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Random walk on fractals: numerical studies in two dimensions

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Abstract. Monte Carlo calculations are used to investigate some statistical properties of random walks on fractal structures. Two kinds of lattices are used: the Sierpinski gasket and the infinite percolation cluster, in two dimensions. Among other problems, we study: (i) the range R_N of the walker (number of distinct visited sites during N steps): average value S_N, variance σ_N and asymptotic distribution; (ii) renewal theory (return to the original site): probability of return $P_0(N)$, mean number of returns ν_N.

The probability distribution of the walker position $P(N, \mathbf{R})$ after N steps is discussed. The asymptotic behaviour ($N \gg 1$) of these quantities exhibits power laws, with associated exponents. The numerical values of these exponents are in good agreement with recent theoretical predictions (Alexander/Orbach and Rammal/Toulouse).

1. Introduction

Recently, there has been a good deal of interest in fractal structures (Mandelbrot 1977), mainly due to their dilation symmetry (scaling invariance) in contrast to translationally invariant systems. A random walk in free space, linear polymers and percolation clusters at threshold are just a few examples of fractals. In the latter case, recent observations by several authors (Voss *et al* 1982, Kapitulnik and Deutscher 1982) show the importance of fractals in representing these disordered structures. From this viewpoint, fractals are expected to bridge the gap between crystalline materials and disordered systems. In general, two kinds of fractals can be distinguished: deterministic fractals (self-similar) such as the family of the Sierpinski gaskets (Gefen *et al* 1981, Rammal and Toulouse 1982) and random fractals (statistically self-similar) such as the percolation clusters (Stauffer 1979, 1981, Deutscher 1981).

Whereas Euclidean space is well characterised by one space dimension d, fractals require the definition of (at least) three dimensions: d, the dimension of the embedding Euclidean space, $\bar{d}$, the fractal dimension (Mandelbrot 1977), $\tilde{d}$, the spectral dimension (Alexander and Orbach 1982). In general, $\tilde{d}$ describes the power law $\rho(\omega) \sim \omega^{\tilde{d}-1}$ of harmonic excitations (elastic vibrations for instance) close to $\omega \sim 0$. In a recent publication (Rammal and Toulouse 1983, referred to hereafter as I) the importance of the spectral dimensionality $\tilde{d}$ in many physical problems was investigated: classical diffusion, quantum localisation, self avoidance etc. In particular, a number of interesting predictions relative to the diffusion on fractal spaces were stated in I: range of a random walk, renewal theory, etc. One of the objects of the present work is to check numerically some of the predictions given in I. New results relative to random walk statistics (RW) on fractals, such as the spatial distribution, mean first-passage time, zero-crossing set, etc are also stated, as an extension of those of I.

This paper is organised as follows. In § 2, numerical results obtained for RWs on the Sierpinski gasket in two dimensions are reported. We focus our attention on three statistical properties of RWs. The first is the range R_N of the walk (number of distinct visited sites during N steps). In particular, we investigate the average $S_N = \langle R_N \rangle$, the variance σ_N^2 of R_N and the asymptotic distribution of R_N, at $N \gg 1$. For instance, we find that S_N and σ_N are given at $N \gg 1$ by the same power of N with the exponent $\tilde{d}/2$. It is argued that this particular behaviour of R_N is common to all fractals with $\tilde{d} < 2$. The second property of the RWs is relative to the renewal theory: probability $P_0(N)$ and number ν_N of returns to the original site of the walk, after N steps. The result stated in paper I, $P_0(N) \sim N^{-\tilde{d}/2}$, is verified with a good numerical accuracy (less than 1%). The third property investigated in § 2 concerns the spatial distribution of the walk. We show in particular the consistency of the scaling behaviour of this distribution (Rammal and Vannimenus 1983) with the obtained numerical results. New results, relative to the mean first-passage time, zero-crossing set, etc, are deduced from the spatial distribution of the RW.

In § 3, we investigate the same properties as in § 2 for RWs on the percolation clusters (at threshold) on a square lattice. The scaling behaviour for S_N, $P_0(N)$, etc, stated in I, is recovered, from which we deduce the value of the exponent $\tilde{d}$. Within numerical accuracy, a relatively good agreement with the conjecture (Alexander and Orbach 1982) $\tilde{d} = \frac{4}{3}$ is observed. In addition to the above statistics, we investigate also the behaviour of the 'open frontier' defined in I, in the asymptotic regime $N \gg 1$. Section 4 is mainly devoted to discussion of open problems, and the possible extension of this work. In particular, we argue that RW statistics provide an alternative method for the calculation of dynamical exponents, such as the conductivity exponent t, in percolation problems.

2. Random walk statistics on the Sierpinski gasket

The Sierpinski gaskets form a family of self-similar structures, which can be built in any Euclidean dimension d (see figure 1 for $d = 2$) and which have been applied to studies (Gefen *et al* 1981, Rammal and Toulouse 1982, Alexander 1983), because it lends itself particularly conveniently to scaling computations. At stage n, the total

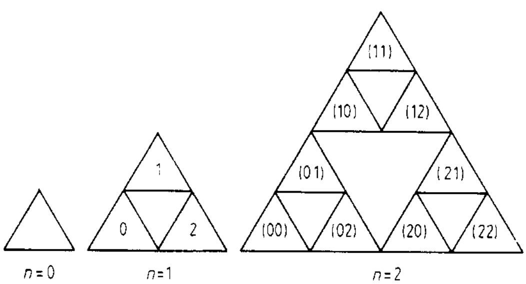

Figure 1. The Sierpinski gasket ($d = 2$) at successive stages: $n = 0$, 1 and 2. Stage $n + 1$ is obtained by the juxtaposition of three n-stage structures. For $n = 1$ and $n = 2$, we have shown the labelling of the plaquettes (up triangles) used in the site coding.

number of sites in the gasket is given by $N_n = (d+1)(1+(d+1)^n)/2$. The scaling factor of the gasket is $b=2$ and the fractal dimensionality is (Gefen *et al* 1981) $\bar{d} = \ln(d+1)/\ln 2$. The spectral dimensionality $\tilde{d}$ was found in I: $\tilde{d} = 2\ln(d+1)/\ln(d+3)$. In the following, we shall study the RWs on the gasket at $d=2$, in order to check some statistical properties of RWs on this non-trivial fractal structure.

2.1. Labelling procedure

Starting from the lowest left corner of the gasket, a natural coding of sites is easily obtained, using p-adic numbers (Bachman 1964, Mahler 1973). We shall illustrate briefly this coding at $d=2$. Each up triangle (plaquette) is labelled on the gasket at stage n by a word of n elements: $(x_1 x_2 x_3 \ldots x_n)$ where $x_i = 0, 1$ or 2 $(1 \leqslant i \leqslant n)$ are defined recursively from stage n to stage $n+1$. Beside the corners, every plaquette has three neighbouring ones, which labels are trivially found using the above coding. In this way, the sites of the gasket are labelled with the two plaquette labels sharing the considered site. These labels are the 'normal' coordinates on the gasket, in opposition to Cartesian coordinates usually used in Euclidean lattices.

Starting from an arbitrary site on the lattice, a RW trajectory is generated by a standard Monte Carlo procedure. The lattice size (i.e. n) was chosen to be much larger than the span of the walks ($n > 10$), so as to avoid end effects. A large number of steps N for each trajectory was needed to reach the asymptotic regime, and sampling over a very large number of trajectories was used to obtain accurate results.

2.2. Range of the random walk

The range of a RW R_N is defined as the number of distinct sites visited up to step N. The behaviour of the expectation value $S_N = \langle R_N \rangle$ is well known on Euclidean lattices of dimension d (Montroll and West 1979). For instance,

$$S_N \simeq (8/\pi)^{1/2} N^{1/2} \quad \text{at } d=1, \qquad S_N \simeq N/\ln N \quad \text{at } d=2$$
$$S_N \simeq N \quad \text{at } d \geqslant 3. \tag{1}$$

The numerical results obtained for the gasket at $d=2$ are shown in figure 2. Averages are taken over 4×10^3 walks of $N \leqslant 500$ steps. The asymptotic region is reached for

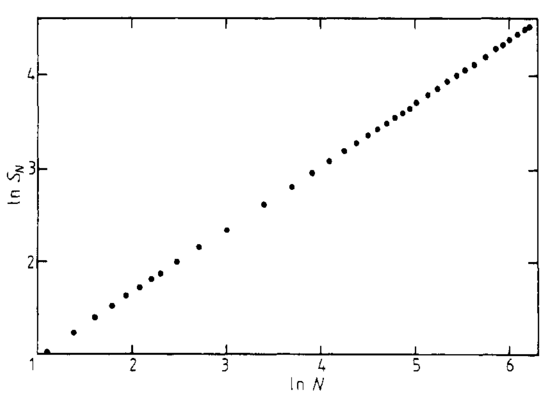

Figure 2. Plot of ln S_N against ln N for a set of 4×10^4 walks of $N \leqslant 500$ steps each, for the Sierpinski gasket (SG).

$N \geq 50$, where a power law is observed. The deduced value of the exponent is 0.682 ± 0.005, in very good agreement with the predicted value ($S_N \sim N^{\tilde{d}/2}$) given in I: $\tilde{d}/2 = 0.68260$.

The same results were used to extract the behaviour of the variance of the random variable R_N:

$$\sigma_N^2 \equiv \mathrm{Var}\, R_N = \langle R_N^2 \rangle - \langle R_N \rangle^2. \tag{2}$$

The variance of R_N increases with N, showing a power law as a function of N. More precisely, we have used the ratio ρ defined by

$$\rho(N) = (\mathrm{Var}\, R_N)^{1/2} / S_N \tag{3}$$

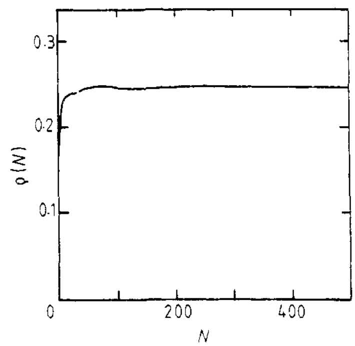

Figure 3. Plot of the ratio $\rho(N) = (\mathrm{Var}\, R_N)^{1/2}/S_N$ as a function of N, for the SG.

to study the asymptotic behaviour of Var R_N. The obtained results are shown in figure 3. After a transient regime, ρ as a function of N converges towards a limiting value given by $\rho = 0.244 \pm 0.002$. This special behaviour

$$\mathrm{Var}\, R_N \sim N^{\tilde{d}} \qquad N \gg 1 \tag{4}$$

is known to occur at $d = 1$ at long times (Feller 1951). The obtained numerical value of the ratio ρ at $d = 1$ is $\rho_1 = 0.298 \pm 0.002$. Such behaviour of the RW on fractals with $\tilde{d} < 2$ is to be contrasted with that on Euclidean space of dimension $d \geq 2$. In fact, the asymptotic behaviour of Var R_N in these cases is well known (Jain and Pruitt 1971):

$$\mathrm{Var}\, R_N \simeq N^2/\ln^4(N) \qquad \text{and } \rho_2(N) \simeq 1/\ln N \qquad \text{at } d = 2 \tag{5}$$

$$\mathrm{Var}\, R_N \simeq N \ln N \qquad \text{and } \rho_3(N) \simeq (\ln N/N)^{1/2} \qquad \text{at } d = 3. \tag{6}$$

The common behaviour at $d = 1$ and $d = 2$ of the family of the Sierpinski gaskets is expected to occur for every value of d. The constant value of the ratio ρ is expected to be a decreasing function of d, reaching the value $\rho = 0$ only at $d = \infty$ where $\tilde{d} = 2$. We conjecture that $\lim_{N \to \infty} \rho(N)$ exists and remains finite, for every fractal structure with a spectral dimensionality $\tilde{d} < 2$. For structures with $\tilde{d} \geq 2$, this limiting value of $\rho(N)$ is expected to be zero. It will be interesting to check this conjecture numerically on fractals with different values of $\tilde{d}$.

The conjectured behaviour of $\rho(N)$ suggests that the limiting law of R_N is not normal at $\tilde{d} < 2$. Such behaviour is known in the case of a linear chain ($\tilde{d} = \bar{d} = d = 1$). The reduced variable $x = R_N/S_N$ is known (Feller 1951, Jain and Pruitt 1972) to

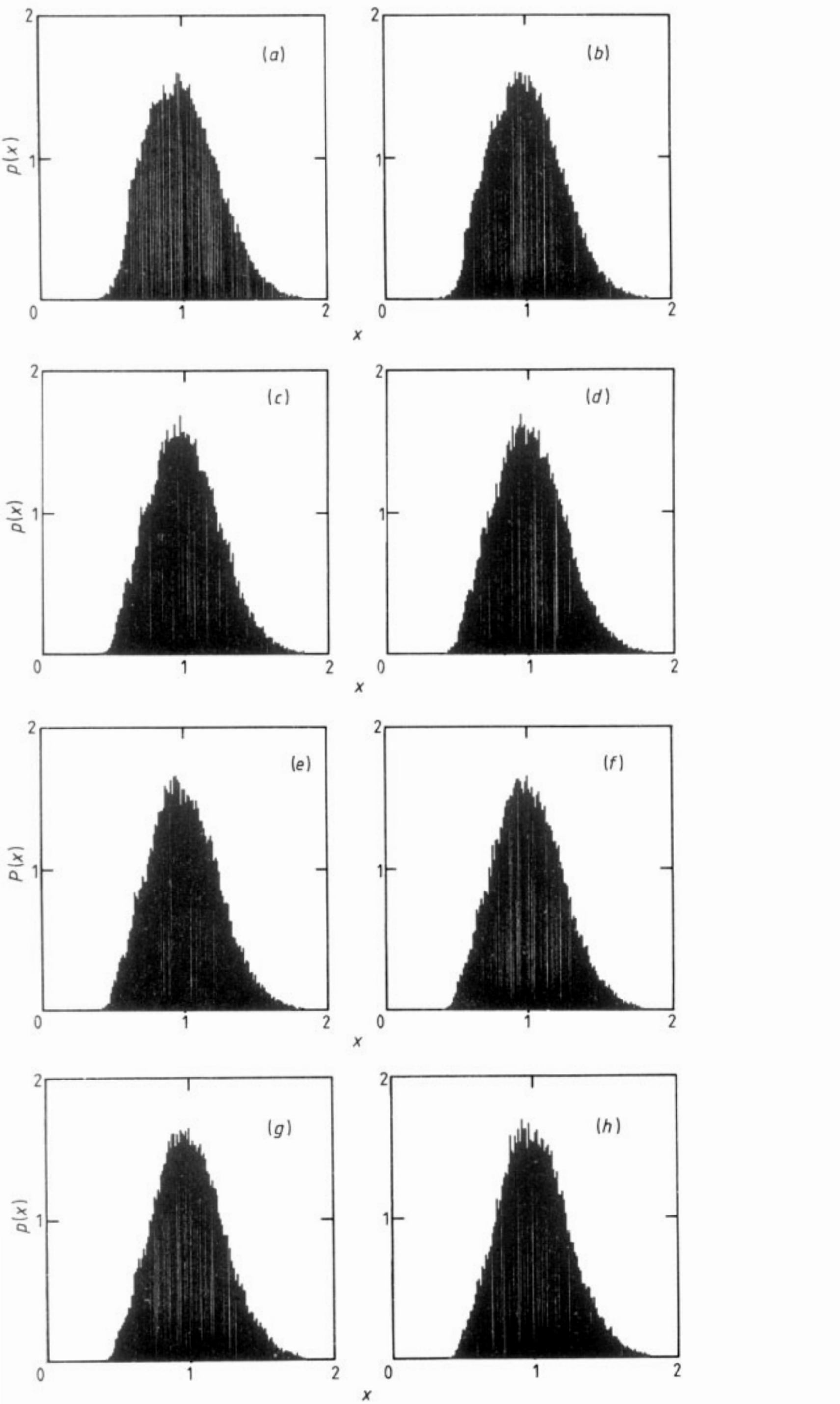

Figure 4. Successive distributions $P(x)$ of the ratio $x = R_N/S_N$ for increasing values of N. (a), $N = 300$, $S_N = 65.16$; (b), $N = 350$, $S_N = 72.38$; (c), $N = 400$, $S_N = 79.44$; (d), $N = 420$, $S_N = 81.85$; (e), $N = 440$, $S_N = 84.44$; (f), $N = 460$, $S_N = 87.02$; (g) $N = 480$, $S_N = 89.46$; (h), $N = 500$, $S_N = 92.06$; for the SG.

converge in distribution to a proper law. The limiting law is that of

$$(\tfrac{1}{8}\pi)^{1/2}\left\{\max_{0\leqslant t\leqslant 1} Y(t) - \min_{0\leqslant t\leqslant 1} Y(t)\right\} \tag{7}$$

where $Y(t)$ is standard one-dimensional Brownian motion.

According to the above conjecture relative to Var R_N at $\tilde{d}<2$, we expect that a similar behaviour of the distribution of x must occur on fractals with $\tilde{d}<2$. More precisely, for these structures, the reduced variable x is expected to converge in distribution to a proper law. In figure 4, we show the obtained distribution $P(x)$ of the variable x on the gasket for different values of N. This figure exhibits the limiting distribution of the ratio x (as for $d=1$), in contrast with the Euclidean lattice case, where R_N/S_N converges to 1 with a probability of unity at $d\geqslant 2$. The presence of a distribution for the ratio x reveals the strong fluctuations of the random variable R_N at $N \gg 1$. Such a behaviour of the RW range is then also expected to occur on any fractal structure where $\tilde{d}<2$. A qualitative change must occur at $\tilde{d}\geqslant 2$ as for Euclidean lattices.

To summarise, the value $\tilde{d}=2$ is not only associated with a modification in the asymptotic behaviour ($N\gg 1$) of the expected value $S_N=\langle R_N\rangle$ as was argued in I:

$$S_N \sim N^{\tilde{d}/2} \quad \text{at } \tilde{d}<2, \qquad S_N\sim N \quad \text{at } \tilde{d}>2. \tag{8}$$

A profound modification of the distribution of the variable R_N seems also to occur at this 'critical' value of $\tilde{d}$. For $\tilde{d}=2$, logarithmic corrections to simple power laws are expected in the expression of S_N, Var $R_N, \ldots$, as for Euclidean lattices.

2.3. *Renewal theory*

As was shown above, the study of the range of the RW provides a direct measurement of the spectral dimensionality $\tilde{d}$. Another measurement of $\tilde{d}$ is given by the renewal theory (Prohorov and Rozanov 1969), which is related to the spatial distribution of the position of the walk. Let τ_k be the time of the kth return to the origin (i.e. starting point of the walk), and let $\tau_0=0$ and $\Delta_k=\tau_k-\tau_{k-1}$ be the time between the $(k-1)$th and kth return; thus

$$\tau_n = \sum_{k=1}^{n} \Delta_k. \tag{9}$$

For a simple RW, the time intervals between two successive returns are independent identically distributed random variables. Then, the time until the nth return is the sum of n of these variables. When properly normalised, we then expect an asymptotically stable distribution $p(x;\alpha,\beta)$ for this variable τ_n (Levy 1937, Gnedenko and Kolmogorov 1968). This stable law is closely related to the value of the spectral dimensionality $\tilde{d}$ of the space. A direct connection is provided by the number ν_t of returns to the original site ω within the time $t\equiv N$. The obvious relation

$$\Pr\{\nu_t\geqslant n\}=\Pr\{\tau_n\leqslant t\} \tag{10}$$

allows us to obtain the asymptotic distribution of ν_t for $t\to\infty$. The left-hand side of equation (10) is related to the cumulative time spent at the origin between time 0 and t. The last is expected (see equation (19) below) to scale like $\nu_t\sim t^\alpha$ $(t\to\infty)$ with

$$\alpha = 1-\tilde{d}/2 \quad \text{at } \tilde{d}<2, \qquad \alpha=0 \quad \text{at } \tilde{d}>2. \tag{11}$$

From these considerations, we expect the following asymptotic behaviour of τ_n:

$$\tau_n \sim n^{1/\alpha}. \tag{12}$$

This means that the time until the nth return increases approximately as $n^{1/\alpha}$; more precisely,

$$\Pr\{\tau_n/n^{1/\alpha} \geqslant x\} \simeq x^{-\alpha} \qquad \text{at } x \gg 1 \text{ and } n \gg 1 \tag{13a}$$

$$\Pr\{\nu_N/N^{\alpha} \leqslant x\} \simeq \mathrm{O}(1) \qquad \text{at } x \gg 1 \text{ and } N \gg 1. \tag{13b}$$

Equations (11)–(13) are simply the generalisation of the corresponding laws at $d=1$ (Prohorov and Rozanov 1969) to fractals with $\tilde{d}<2$. The known results for ν_n are recovered on the Euclidean lattices. Only the parameter α of the stable law is expected to be different. $\tau_n/n^{1/\alpha}$ is then expected to converge to the stable law with density $p(x;\alpha,1)$. For $d=1$, we recover the known results for the renewal theory, where $\alpha=\frac{1}{2}$ and $p(x;\frac{1}{2},1)$ is given by

$$\begin{aligned} p(x;\tfrac{1}{2},1) &= (2\pi)^{-1/2}x^{-3/2}\,\mathrm{e}^{-1/2x} \qquad && x>0 \\ &= 0 && x<0. \end{aligned} \tag{14}$$

According to (12) we deduce

$$\lim_{n\to\infty}(\tau_n/n) = \lim n^{-1}\sum_{k=1}\Delta_k = \infty \qquad \text{at } \tilde{d}<2. \tag{15}$$

If τ is the time of the first return to the original site, then τ is finite with probability one, whereas its mean value given by (15) is infinite. This particular situation is common to all fractal structures with $\tilde{d}<2$.

In the same way, according to (13b), it appears that for $N\to\infty$, the number of returns to the origin in N steps increases proportionally to N^{α} and not N, as might be expected from the recurrence property of the walk. Finally, the probability of return to the origin after N steps, i.e. the probability of closed walks of length N, is given by (Alexander and Orbach 1982, see also I)

$$P_0(N) \sim 1/N^{\tilde{d}/2}. \tag{16}$$

In order to check the above renewal theory, we have calculated separately the average number of returns ν_N given by $\nu_N \sim N^{\alpha}$ ($N \gg 1$) and the probability of return after N steps $P_0(N)$ given by (16). The results obtained for ν_N (average over 4×10^3 walks with different origins ω) are shown in figure 5. As can be seen, the asymptotic power law is reached at $N \geqslant 50$, and the deduced value of the corresponding exponent is 0.3229 ± 0.0003, in good agreement with the predicted value $\alpha = 1-\tilde{d}/2$.

In figure 6, we have shown the results obtained for $P_0(N)$, in the same range of N values. The power law (16) is well reproduced, and the extracted exponent is 0.681 ± 0.003, which is in agreement with (16). In addition to the known results for $d=1$, the good agreement between the predictions of the above renewal theory and the obtained numerical results on the gasket support the validity of this theory on fractals. Therefore, the renewal theory provides a new method for the measurement of the spectral dimensionality $\tilde{d}$. The consistency of the results obtained from the renewal theory and those from the range theory, support fully the predictions of I and those given here.

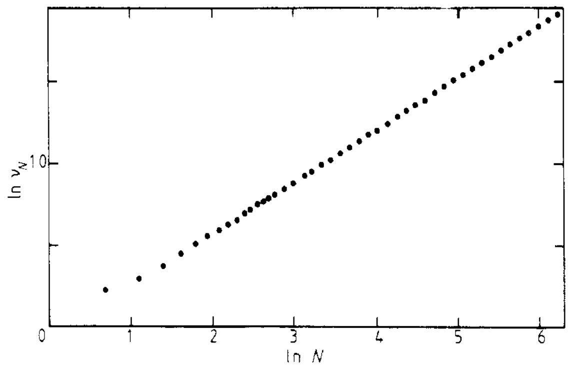

Figure 5. Plot of ln ν_N against ln N for a set of 4×10^4 walks of $N \leq 500$ steps. Here, ν_N denotes the mean number of returns to the origin in N steps for the SG.

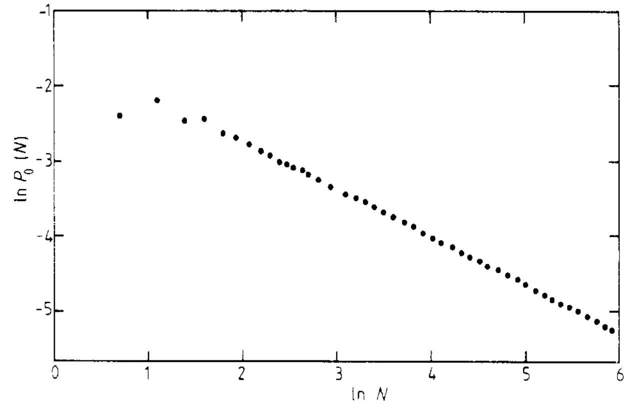

Figure 6. Plot of ln $P_0(N)$ against ln N as in figure 5. $P_0(N)$ is the probability of return to the origin at step N.

*2.4. Spatial distribution P(N, **R**) and mean first-passage time*

In addition to the range and renewal theories discussed in §§ 2.2 and 2.3, the spatial distribution of the walk is another statistical property of the RW. A lot of relatively interesting quantities are extracted from this distribution. In the following, the distances on fractals are measured with the Euclidean metric of the d-dimensional embedding Euclidean space. Using this measure, the normalised spatial distribution of the RW position $\boldsymbol{r}$, starting from a chosen site $\boldsymbol{r}=0$, is given by (Rammal and Vannimenus 1983)

$$P(N,\boldsymbol{r}) = (1/R_0^{\bar{d}})f(r/R_0) \tag{17}$$

where $r=|\boldsymbol{r}|$ denotes the distance from the origin at step N. In (17), R_0 is the 'radial' extension of the walk (Alexander and Orbach 1982) $R_0 \sim N^{\nu_{\mathrm{RW}}}$ where $\nu_{\mathrm{RW}}=\tilde{d}/2\bar{d}$ is the exponent of the RW. The universal function $f(u)$ of the reduced variable in (17) is expected to scale like

$$f(u) \sim \exp(-u^{\delta}) \qquad \text{at } u \gg 1 \tag{18}$$

where δ is given (des Cloizeaux 1982) by $\delta = 1/(1-\nu_{\mathrm{RW}})$. In particular, $\Pr(r>\xi) \sim \xi^{-1/\nu_{\mathrm{RW}}}$ at $\xi \to \infty$.

From (17), we recover easily the expression $P_0(N) \sim N^{-\tilde{d}/2}$ given by (16). The cumulative time spent at the origin between step 0 and N is also given by

$$\int_0^N P(N, r=0)\,\mathrm{d}N \sim N^{-\tilde{d}/2+1} \equiv N^{\alpha}. \tag{19}$$

Accordingly, the fractal dimensionality D of the zero-crossing set is given by

$$D = 1 - \tilde{d}/2 \qquad \text{if } \tilde{d} < 2, \qquad D = 0 \qquad \text{if } \tilde{d} > 2. \tag{20}$$

An interesting quantity which can be extracted from the spatial distribution $P(N, \boldsymbol{r})$ is given by the mean first-passage time $T_1(\xi)$ at a distance ξ from the origin (Seshardi and West 1982). Let $\pi(N, \boldsymbol{r})$ be the probability that the walker has not crossed the boundaries $|\boldsymbol{r}| = \xi$, given that he starts at the origin at time $N = 0$: $\pi(N=0, N) = \delta(r)$ and $\pi(N, r=\xi) = 0$. The expression of $T_1(\xi)$ is given by

$$T_1(\xi) = \int_0^{\infty} \mathrm{d}N \int_0^{\xi} r^{\bar{d}-1}\pi(N, r)\,\mathrm{d}r. \tag{21}$$

The scaling behaviour of $T_1(\xi)$ is also given by the approximation $(\xi \to \infty)$: $\pi(N, r) \simeq P(N, r)$. Therefore,

$$T_1(\xi) = \int_0^{\infty} \mathrm{d}N \int_0^{\xi} r^{\bar{d}-1} P(N, r)\,\mathrm{d}r$$

$$T_1(\xi) \sim \xi^{1/\nu_{\mathrm{RW}}} \qquad \text{at } \xi \to \infty. \tag{22}$$

The result (22) which generalises the known results for Euclidean spaces (Barber and Ninham 1970, Feller 1971) provides a useful method for the measurement of the exponent ν_{RW}. No numerical results for $T_1(\xi)$ are given here. It will be interesting to check (22) for various values of ξ, because it provides an indirect verification of the result given by (17).

To summarise, we have investigated in this section some statistical properties relative to RWs on the $d = 2$ Sierpinski gasket. All the numerical results obtained are in very good agreement with theoretical predictions. This preliminary 'exercise' is to be considered as a first step in the study of dynamical behaviour of more complicated fractals, such as the percolation clusters studied in § 3.

3. Application: random walks on 2D percolation clusters

In the following, we present the results of Monte Carlo calculations of RW statistics on percolation clusters in two dimensions at the critical point p_c. For convenience, we have used the bond percolation clusters, where $p_c = \frac{1}{2}$ is known exactly (Sykes and Essam 1963). The clusters were generated by a cluster-growth method (Leath 1976). A RW was generated on these clusters. The clusters were chosen to be much larger than the span of the walks so as to avoid end effects. The diffusion was restricted to clusters larger than the span of each walk. In order to take into account this restriction, the notion of 'open frontier' (see below) was used in our simulation.

As mentioned in the introduction, the infinite cluster (at a percolation threshold) is a fractal object, of fractal dimensionality (Stauffer 1981)

$$\bar{d} = d - \beta_{\mathrm{p}}/\nu_{\mathrm{p}}$$

and spectral dimensionality (Alexander and Orbach 1982)

$$\tilde{d} = 2(d\nu_p - \beta_p)/(t - \beta_p + 2\nu_p) \tag{23}$$

where d is the Euclidean dimension of the lattice, ν_p, β_p are the critical exponents of the percolation transition, and t is the conductivity exponent. As stated in I, $\bar{d}$ and $\tilde{d}$ are the only ingredients needed to characterise the RW statistics.

Averages over unrestricted RWs traced on a distribution of finite clusters imply a modification of the above results (Ben-Avraham and Havlin 1982, Gefen *et al* 1983). In order to check the conjecture $\tilde{d} = \frac{4}{3}$ (Alexander and Orbach 1982), we limit ourselves here to the restricted RW only.

3.1. *Range of the random walk*

The average value S_N of the RW range R_N after N steps is shown in figure 7. The statistics were taken over 35×10^3 walks, with different starting points, on bond percolation clusters restricted to be larger than the span of each walk. The power law, predicted in I, is reached after $N \geq 100$. The extracted value of the slope of this curve, given by the best fit, yields the value 0.65 ± 0.01 for the exponent of S_N against N.

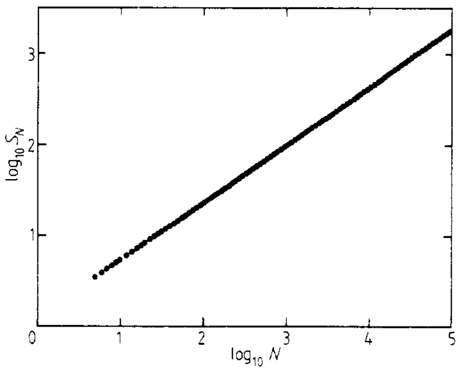

Figure 7. Plot of $\log_{10} S_N$ against $\log_{10} N$ for a set of 35×10^3 walks of N steps traced on bond percolation clusters restricted to be larger than the span of each walk.

This value is in fair agreement with the conjectured one: $\tilde{d}/2 = \frac{2}{3}$. Using the same data, the dispersion around the mean value S_N is shown in figure 8. $(\mathrm{Var}\, R_N)^{1/2}$ exhibits a power law behaviour at large N, as predicted (equation (4)). The deduced value of the exponent of $(\mathrm{Var}\, R_N)^{1/2}$ is in good agreement with that of S_N. In order to show the accuracy reached in our calculation, we have plotted the ratio

$$\rho(N) = (\mathrm{Var}\, R_N)^{1/2}/S_N$$

as a function of N, in figure 9. The fluctuation of this ratio over the large interval of N values is due mainly to the fluctuations in the percolation cluster shapes at threshold. Such a fluctuation was not present in the case of the gasket studied in § 2. Nevertheless, for large values of N, $\rho(N)$ seems to fluctuate around a fixed value: $\rho(N \gg 1) \simeq 0.1$, in good agreement with the conjecture of § 2.

In I, the concept of 'open frontier' was introduced in the context of the range theory of the RW. This quantity is defined as the product of the probability of access

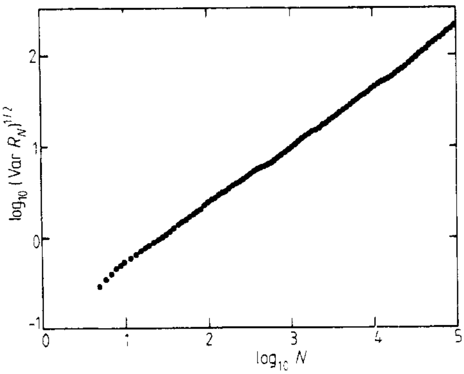

Figure 8. Same plot as in figure 7 for the variance $(\mathrm{Var}\ R_N)^{1/2}$ of the random walk range R_N.

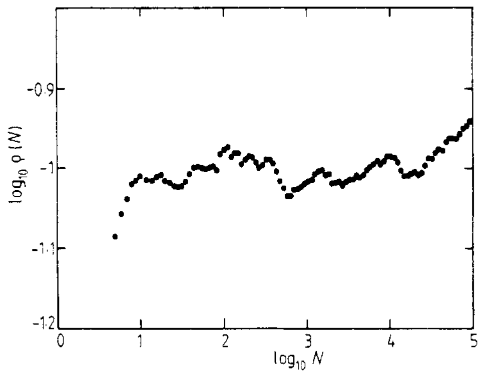

Figure 9. Variation of the ratio $\rho(N) = (\mathrm{Var}\ R_N)^{1/2}/S_N$ as a function of N.

to a fresh site $\mathrm{d}S_N/\mathrm{d}N$, by the number of accessible sites after N steps Σ_N. Given a RW on a fractal with $\tilde{d} < 2$, we have $\Sigma_N \sim S_N \sim N^{\tilde{d}/2}$ (at $N \gg 1$) and then the open frontier F_N after N steps is given by

$$F_N \simeq S_N \mathrm{d}S_N/\mathrm{d}N \tag{24}$$

which must scale like

$$F_N \sim N^{\tilde{d}-1}. \tag{25}$$

In figure 10, we have shown the obtained results for F_N in the same interval of N values. As can be seen, the values of F_N are smaller when compared with N. The best fit of the data gives the slope 0.31 ± 0.02 in good agreement with the conjectured value $\tilde{d} - 1 = \frac{1}{3}$. The deviation due mainly to statistical fluctuation is of the same sign as that obtained in the determination of $\tilde{d}$ from S_N against N.

3.2. Renewal theory

As for the Sierpinski gasket, we have used renewal theory to obtain another estimation of the spectral dimensionality of percolation clusters $\tilde{d}$. In figure 11, the probability of return to the origin $P_0(N)$, after N steps, is plotted against N. Averages were taken

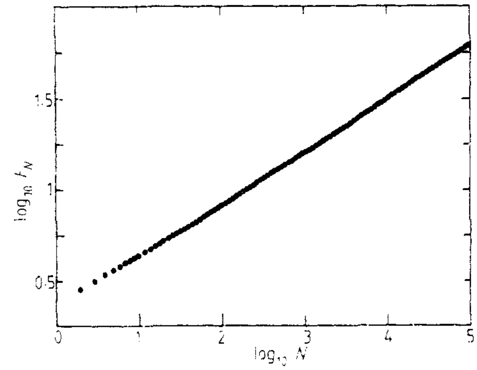

Figure 10. Plot of $\log_{10} F_N$ as a function of $\log_{10} N$ for the same set of walks as in figure 7 (F_N denotes the open frontier at step N).

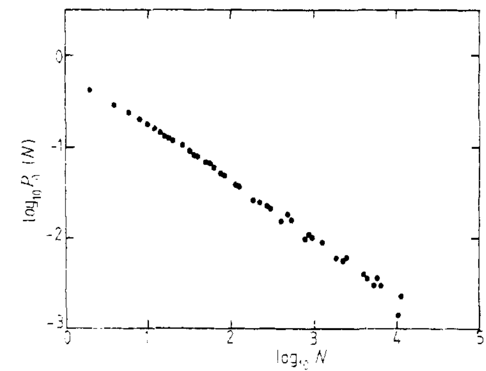

Figure 11. Behaviour of the probability of return to the origin at step N taken from 15×10^3 walks.

over 15×10^3 walks. Clearly $P_0(N)$ exhibits the predicted power law as a function of N. The best fit gives the value 0.65 ± 0.02 of the corresponding exponent. As for S_N, F_N and Var R_N, the calculated value of the associated exponent is smaller than the predicted value, implied by the conjecture $\tilde{d}=\frac{4}{3}$. The differences in all these cases have the same signs, but the relative accuracy reached in our calculations is not sufficient to prove the exactness of the conjecture $\tilde{d}=\frac{4}{3}$. Accuracy better than 1% in the values of this exponent is needed for this task.

4. Conclusion

In this paper, the second in a series, we have investigated some statistical properties of the RW on fractal structures. New analytical results were given for fractals in general: range theory, renewal theory and spatial distribution. Some of these results have been checked numerically on the Sierpinski gasket at $d=2$. Using the range theory, or the renewal theory, we have proposed a new numerical method, in order to extract directly the spectral dimensionality $\tilde{d}$.

Our method provides a direct measurement of the spectral dimensionality in contrast with previous methods (Ben-Avraham and Havlin 1982) using the spatial extension

R_0 of the RW, which involves together the fractal and spectral dimensionalities. As an application, we have investigated the case of the bond percolation clusters at threshold p_c in two dimensions ($d = 2$). Up to the numerical accuracy of our calculation, the obtained numerical value of the spectral dimensionality $\tilde{d}$ is in relatively good agreement with the conjectured value (Alexander and Orbach 1982) $\tilde{d} = \frac{4}{3}$. The extension of this study (in progress) to other values $d = 3$, 4, 5 and 6 provides a direct check of this conjecture on these spaces. Using the method proposed in this paper, new precise determinations of the conductivity exponent t, given by (23) at $d = 3, 4, \ldots$, are expected from these calculations. At $d = 2$, the direct measurement of the exponent t (Derrida and Vannimenus 1982) is in good agreement with the value $\tilde{d} = \frac{4}{3}$.

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References

Alexander S 1983 *Phys. Rev.* B **27** 1541
Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
Bachman G 1964 *Introduction to p-adic numbers and valuation theory* (New York: Academic)
Barber M N and Ninham B W 1970 *Random and restricted walks: theory and applications* (New York: Gordon and Breach)
Ben-Avraham D and Havlin S 1982 *J. Phys. A: Math. Gen.* **15** L691
des Cloizeaux J 1982 *Phase transition: Cargèse 1980* ed M Levy, J C Le Guillou and J Zinn-Justin (New York: Plenum) pp 371–94
Derrida B and Vannimenus J 1982 *J. Phys. A: Math. Gen.* **15** L557–64
Deutscher G 1981 *Lecture notes in physics* vol 49, p 26 (Berlin: Springer)
Feller W 1951 *Ann. Math. Stat.* **22** 427–32
—— 1971 *An introduction to probability theory and its applications* vol 2, 2nd edn (New York: Wiley)
Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
Gefen Y, Aharony A, Mandelbrot B B and Kirkpatrick S 1981 *Phys. Rev. Lett.* **47** 1771–4
Gnedenko B V and Kolmogorov 1968 *Limit distribution of sums of independent random variables* revised edn (Reading, MA: Addison-Wesley)
Jain N C and Pruitt W E 1971 *J. Anal. Math.* **24** 369–93
—— 1972 in *Proc. sixth Berkeley Symp. on mathematical statistical and probability* vol III, ed L M Le Cam, J Neyman and E L Scott (University of California Press) pp 31–50
Kapitulnik A and Deutscher G 1982 *Phys. Rev. Lett.* **49** 1444–7
Leath P L 1976 *Phys. Rev.* B **14** 5046
Levy P 1937 *Théorie de l'addition des variables aléatoires* (Paris: Gauthiers-Villars)
Mahler K 1973 *p-adic numbers and their functions* (Cambridge: CUP)
Mandelbrot B B 1977 *Fractals: form, chance and dimension* (San Francisco: Freeman)
Montroll E W and West B J 1979 in *Fluctuation phenomena* ed E W Montroll and J L Lebowitz (Amsterdam: North-Holland) ch 2
Prohorov Y V and Rozanov Y A 1969 *Probability theory* (Berlin: Springer) pp 186–90, 51–5
Rammal R and Toulouse G 1982 *Phys. Rev. Lett.* **49** 1194
—— 1983 *J. Physique Lett.* **44** L13–22
Rammal R and Vannimenus J 1983 *Preprint*
Seshardi V and West B J 1982 *Proc. Natl Acad. Sci. USA* **79** 4501–5
Stauffer D 1979 *Phys. Rep.* **54** 1
—— 1981 *Lecture notes in physics* vol 49 (Berlin: Springer) p 9
Sykes M F and Essam J W 1963 *Phys. Rev. Lett.* **10** 3
Voss R F, Laibowitz R B and Allessandrini E I 1982 *Phys. Rev. Lett.* **49** 1441–4