_j-1} \mu^w(\phi S_\alpha^n S^k \tilde{g}_w) \\ &= \sum_{k=0}^{n_l-1} [\mu^e(S^k \tilde{g}_w) \mu^w(\phi) + \mathcal{O}(C_\phi \eta^n k^d)] + \lim_{j \rightarrow \infty} \sum_{k=n_l}^{n_j-1} \mathcal{O}(C_\phi \eta^k) \\ &= \sum_{k=0}^{n_l-1} \mu^e(S^k \tilde{g}_w) \mu^w(\phi) + C_\phi \mathcal{O}(\eta^{n_l} + \eta^n n_l^d). \end{aligned}$$

²²Indeed, $\mu^w(\pi_z) \leq a_z(1+\varepsilon)$, thus $\pi_z \geq (1-\varepsilon)a_z \geq (1-\varepsilon)(1+\varepsilon)^{-1}\alpha_z$.

²³Note that (3.7) imply $\|S'_\alpha \mu\| \leq \sum_z \alpha_z \|F' \mu\| \leq (2\lambda)^{-1} \|\mu\| + B\|\mu\|$ and $|S_\alpha \mu| \leq |\mu|$. Thus, iterating, for each $n \in \mathbb{N}$, $\|(S_\alpha^n)' \mu\| \leq C \|\mu\|$.

Taking first the limit for $n \rightarrow \infty$ and the one $l \rightarrow \infty$ yields $\mu^w(\phi\psi) = \mu^w(\phi)\mu^e(\psi)$. That is ψ is μ^w almost surely constant. This implies that $\tilde{g}_w = 0$ and hence $\langle w, z - v \rangle \pi_z = 0$, μ^w a.s.. This is equivalent to saying that the vectors in the set $\{(\langle e_1, z \rangle, \dots, \langle e_d, z \rangle)\}_{z \in \Lambda} \cup \{(1, \dots, 1)\}$ are linearly dependent over \mathbb{R} , but this implies that they are linearly dependent over \mathbb{Z} . In other words we can assume that $w \in \mathbb{Z}^d$. Accordingly,

$$\left| \sum_{z \in \Lambda} \pi_z e^{i\langle w, z \rangle} \right| = \left| \sum_{z \in \Lambda} \pi_z e^{i\langle w, v \rangle} \right| = 1 \quad \mu^w - \text{a.s.}$$

Which, since $\mu^w \in \mathcal{B}$ (hence its marginal on I^Λ is absolutely continuous w.r.t. Lebesgue) and $\pi_z \in \mathcal{C}^1(I^\Lambda, \mathbb{R})$ (assumption 2), contradicts assumption 4. \square

3.4. Variation bounds for conditional measures. In the previous subsection we obtained several results for random walks provided that we start the environment in a measure with “density” of bounded variation. Here we show why such measures constitute a natural class for the problem at hand. More precisely we shall show that if we start with a nice measure and condition on a behavior of a walk during an initial time interval we still have a good control on the variation of densities.

For future needs we consider two random walks (X_t, Y_t) evolving in the same environment starting respectively at $a, b \in \mathbb{Z}^d$ and with the environment at time zero distributed according to the measure $\nu \in \mathcal{B}$. Let $\mathbb{P}_{a,b,\nu}^2$ be the measure on $(\Theta \times \mathbb{Z}^{2d})^\mathbb{N}$ associated to such a process and $\mathbb{E}_{a,b,\nu}^2$ the corresponding expectation $\mathbb{P}_\nu^2 := \mathbb{P}_{0,0,\nu}^2$ and $\mathbb{E}_\nu^2 := \mathbb{E}_{0,0,\nu}^2$.

Let $m \in \mathbb{N}$ and consider the σ -algebra $\mathcal{F}_m^{XY} = \sigma\{X_1, Y_1, \dots, X_m, Y_m\}$. We are interested in computing $\mathbb{E}_{a,b,\nu}^2(f(X, Y, \theta^m) \mid \mathcal{F}_m^{XY})$ for each local $\mathcal{F}_m^{XY} \otimes \mathcal{T}$ -measurable function f and probability measure $\nu \in \mathcal{B}$. Thus, we are interested in the measures $\nu_{a,b,m}^{XY}$ defined by

$$\mathbb{E}_{a,b,\nu}^2(f(X, Y, \theta^m) \mid \mathcal{F}_m^{XY}) =: \int_{\Theta} f(X, Y, \theta) \nu_{a,b,m}^{XY}(d\theta).$$

Lemma 3.5. *There exists $C_6 > 0$ and $0 < \varepsilon_1 \leq \varepsilon_0$ such that, if assumption 3 is satisfied for ε_1 , then for each $m \in \mathbb{N}$, $a, b \in \mathbb{Z}^d$ and probability measure $\nu \in \mathcal{B}$ the following holds*

$$\|\nu_{a,b,m}^{XY}\| \leq C_6 \|\nu\|.$$

Proof. Given two random walks realizations $X, Y : \mathbb{N} \rightarrow \mathbb{R}^d$, let us define the operators

$$S_{X,Y,k} f(\theta) := \pi_{z_k}(\tau^{X_k} \theta) \pi_{w_k}(\tau^{Y_k} \theta) f \circ F(\theta),$$

where $z_k = X_{k+1} - X_k$ and $w_k = Y_{k+1} - Y_k$. With such a notation we can write

$$\nu_{a,b,m}^{XY} = \frac{S'_{X,Y,m-1} \cdots S'_{X,Y,0} \nu}{S'_{X,Y,m-1} \cdots S'_{X,Y,0} \nu(1)}.$$

Recalling (3.8) and the Lasota-Yorke inequality for the map F (see (3.7)), and using assumption 3 we have

$$\|S'_{X,Y,k} \nu\| \leq 2\lambda^{-1}(1 + \varepsilon_1)^2 \|\nu\| a_{z_k} a_{w_k} + B S'_{X,Y,k} \nu(1). \quad (3.17)$$

Hence, for ε_1 such that $2\lambda^{-1}(1 + \varepsilon_1)^2 \leq \eta(1 - \varepsilon_1)^{-2} < 1$ we can iterate the above inequality and obtain

$$\begin{aligned} \|S'_{X,Y,m-1} \cdots S'_{X,Y,0} \nu\| &\leq \eta^m (1 - \varepsilon_1)^{2m} \|\nu\| \prod_{k=0}^{m-1} a_{z_k} a_{w_k} \\ &+ B \sum_{j=0}^{m-1} \eta^j (1 - \varepsilon_1)^{2j} \nu(S_{X,Y,0} \cdots S_{X,Y,m-1-j} 1) \prod_{k=m-j}^{m-1} a_{z_k} a_{w_k} \\ &\leq [\eta^m \|\nu\| + (1 - \eta)^{-1} B] \nu(S_{X,Y,0} \cdots S_{X,Y,m-1} 1), \end{aligned}$$

which proves the Lemma with $C_6 = 1 + (1 - \eta)^{-1} B$.²⁴ \square

4. Annealed Invariance Principle.

This section is devoted to proving an averaged invariance principle. This result is used in Section 5 to prove Lemma 2.8.

Consider the process

$$\hat{X}_t^N = \frac{1}{\sqrt{N}} \left\{ \tilde{X}_{\lceil tN \rceil} + (tN - \lceil tN \rceil) \tilde{\Delta}_{\lceil tN \rceil} \right\}. \quad (4.1)$$

Note that $\hat{X}_t^N \in \mathcal{C}^0([0, 1], \mathbb{R}^d)$, by construction. In fact, Lemma 2.6 implies higher regularity.

Lemma 4.1. *The family of processes $\{\hat{X}^N\} \subset \mathcal{C}^0([0, 1], \mathbb{R}^d)$ is tight.*

Proof. Let $\varsigma \in (0, 1/2)$,

$$L_\varsigma(f) := \sup_{t,s \in [0,1]} \frac{\|f(t) - f(s)\|}{|t - s|^\varsigma},$$

and $K_L^\varsigma := \{f \in \mathcal{C}^0([0, 1], \mathbb{R}^d) : f(0) = 0, L_\varsigma(f) \leq L\}$.

By Lemma 2.6 it follows that, for each $N \in \mathbb{N}$, $t \in [0, 1]$ and $h \in [-t, 1 - t]$,

$$\mathbb{P} \left(\left\{ \|\hat{X}_{t+h}^N - \hat{X}_t^N\| \geq Lh^\varsigma \right\} \right) \leq e^{-CL^2 h^{2\varsigma-1}}. \quad (4.2)$$

In addition, if $\|\hat{X}_t^N\| + \|\hat{X}_{t+h}^N\| \leq L^{1-\varsigma}$, then the set in (4.2) is empty for all $h > L^{-1}$. Now the result follows in complete analogy with the usual proof of the Hölder continuity of the Brownian motion, based on applying the above estimates to the dyadic rationals, yielding

$$\mathbb{P} \left(\left\{ \hat{X}^N \notin K_L^\varsigma \right\} \right) \leq e^{-CL}.$$

Since K_L^ς are compact in $\mathcal{C}^0([0, 1], \mathbb{R}^d)$ the tightness follows. \square

Lemma 4.1 also allows us to prove the invariance principle.

Lemma 4.2. *For each probability measure $\nu \in \mathcal{B}$ the process $\{\hat{X}_t^N\}$ converges in law to the Brownian motion with diffusion matrix Σ^2 .*

²⁴Note that, for a probability measure, $1 = \nu(1) = \nu(\partial_{\theta_i} \theta_i) \leq \|\nu\|$.

Proof. In view of Lemma 4.1 we only need to check the convergence of finite dimensional distributions. We consider two dimensional distributions, the general case being very similar. Accordingly, let $t_1 < t_2$ and fix ξ_1, ξ_2 . We have

$$\begin{aligned} \mathbb{E}_\nu \left(\exp(i\langle \xi_1, \widehat{X}_{t_1}^N \rangle + i\langle \xi_2, \widehat{X}_{t_2}^N \rangle) \right) &= \mathbb{E}_\nu \left(\exp(i\langle [\xi_1 + \xi_2], \widehat{X}_{t_1}^N \rangle) \exp(i\langle \xi_2, [\widehat{X}_{t_2}^N - \widehat{X}_{t_1}^N] \rangle) \right) \\ &= \mathbb{E}_\nu \left(\mathbb{E} \left(\exp(i\langle \xi_2, [\widehat{X}_{t_2}^N - \widehat{X}_{t_1}^N] \rangle) \middle| \mathcal{F}_{[t_1, N]} \right) \exp(i\langle [\xi_1 + \xi_2], \widehat{X}_{t_1}^N \rangle) \right). \end{aligned}$$

By Lemma 3.5 and Proposition 2.4 we have²⁵

$$\mathbb{E} \left(\exp(i\langle \xi_2, [\widehat{X}_{t_2}^N - \widehat{X}_{t_1}^N] \rangle) \middle| \mathcal{F}_{[t_1, N]} \right) = \exp \left(-\frac{1}{2} \langle \xi_2, \Sigma^2 \xi_2 \rangle (t_2 - t_1) (1 + o(1)) \right)$$

and so using Proposition 2.4 again we obtain

$$\mathbb{E}_\nu \left(\exp(i\langle \xi_1, \widehat{X}_{t_1}^N \rangle + i\langle \xi_2, \widehat{X}_{t_2}^N \rangle) \right) \sim e^{-\frac{1}{2} [\langle \xi_2, \Sigma^2 \xi_2 \rangle (t_2 - t_1) + \langle (\xi_1 + \xi_2), \Sigma^2 (\xi_1 + \xi_2) \rangle t_1]}$$

Thus, $(\widehat{X}_{t_1}^N, \widehat{X}_{t_2}^N)$ is asymptotically Gaussian with zero mean and the variance predicted by the Brownian Motion. \square

5. Proofs: Quenched CLT via the study of two random walks

The goal of this section is to establish Lemma 2.8.

Lemma 4.1 shows that the distributions of the processes $(\widehat{X}_t^N, \widehat{Y}_t^N)$ are tight, hence they have accumulation points. Our next task is to characterize such accumulation points. Let us consider any accumulation point $(\widehat{X}_t^\infty, \widehat{Y}_t^\infty)$. We will see that $(\widehat{X}_t^\infty, \widehat{Y}_t^\infty)$ is a centered Gaussian random variables with variance

$$\Sigma_2^2 := t \begin{pmatrix} \Sigma^2 & 0 \\ 0 & \Sigma^2 \end{pmatrix}. \quad (5.1)$$

More precisely, if we define the second order differential operator $\Delta_{\Sigma_2^2} := \sum_{i,j} (\Sigma_2^2)_{ij} \partial_i \partial_j$ we have the following.

Proposition 5.1. *For any $\psi \in \mathcal{C}^3(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ we have*

$$\frac{d}{dt} \mathbb{E}^2(\psi(t, \widehat{X}_t^\infty, \widehat{Y}_t^\infty)) = \mathbb{E}^2(\partial_t \psi(t, \widehat{X}_t^\infty, \widehat{Y}_t^\infty) + \frac{1}{2} \Delta_{\Sigma_2^2} \psi(t, \widehat{X}_t^\infty, \widehat{Y}_t^\infty)).$$

More precisely, there exists $\beta \in (0, \frac{1}{6})$ and $\vartheta \in (0, 1 - 2\beta)$ such that, for all $N \in \mathbb{N}$ and $t, h \in [0, 1]$ such that $h > N^{\vartheta-1}$ we have

$$\begin{aligned} & \left| \mathbb{E}^2 \left(\psi(t+h, \widehat{X}_{t+h}^N, \widehat{Y}_{t+h}^N) - \psi(t, \widehat{X}_t^N, \widehat{Y}_t^N) - h \left[\partial_t \psi(t, \widehat{X}_t^N, \widehat{Y}_t^N) + \frac{1}{2} \Delta_{\Sigma_2^2} \psi(t, \widehat{X}_t^N, \widehat{Y}_t^N) \right] \right) \right| \\ & \leq C \|\psi\|_{\mathcal{C}^3} (N^{-\beta} h + h^{\frac{3}{2}} + N^{-\frac{1}{2}}). \end{aligned}$$

²⁵In fact, Lemma 3.5 considers two walks, yet the corresponding result for one walk can be obtained by integrating over the second walk. Moreover, for each smooth function $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathbb{E}_\nu(f(\widehat{X}_{t_2}^N, \widehat{X}_{t_1}^N) \mid \mathcal{F}_{[t_1, N]}) &= \mathbb{E}_\nu \left(\mathbb{E}(f(\widehat{X}_{t_2}^N, \widehat{X}_{t_1}^N) \mid \tilde{X}_{[t_1, N]}, \theta^{[t_1, N]}) \mid \mathcal{F}_{[t_1, N]} \right) \\ &= \nu_{\tilde{X}_{[t_1, N]}, [t_1, N]}^{\tilde{X}} \left(\mathbb{E}(f(\widehat{X}_{t_2}^N, \widehat{X}_{t_1}^N) \mid \tilde{X}_{[t_1, N]}, \theta^{[t_1, N]}) \right) \\ &= \mathbb{E}_{X_{t_1}^N, \nu_{\tilde{X}_{[t_1, N]}, [t_1, N]}^{\tilde{X}}} \left(\mathbb{E}(f(\widehat{X}_{t_2}^N - t_1, \widehat{X}_0^N) \mid \tilde{X}_0, \theta^0) \right) = \mathbb{E}_{X_{t_1}^N, \nu_{\tilde{X}_{[t_1, N]}, [t_1, N]}^{\tilde{X}}} \left(f(\widehat{X}_{t_2}^N - t_1, \widehat{X}_0^N) \right). \end{aligned}$$

Proposition 2.4 can then be applied after translating $\nu_{\tilde{X}_{[t_1, N]}, [t_1, N]}^{\tilde{X}}$ by $\tilde{X}_{[t_1, N]}$.

Thanks to the above Proposition for each $\phi \in \mathcal{C}_0^3(\mathbb{R}^d \times \mathbb{R}^d)$ we can define ψ by

$$\begin{aligned} \partial_t \psi + \frac{1}{2} \Delta_{\Sigma_2^2} \psi &= 0 \\ \psi(1, x, y) &= \phi(x, y) \end{aligned} \quad (5.2)$$

and, by applying Proposition 5.1 with the choice $h = \lceil N^{2\beta} \rceil^{-1}$, obtain the wanted result:

$$\begin{aligned} \mathbb{E}^2(\phi(\hat{X}_1^N, \hat{Y}_1^N)) &= \mathbb{E}^2(\psi(1, \hat{X}_1^N, \hat{Y}_1^N)) \\ &= \mathbb{E}^2(\psi(0, \hat{X}_0^N, \hat{Y}_0^N)) + \sum_{i=0}^{h^{-1}-1} \mathbb{E}^2(\psi((i+1)h, \hat{X}_{(i+1)h}^N, \hat{Y}_{(i+1)h}^N) - \psi(ih, \hat{X}_{ih}^N, \hat{Y}_{ih}^N)) \\ &= \mathbb{E}^2(\psi(0, 0, 0)) + \mathcal{O}(\|\psi\|_{\mathcal{C}^3} N^{-\beta}) = \mathbb{E}_{\mathcal{N}(0, \Sigma_2^2)}(\phi) + \mathcal{O}(\|\phi\|_{\mathcal{C}^3} (N^{-\beta} + N^{-\frac{1}{2}+2\beta})), \end{aligned}$$

where we have used the explicit solution of (5.2).²⁶ Remembering the form of Σ_2^2 (see (5.1)), Lemma 2.8, and hence Theorem 2.7, follow.

Proof of Proposition 5.1. We start by the following Taylor expansion

$$\begin{aligned} \mathbb{E}^2(\psi(t+h, \hat{X}_{t+h}^N, \hat{Y}_{t+h}^N)) - \mathbb{E}^2(\psi(t, \hat{X}_t^N, \hat{Y}_t^N)) &= \mathbb{E}^2(\partial_t \psi(t, \hat{X}_t^N, \hat{Y}_t^N))h \\ &+ \mathbb{E}^2(\partial_x \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N) + \partial_y \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N)) \\ &+ \frac{1}{2} \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N) \cdot \partial_x^2 \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N)) \\ &+ \mathbb{E}^2((\hat{Y}_{t+h}^N - \hat{Y}_t^N) \cdot \partial_{xy} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N)) \\ &+ \frac{1}{2} \mathbb{E}^2((\hat{Y}_{t+h}^N - \hat{Y}_t^N) \cdot \partial_y^2 \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N)) \\ &+ \frac{1}{2} \mathbb{E}^2(\partial_{tx} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N))h \\ &+ \frac{1}{2} \mathbb{E}^2(\partial_{ty} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N))h \\ &+ \mathcal{O}\left(|\partial_t^2 \psi|_{\infty} h^2 + \|\psi\|_{\mathcal{C}^3} \left[\mathbb{E}(\|\hat{X}_{t+h}^N - \hat{X}_t^N\|^3) + \mathbb{E}(\|\hat{Y}_{t+h}^N - \hat{Y}_t^N\|^3) \right]\right). \end{aligned} \quad (5.3)$$

Next, we will analyze the terms in equation (5.3) one by one.

First of all note that the remainders are of order $h^{\frac{3}{2}}$.²⁷ Let us start the estimates with the term $\mathbb{E}(\partial_x \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N))$. To this end we consider the σ -algebra \mathcal{F}_m^{XY} generated by $\{X_1, \dots, X_m, Y_1, \dots, Y_m\}$. Setting $\tilde{g} = g - v$, $\ell_t :=$

²⁶Indeed, equation (5.2) is just the backward heat equation, thus for $t \in (0, 1)$

$$\psi(t, x, y) = \frac{1}{(4\pi)^d \det(\Sigma_2^{-1})(1-t)^d} \int_{\mathbb{R}^{2d}} e^{-\frac{\langle (x-z, y-w), \Sigma_2^{-2}(x-z, y-w) \rangle}{2(1-t)}} \phi(z, w) dz dw.$$

²⁷In fact Lemma 2.6 implies, for each $p \in \mathbb{N}$, that

$$\begin{aligned} \mathbb{E}(\|\hat{X}_{t+h}^N - \hat{X}_t^N\|^p) &\leq N^{\frac{p}{2}} h^p \mathbb{E}\left(\left\| \frac{1}{|hN|} (\tilde{X}_{\lceil (t+h)N \rceil} - \tilde{X}_{\lceil tN \rceil}) \right\|^p\right) \\ &\leq Ch^{\frac{p}{2}} + C_p N^{\frac{p}{2}} h^p \int_{(Nh)^{-\frac{1}{2}}}^{\infty} x^{p-1} \mathbb{P}\left(\left\| \frac{1}{|hN|} (\tilde{X}_{\lceil (t+h)N \rceil} - \tilde{X}_{\lceil tN \rceil}) \right\| \geq x\right) dx \\ &\leq Ch^{\frac{p}{2}} + C_p N^{\frac{p}{2}} h^p \int_{(Nh)^{-\frac{1}{2}}}^{\infty} x^{p-1} e^{-Cx^2 hN} dx \leq C_p h^{\frac{p}{2}}. \end{aligned}$$

$[tN] + 1$ we can write

$$\begin{aligned} \mathbb{E}((\hat{X}_{t+h}^N - \hat{X}_t^N) | \mathcal{F}_{\ell_t}^{XY}) &= N^{-\frac{1}{2}} \sum_{k=\ell_t}^{\ell_{t+h}} \mathbb{E}(\tilde{\Delta}_k^X | \mathcal{F}_{\ell_t}^{XY}) + \mathcal{O}(N^{-\frac{1}{2}}) \\ &= N^{-\frac{1}{2}} \sum_{k=\ell_t}^{\ell_{t+h}} \mathbb{E}((S^{k-\ell_t} \tilde{g}) \circ \tau^{X_{\ell_t}} | \mathcal{F}_{\ell_t}^{XY}) + \mathcal{O}(N^{-\frac{1}{2}}) \\ &= N^{-\frac{1}{2}} \sum_{k=\ell_t}^{\ell_{t+h}} [(S^{k-\ell_t} \tau^{X_{\ell_t}})'(\mu^e)_{\ell_t}^{XY}](\tilde{g}) + \mathcal{O}(N^{-\frac{1}{2}}). \end{aligned}$$

where, by Lemma 3.5, $\|(\mu^e)_{\ell_t}^{XY}\| \leq C$ and hence $\|(\tau^{X_{\ell_t}})'(\mu^e)_{\ell_t}^{XY}\| \leq C$. From Theorem 2.2 and the fact that $\mu^w(\tilde{g}) = 0$ it follows that

$$|\mathbb{E}^2(\partial_x \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{X}_{t+h}^N - \hat{X}_t^N))| \leq CN^{-\frac{1}{2}} \|\psi\|_{\mathcal{C}^1}. \quad (5.4)$$

In complete analogy we have

$$|\mathbb{E}^2(\partial_y \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N))| \leq CN^{-\frac{1}{2}} \|\psi\|_{\mathcal{C}^1}. \quad (5.5)$$

The quadratic terms involving ∂_{tx} and ∂_{ty} are estimated in the same manner yielding terms of order $N^{-\frac{1}{2}} h \|\psi\|_{\mathcal{C}^2}$. The quadratic terms involving only X or only Y yield the following

$$\begin{aligned} \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i (\hat{X}_{t+h}^N - \hat{X}_t^N)_j \partial_{x_i x_j} \psi) &= (\Sigma^2)_{ij} \mathbb{E}^2(\partial_{x_i x_j} \psi) h + \mathcal{O}\left(\frac{\|\psi\|_{\mathcal{C}^2}}{\sqrt{N}}\right) \\ \mathbb{E}^2((\hat{Y}_{t+h}^N - \hat{Y}_t^N)_i (\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j \partial_{y_i y_j} \psi) &= (\Sigma^2)_{ij} \mathbb{E}^2(\partial_{y_i y_j} \psi) h + \mathcal{O}\left(\frac{\|\psi\|_{\mathcal{C}^2}}{\sqrt{N}}\right). \end{aligned} \quad (5.6)$$

Indeed, in analogy with what we have done before, we can condition with respect to $\mathcal{F}_{\ell_t}^{XY}$ and

$$\begin{aligned} \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i (\hat{X}_{t+h}^N - \hat{X}_t^N)_j | \mathcal{F}_{\ell_t}^{XY}) &= \frac{1}{N} \left\{ \sum_{k=\ell_t}^{\ell_{t+h}-1} \mathbb{E}^2((\tilde{\Delta}_k^X)_i (\tilde{\Delta}_k^X)_j | \mathcal{F}_{\ell_t}^{XY}) \right. \\ &\quad \left. + \sum_{k=\ell_t}^{\ell_{t+h}-1} \sum_{l=\ell_t}^{k-1} \mathbb{E}^2((\tilde{\Delta}_k^X)_i (\tilde{\Delta}_l^X)_j + (\tilde{\Delta}_l^X)_i (\tilde{\Delta}_k^X)_j | \mathcal{F}_{\ell_t}^{XY}) \right\} \\ &\quad + N^{-1} \mathcal{O}\left(1 + \sum_{l=\ell_t+1}^{\ell_{t+h}-1} \mathbb{E}^2((\tilde{\Delta}_{\ell_{t+h}-1}^X)_i (\tilde{\Delta}_l^X)_j + (\tilde{\Delta}_{\ell_{t+h}-1}^X)_j (\tilde{\Delta}_l^X)_i | \mathcal{F}_{\ell_t}^{XY})\right), \end{aligned} \quad (5.7)$$

where the boundary terms of the type $\mathbb{E}^2((\tilde{\Delta}_{\ell_t-1}^X)_i (\tilde{\Delta}_l^X)_j | \mathcal{F}_{\ell_t}^{XY})$ have been estimated as in (5.4). To estimate such an expression note that

$$\mathbb{E}^2((\tilde{\Delta}_k^X)_i (\tilde{\Delta}_l^X)_j | \mathcal{F}_{\ell_t}^{XY}) = (S^{j-\ell_t} \tau^{X_{\ell_t}})' \nu_{\ell_t}^{XY}(\tilde{g}_j S^{k-j} \tilde{g}_k).$$

Since Lemma 3.5 and Theorem 2.2 imply $\|(S^{j-\ell_t}\tau^{X_{\ell_t}})'\nu_{\ell_t}^{XY}\| \leq C\|\mu^e\|$ and also that $\|\tilde{g}_j(S^{j-\ell_t}\tau^{X_{\ell_t}})'\nu_{\ell_t}^{XY}\| \leq C\|\mu^e\|$ we can apply Theorem 2.2

$$\begin{aligned} \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_l^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right| &= \nu_{\ell_t}^{XY}(S^{j-\ell_t}\tau^{X_{\ell_t}}\tilde{g}_j)\mu^w(\tilde{g}_k) + \mathcal{O}(\eta^{k-j}) = \mathcal{O}(\eta^{k-j}) \\ \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_l^X)_j \mid \mathcal{F}_{\ell_t}^{XY}) \right| &= \nu_{\ell_t}^{XY}(1)\mu^w(\tilde{g}_j S^{k-j}\tilde{g}_k) + \mathcal{O}(\eta^{j-\ell_t}) \\ &= \mu^w(\tilde{g}_j S^{k-j}\tilde{g}_k) + \mathcal{O}(\eta^{j-\ell_t}). \end{aligned}$$

Using such estimates in (5.7) and remembering formula (3.16) (for $\ddot{\alpha}_0 = \Sigma^2$) the first equation of (5.6) follows. The second equation of (5.6) is proven in complete analogy.

Finally, we must deal with the mixed quadratic term.

$$\mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i(\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j \mid \mathcal{F}_{\ell_t}^{XY}) = \sum_{k,m=\ell_t-1}^{\ell_t+h} \frac{\mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{\ell_t}^{XY})}{N} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

If $|k-m| \geq A \ln N$, then, for $A > \ln \eta^{-1}$, by Lemma 3.5 and Theorem 2.2 it follows that

$$\begin{aligned} \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i(\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j \mid \mathcal{F}_{\ell_t}^{XY}) &= \sum_{\substack{\ell_t+2A \ln N \leq k \leq \ell_t+h \\ |m-k| \leq A \ln N}} \frac{\mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{\ell_t}^{XY})}{N} \\ &\quad + \mathcal{O}(N^{-\frac{1}{2}}). \end{aligned}$$

Next, suppose that $|m-k| \leq A \ln N$ and $|X_{k-A \ln N} - Y_{k-A \ln N}| > 4C_0 A \ln N$.

Assume, to fix our ideas, that $k \leq m$. Then, for all times l such that $|l-k| \leq A \ln N$, the two walks explore disjoint parts of the environment. Thus, we can consider the process started at time $k - A \ln N$ with the conditional measure $(\mu^e)_{k-A \ln N}^{XY}$ and with the walks starting from $a = X_{k-A \ln N}$, $b = Y_{k-A \ln N}$, $\|a-b\| > 4C_0 A \ln N$. If we set $\mathbf{f} = S^{A \ln N} \tau^{X_k} \tilde{g}_i$ and $\mathbf{h} = S^{m-k+A \ln N} \tau^{Y_k} \tilde{g}_j$ we have that the two functions depend on different sets of variables (let $B \subset \mathbb{Z}^d$ be the set of variables on which \mathbf{f} depends and B' the ones relative to \mathbf{h}) and

$$\mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{k-A \ln N}^{XY}) = (\mu^e)_{k-A \ln N}^{XY}(\mathbf{f} \mathbf{h}).$$

We can then define the Newtonian potential of \mathbf{f} ,

$$\begin{aligned} \Psi(\theta) &= \frac{1}{|B|(|B|-2)\alpha_{|B|}} \int_{I^{\mathbb{Z}^d}} \|\theta^B - \vartheta^B\|^{-|B|+2} \mathbf{f}(\vartheta) \, d\vartheta^B \otimes_{j \notin B} \mu_0(d\vartheta_j) \\ &= \frac{1}{|B|(|B|-2)\alpha_{|B|}} \int_{I^B} \|\theta^B - \vartheta^B\|^{-|B|+2} \mathbf{f}(\vartheta^B) \, d\vartheta^B. \end{aligned} \tag{5.8}$$

where $\theta^B = (\theta_l)_{l \in B}$ and α_l is the volume of the unit ball in \mathbb{R}^l . It is well known that, for θ^B in the interior of I^B

$$\sum_{l \in B} \partial_{\theta_l} \Psi = \mathbf{f}.$$

Thus, remembering Lemma 3.5, we can write²⁸

$$\begin{aligned} \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{k-A \ln N}^{XY}) \right| &\leq \sum_{l \in B} |(\mu^e)_{k-A \ln N}^{XY}(\partial_{\theta_l}(\partial_{\theta_l} \Psi \cdot \mathbf{h}))| \\ &\leq |B| \cdot \|(\mu^e)_{k-A \ln N}^{XY}\| \cdot \sup_{l \in B} |\partial_{\theta_l} \Psi \cdot \mathbf{h}|_\infty \leq CA^d C_0^d |\ln N|^d \cdot |g|_\infty \cdot \sup_{l \in B} |\partial_{\theta_l} \Psi|_\infty. \end{aligned}$$

By (5.8) we have, for $l \in B$,

$$\partial_{\theta_l} \Psi(\theta) = \int_{I^d} \frac{\theta_l - \vartheta_l}{|B|(|B| - 2)\alpha_{|B|} \cdot \|\theta^B - \vartheta^B\|^{|B|}} \mathbf{f}(\vartheta) d\vartheta^B \otimes_{j \notin B} \mu_0(d\vartheta_j) =: \nu_l^\theta(\mathbf{f}).$$

Unfortunately, $\nu_l^\theta \notin \mathcal{B}$ due to the singularity of the kernel. To take care of this problem we need to isolate the singularity. For each $r > 0$ let $\chi_r \in C^\infty(\mathbb{R}^B, [0, 1])$ such that $\chi_r(\theta^B) = 0$ for all $\|\theta^B\| \leq r$ and $\chi_r(\theta^B) = 1$ for all $\|\theta^B\| \geq 2r$. Clearly χ_r can be chosen radial and so that $\sup_l |\partial_{\theta_l} \chi_r|_\infty \leq Cr^{-1}$. We then define

$$\nu_{l,r}^\theta(\phi) := \int_{I^d} \frac{(\theta_l - \vartheta_l) \cdot \chi_r(\theta - \vartheta)}{|B|(|B| - 2)\alpha_{|B|} \cdot \|\theta^B - \vartheta^B\|^{|B|}} \phi(\vartheta) d\vartheta^B \otimes_{j \notin B} \mu_0(d\vartheta_j)$$

and $\mu_{l,r}^\theta(f) := \nu_l^\theta(\phi) - \nu_{l,r}^\theta(\phi)$. A direct computation shows that $|\mu_{l,r}^\theta| \leq Cr(AC_0 \ln N)^{-d}$ and $\|\nu_{l,r}^\theta\| \leq C(AC_0 \ln N)^{-d} \ln r^{-1}$. Since $|\mathbf{f}| \leq 2|g|_\infty$ we can finally use Theorem 2.2 to estimate

$$\begin{aligned} \left| \mathbb{E}^2((\tilde{\Delta}_k^X)_i(\tilde{\Delta}_m^Y)_j \mid \mathcal{F}_{k-A \ln N}^{XY}) \right| &\leq C \left\{ r + \sup_{\substack{l \in B \\ \theta \in I^B}} |(\tau^{X_k})' \nu_{l,r}^\theta(S^{A \ln N} \tilde{g}_i)| \right\} \\ &\leq C \left\{ r + \mu^w(\tilde{g}_i) + \sup_{\substack{l \in B \\ \theta \in I^B}} \|(\tau^{X_k})' \nu_{l,r}^\theta\| \eta^{A \ln N} \right\} \\ &\leq C \{ r + N^{-1} \ln r^{-1} \} \leq CN^{-1} \ln N, \end{aligned}$$

where we have chosen $r = N^{-1}$.

In conclusion,

$$\begin{aligned} \left| \mathbb{E}^2((\hat{X}_{t+h}^N - \hat{X}_t^N)_i \cdot \partial_{x_i y_j} \psi(t, \hat{X}_t^N, \hat{Y}_t^N) \cdot (\hat{Y}_{t+h}^N - \hat{Y}_t^N)_j) \right| &\leq CN^{-\frac{1}{2}} \|\psi\|_{\mathcal{C}^2} \\ &+ A \ln N N^{-1} \mathbb{E}^2(\text{Card}\{t \leq N : \|X_t - Y_t\| \leq 4C_0 A \ln N\}) \|\psi\|_{\mathcal{C}^2}. \end{aligned} \quad (5.9)$$

In Section 6 we prove the following bound.

Lemma 5.2. *Let A be a large constant and set $L_N := A \ln N$. There exists $\delta_0 \in (0, 1)$ such that*

$$\mathbb{E}^2(\text{Card}\{t \leq N : \|X_t - Y_t\| \leq L_N\}) \leq CN^{\delta_0} \quad (N \in \mathbb{N}). \quad (5.10)$$

Lemma 5.2 allows to estimate the last term in the right hand side of (5.9) by

$$CN^{\delta_0-1} \ln N \|\psi\|_{\mathcal{C}^2} = Ch(h^{-1}N^{\delta_0-1}) \ln N \|\psi\|_{\mathcal{C}^2},$$

proving the proposition by choosing $\beta = \frac{1-\delta_0}{6}$ and $\vartheta = 1 - 3\beta$. \square

²⁸Remember that the marginal of $(\mu^e)_l^{XY}$ on $B \cup B'$ is absolutely continuous with respect to Lebesgue, hence the boundary of I^B has zero measure, moreover $\partial_{\theta_l} \Psi$ is a continuous function on I^d .

6. Two walks estimates

In Section 5 we proved that Lemma 2.8 (and hence Theorem 2.7) holds provided the average number of times two walks come closer than $A \ln N$ in time N is smaller than N^{δ_0} for some $\delta_0 \in (0, 1)$. The purpose of this section is to prove such an estimate and therefore conclude the argument.

6.1. *On the number of close encounters.* The proof of inequality (5.10) can be reduced to the following simpler inequality.

Lemma 6.1. *There exist $\rho \in (0, 1)$, $C_7 > 0$ such that for any $m \in \mathbb{N}$ and for any a, b such that $\|a - b\| > L_N$, we have*

$$\mathbb{P}^2(\{\|X_j - Y_j\| > L_N \quad \forall j \in \{m, \dots, m + N\}\} \mid X_m = a, Y_m = b) \geq \frac{C_7}{N^\rho}, \quad (6.1)$$

(Here \mathbb{P}^2 is the underlying probability for the process (θ_t, X_t, Y_t) started with θ_0 distributed according to μ^ε).

We postpone the proof of the above Lemma until finishing the proof of (5.10).

Proof of Lemma 5.2. Notice that assumption 4 implies that the walks can move in different directions with positive probability. In particular, there exists $\gamma > 0$ such that for each $a, b \in \mathbb{Z}^d$, $m \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{P}^2\left(\left\{\|X_{m+L_N} - Y_{m+L_N}\| \geq L_N\right\} \mid X_m = a, Y_m = b\right) \geq c(\delta)\gamma^{\delta L_N}, \quad (6.2)$$

Indeed

$$\mathbb{P}^2\left(\left\{\|X_{m+\delta L_N} - Y_{m+\delta L_N}\| \geq \delta L_N\right\} \mid X_m = a, Y_m = b\right) \geq \gamma^{\delta L_N}$$

the latter being the probability of one fixed path in which X_i, Y_i get further and further apart at each step. On the other hand

$$\mathbb{P}^2\left(\left\{\|X_{m+L_N-\delta L_N} - Y_{m+L_N-\delta L_N}\| \geq L_N\right\} \mid \|X_m - Y_m\| \geq \delta L_N\right) \geq c(\delta), \quad (6.3)$$

To verify (6.3) let $W^{(1)}(t)$ and $W^{(2)}(t)$ be independent Brownian Motions such that $W^{(2)}(0) - W^{(1)}(0) = \mathbf{v}$, where $\|\mathbf{v}\| = \delta$. Let $\hat{\mathbf{v}} := \mathbf{v} \|\mathbf{v}\|^{-1}$, and

$$c(\delta) := \frac{1}{2}P\left(\|W^{(2)}(1) - W^{(1)}(1)\| > 1, \quad \text{and for all } t \in [0, 1]\right)$$

$$\langle W^{(2)}(t), \hat{\mathbf{v}} \rangle > \langle W^{(2)}(0), \hat{\mathbf{v}} \rangle - \frac{\delta}{3} \quad \text{and} \quad \langle W^{(1)}(t), \hat{\mathbf{v}} \rangle < \langle W^{(1)}(0), \hat{\mathbf{v}} \rangle - \frac{\delta}{3}$$

Observe that the invariance principle established in Lemma 4.2, Lemma 3.5 and the fact that local dynamics are independent implies a two particle invariance principle as long as the walkers explore disjoint regions in the phase space. Therefore the probability that two walkers grow L_N apart exploring disjoint regions of the phase space is at least $c(\delta)$ for large N proving (6.3). Choose $\varrho < 1 - \rho$. Then,

$$\begin{aligned} & \mathbb{P}^2\left(\left\{\sup_{m \leq i \leq m+N^e} \|X_i - Y_i\| \leq L_N\right\} \mid X_m = a, Y_m = b\right) \\ & \leq \prod_{j=1}^{N^e L_N^{-2}} (1 - c(\delta)\gamma^{\delta L_N}) \leq e^{-c(\delta)\gamma^{\delta L_N} L_N^{-2} N^e} \leq e^{-C N^{\varrho/2}}, \end{aligned} \quad (6.4)$$

provided that δ is sufficiently small.

Next, consider the sets $B_R^- := \{(x, y) : \|x - y\| \leq R\}$, $B_R^+ := \{\|x - y\| > R\}$ and the stopping times, for $k > 0$,

$$\begin{aligned} s_0 &:= \inf \{j \in \mathbb{N} : j > 0, (X_{j-1}, Y_{j-1}) \in B_{L_N}^-, (X_j, Y_j) \in B_{L_N}^+\}, \\ s_{2k} &:= \inf \{j \in \mathbb{N} : j > s_{2k-2}, (X_{j-1}, Y_{j-1}) \in B_{L_N}^-, (X_j, Y_j) \in B_{L_N}^+\}, \\ s_1 &:= \inf \{j \in \mathbb{N} : j > s_0, (X_{j-1}, Y_{j-1}) \in B_{L_N}^+, (X_j, Y_j) \in B_{L_N}^-\}, \\ s_{2k+1} &:= \inf \{j \in \mathbb{N} : j > s_{2k-1}, (X_{j-1}, Y_{j-1}) \in B_{L_N}^+, (X_j, Y_j) \in B_{L_N}^-\}. \end{aligned}$$

Clearly, $s_{2k} < s_{2k+1} < s_{2k+2}$ and $s_k > k$. As $X_0 = Y_0$, these stopping times are adapted to the filtration \mathcal{F}_t^{XY} . Note that the s_{2k} are upcrossing times hence $\|X_t - Y_t\| \leq L_N$ for all the $t \in \{s_{2k-1}, \dots, s_{2k} - 1\}$. With this notation, (6.4) implies

$$\mathbb{P}^2 \left(\left\{ \sup_{i \leq N} (s_{2i} - s_{2i-1}) > N^e \right\} \right) \leq N \sup_{i \leq N} \mathbb{P}^2(\{s_{2i} - s_{2i-1} > N^e\}) \leq N e^{-N^{e/2}}.$$

Let us set $J := \inf\{k \in \mathbb{N} : s_k \geq N\}$, clearly $J \leq N$,

$$\mathbb{E}^2(\text{Card}\{n < N : \|X_n - Y_n\| \leq L_N\}) \leq N^2 e^{-N^{e/2}} + N^e \mathbb{E}^2(J). \quad (6.5)$$

It remains to investigate the length of the intervals of time in which the two walks are further apart than L_N . Let $S_n := \{\sup_{k \leq n} (s_{2k+1} - s_{2k}) < N\}$, and denote by $\mathcal{F}_{s_{2k}}^{XY}$ the σ -algebra associated to the filtration \mathcal{F}_t^{XY} and the stopping time s_{2k} . Then, by (6.1),

$$\begin{aligned} \mathbb{P}^2(\{J > n\}) &\leq \mathbb{E}^2(\mathbb{1}_{S_n}) = \mathbb{E}^2(\mathbb{1}_{S_{n-1}} \mathbb{P}^2(\{s_{2n+1} - s_{2n} < N\} | S_{n-1})) \\ &= \mathbb{E}^2(\mathbb{1}_{S_{n-1}} \mathbb{P}^2(\{s_{2n+1} - s_{2n} < N\} | \mathcal{F}_{s_{2n}}^{XY})) \\ &\leq \left(1 - \frac{C_0}{N^\rho}\right) \mathbb{E}^2(\mathbb{1}_{S_{n-1}}) \leq \dots \leq \left(1 - \frac{C_0}{N^\rho}\right)^n. \end{aligned}$$

Thus, letting $1 - \varrho > \alpha > \rho$, it follows that

$$\mathbb{P}^2(\{J > N^\alpha\}) \leq C e^{-C_0 N^{\alpha-\rho}}.$$

which means that $\mathbb{E}^2(J) \leq N^\alpha + N \mathbb{P}^2(\{J > N^\alpha\}) \leq C N^\alpha$. In view of (6.5) this proves (5.10) provided we have chosen δ_0 so that $\varrho + \alpha < \delta_0$. \square

Our program is thus completed once we prove (6.1). To this end an intermediate result is needed.

Lemma 6.2. *Given $R > 0$, take two points a_R and b_R such that $\|a_R - b_R\| = R$. Consider two walks starting at a_R and b_R respectively with the environment given by the probability distribution $\nu \in \mathcal{B}$ and define the stopping time $\tau_{\delta, R}$ as the first time $n > 0$ such that*

$$\|X_n - Y_n\| \leq \frac{R}{1 + \delta} \text{ or } \|X_n - Y_n\| \geq (1 + \delta)R.$$

For each $C_8 > 0$ there exist $R_\delta \in \mathbb{R}_+$, $c_1, c_2 > 0$ such that for each $\|\nu\| \leq C_8$, for each $R \geq R_\delta$

$$\mathbb{P}_\nu^2(\{\|X_{\tau_{\delta, R}} - Y_{\tau_{\delta, R}}\| \geq (1 + \delta)R\}) \geq \frac{1}{2 + \delta} - c_1 e^{-c_2/\delta}.$$

Proof. Let T_R be the first time $n > 0$ such that

$$\max(\|X_n - X_m\|, \|Y_n - Y_m\|) \geq \frac{R}{2}.$$

Then, by Section 4 the pair

$$\left(\frac{X_{\min(T_R, tR^2)} - X_m}{R}, \frac{Y_{\min(T_R, tR^2)} - X_m}{R} \right)$$

is asymptotic, when $R \rightarrow \infty$, to a pair of independent Brownian Motions

$$(W^{(1)}(t), W^{(2)}(t)) \in \mathbb{R}^d \times \mathbb{R}^d \quad W^{(1)}(0) = 0, \quad \|W^{(2)}(0)\| = 1$$

stopped at time T when one of them wanders more than $1/2$ from its starting position.²⁹ Let τ be the first time $\|W^{(1)}(t) - W^{(2)}(t)\| = (1 + \delta)$ or $\|W^{(1)}(t) - W^{(2)}(t)\| = (1 + \delta)^{-1}$. Recall that

$$P(\{\|W^{(1)}(\tau) - W^{(2)}(\tau)\| = (1 + \delta)\}) \geq \frac{1}{2 + \delta}$$

(the worst case is then $d = 1$, see e.g. Revuz and Yor, 1999, Section XI.1). On the other hand

$$P(T < \tau) \leq P(\tau > \delta) + P(T < \delta) \tag{6.6}$$

and both terms are $O(e^{-c/\delta})$, the first one because

$$P(\tau > (k + 1)\delta^2 | \tau > k\delta^2) \leq \gamma < 1$$

and the second one by Hoeffding's inequality (see e.g. Grimmett and Stirzaker, 2001). Now³⁰

$$\begin{aligned} & \mathbb{P}(\{\|X_{\tau_{\delta,R}} - Y_{\tau_{\delta,R}} = R(1 + \delta)\}) \\ & \geq \mathbb{P}(\{\|X_{\tau_{\delta,R}} - Y_{\tau_{\delta,R}} = R(1 + \delta)\} \text{ and } \tau_{\delta,R} < T_R) \geq \frac{1}{2 + \delta} - c_1 e^{-c_2/\delta}. \end{aligned}$$

□

We now use the following comparison criterion (proved in section 6.2).

Lemma 6.3. *Suppose $\xi_1, \xi_2 \dots \xi_n \dots$ is a random process such that $\xi_n = \pm 1$ and for all n*

$$P(\xi_n = 1 | \xi_1 \dots \xi_{n-1}) \geq p.$$

Let $\tilde{\xi}_1, \tilde{\xi}_2 \dots \tilde{\xi}_n \dots$ be iid random variables such that $\tilde{\xi}_n = \pm 1$, and $P(\tilde{\xi}_n = 1) = p$. Let

$$\mathcal{X}_n = \sum_{j=1}^n \xi_j + \mathcal{X}_0 \quad \tilde{\mathcal{X}}_n = \sum_{j=1}^n \tilde{\xi}_j + \tilde{\mathcal{X}}_0.$$

Then for any $\alpha_1 < \alpha < \alpha_2$

$$P(\mathcal{X}_k \text{ reaches } \alpha_2 \text{ before } \alpha_1 | \mathcal{X}_0 = \alpha) \geq P(\tilde{\mathcal{X}}_k \text{ reaches } \alpha_2 \text{ before } \alpha_1 | \tilde{\mathcal{X}}_0 = \alpha)$$

²⁹More precisely, the fact that each component is approximately Brownian comes from Section 4 while independence is due to the fact that the walkers explore non-intersecting regions in the phase space and can be proven by the same arguments used to estimate (5.9).

³⁰Here c_1 should be taken a little bit larger than the implied constants in (6.6) to take into account that our process is only approximated by Brownian Motion.

Recall that by Gambler's Ruin Formula for $p \neq 1/2$

$$P(\tilde{\mathcal{X}}_k \text{ reaches } \alpha_2 \text{ before } \alpha_1 | \tilde{\mathcal{X}}_0 = \alpha) = \frac{\left(\frac{p}{1-p}\right)^{\alpha_2-\alpha} - \left(\frac{p}{1-p}\right)^{\alpha_2-\alpha_1}}{1 - \left(\frac{p}{1-p}\right)^{\alpha_2-\alpha_1}} \quad (6.7)$$

Proof of Lemma 6.1. Let $X_m = a$ and $Y_m = b$ with $\|a-b\| \geq L_N$ and $\kappa \in (\frac{1}{2}, 1)$.

Using ellipticity of assumption 4 for the first δL_N steps we see that with probability greater than $N^{-c\delta}$ our walkers move distance $(1+\delta)L_N$ apart without getting within distance L_N from each other. Let τ_1 be the first time after m when our walkers move distance $(1+\delta)L_N$ apart and let τ_{n+1} be the first time after τ_n when

$$\|X_j - Y_j\| \geq (1+\delta)\|X_{\tau_n} - Y_{\tau_n}\| \text{ or } \|X_j - Y_j\| \leq (1+\delta)^{-1}\|X_{\tau_n} - Y_{\tau_n}\|.$$

Applying Lemma 6.3 to $\mathcal{X}_n = \frac{\ln \|X_{\tau_n} - Y_{\tau_n}\|}{\ln(1+\delta)}$ with $\alpha_1 = \frac{\ln L_N}{\ln(1+\delta)}$, $\alpha = \alpha_1 + 1$, $\alpha_2 = \frac{\kappa \ln N}{\ln(1+\delta)}$ and using Lemma 6.2, taking into account Lemma 3.5, to estimate the probability of moving apart we conclude from (6.7) that for each $\epsilon > 0$, by choosing δ small and N large enough, the probability that the walkers move distance $N^\kappa L_N$ apart without getting within distance L_N from each other is at least $c\delta N^{-\kappa-\epsilon}$.

Hence there is a polynomially small probability of making an excursion of size $N^\kappa L_N$ before returning to a distance L_N . On the other hand once we have such a big excursion Lemma 2.6 implies that it will take more than N steps to come back, indeed

$$\begin{aligned} \mathbb{P}^2 \left(\left\{ \inf_{\ell+1 \leq j \leq \ell+N} \|X_j - Y_j\| \leq L_N \right\} \mid \|X_\ell - Y_\ell\| \geq N^\kappa L_N \right) \\ \leq CN e^{-CN^{2\kappa-1}}. \end{aligned}$$

The last two estimates imply Lemma 6.1 □

6.2. Comparison Lemma.

Proof of Lemma 6.3. Let $U_1, U_2 \dots U_n \dots$ be random variables which are independent and uniformly distributed on $[0, 1]$. Define

$$\xi_n^* = -1 \text{ if } U_n < P(\xi_n = -1 | \xi_1 = \xi_1^*, \dots, \xi_{n-1} = \xi_{n-1}^*) \text{ and } \xi_n = 1 \text{ otherwise.}$$

Also let $\tilde{\xi}_n^* = -1$ if $U_n < 1-p$ and $\tilde{\xi}_n^* = 1$ otherwise. Let

$$\mathcal{X}_n^* = \sum_{j=1}^n \xi_j^*, \quad \tilde{\mathcal{X}}_n^* = \sum_{j=1}^n \tilde{\xi}_j^*.$$

Then $\{\mathcal{X}_n^*\}$ has the same distribution as $\{\mathcal{X}_n\}$, $\{\tilde{\mathcal{X}}_n^*\}$ has the same distribution as $\{\tilde{\mathcal{X}}_n\}$ and $\mathcal{X}_n^* \geq \tilde{\mathcal{X}}_n^*$. □

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References

- V. Baladi. *Positive Transfer Operators and Decay of Correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific, Singapore (2000).
- A. Bandyopadhyay and O. Zeitouni. Random walk in dynamic markovian random environment. *ALEA (Latin American Journal of Probability and Statistics)* **1**, 205–224 (2006).
- J.-B. Bardet, S. Gouëzel and G. Keller. Limit theorems for coupled interval maps. *Stochastics and Dynamics* **7**, 17–36 (2007).
- C. Boldrighini, R. A. Minlos and A. Pellegrinotti. Random Walk in a Fluctuating Random Environment with Markov Evolution. In R.A. Minlos, S. Shlosman and Yu.M. Suhov, editors, *On Dobrushin's way. From probability theory to statistical physics*, volume 198 of *AMS Transl. Ser. 2*, pages 13–35. AMS, Providence, RI (2000).
- D. Dolgopyat, G. Keller and C. Liverani. Random walk in markovian environment (2007). To appear in *Annals of Probability* (preprint).
- G. Grimmett and D. Stirzaker. *Probability and Random Processes*. Oxford Univ. Press, 3rd edition (2001).
- T. Kato. *Perturbation theory for linear operators*. Springer (1966).
- G. Keller and C. Liverani. A spectral gap for a one-dimensional lattice of coupled piecewise expanding interval maps. In J. Chazottes and B. Fernandez, editors, *Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems*, volume 671 of *Lecture Notes in Physics*, pages 115–151. Springer (2005).
- G. Keller and C. Liverani. Uniqueness of the SRB measure for piecewise expanding weakly coupled map lattices in any dimension. *Communications in Mathematical Physics* **262** (1), 33–50 (2006).
- C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Communications in Mathematical Physics* **104**, 1–19 (1986).
- D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 3d edition (1999).
- A.-S. Sznitman. Topics in random walk in random environment. In *Notes of course at School and Conference on probability*, ICPT Lecture Series, pages 203–266. Trieste (2004).
- O. Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin (2004).