RANDOM WALK LOOP SOUP

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ABSTRACT. The Brownian loop soup introduced by Lawler and Werner (2004) is a Poissonian realization from a σ-finite measure on unrooted loops. This measure satisfies both conformal invariance and a restriction property. In this paper, we define a random walk loop soup and show that it converges to the Brownian loop soup. In fact, we give a strong approximation result making use of the strong approximation result of Komlós, Major, and Tusnády. To make the paper self-contained, we include a proof of the approximation result that we need.

1. INTRODUCTION

The Brownian loop soup with intensity λ, which we define below, is a Poissonian realization from a particular measure on unrooted loops in $\mathbb{C}$ that satisfies both conformal invariance and a property called the restriction property. A realization of the loop soup consists of a countable collection of loops. In a fixed bounded domain D, there is an infinite number of loops that stay in D; however, the number of loops of diameter at least ϵ in the bounded domain is finite. A corollary of conformal invariance is scale invariance: if $\mathcal{A}$ is a realization of the Brownian loop soup and each loop is scaled in space by $1/N$ and in time by $1/N^2$, the resulting configuration also has the distribution of the Brownian loop soup. In this paper, we will show that the Brownian soup is a limit of random walk soups. There are two natural approaches to showing this. One is a "weak" limit to show that the Brownian loop measure is a weak limit of random walk measures (this requires some care since the measures are infinite). However, we choose the more direct "coupling" approach of defining the random walk loop soup and the Brownian loop soup on the same probability space so that the realizations are close. Since a realization is a countable collection of loops, it is a little tricky to say what it means for the realizations to be close. We will prove, in fact, that in a bounded domain D, except for an event of small probability, there is a one-to-one correspondence between the Brownian loops and the random walk loops if we restrict to loops that are not "too small". The Brownian loops and random walk loops that correspond with each other will be very close. We will use the dyadic approximation scheme as in [3] to establish the strong approximation of the two soups.

We start by defining the Brownian loop measure. It is easier to define the loop measure first on *rooted* loops. A *(rooted) loop* is a continuous function $\gamma : [0, t_\gamma] \to \mathbb{C}$

with $\gamma(0) = \gamma(t_\gamma)$. We will only consider loops with $0 < t_\gamma < \infty$. Let $\mathcal{C}$ denote the set of all loops and $\mathcal{C}_t$ the set of loops γ with $t_\gamma = t$ and $\gamma(0) = \gamma(t_\gamma) = 0$. The *Brownian bridge* measure μ^{br} is the probability measure on loops induced by a Brownian bridge, i.e., by $B_t := W_t - tW_1, 0 \le t \le 1$, where W_t is a standard two-dimensional Brownian motion. The measure μ^{br} is supported on $\mathcal{C}_1$. The *(rooted) Brownian loop measure* is the measure μ on $\mathcal{C}_1 \times \mathbb{C} \times (0, \infty)$ given by

$$\mu = \mu^{\mathrm{br}} \times \text{area} \times [\frac{1}{2\pi t^2} \, dt].$$

The measure μ induces a measure on $\mathcal{C}$ (which we also denote by μ) by the function $(\gamma, z, t) \mapsto \tilde{\gamma}$, where $\tilde{\gamma}$ is γ scaled (using Brownian scaling) to have time duration t and translated to have root z. In other words,

$$\tilde{\gamma}(s) = z + t^{1/2} \, \gamma(s/t), \quad 0 \le s \le t.$$

This measure is clearly translation invariant, and it is straightforward to check that if $r > 0$, then μ is invariant under the Brownian scaling map $(\gamma, z, t) \mapsto (\gamma, rz, r^2t)$.

Let us denote by $\mu_t^{\mathrm{br}}(z)$ the probability measure on loops induced by a Brownian bridge of time duration t rooted at z. Then the measure μ (as a measure on $\mathcal{C}$) can be written as

$$\int_{\mathbb{C}} \int_0^\infty \frac{1}{2\pi t^2} \, \mu_t^{\mathrm{br}}(z) \, dt \, dA(z),$$

where A denotes area.

An *unrooted loop* is an equivalence class of (rooted) loops under the equivalence $\gamma \sim \theta_r \gamma$ for every $r \in \mathbb{R}$, where $\theta_r \gamma(s) = \gamma(s + r)$ (here we consider a rooted loop γ of time duration t_γ as a continuous function $\gamma : \mathbb{R} \to \mathbb{C}$ with $\gamma(s + t_\gamma) = \gamma(s)$ for all s). The unrooted loop measure μ^{u} is the measure obtained from μ by "forgetting the root". A *rooted Brownian loop soup* with intensity λ is a Poissonian realization from $\lambda\mu$. An *(unrooted) Brownian loop soup* is a realization from $\lambda\mu^{\mathrm{u}}$. One can obtain an unrooted loop soup by starting with a rooted loop soup and forgetting the root.

If D is a domain in $\mathbb{C}$ we let $\mu_D, \mu^{\mathrm{u}}{}_D$ denote μ, μ^{u} restricted to loops that lie in D. The family of measure $\{\mu^{\mathrm{u}}{}_D\}$ clearly satisfy the restriction property, i.e., if $D' \subset D$, then $\mu^{\mathrm{u}}{}_{D'}$ is $\mu^{\mathrm{u}}{}_D$ restricted to curves lying in D'. It is also shown in [4] that the family satisfies a conformal invariance property, i.e., if $f : D \to D'$ is a conformal transformation, then $f \circ \mu^{\mathrm{u}}{}_D = \mu^{\mathrm{u}}{}_{D'}$, if the quantities are suitably interpreted. In particular, if γ is a curve lying in D, we define $f \circ \gamma$ to be the curve in D', reparametrized by the conformal map; see [4] for details. The measure μ on rooted loops is not conformally invariant.

In this paper we study the loop measure for simple (nearest neighbor) random walks on the integer lattice $\mathbb{Z}^2$, which we can consider as a subset of $\mathbb{C}$. The rooted loop measure μ^{rw} gives each (nearest neighbor) random walk loop in $\mathbb{Z}^2$ of length $2n$ measure $(2n)^{-1} 4^{-2n}$. The unrooted loop measure is obtained from the rooted loop measure by "forgetting the root". It is almost the same measure as that obtained by giving measure 4^{-2n} to every unrooted loop of length $2n$. (If a loop of length $2n$ is obtained by taking a loop of length n and repeating the same loop again, then this unrooted loop does not get full measure 4^{-2n} under our random walk loop measure; these exceptional loops are an exponentially small subset of the set of all loops, so it is not important whether we give these unrooted loops measure 4^{-2n} or $(1/2) \, 4^{-2n}$.) We will focus on the rooted measure in this paper. A rooted random

walk loop of length $2n$ can also be considered as a continuous path $\gamma : [0, 2n] \to \mathbb{C}$ by linear interpolation. We will call a Poissonian realization from $\lambda \mu^{\mathrm{rw}}$ a *rooted random walk loop soup* (with intensity λ).

In this paper, we make a precise statement that the random walk loop soup, appropriately scaled, approaches the Brownian loop soup. We will define $(\mathcal{A}_\lambda, \tilde{\mathcal{A}}_\lambda)$ on the same probability space so that $\mathcal{A}_\lambda$ is a realization of the Brownian loop soup with intensity λ, and $\tilde{\mathcal{A}}_\lambda$ is a realization of the random walk loop soup with intensity λ. We consider the loops in $\tilde{\mathcal{A}}_\lambda$ as curves in $\mathcal{C}$ by linear interpolation. Note that $\mathcal{A}_\lambda$ is a (random) countable set of curves and $\tilde{\mathcal{A}}_\lambda$ is a (random) multi-set (i.e., a set where some elements can appear more than once) of lattice curves. For each positive integer N we define $\mathcal{A}_{\lambda,N}$ to be the collection of loops obtained from $\mathcal{A}_\lambda$ by scaling space by $1/N$. More precisely,

$$\mathcal{A}_{\lambda,N} = \{\Phi_N \gamma : \gamma \in \mathcal{A}_\lambda\},$$

where $t_{\Phi_N \gamma} = t_\gamma / N^2$ and

$$\Phi_N \gamma(t) = N^{-1} \gamma(tN^2), \quad 0 \le t \le t_\gamma / N^2.$$

Note that the scaling rule implies that $\mathcal{A}_{\lambda,N}$ is a realization of the Brownian loop soup with parameter λ. We define

$$\tilde{A}_{\lambda,N} = \{\tilde{\Phi}_N \gamma : \gamma \in \tilde{\mathcal{A}}_\lambda\},$$

where $t_{\tilde{\Phi}_N \gamma} = t_\gamma / (2N^2)$ and

$$\tilde{\Phi}_N \gamma(t) = N^{-1} \gamma(t2N^2), \quad 0 \le t \le t_\gamma / (2N^2).$$

The scaling is slightly different for the random walk loops because the covariance of a simple two-dimensional random walk in $2n$ steps is nI as opposed to $2nI$ for a Brownian motion at time $2n$; roughly speaking, this is because in $2n$ steps the random walk moves about n steps horizontally and n steps vertically.

We will prove the theorem below. The ideas in the proof are simple and flexible. However, due to discretization, stating the result is a little unwieldy. To aid this, we introduce the following auxiliary functions. For $t \ge (5/8)N^{-2}$ and positive integer k, we let

$$\varphi_N(t) = \frac{k}{N^2} \quad \text{if} \quad \frac{k}{N^2} - \frac{3}{8N^2} \le t < \frac{k}{N^2} + \frac{5}{8N^2}.$$

Also, for $z \in \mathbb{C}, z_0 \in \mathbb{Z}^2$ we define

$$\psi_N(z) = \frac{z_0}{N} \quad \text{if} \quad \max\{|\mathrm{Re}\{Nz - z_0\}|, |\mathrm{Im}\{Nz - z_0\}|\} < \frac{1}{2}.$$

The definition of ψ_N if Nz happens to fall on a bond of the dual lattice of $\mathbb{Z}^2$ is irrelevant for our theorem.

Theorem 1.1. *One can define on the same probability space $\mathcal{A}_\lambda$ and $\tilde{\mathcal{A}}_\lambda$ such that:*

- *For each $\lambda > 0$, $\mathcal{A}_\lambda$ is a realization of the Brownian loop soup; the realizations are increasing in λ.*
- *For each $\lambda > 0$, $\tilde{\mathcal{A}}_\lambda$ is a realization of the random walk loop soup; the realizations are increasing in λ,*

and such that the following holds. Let $\mathcal{A}_{\lambda,N}$ and $\tilde{\mathcal{A}}_{\lambda,N}$ be as defined above. Then there exists a $c > 0$ such that for every $r \ge 1, N, \lambda$ and every $2/3 < \theta < 2$, except perhaps on an event of probability at most $c(\lambda + 1) r^2 N^{2-3\theta}$, there is a one-to-one correspondence between $\{\tilde{\gamma} \in \tilde{\mathcal{A}}_{\lambda,N} : t_{\tilde{\gamma}} > N^{\theta-2}, |\tilde{\gamma}(0)| < r\}$ and $\{\gamma \in \mathcal{A}_{\lambda,N} :$

$\varphi_N(t_\gamma) > N^{\theta-2}, |\psi_N(\gamma(0))| < r\}$. *If* $\tilde{\gamma} \in \tilde{\mathcal{A}}_{\lambda,N}$ *and* $\gamma \in \mathcal{A}_{\lambda,N}$ *are paired in this correspondence, then*

$$|t_\gamma - t_{\tilde{\gamma}}| \leq 5/8N^{-2},$$

$$\sup_{0 \leq s \leq 1} |\gamma(st_\gamma) - \tilde{\gamma}(st_{\tilde{\gamma}})| \leq c\, N^{-1} \log N.$$

The outline of the paper is as follows. In the next three sections, we define the random walk loop soup, state the strong approximation result between random walk bridges and Brownian bridges that we need, construct the probability space on which both the random walk and Brownian loop soups are defined, and verify that the construction satisfies Theorem 1.1. The next section concerns the soups in bounded domains. Here we establish a similar result to the theorem above, although the error terms are somewhat larger. The remainder of the paper gives a self-contained proof of the strong approximation result that we need.

2. RANDOM WALK LOOPS AND RANDOM WALK SOUP

A *(rooted) random walk loop* of length $2n$ in $\mathbb{Z}^2$ (which will be considered as a subset of $\mathbb{C}$) is a $(2n+1)$-tuple $\omega = [\omega_0, \ldots, \omega_{2n}]$ with $|\omega_j - \omega_{j-1}| = 1$ and $\omega_0 = \omega_{2n}$. A loop can be considered as a curve $\gamma : [0, 2n] \to \mathbb{C}$; here $\gamma(m) = \omega_m$ for integer m and $\gamma(t)$ is defined for other t by linear interpolation. Let $\mathcal{L}_n$ denote the set of random walk loops of length $2n$ and let $\mathcal{L}_n^z$ be the set of such loops with $\omega_0 = z$. There is a natural one-to-one correspondence between $\mathcal{L}_n^0$ and $\mathcal{L}_n^z$ given by $\omega \leftrightarrow z + \omega$. We let $\mathcal{L} = \bigcup_{n \geq 1} \mathcal{L}_n$, $\mathcal{L}^z = \bigcup_{n \geq 1} \mathcal{L}_n^z$. We let ν denote the random walk loop measure $\mathcal{L}$, i.e., the measure that assigns measure 4^{-2n} to each $\omega \in \mathcal{L}_{2n}$. We write ν_n^z for ν restricted to $\mathcal{L}_n^z$. For fixed z, it is straightforward to show

$$\nu(\mathcal{L}_n^z) = \nu(\mathcal{L}_n^0) = \left[2^{-2n} \binom{2n}{n} \right]^2 = \frac{1}{\pi\, n} - \frac{1}{4\pi n^2} + O(\frac{1}{n^3}).$$

The second equality holds because the probability that a two-dimensional simple random walk returns to the origin at time $2n$ is the square of the probability that a one-dimensional simple random walk returns to the origin; see (3.1) below to see why this is true. The final equality is derived from Stirling's formula with error (see [2, Chapter II, (9.14)]):

$$(2.1) \qquad\qquad n! = \sqrt{2\pi}\, n^{n+(1/2)}\, e^{-n} \left[1 + \frac{1}{12n} + O(\frac{1}{n^2}) \right].$$

We define the *(rooted) random walk loop measure* μ^{rw} to be the measure that assigns measure $(2n)^{-1}\nu(\omega) = (2n)^{-1}\, 4^{-2n}$ to each $\omega \in \mathcal{L}_n$. A *rooted random walk loop soup of intensity* λ is a Poissonian realization from the measure $\lambda\, \mu^{\mathrm{rw}}$. We will obtain a realization by first using a Poisson point process to select a multi-set of ordered pairs (n, z), where the length of the loop is $2n$ and it is rooted at z. Then, given (n, z), we choose a loop from the appropriate random walk bridge measure. In other words, we write

$$\mu^{\mathrm{rw}} = \sum_{z \in \mathbb{Z}^2} \sum_{n=1}^{\infty} \frac{\nu_n^z}{2n} = \sum_{z \in \mathbb{Z}^2} \sum_{n=1}^{\infty} \frac{\nu(\mathcal{L}_n^z)}{2n}\, \frac{\nu_n^z}{\nu(\mathcal{L}_n^z)}.$$

Let

$$\tilde{N}(n, z; t), \quad n \in \{1, 2, 3, \ldots\},\ z \in \mathbb{Z}^2,$$

be independent Poisson processes (in the variable t) with parameter

$$\tilde{q}_n := \frac{1}{2n}\nu(\mathcal{L}_n^z) = \frac{1}{2\pi n^2} - \frac{1}{8\pi n^3} + O(\frac{1}{n^4}).$$

Let

$$\tilde{L}(n, z; m), \quad n \in \{1, 2, 3, \ldots\}, \; z \in \mathbb{Z}^2, \; m \in \{1, 2, 3, \ldots\},$$

be independent random variables, independent of the $\tilde{N}(n, z; t)$, taking values in $\mathcal{L}^0$; the distribution of $\tilde{L}(n, z; m)$ is the probability measure of a random walk bridge of $2n$ steps, i.e., $\nu_n^0/\nu(\mathcal{L}_n^0)$, which is the uniform probability measure on $\mathcal{L}_n^0$. If $\omega \in \mathcal{L}_n^z$, let

$$J_t(\omega) = \sum_{k=1}^{\tilde{N}(n,z;t)} \mathbf{1}_{\{L(n,z;k)+z=\omega\}}.$$

Note that $\{J_t(\omega) : \omega \in \mathcal{L}\}$ is a collection of independent Poisson processes; the process $J_t(\omega)$ has parameter $(2n)^{-1}4^{-2n}$ if $\omega \in \mathcal{L}_n$. We could equally well have constructed the loop soup starting with these Poisson processes. We have chosen the longer construction because it will be useful for coupling the loop soup with the Brownian loop soup.

Although we have used t for the time parameter of the Poisson processes, by choosing $t = \lambda$ we get an increasing family of realizations of the loop soup $\tilde{\mathcal{A}}_\lambda$ parametrized by λ. We think of the loop soup of intensity λ as a multi-set $\tilde{\mathcal{A}}_\lambda$ of loops, where loop ω appears $J_\lambda(\omega)$ times in $\tilde{\mathcal{A}}_\lambda$.

3. Strong approximation

If S_n denotes a simple random walk, we define S_t, $0 \le t < \infty$, by linear interpolation. The key to the coupling is the following result, due to Komlós, Major, and Tusnády, which shows that a simple random walk bridge and a Brownian bridge can be coupled very closely. Because the form of the result we need is slightly different than that proved in [3], we have included a proof in the final section. By a simple random walk bridge of time duration $2n$, we will mean a process X_t, $0 \le t \le 2n$, that has the law of S_t, $0 \le t \le 2n$, conditioned to have $S_{2n} = 0$.

Lemma 3.1 (Dyadic approximation). *There exists a $c < \infty$ such that for every positive integer n, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which are defined a one-dimensional Brownian bridge $B_t, 0 \le t \le 1$, and a one-dimensional simple random walk bridge X_t, $0 \le t \le 2n$, such that*

$$\mathbb{P}\{\sup_{0 \le s \le 1} |(2n)^{-1/2}X_{2ns} - B_s| \ge c\,n^{-1/2}\log n\} \le c\,n^{-30}.$$

Proof. This is a special case of Theorem 6.4.   □

Remark. The choice of 30 as the exponent on the right-hand side is arbitrary. The same result holds with error $O(n^{-r})$ for any $r > 0$, with suitably chosen $c = c_r$.

Corollary 3.2. *There exists a $c < \infty$ such that for every positive integer n, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which are defined a two-dimensional Brownian bridge $B_t, 0 \le t \le 1$, and a two-dimensional simple random walk bridge X_t, $0 \le t \le 2n$, such that for each n,*

$$\mathbb{P}\{\sup_{0 \le s \le 1} |n^{-1/2}X_{2ns} - B_s| \ge c\,n^{-1/2}\log n\} \le c\,n^{-30}.$$

Proof. If S_j^1, S_j^2 are independent one-dimensional simple random walks, then

$$(3.1) \qquad S_j := \frac{S_j^1 + i\, S_j^2}{1 + i}$$

is a two-dimensional simple random walk (written in complex form). Conditioning on $S_{2n} = 0$ is the same as conditioning on $S_{2n}^1 = S_{2n}^2 = 0$. In other words, we can obtain a two-dimensional random walk bridge as the product of two independent one-dimensional random walk bridges. Hence we can construct the probability space as the product of two probability spaces as in the lemma. $\qquad\square$

The following corollary is in the form that we will need in the rest of the paper. We have chosen to write it out in detail because of this. Recall that if a process is defined only for integer times, we extend its definition to non-integer times by linear interpolation.

Corollary 3.3. *There exists a c and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which are defined processes $B^{n,z,m}$, $S^{n,z,m}$, $n = 1, 2, \ldots, z \in \mathbb{Z}^2, m = 1, 2, \ldots$, such that*

- *the processes*

  $$B_t^{n,z,m}, \quad 0 \le t \le 1, \ \ n = 0, 1, \ldots, \ \ z \in \mathbb{Z}^2, \ \ m = 1, 2, \ldots,$$

  are independent two-dimensional Brownian bridges;
- *the processes*

  $$S_j^{n,z,m}, \quad n = 0, 1, \ldots, \ \ z \in \mathbb{Z}^2, j = 0, \ldots, 2n, \ \ m = 1, 2, \ldots,$$

  are independent and $S_j^{n,z,m}, j = 0, \ldots, 2n$, has the distribution of a two-dimensional simple random walk conditioned so that $S_{2n}^{n,z,m} = 0$.
-
  $$\mathbb{P}\{ \sup_{0 \le s \le 1} |n^{-1/2} S_{2ns}^{n,z,m} - B_s^{n,z,m}| \ge c\, n^{-1/2} \log n \} \le c\, n^{-30}.$$

Proof. Take products of the probability spaces in the previous corollary. $\qquad\square$

For the remainder of this paper we fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as in the previous corollary. On this probability space are defined the independent $\mathcal{L}^0$-valued random variables $\tilde{L}(n, z; m)$ as in Section 2,

$$\tilde{L}(n, z; m) = [S_0^{n,z,m}, S_1^{n,z,m}, \ldots, S_{2n}^{n,z,m}].$$

Also, we have independent, identically distributed random variables $L(n, z; m)$ taking values in $\mathcal{C}_1$:

$$L(n, z; m) = B^{n,z,m}.$$

These have the distribution of a two-dimensional Brownian bridge of time duration 1. We assume that these are coupled as in the proposition.

4. Constructing the Brownian loop soup

In this section we will show how to construct the Brownian loop soup in a way that is highly correlated with the random walk loop soup. We will restrict ourselves to the rooted Brownian loop soup restricted to loops of time duration at least 5/8; to get a complete realization one can attach an independent realization of the loop soup with loops of time duration less than 5/8. These small loops will not be

coupled with the random walk loops. Let $N(n, z; t)$ be a collection of independent Poisson processes with parameter

$$q_n := \int_{n-(3/8)}^{n+(5/8)} \frac{ds}{2\pi s^2} = \frac{1}{2\pi(n+(5/8))\,(n-(3/8))} = \frac{1}{2\pi n^2} - \frac{1}{8\pi n^3} + O(\frac{1}{n^4}).$$

Recall the definition of $\tilde{N}(n, z; t)$ from Section 2, and note that $q_n - \tilde{q}_n = O(\frac{1}{n^4})$. (We have chosen to couple random walk loops of $2n$ steps with Brownian loops of time duration $n - (3/8)$ to $n + (5/8)$. The particular choice of interval $[-3/8, 5/8]$ was chosen so that q_n and $\tilde{q}_n$ agree up to an error of size $O(n^{-4})$.) It is easy to see that we can couple $N(n, z; t), \tilde{N}(n, z, t)$ on the same probability space so that:

- $\{N(n, z; t)\}$ are independent Poisson processes with parameter q_n;
- $\{\tilde{N}(n, z; t)\}$ are independent Poisson processes with parameter $\tilde{q}_n$;
- there is a c such that for all n, z, t, $\mathbb{P}\{N(n, z; t) \neq \tilde{N}(n, z; t)\} \leq t\,|q_n - \tilde{q}_n| \leq ctn^{-4}$.

In fact, we can let $\hat{N}(n, z; t), n = 1, 2, \ldots, z \in \mathbb{Z}^2$, be independent Poisson processes with parameter 1 and then set

$$N(n, z; t) = \hat{N}(n, z; q_n t), \qquad \tilde{N}(n, z; t) = \hat{N}(n, z, ; \tilde{q}_n t).$$

Assume without loss of generality that on this probability space we have independent copies of the coupled processes

$$L(n, z; m), \quad \tilde{L}(n, z; m)$$

as in Section 3, independent complex-valued random variables $Y(n, z; m)$ that are uniformly distributed on the square $\{x + iy : |x| \leq 1/2, |y| \leq 1/2\}$, and independent real-valued random variables $T(n, z; m)$ with density

$$\frac{(n + \frac{5}{8})(n - \frac{3}{8})}{s^2}, \quad n - \frac{3}{8} \leq s \leq n + \frac{5}{8}.$$

We construct the rooted Brownian loop soup (restricted to loops of time duration at least $5/8$) as follows:

- $N(n, z; t)$ will be the number of rooted loops that have appeared by time t whose root is in the unit square centered at z and whose time duration is between $n - (3/8)$ and $n + (5/8)$;
- scale the bridge (of time duration 1 and rooted at 0) $L(n, z, m)$ so that it has time duration $T(n, z, m)$, and then translate it so that its root is $z + Y(n, z, m)$; we call this final loop $L^*(n, z, m)$.

Then it is easy to see from the definition that the collection of loops

$$\mathcal{A}_\lambda = \{L^*(n, z; m) : N(n, z; \lambda) \geq m, \quad n \in \mathbb{Z}^+, z \in \mathbb{Z}^2\}$$

is a realization of the Brownian loop soup with intensity λ (restricted to loops of time duration at least $5/8$). We can then extend $\mathcal{A}_\lambda$ to a realization of the Brownian loop soup by adding an independent realization of loops of time duration less than $5/8$. Recall from the discussion in the Introduction that $\mathcal{A}_{\lambda, N}$ is also a realization of the Brownian loop soup. On this space we also have the scaled random walk soup $\tilde{\mathcal{A}}_{\lambda, N}$.

We will now show that this coupling satisfies the conclusions of Theorem 1.1. Without loss of generality, we may assume that $\lambda r^2 N^{2-3\theta} \leq 1$; in particular, $\lambda \leq$

N^4. First, note that

$$\mathbb{P}\{N(n, z; \lambda) \neq \tilde{N}(n, z; \lambda) \text{ for some } N^\theta \leq n < \infty, |z| \leq rN\}$$
$$\leq c \sum_{|z| \leq rN} \sum_{n \geq N^\theta} \lambda\, n^{-4} \leq c\, \lambda\, r^2\, N^{2-3\theta}.$$

Hence, except for an event of probability $O(\lambda r^2 N^{2-3\theta})$, $N(n, z; \lambda) = \tilde{N}(n, z; \lambda)$, for $n \geq N^\theta, |z| \leq rN$. Also,

$$\mathbb{P}\{N(n, z; \lambda) \geq N^5 \text{ for some } N^{2/3} \leq n \leq N^6, |z| \leq r\, N\}$$
$$\leq c\, r^2\, N^8\, \mathbb{P}\{Y \geq N^5\} \leq c\, r^2\, N^{-5},$$

where Y is a Poisson random variable with expectation $c\, N^4$. The last estimate uses an easy estimate on Poisson random variables; in fact for positive integers N, $\mathbb{P}\{Y \geq N^5\} \leq \mathbb{P}\{Y \geq N^4\}^N \leq e^{-aN}$.

Let

$$Z = Z_{N,r,\theta,\beta} = \sum_{|z| \leq rN} \sum_{n \geq N^6} N(n, z, \lambda).$$

Then Z is Poisson with

$$\mathbb{E}[Z] \leq c\, \lambda \sum_{|z| \leq rN} \sum_{n \geq N^6} \frac{1}{n^2} \leq c\, \lambda\, r^2\, N^{-4}.$$

Hence, $\mathbb{P}\{Z \neq 0\} \leq c\, \lambda\, r^2 N^{-4} \leq c\, \lambda\, r^2\, N^{2-3\theta}$. Therefore,

$$\mathbb{P}\{N(n, z, \lambda) > 0 \text{ for some } n \geq N^6, |z| \leq rN\} \leq c\, \lambda\, r^2\, N^{2-3\theta},$$

and similarly the same estimate holds with $\tilde{N}(n, z, \lambda)$ replacing $N(n, z, \lambda)$.

Let us denote the loops $L^*(n, z; m)$ and $\tilde{L}(n, z; m) + z$ by $\gamma_{n,z,m}$ and $\tilde{\gamma}_{n,z,m}$. Let $A = A_{N,r}$ be the event

$$A = \{ \sup_{0 \leq s \leq 1} |\gamma_{n,z,m}(st_\gamma) - \tilde{\gamma}_{n,z,m}(st_{\tilde{\gamma}})| \geq c_2 \log N^6$$

$$\text{for some } |z| < rN, N^{2/3} \leq n \leq N^6, m \leq N^5\}.$$

Here we use c_2 for the constant c from Corollary 3.3. Then the corollary tells us that

$$\mathbb{P}(A) \leq r^2\, N^2\, N^6\, N^5\, O((N^{2/3})^{-30}) \leq c\, r^2\, N^{2-3\theta}.$$

On the intersection of A^c and the events that

$$N(n, z; \lambda) = \tilde{N}(n, z; \lambda), \quad n \geq N^\theta,\ |z| \leq rN,$$
$$N(n, z; \lambda) \leq N^5, \quad N^{2/3} \leq n \leq N^6,\ |z| < rN,$$
$$N(n, z; \lambda) = \tilde{N}(n, z, \lambda) = 0, \quad n \geq N^6,\ |z| \leq rN,$$

the coupling satisfies the conclusion of Theorem 1.1.

5. Bounded domains

In this section, we let D denote a simply connected domain in $\mathbb{C}$ containing the origin, contained in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. If $\epsilon > 0$, let $D_\epsilon = \{z \in D : \text{dist}(z, \partial D) > \epsilon\}$. Recall the definition of the loop measure μ from the Introduction. We say that a loop γ is in D if $\gamma[0, t_\gamma] \subset D$.

Proposition 5.1. *There is a $c < \infty$ such that the μ measure of the set of loops in $\mathbb{D}$ of time duration at least t_0 that are not in $\mathbb{D}_\epsilon$ is bounded above by $c\,\epsilon\,t_0^{-3/2}$.*

Proof. The measure we are interested in is given by

$$\int_{\mathbb{D}} \int_{t_0}^{\infty} \frac{1}{2\pi t^2}\, p_t(z)\, dt\, dA(z),$$

where $p_t(z) = \mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,t], \partial\mathbb{D}) \le \epsilon\}$ and B_z^{br} denotes a Brownian bridge of time duration t rooted at z. By time reversal, we can see that

$$\mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,t], \partial\mathbb{D}) \le \epsilon\} \le 2\,\mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,\tfrac{t}{2}], \partial\mathbb{D}) \le \epsilon, B_z^{\mathrm{br}}[0,t] \subset \mathbb{D}\}.$$

But,

$$\mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,\tfrac{t}{2}], \partial\mathbb{D}) \le \epsilon, B_z^{\mathrm{br}}[0,t] \subset \mathbb{D}\}$$
$$= \lim_{\delta \to 0^+} \frac{2t}{\delta^2} \mathbb{P}^z\{0 < \mathrm{dist}(B[0,\tfrac{t}{2}], \partial\mathbb{D}) \le \epsilon, B[0,t] \subset \mathbb{D}, |B_t - z| < \delta\},$$

where B_t denotes a standard Brownian motion. By the strong Markov property and the "gambler's ruin" estimate,[1]

$$\mathbb{P}^z\{0 < \mathrm{dist}(B[0,\tfrac{t}{2}], \partial\mathbb{D}) \le \epsilon; B[0,\tfrac{3}{4}t] \subset \mathbb{D}\} \le c\,\epsilon\,t^{-1/2}.$$

Given this event, the probability that $|B_t - z| < \delta$ is bounded above by $c\delta^2/t$. Hence

$$p_t(z) \le c\,\epsilon\,t^{-1/2},$$

and the result follows by integrating. $\qquad\square$

Proposition 5.2. *Suppose D is a simply connected domain contained in the unit disk. There is a $c < \infty$ such that the μ measure of the set of loops in D of time duration at least t_0 that are not in D_ϵ is bounded above by $c\,\epsilon^{1/2}\,t_0^{-5/4}$.*

Proof. The measure we are interested in is

$$\int_{D} \int_{t_0}^{\infty} \frac{1}{2\pi t^2}\, p_t(z)\, dt\, dA(z),$$

where $p_t(z) = \mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,t], \partial D) \le \epsilon\}$ and B_z^{br} denotes a Brownian bridge of time duration t rooted at z. By time reversal, we can see that

$$\mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,t], \partial D) \le \epsilon\} \le 2\mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,\tfrac{t}{2}], \partial D) \le \epsilon, B_z^{\mathrm{br}}[0,t] \subset D\}.$$

But,

$$\mathbb{P}\{0 < \mathrm{dist}(B_z^{\mathrm{br}}[0,\tfrac{t}{2}], \partial D) \le \epsilon, B_z^{\mathrm{br}}[0,t] \subset D\}$$
$$= \lim_{\delta \to 0^+} \frac{2t}{\delta^2}\, \mathbb{P}^z\{0 < \mathrm{dist}(B[0,\tfrac{t}{2}], \partial D) \le \epsilon, B[0,t] \subset D, |B_t - z| < \delta\},$$

where B_t denotes a standard Brownian motion. By the strong Markov property and the Beurling estimate (see Lemma 5.3 below),

$$\mathbb{P}^z\{0 < \mathrm{dist}(B[0,\tfrac{t}{2}], \partial D) \le \epsilon; B[0,\tfrac{3}{4}t] \subset D\} \le c\,\epsilon^{1/2}\,t^{-1/4}.$$

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[1] The gambler's ruin estimate states that the probability that a one-dimensional standard Brownian motion starting at $\epsilon > 0$ stays positive up to time t is bounded above by $c\,\epsilon\,t^{-1/2}$.

Given this event, the probability that $|B_t - z| < \delta$ is bounded above by $c\delta^2/t$. Hence

$$p_t(z) \leq c\,\epsilon^{1/2}\,t^{-1/4},$$

and the result follows by integrating. □

Lemma 5.3 (Beurling estimate). *Let B_t denote a standard two-dimensional Brownian motion. There is a c such that if $\gamma : [0,\infty) \to \mathbb{C}$ is any continuous curve with $|\gamma(0)| = r$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$, then*

$$\mathbb{P}^z\{B[0,t] \cap \gamma[0,\infty) = \emptyset\} \leq c\left(\frac{r}{\sqrt{t}}\right)^{\frac{1}{2}}, \qquad |z| \leq r.$$

Proof. If t is replaced by $\sigma_t = \inf\{s : |B_s| = \sqrt{t}\}$, then this lemma follows from the Beurling estimate which is a corollary of the Beurling Projection Theorem [1, Theorem V.4.1]. We will assume that estimate and show how the estimate for fixed times t can be deduced from the result for the stopping times σ_t. By scaling it is enough to do $t = 1$. Let $\tau_n = \inf\{t : |B_t - B_0| = 2^{-n}\}$. We will show that $\mathbb{P}\{B[0, 1 \wedge \tau_0] \cap \gamma[0,\infty) = \emptyset\} \leq cr^{\frac{1}{2}}$.

First, note that

$$\mathbb{P}^z\{B[0, 1 \wedge \tau_0] \cap \gamma[0,\infty) = \emptyset\}$$
$$\leq \mathbb{P}^z\{B[0,\tau_0] \cap \gamma[0,\infty) = \emptyset\} + \mathbb{P}^z\{B[0,1] \cap \gamma[0,\infty) = \emptyset, \tau_0 > 1\}.$$

The Beurling estimate gives us that the first summand is bounded by $cr^{1/2}$. Thus, we only need to bound the second summand. To do this we let A_k be the event

$$A_k = \{\tau_k - \tau_{k+1} > 3^{-(k+1)} \text{ and } \tau_j - \tau_{j+1} \leq 3^{-(j+1)}, \text{ for all } j > k\}.$$

Note that

$$(5.1) \quad \mathbb{P}^z\{\tau_k - \tau_{k+1} > 3^{-(k+1)}\} = \mathbb{P}^0\{\tau_0 - \tau_1 > (4/3)^k/3\} \leq c_1 \exp\{-c_2\,(4/3)^k\}.$$

The last estimate is a standard estimate for Brownian motion and implies that $\mathbb{P}\{\tau_k - \tau_{k+1} > 3^{-(k+1)} \text{ i.o.}\} = 0$. Since $\sum_{k=1}^{\infty} 3^{-k} < 1$, we see that the event $\{\tau_0 > 1\}$ is contained in $\bigcup_{k\geq 0} A_k$ up to an event of probability zero. Using the Beurling estimate, we see that

$$\mathbb{P}^z\{B[0,1] \cap \gamma[0,\infty) = \emptyset, \tau_0 > 1\}$$

$$\leq \sum_{k=0}^{\infty} \mathbb{P}^z\{B[0,1] \cap \gamma[0,\infty) = \emptyset, A_k\}$$

$$\leq \sum_{k=0}^{\infty} \mathbb{P}^z\{B[0,\tau_{k+1}] \cap \gamma[0,\infty) = \emptyset, \tau_k - \tau_{k+1} \geq 3^{-(k+1)}\}$$

$$\leq \sum_{k=0}^{\infty} c\left(\frac{r}{2^{-(k+1)}}\right)^{1/2} \mathbb{P}^z\{\tau_k - \tau_{k+1} > 3^{-(k+1)}\} \leq cr^{1/2}.$$

The last inequality uses (5.1). □

Corollary 5.4. *There is a c such that for every N, λ and $\theta < 2$, there exists a coupling of the Brownian loop soup restricted to $\mathbb{D}[D]$ and a $(1/N)$-random walk soup restricted to $\mathbb{D}[D]$ such that the one-to-one correspondence of Theorem 1.1 holds except on an event of probability at most $c(\lambda+1) \log N\, N^{2-(3/2)\theta}$ [respectively, $c(\lambda + 1) \log N^{(1/2)}\, N^{2-(5/4)\theta}$].*

Proof. It follows from Proposition 5.1 that the probability that a realization of the Brownian loop soup has at least one loop of time duration greater than $cN^{\theta-2}$ staying in $\mathbb{D}$, but not in $\mathbb{D}_\epsilon$ for $\epsilon = c(\log N/N)$ is $O(\lambda \log N \, N^{2-(3/2)\theta})$. For general domains we get $O(\lambda(\log N)^{(1/2)} N^{2-(5/4)\theta})$ upon using Proposition 5.2. Therefore, if we consider the coupling of a $(1/N)$-random walk soup and a Brownian soup and if we restrict to those loops in a domain D, then we get the one-to-one correspondence of the loops as before, except on an event of probability $c(\lambda+1) N^{2-(3/2)\theta} \log N$ if $D = \mathbb{D}$ or an event of probability $c(\lambda+1) N^{2-(5/4)\theta} \log N^{1/2}$ for general simply connected D contained in the unit disk. $\square$

Remark. The ideas in this section extend to a wide class of non-simply connected domains, e.g., domains whose boundary consists of a finite number of connected components larger than a single point. The main fact that we need is that the connected components of the boundary are non-trivial so that the Beurling estimates can be used to quantify the notion "if the loop gets near the boundary, then it probably hits the boundary".

6. THE DYADIC APPROXIMATION

6.1. **Introduction.** In this note we give a proof of the "dyadic" strong approximation for random walk bridges by Brownian bridges using the methods in [3]. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = 1/2$ and let $S_n = X_1 + \cdots + X_n$. For positive integers n, let $L_n = \{z \in \mathbb{Z} : \mathbb{P}\{S_n = z\} > 0\}$. If $z \in L_n$, $\{S_m^{(n,z)}\}_{m=0}^n$ will denote a process with the law of $\{S_m\}_{m=0}^n$ conditioned so that $S_n = z$.

We start with a definition. Suppose Z is a continuous random variable with strictly increasing distribution function F and G is the distribution function of a discrete random variable whose support is $\{a_1, a_2, \ldots\}$. Then (Z, W) are *quantile-coupled (with distribution functions (F, G))* if W is defined by

$$W = a_j \quad \text{if} \quad r_{j-} < Z \le r_j,$$

where r_{j-}, r_j are defined by

$$F(r_{j-}) = G(a_j-), \qquad F(r_j) = G(a_j).$$

The quantile-coupling has the following property. If

$$F(a_k - x) \le G(a_k-) < G(a_k) \le F(a_k + x),$$

then

(6.1) $$|Z - W| = |Z - a_k| \le x \quad \text{on the event} \quad \{W = a_k\}.$$

We will need the following lemmas about the random walk; we will prove them in §6.4.

Lemma 6.1. *There exists $\epsilon_0 > 0$ such that for every $b_1 > 0$ there exist $0 < c, a < \infty$ such that the following holds. Let N be an $N(0,1)$ random variable. For each integer $n > 1$, each integer m with $|2m - n| \le 1$, and every $z \in L_n$, let*

$$Z = Z^{(m,n,z)} = \frac{m}{n} z + \sqrt{m(1 - \frac{m}{n})} \, N,$$

so that $Z \sim N(\frac{m}{n}z, m(1 - \frac{m}{n}))$. Let $W = W^{(m,n,z)}$ be the random variable with $\mathcal{L}(W) = \mathcal{L}(S_m^{(n,z)})$ that is quantile-coupled with Z. Then if $|z| \leq \epsilon_0 n$ and $\mathbb{P}\{W = w\} > 0$,

$$(6.2) \qquad \mathbb{E}[e^{a|Z-W|} \mid W = w] \leq c \exp\left\{b_1 \frac{w^2 + z^2}{n}\right\}.$$

Remark. For a simple random walk, it is easy to show that (6.2) holds for $\epsilon_0 n \leq |z| \leq n$ (with perhaps different a, c), so it follows that the result holds for all $|z|$. However, we state the lemma only for $|z| \leq \epsilon_0 n$ because this is all that we use.

Lemma 6.2. *There exist c_2, b_2, ϵ_0 such that for every integer $n \geq 2$, every integer m with $|2m - n| \leq 1$, every $z \in L_n$ with $|z| \leq \epsilon_0 n$, and every $w \in \mathbb{Z}$,*

$$\mathbb{P}\{S_m = w \mid S_n = z\} \leq c_2\, n^{-1/2} \exp\left\{-b_2 \frac{(w - (z/2))^2}{n}\right\}.$$

Remark. We can actually show that this holds for any $\epsilon_0 < 1$ (with the constants c_2, b_2 depending on ϵ_0).

Let B denote a Brownian bridge, i.e., a Brownian motion in $\mathbb{R}$ conditioned so that $B_0 = B_1 = 0$ (see §6.2 for a more precise definition). If $z_1, z_2 \in \mathbb{R}$, $n > 0$, then

$$(6.3) \qquad Y_t^{(n,z_1,z_2)} := \sqrt{n}\, B_{t/n} + \frac{n-t}{n} z_1 + \frac{t}{n} z_2, \quad 0 \leq t \leq n,$$

is the Brownian bridge conditioned so that $B_0 = z_1$, $B_n = z_2$. We write $Y_t^{(n,z)}$ for $Y_t^{(n,0,z)}$. If $(S^{(n,z)}, B)$ are defined on the same probability space, we define

$$\Delta(n, z) = \Delta(n, z, S^{(n,z)}, B) = \sup_{0 \leq t \leq n} |Y_t^{(n,z)} - S_t^{(n,z)}|.$$

(Recall that $S_t^{(n,z)}$ is defined for non-integer t by linear interpolation.) In §6.3, we will prove the following.

Theorem 6.3. *For every $b > 0$, there exist $0 < c, a, \alpha < \infty$ such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B and the family of processes $\{S^{(n,z)} : z \in L_n\}$ such that*

$$(6.4) \qquad \mathbb{E}[e^{a\Delta(n,z)}] \leq c\, n^\alpha\, e^{b|z|^2/n}.$$

Using Chebyshev's inequality, we get the following theorem as a corollary.

Theorem 6.4. *For every $b > 0$ there exist $0 < c, \alpha < \infty$, such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B and the family of processes $\{S^{(n,z)} : z \in L_n\}$ such that for all $r > 0$,*

$$\mathbb{P}\{\Delta(n, z) > r\, c \log n\} \leq c\, n^{\alpha-r}\, e^{bz^2/n}.$$

6.2. **Brownian bridge.** If W_t denotes a standard one-dimensional Brownian motion, then the process

$$B_t = W_t - t\, W_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (conditioned so that $B_0 = 0, B_1 = 0$)*. (See [6, IV.40] for more detail on Brownian bridges.) It can also be characterized as the continuous Gaussian process $B_t, 0 \leq t \leq 1$, with

$$\mathbb{E}[B_t] = 0, \qquad \mathbf{Cov}[B_s, B_t] = \mathbb{E}[B_s B_t] = s(1 - t), \quad 0 \leq s \leq t \leq 1.$$

More generally, if B_t is a Brownian bridge, then

$$X_t = \sqrt{s_2 - s_1}\, B_{(t-s_1)/(s_2-s_1)} + x_1 + \left(\frac{t-s_1}{s_2-s_1}\right)(x_2 - x_1), \quad s_1 \leq t \leq s_2,$$

is a Brownian bridge, in the time interval $[s_1, s_2]$, conditioned so that $X_{s_1} = x_1, X_{s_2} = x_2$. It is the continuous Gaussian process $X_t, s_1 \leq t \leq s_2$, with

$$\mathbb{E}[X_t] = x_1 + \left(\frac{t-s_1}{s_2-s_1}\right)(x_2 - x_1), \quad \mathbf{Cov}[X_s X_t] = \frac{(s-s_1)(s_2-t)}{s_2-s_1},$$

for $s_1 \leq s \leq t \leq s_2$.

Lemma 6.5. *Suppose* $B, \tilde{B}$ *are independent Brownian bridges and* N *is an independent* $N(0,1)$ *random variable. Suppose* $0 < s < 1$, *and define* $X_t, 0 \leq t \leq 1$, *by*

$$X_s = \sqrt{s(1-s)}\, N,$$

$$X_t = \sqrt{s}\, B_{t/s} + \frac{t}{s} X_s, \quad 0 \leq t \leq s,$$

$$X_t = \sqrt{1-s}\, \tilde{B}_{(t-s)/(1-s)} + \frac{1-t}{1-s} X_s, \quad s \leq t \leq 1.$$

Then X_t *is a Brownian bridge conditioned so that* $X_0 = X_1 = 0$.

Proof. This can be easily checked using the Gaussian characterization of Brownian bridges. The formulas are not mysterious. What we are doing is defining $X_t, 0 \leq t \leq 1$, by first choosing X_s (using the appropriate distribution on X_s), and then defining $X_t, 0 \leq t \leq s$, and $X_t, s \leq t \leq 1$, as appropriate Brownian bridges. $\square$

We will need the following easy estimate for Brownian bridges. Let

$$M = \sup_{0 \leq t \leq 1} |B_t|.$$

Then there exist $\tilde{c}, u$ such that for all $a > 0$,

$$(6.5) \qquad\qquad \mathbb{E}[e^{aM}] \leq \tilde{c}\, e^{ua^2}.$$

If B_t is replaced by a Brownian motion W_t, this estimate is standard using the reflection principle. That argument can easily be adapted to establish (6.5), perhaps with different $\tilde{c}, u$. (In fact, the maximum for Brownian motion stochastically dominates M so (6.5) holds with the same $\tilde{c}, u$, but we will not need this stronger fact.)

6.3. **Proof of Theorem 6.3.** It suffices to prove the result for b sufficiently small. We fix positive $b < b_2/37$, where b_2 is the constant from Lemma 6.2. We let ϵ_0 be the smaller of the two values of ϵ_0 in Lemmas 6.1 and 6.2.

In this proof, by an *n-coupling* we will mean a probability space on which are defined a Brownian bridge B and the family of processes $\{S^{(n,z)} : z \in L_n\}$.

Note that for any n-coupling, if $z \in L_n$, $S_t = S_t^{(n,z)}$, and $Y_t = Y_t^{(n,z)}$ as in (6.3), then

$$\Delta(n,z) = \sup_{0 \leq t \leq n} |S_t - Y_t| \leq \sup_{0 \leq t \leq n} |S_t| + \sup_{0 \leq t \leq n} |Y_t| \leq n + \left[n + \sqrt{n} \sup_{0 \leq t \leq 1} |B_t|\right].$$

Hence,

$$\mathbb{E}[e^{a\Delta(n,z)}] \leq e^{2an}\, \mathbb{E}[\exp\{a\sqrt{n}[\sup_{0 \leq t \leq 1} |B_t|]\}] \leq \tilde{c}\, e^{(2a+ua^2)n},$$

where $u, \tilde{c}$ are as in (6.5). Clearly, there exists $a_0 = a_0(b) > 0$ such that if $a \in (0, a_0)$, then $2a + ua^2 \leq b\,\epsilon_0^2$.

Therefore, for any n-coupling inequality (6.4) will hold with $c = \tilde{c}, \alpha = 0$ and $a \in (0, a_0)$ for all $z \in L_n$ with $|z| \geq n\epsilon_0$. For the remainder of this section, we will assume $a < a_0$. We will also assume that a is sufficiently small so that (6.2) holds with $b_1 = b/20$. We now fix such a value of a, and we will show how to construct the n-couplings so that (6.4) holds for some c, α.

We will use an induction. Clearly, we can choose n-couplings for $n \leq 2$ such that

$$\mathbb{E}[e^{a\Delta(n,z)}]e^{-b|z|^2/n} \leq C \qquad \forall z \in L_n,\, n \leq 2.$$

We can assume without loss of generality that $C \geq 1$. We will show that there exists a constant c (which without loss of generality we can assume is greater than both 1 and the $\tilde{c}$ in (6.5)) such that for every positive integer s, if there exist n-couplings for all $n \leq 2^s$ such that

$$(6.6) \qquad \mathbb{E}[e^{a\Delta(n,z)}]e^{-b|z|^2/n} \leq C(s),$$

then there exist n-couplings for all $n \leq 2^{s+1}$ such that

$$(6.7) \qquad \mathbb{E}[e^{a\Delta(n,z)}]e^{-b|z|^2/n} \leq c\,C(s).$$

The theorem follows easily from this claim.

In order to prove the claim above, let $2^s < n \leq 2^{s+1}$. We will show how to construct a probability space on which are defined a Brownian bridge and a family of processes $\{S^{(n,z)} : z \in L_n, |z| \leq n\epsilon_0\}$ satisfying (6.7). Once this is done, we can adjoin, possibly after enlarging the probability space, the processes for $|z| > n\epsilon_0$. Since $c \geq \tilde{c}$, (6.7) will also hold for these processes. Hence, we assume $|z| \leq \epsilon_0 n$. For notational ease we will assume that n is even and we write $n = 2k$. Note that k is an integer with $2^{s-1} < k \leq 2^s$. (If n is odd we write $n = k + (k+1)$ and do a similar argument.)

We define the n-coupling as follows:

- Choose two independent k-couplings

$$\left(\{S^{1\,(k,z)}\}_{z\in L_k}, B^1\right), \qquad \left(\{S^{2\,(k,z)}\}_{z\in L_k}, B^2\right),$$

  satisfying (6.6).
- Let $N \sim N(0,1)$ and define the translated *normal* random variables $Z^z = \sqrt{n/4}\,N + \frac{z}{2}$. Define the quantile-coupled random variables W^z as in Lemma 6.1. Assume, as we may, that all these random variables are independent of the two k-couplings chosen above. Note that a has been chosen sufficiently small so that (6.2) holds with $b_1 = b/20$; i.e.,

$$\mathbb{E}[e^{a|Z^z - W^z|} \mid W^z = w] \leq c\,\exp[\frac{b}{20}\,\frac{w^2 + z^2}{n}].$$

- Let

$$(6.8) \qquad B_t = \begin{cases} \frac{1}{\sqrt{2}}B^1_{2t} + t\,N, & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2}}B^2_{2(t-\frac{1}{2})} + (1-t)\,N, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

  By Lemma 6.5, B_t is a Brownian bridge.

- Let $S_k^{(n,z)} = W^z$, and

$$S_m^{(n,z)} = \begin{cases} S_m^{1\,(k,W^z)}, & 0 \le m \le k, \\ W^z + S_{m-k}^{2\,(k,z-W^z)}, & k \le m \le n. \end{cases}$$

What we have done is to first choose the value of $S_k^{(n,z)}$ from the appropriate conditional distribution of S_k given $S_n = z$ and then to define the other values of $S_m^{(n,z)}$ from the conditional distribution of S_m given $S_n = z, S_k = W^z$.

This gives us our coupling; we need to show that it satisfies (6.7). Note that

$$\Delta(n, z, S^{(n,z)}, B)$$

$$\le |Z^z - W^z| + \max\left\{ \Delta(k, W^z, S^{1\,(k,W^z)}, B^1),\, \Delta(k, z - W^z, S^{2\,(k,z-W^z)}, B^2) \right\}.$$

Therefore,

$$\mathbb{E}[e^{a\Delta(n,z)} \mid W^z = w] \le \mathbb{E}[e^{a|Z^z - W^z|} \mid W^z = w]\, C(s)\, (e^{b|w|^2/k} + e^{b|z-w|^2/k}).$$

Here we have used the the fact that our two k-couplings satisfy (6.6) and the simple inequality, $\mathbb{E}[e^{\max\{Z_1, Z_2\}}] \le \mathbb{E}[e^{Z_1}] + \mathbb{E}[e^{Z_2}]$. Therefore,

$$\frac{\mathbb{E}[e^{a\Delta(n,z)}]}{C(s)} \le c \sum_w \mathbb{P}\{W^z = w\}\, \exp\left\{ \frac{b}{20} \frac{w^2 + z^2}{n} \right\} \exp\left\{ b \frac{\max\{w^2, (z-w)^2\}}{k} \right\}.$$

Since $w^2 + z^2 \le 5 \max\{w^2, (z-w)^2\}$ and $k = n/2$, this sum is bounded by

$$c \sum_w \mathbb{P}\{W^z = w\}\, \exp\left\{ \frac{9}{4} b \frac{\max\{w^2, (z-w)^2\}}{n} \right\}.$$

We now split this sum into two pieces: $|w - \frac{z}{2}| \le |z|/6$ and $|w - \frac{z}{2}| > |z|/6$. If $|w - \frac{z}{2}| \le |z|/6$, then $\max\{w^2, (z-w)^2\} \le (2z/3)^2$; hence,

$$c \sum_{|w - \frac{z}{2}| \le |z|/6} \mathbb{P}\{W^z = w\}\, \exp\left\{ \frac{9}{4} b \frac{\max\{w^2, (z-w)^2\}}{n} \right\} \le c\, \exp\{\frac{bz^2}{n}\}.$$

To handle the other piece we make use of Lemma 6.2. Since $b_2 > 37b$, we have

$$\mathbb{P}\{W^z = w\} = \mathbb{P}\{S_k = w \mid S_n = z\} \le c\, n^{-1/2}\, \exp\{-37b \frac{(w - \frac{z}{2})^2}{n}\},$$

and so, the sum over the second piece is bounded by

$$c \sum_{|w - \frac{z}{2}| > |z|/6} n^{-1/2}\, \exp\{-37b \frac{(w - \frac{z}{2})^2}{n}\}\, \exp\{\frac{9}{4} b \frac{\max\{w^2, (z-w)^2\}}{n}\}.$$

For $|w - \frac{z}{2}| > |z|/6$, we have $(w - \frac{z}{2})^2 > \frac{1}{16} \max\{w^2, (z-w)^2\}$. Hence, the sum is bounded by

$$c \sum_w n^{-1/2}\, \exp\{-\frac{b}{16} \frac{w^2}{n}\},$$

which is clearly bounded by a constant. Therefore,

$$\frac{\mathbb{E}[e^{a\Delta(n,z)}]}{C(s)} \le c\, \exp\{\frac{bz^2}{n}\}.$$

That is, our n-coupling satisfies (6.7).

6.4. **Local central limit theorem.** We will derive the local central limit theorem for conditioned simple random walks which is essentially a normal approximation for hypergeometric random variables. Although this is standard, we derive it here because the size of the error term is important for us. Our starting point is Stirling's formula with error

$$(6.9) \qquad n! = \sqrt{2\pi}\, n^{n+(1/2)}\, e^{-n} \left[1 + O(n^{-1})\right].$$

Lemma 6.6. *Suppose* l, m, j *are integers with* $m > 0, |l| \le m/2, |j| \le m/8$. *Then*

$$\mathbb{P}\{S_{2m} = 2j + 2l \mid S_{4m} = 4l\}$$

$$= 2\sqrt{\frac{1}{2\pi\, m(1 - (l/m)^2)}}\, \exp\left\{-\frac{(2j)^2}{2\, m(1 - (l/m)^2)} + O(\frac{1}{m}) + O(\frac{j^4}{m^3})\right\}.$$

Proof. Throughout this proof we assume that $|l| \le m/2, |j| \le m/8$. Without loss of generality we may assume that $l \ge 0$. Let

$$p(2m, 2j \mid 4l) = \mathbb{P}\{S_{2m} = 2j \mid S_{4m} = 4l\} = \frac{\binom{2m}{m+j}\binom{2m}{m+2l-j}}{\binom{4m}{2m+2l}}.$$

Then,

$$p(2m, 2l + 2(j+1) \mid 4l) = p(2m, 2l + 2j \mid 4l)\, \frac{(m - j)^2 - l^2}{(m + j + 1)^2 - l^2},$$

and if $j > 0$,

$$\begin{aligned}
p(2m, 2l + 2j \mid 4l) &= p(2m, 2l \mid 4l) \prod_{i=0}^{j-1} \frac{(m - i)^2 - l^2}{(m + i + 1)^2 - l^2} \\
&= p(2m, 2l \mid 4l) \prod_{i=1}^{j}[1 - \frac{4im + 2i - 2m - 1}{(m + i)^2 - l^2}].
\end{aligned}$$

By symmetry, $p(2m, 2l - 2j \mid 4l) = p(2m, 2l + 2j \mid 4l)$.

Stirling's formula (6.9) gives

$$(6.10) \qquad p(2m, 2l \mid 4l) = \frac{\binom{2m}{m+l}^2}{\binom{4m}{2m+2l}} = 2\sqrt{\frac{1}{2\pi\, m(1 - (l/m)^2)}} \left[1 + O(\frac{1}{m})\right].$$

This gives the result for $j = 0$. To finish the proof of the result we need to show that

$$\sum_{i=1}^{j} \log[1 - \frac{4im + 2i - 2m - 1}{(m + i)^2 - l^2}] = -\frac{(2j)^2}{2\, m(1 - (l/m)^2)} + O(\frac{1}{m}) + O(\frac{j^4}{m^3}).$$

Note that if $l \le m/2$, $j \le m/8$ and $1 \le i \le j$, then

$$0 \le \frac{4im + 2i - 2m - 1}{(m + i)^2 - l^2} \le \frac{2}{3}.$$

There exists c_1 such that

$$|\log(1 - x) + x + \frac{x^2}{2}| \le c_1 x^3, \quad 0 \le x \le 2/3.$$

Therefore,

$$\sum_{i=1}^{j} \log[1 - \frac{4im + 2i - 2m - 1}{(m+i)^2 - l^2}]$$

$$= O(\frac{j^4}{m^3}) - \sum_{i=1}^{j} \frac{4im + 2i - 2m - 1}{(m+i)^2 - l^2} - \frac{1}{2} \sum_{i=1}^{j} (\frac{4im + 2i - 2m - 1}{(m+i)^2 - l^2})^2.$$

Note that

$$\frac{1}{(m+i)^2 - l^2} = \frac{1}{m^2 - l^2} - \frac{2im}{(m^2 - l^2)^2} + O(\frac{i^2}{m^4}).$$

Hence,

$$\sum_{i=1}^{j} \frac{4im + 2i - 2m - 1}{(m+i)^2 - l^2} = O(\frac{1}{m} + \frac{j^4}{m^3}) + \sum_{i=1}^{j} \frac{4im - 2m}{m^2 - l^2} - \sum_{i=1}^{j} \frac{8i^2 m^2}{(m^2 - l^2)^2}$$

$$= O(\frac{1}{m} + \frac{j^4}{m^3}) + \frac{2j(j+1)m - 2jm}{m^2 - l^2} - \frac{(8/3)j^3 m^2}{(m^2 - l^2)^2}$$

$$= O(\frac{1}{m} + \frac{j^4}{m^3}) + \frac{2j^2 m}{m^2 - l^2} - \frac{(8/3)j^3 m^2}{(m^2 - l^2)^2}.$$

Also,

$$\frac{1}{2} \sum_{i=1}^{j} (\frac{4im + 2i - 2m - 1}{(m+i)^2 - l^2})^2 = O(\frac{1}{m} + \frac{j^4}{m^3}) + \frac{(8/3)j^3 m^2}{(m^2 - l^2)^2}.$$

The result follows immediately. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

One can deduce the following more general case from the result above.

Lemma 6.7. *There is a c such that for $n \geq 2$ an integer and m an integer with $|2m - n| \leq 1$, if $|z|, |w| \leq c/n$, $z \in L_n$ and $w + \frac{m}{n}z \in L_m$, then*

$$\mathbb{P}\{S_m = \frac{m}{n}z + w | S_n = z\}$$

$$= 2\sqrt{\frac{1}{2\pi(m - \frac{m^2}{n})(1 - (\frac{z}{n})^2)}} \exp\left\{-\frac{w^2}{2(m - \frac{m^2}{n})(1 - (\frac{z}{n})^2)} + O(\frac{1}{n} + \frac{w^4}{n^3})\right\}.$$

We now state without proof an easy large deviation estimate that follows from large deviations for binomial random variables.

Lemma 6.8. *There exists an $\eta > 0$ such that, for any $a > 0$, there exist $C = C(a) < \infty$, and $\gamma = \gamma(a) > 0$, such that for all z with $|z|/n < \eta$*

$$\mathbb{P}\{|S_m - \frac{m}{n}z| > am | S_n = z\} \leq Ce^{-\gamma m}.$$

Lemma 6.2 follows easily from Lemmas 6.7 and 6.8.

6.5. **Coupling of conditioned distribution and normal distribution.** In the remainder of this section we prove Lemma 6.1. Note that we only need to prove the lemma for n sufficiently large. In order to simplify the notation we will assume that n is even and hence $m = n/2$. If n is odd, one can do the same argument. We will use a slightly weaker form of Lemma 6.7; more precisely, we will assume that

$$(6.11) \quad \mathbb{P}\{S_m = \frac{z}{2} + w | S_n = z\} = 2\sqrt{\frac{1}{2\pi\sigma_{n,z}^2}} \exp\left\{-\frac{w^2}{2\sigma_{n,z}^2} + O(\frac{1}{\sqrt{n}} + \frac{|w|^3}{n^2})\right\},$$

where $\sigma_{n,z}^2 = (n/4)\,[1 - (z/n)^2]$. We have replaced the $O(1/n), O(w^4/n^3)$ terms with the larger $O(1/\sqrt{n}), O(|w|^3/n^2)$ terms, respectively. Our reason for doing this is that more general random walks satisfy local central limit theorems with this weaker error term, and it is useful to know that the arguments in this section only use the weaker form.

Let X denote a $N(0,1)$ random variable, let

$$Z = Z_{n,z} = \frac{\sqrt{n}}{2}\,X + \frac{z}{2}, \qquad \hat{Z} = \hat{Z}_{n,z} = \sigma_{n,z}\,X + \frac{z}{2},$$

and let $W = W_{n,z}$ be the random variable with distribution of $S_{n/2}^{(n,z)}$ that is quantile-coupled with X. Note that W is also quantile-coupled with Z and $\hat{Z}$. Let us write $F = F_{n,z}$ for the distribution function of $\hat{Z}$ and $G = G_{n,z}$ for the distribution function of W. It follows from Lemma 6.9 below that there exist c, ϵ and N such that for all $n > N$, for all $z \in L_n$ with $|z|/n < \epsilon$, and all w with $|w - (z/2)|/n < \epsilon$,

$$(6.12) \quad F(w-c(1+\frac{(w - (z/2))^2}{n})) \le G(w-) \le G(w) \le F(w+c(1+\frac{(w - (z/2))^2}{n})).$$

It follows from (6.1) and (6.12) that the quantile-coupling satisfies

$$|\hat{Z} - W| \le c\left[1 + \frac{(W - (z/2))^2}{n}\right]$$

for all $n > N$, provided that $|z|, |W - (z/2)| < \epsilon n$. Also

$$|Z - \hat{Z}| = \left[1 - \sqrt{1 - (\frac{z}{n})^2}\right]\,|Z - \frac{z}{2}| \le c\,\frac{z^2}{n},$$

provided that $|z|, |W - (z/2)| < \epsilon n$ (which implies the estimate $|Z| \le c'\,n$). Therefore, for n large enough the quantile-coupling satisfies

$$|Z - W| \le c\left[1 + \frac{(W - (z/2))^2}{n} + \frac{z^2}{n}\right], \qquad \text{for } |z|, |W - (z/2)| \le \epsilon n.$$

For any $\epsilon' > 0$, straightforward exponential estimates show that there exists an $a > 0$ such that (6.2) holds for $|w| \ge \epsilon'\,n$. Hence to get Lemma 6.1, it suffices to prove the following estimate. We have written the estimate for random variables without the odd/even parity issues of simple random walk and hence have dropped a factor of 2. If we have a random variable that satisfies (6.11), i.e., that is supported only on even or only on odd integers, we can convert it to a random variable on all the integers by dividing the mass at k equally between k and $k - 1$.

Lemma 6.9. *For every $\tilde{c}, \tilde{\epsilon}$ there exist c_1, ϵ_1, N_1 such that the following holds for every positive integer $n > N_1$ and every $\sigma^2 \in [1/\tilde{c}, \tilde{c}]$. Suppose S is an integer random variable such that for every integer $|j| \le \tilde{\epsilon}\,n$,*

$$(6.13) \qquad \mathbb{P}\{S = j\} = \frac{1}{\sqrt{2\pi\sigma^2 n}}\,\exp\left\{-\frac{j^2}{2\sigma^2 n} + \delta(j)\right\},$$

where

$$|\delta(j)| \le \tilde{c}\left[\frac{1}{\sqrt{n}} + \frac{|j|^3}{n^2}\right].$$

Assume additionally that for every a there exist positive c, b such that

$$\mathbb{P}\{|S| \geq a\,n\} \leq c\,e^{-bn},$$

where c and b do not depend on S.

Then if G denotes the distribution function of S and F denotes the distribution function of an $N(0, \sigma^2 n)$ random variable,
(6.14)
$$F\left(x - c_1\,[1 + \frac{x^2}{n}]\right) \leq G(x-1) \leq G(x+1) \leq F\left(x + c_1\,[1 + \frac{x^2}{n}]\right), \qquad |x| \leq \epsilon_1 n.$$

Proof. This is a straightforward estimate of sums and integrals. It suffices to prove (6.14) for integer x, and by symmetry we can assume $x \geq 0$. Since we only need to establish the result for n large when we write inequalities in this proof, we will only be asserting that they are valid for n large enough.

Let $\bar{F} = 1 - F, \bar{G} = 1 - G, \bar{\Phi} = 1 - \Phi$, where Φ is the distribution function of an $N(0, 1)$ random variable. Let $q(y) = q(y; n, \sigma^2) = (2\pi\sigma^2 n)^{-1/2}\,e^{-y^2/(2\sigma^2 n)}$.

We will deal with the somewhat easier case $|x| \leq \sqrt{3\tilde{c}}\sqrt{n}$, at once. Note that (6.13) implies $c'/\sqrt{n} \leq G(x) - G(x-1) \leq c'/\sqrt{n}$ for $|x| \leq \sqrt{3\tilde{c}}\sqrt{n}$. By definition,

$$\bar{G}(x) = \sum_{j>x} \mathbb{P}\{S_n = j\} = \sum_{j>x} q(j) + \sum_{j>x} [\mathbb{P}\{S_n = j\} - q(j)].$$

Also,

$$\sum_{j>x} q(j) = O(\frac{1}{\sqrt{n}}) + \int_x^\infty q(y)\,dy.$$

Straightforward estimates using the assumptions give

$$\sum_{j>x} |\mathbb{P}\{S_n = j\} - q(j)| = O(\frac{1}{\sqrt{n}}).$$

Hence, we see that (6.14) holds for $|x| \leq \sqrt{3\tilde{c}}\sqrt{n}$.

It remains to prove (6.14) for $x > \sqrt{3\tilde{c}}\sqrt{n}$. To make our strategy more intuitive for the reader we now state the simple fact about the standard normal distribution that lies at the heart of our lemma. The interested reader is referred to [5] for many more details and for further reading on the KMT approximation. For any $A > 0$, there exist a c and an $\epsilon > 0$ such that for $\sigma\sqrt{n} \leq z \leq \epsilon n$,

(6.15)
$$e^{A\frac{z^3}{n^2}} \bar{\Phi}(\frac{z}{\sigma\sqrt{n}}) \leq \bar{\Phi}(\frac{z}{\sigma\sqrt{n}} - c\frac{z^2}{\sigma n^{3/2}}),$$

$$e^{-A\frac{z^3}{n^2}} \bar{\Phi}(\frac{z}{\sigma\sqrt{n}}) \geq \bar{\Phi}(\frac{z}{\sigma\sqrt{n}} + c\frac{z^2}{\sigma n^{3/2}}).$$

Assume now that $\sqrt{3\tilde{c}}\sqrt{n} \leq x \leq n^{5/8}$. The choice of $5/8$ as the exponent is somewhat arbitrary. One could take any exponent strictly less than $2/3$ as the argument below shows. We have

$$\sum_{j>x} |\mathbb{P}\{S = j\} - q(j)| \leq O(\exp\{-c_3 n^{1/3}\}) + c \sum_{j>x} \frac{j^3}{n^2}\,q(j).$$

Using the fact that x is an integer and that $y^3 q(y)$ is decreasing for $y > \sqrt{3\tilde{c}}\sqrt{n}$, we see that

$$\sum_{j>x} \frac{j^3}{n^2} q(j) \leq \int_x^\infty \frac{y^3}{n^2} \frac{1}{\sqrt{2\pi\sigma^2 n}} e^{-y^2/(2\sigma^2 n)} \, dy \leq c \frac{1}{\sqrt{n}} \int_{\frac{x}{\sigma\sqrt{n}}}^\infty z^3 \, e^{-z^2/2} \, dz.$$

Integration by parts gives

$$\int_s^\infty z^3 \, e^{-z^2/2} \, dz = (s^2 + 2) \, e^{-s^2/2}.$$

Therefore,

$$\sum_{j>x} |\mathbb{P}\{S = j\} - q(j)| \leq c \, \frac{x^2}{n^{3/2}} \, \exp\{-\frac{x^2}{2\sigma^2 n}\} \leq c \frac{x^3}{n^2} \bar{\Phi}(\frac{x}{\sigma\sqrt{n}}).$$

Also, using simple estimates we obtain

$$\sum_{j>x} q(j) = \bar{\Phi}(\frac{x}{\sigma\sqrt{n}})(1 + O(\frac{x}{n})).$$

Hence,

$$\bar{G}(x) = \bar{\Phi}(\frac{x}{\sigma\sqrt{n}}) \exp\{O(\frac{x^3}{n^2})\}.$$

The result for x in this range now follows from (6.15).

Assume now that $n^{5/8} \leq x$. Note that there is a constant $\bar{c}$, depending only on $\tilde{c}$, such that $q(y)e^{\frac{2\tilde{c}y^3}{n^2}}$ is decreasing for $x \leq y \leq 2\bar{c}n$, and such that

$$\int_{x-1}^{\bar{c}n} q(y)e^{\frac{2\tilde{c}y^3}{n^2}} dy \leq \int_{x-2}^{2x} q(y)e^{\frac{2\tilde{c}y^3}{n^2}} dy.$$

From this we see that there is an ϵ_2 such that if $x \leq \epsilon_2 n$, then

$$\bar{G}(x) \leq \int_x^{\bar{c}n} q(y)e^{\frac{2\tilde{c}y^3}{n^2}} dy + \bar{G}(\bar{c}n - 1) \leq \int_{x-1}^{\bar{c}n} q(y)e^{\frac{2\tilde{c}y^3}{n^2}} dy \leq \int_{x-2}^{2x} q(y)e^{\frac{2\tilde{c}y^3}{n^2}} dy.$$

Hence,

$$\bar{G}(x) \leq e^{\frac{16\tilde{c}x^3}{n^2}} \bar{\Phi}(\frac{x-2}{\sigma\sqrt{n}}).$$

Similar arguments can be used to obtain

$$\bar{G}(x) \geq^{-\frac{16\tilde{c}x^3}{n^2}} \bar{\Phi}(\frac{x+2}{\sigma\sqrt{n}}).$$

The result for x in this range follows from (6.15), and this concludes the proof of the lemma. $\qquad\square$

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