The Annals of Probability
2015, Vol. 43, No. 3, 992–1044
DOI: 10.1214/13-AOP867

RANDOM WALKS IN CONES[1]

BY DENIS DENISOV AND VITALI WACHTEL

University of Manchester and University of Munich

We study the asymptotic behavior of a multidimensional random walk in a general cone. We find the tail asymptotics for the exit time and prove integral and local limit theorems for a random walk conditioned to stay in a cone. The main step in the proof consists in constructing a positive harmonic function for our random walk under minimal moment restrictions on the increments. For the proof of tail asymptotics and integral limit theorems, we use a strong approximation of random walks by Brownian motion. For the proof of local limit theorems, we suggest a rather simple approach, which combines integral theorems for random walks in cones with classical local theorems for unrestricted random walks. We also discuss some possible applications of our results to ordered random walks and lattice path enumeration.

1. Introduction, main results and discussion.

1.1. *Motivation.* Random walks conditioned to stay in cones is a very popular topic in probability. They appear naturally in many situations. Here, we mention some of them:

- Nonintersecting paths, which can be seen as a multidimensional random walk in one of Weyl chambers, are used in modeling of different physical phenomena; see Fisher [22]. There are also a lot of connections between nonintersecting paths and Young diagrams, domino tiling, random matrices and many other mathematical objects; for an overview, see König [31].
- Random walks in the quarter-plane reflected at the boundary are often used in the queueing theory. For diverse examples, see monographs written by Cohen [12], by Fayolle, Iasnogorodski and Malyshev [19] and a paper by Greenwood and Shaked [29].
- Asymptotic behavior of branching processes and random walks in random environment is closely connected to the behavior of random walks conditioned to stay positive, which are one-dimensional cases of a random walk conditioned to stay in a cone; see [1] and references therein.

The main purpose of the present paper is to propose an approach which determines the asymptotic behavior of exit times and allows one to prove limit theorems for a rather wide class of cones and under minimal moment conditions on

Received November 2012; revised June 2013.
[1]Supported by DFG.
MSC2010 subject classifications. Primary 60G50; secondary 60G40, 60F17.
Key words and phrases. Random walk, exit time, harmonic function, Weyl chamber.

the increments of random walks. For that, we use a strong approximation of multidimensional random walks with multidimensional Brownian motion. This allows to extend the corresponding results for the Brownian motion to the discrete time setting and to study the asymptotic behavior of random walks.

For Brownian motion, the study of exit times from cones was initiated by Burkholder. In [9], he proposed necessary and sufficient conditions for the existence of moments of exit times. Later on, using these results, DeBlassie [13] found an exact formula for $\mathbf{P}(\tau_x > n)$ as an infinite series. This formula allowed him to obtain tail asymptotics for exit times. These results were obtained by Bañuelos and Smits [3] under more general conditions. Garbit [24] defined a Brownian motion started at origin and conditioned to stay in a cone.

For random walks in discrete time, much less is known. A corresponding generalization of Burkholder's results was obtained by McConnell [36]. Namely, he found necessary and sufficient conditions for the existence of moments of exit times. Varopoulos [43, 44] derived upper and lower bounds for the tail probability under an additional assumption that the increments of the random walk are bounded. Moreover, he showed that this upper bound remains valid for Markov chains with zero drifts and bounded increments. MacPhee, Menshikov and Wade [35] gave criteria for the existence of moments of exit times from wedges for Markov chains with asymptotically zero drifts and bounded increments. The exact asymptotic behavior for the exit times of a random walk is known only in some special cases. Shimura [41] and Garbit [25] obtained the asymptotics of the tail and some limit theorems for two-dimensional random walks. There are many results in the literature on random walks in Weyl chambers. We shall mention them later, in a special paragraph devoted to Weyl chambers.

1.2. *Notation and assumptions*. Consider a random walk $\{S(n), n \ge 1\}$ on $\mathbb{R}^d$, $d \ge 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n), n \ge 1\}$ is a family of independent copies of a random variable $X = (X_1, X_2, \dots, X_d)$. Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$ and Σ an open and connected subset of $\mathbb{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, that is, $\Sigma = K \cap \mathbb{S}^{d-1}$.

Let τ_x be the exit time from K of the random walk with starting point $x \in K$, that is,

$$\tau_x = \inf\{n \ge 1 : x + S(n) \notin K\}.$$

In this paper, we study asymptotics for

$$\mathbf{P}(\tau_x > n), \qquad n \to \infty,$$

construct a random walk conditioned to stay in the cone K and prove limit theorems for this random walk.

The essential part of the proof is a coupling with the Brownian motion. Hence, we extensively use related results for the Brownian motion. Let $B(t)$ be a standard Brownian motion on $\mathbb{R}^d$ and let τ_x^{bm} be the exit time of $B(t)$ from the cone K,

$$\tau_x^{\mathrm{bm}} = \inf\{t \ge 0 : x + B(t) \notin K\}.$$

The harmonic function of the Brownian motion killed at the boundary of K can be described as the minimal (up to a constant), strictly positive on K solution of the following boundary problem:

$$\Delta u(x) = 0, \qquad x \in K \text{ with boundary condition } u|_{\partial K} = 0.$$

If such a function exists, then see [13] and [3]. One can show that

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \sim \varkappa \frac{u(x)}{t^{p/2}}, \qquad t \to \infty. \tag{1}$$

The function $u(x)$ and constant p can be found as follows. If $d = 1$, then we have only one nontrivial cone $K = (0, \infty)$. In this case, $u(x) = x$ and $p = 1$. Assume now that $d \ge 2$. Let $L_{\mathbb{S}^{d-1}}$ be the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$ and assume that Σ is regular with respect to $L_{\mathbb{S}^{d-1}}$. With this assumption, there exists a complete set of orthonormal eigenfunctions m_j and corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ satisfying

$$\begin{aligned} L_{\mathbb{S}^{d-1}} m_j(x) &= -\lambda_j m_j(x), \qquad x \in \Sigma, \\ m_j(x) &= 0, \qquad x \in \partial\Sigma. \end{aligned} \tag{2}$$

Then

$$p = \sqrt{\lambda_1 + (d/2 - 1)^2} - (d/2 - 1) > 0$$

and the harmonic function $u(x)$ of the Brownian motion is given by

$$u(x) = |x|^p m_1\left(\frac{x}{|x|}\right), \qquad x \in K. \tag{3}$$

We refer to [3] for further details on exit times of Brownian motion.

Unfortunately, we are not able to determine the asymptotic behavior of exit times for random walks for such a general class of cones. More precisely, we will use the following additional conditions on the cone K:

- We assume that there exists an open and connected set $\widetilde{\Sigma} \subset \mathbb{S}^{d-1}$ with $\mathrm{dist}(\partial\Sigma, \partial\widetilde{\Sigma}) > 0$ such that $\Sigma \subset \widetilde{\Sigma}$ and the function m_1 can be extended to $\widetilde{\Sigma}$ as a solution to (2).
- K is either convex or starlike [there exists $x_0 \in \Sigma$ such that $x_0 + K \subset K$ and $\mathrm{dist}(x_0 + K, \partial K) > 0$] and C^2. (Every convex cone is also starlike, for the proof see Remark 15.)

It is known that if m_1 can be extended then the boundary $\partial\Sigma$ should be piecewise real-analytic. Furthermore, if $\partial\Sigma$ is real-analytic, then m_1 is extendable; see, for example, Theorem A in Morrey and Nirenberg [38].[2] Since the boundary of every two-dimensional cone consists of two points on the unit circle, one can always extend m_1 to a bigger cone in $\mathbb{R}^2$. Furthermore, it is clear that we can extend $u(x)$ to a harmonic function in the cone $\widetilde{K}$ generated by $\widetilde{\Sigma}$ using (3). We impose the following assumptions on the increments of the random walk:

- *Normalization assumption*: We assume that $\mathbf{E}X_j = 0, \mathbf{E}X_j^2 = 1$, $j = 1, \dots, d$. In addition, we assume that $\operatorname{cov}(X_i, X_j) = 0$.
- *Moment assumption*: We assume that $\mathbf{E}|X|^\alpha < \infty$ with $\alpha = p$ if $p > 2$ and some $\alpha > 2$ if $p \le 2$.

1.3. *Tail distribution of τ_x and a conditioned limit theorem.* Let

$$K^\varepsilon = \{y \in \mathbb{R}^d : \operatorname{dist}(y, x) < \varepsilon|x| \text{ for some } x \in K\}.$$

This new region is a cone. It follows from our assumptions that we can pick a sufficiently small $\varepsilon > 0$ which will ensure that $K \subset K^{4\varepsilon} \subset \widetilde{K}$. Recall that $u(x)$ is harmonic on a bigger cone $\widetilde{K}$ and, therefore,

$$u(x) \text{ is harmonic on } K^{4\varepsilon}.$$

Having u we define a new function

$$v(x) = \begin{cases} u(x), & x \in G, \\ |x|^{p-a}, & \text{otherwise}, \end{cases}$$

where

$$G = K^\varepsilon \cap \left(K \cup \{x \in K^c : \operatorname{dist}(x, \partial K) \le |x|^{1-a}\}\right).$$

We will pick a sufficiently small constant $a > 0$ later.

Let

$$f(x) = \mathbf{E}v(x + X) - v(x), \qquad x \in K. \tag{4}$$

Note that if $v(x + S(n))$ is a martingale, then $f(x) = 0$. Then let

$$V(x) = v(x) - \mathbf{E}v(x + S(\tau_x)) + \mathbf{E}\sum_{k=0}^{\tau_x - 1} f(x + S(k)). \tag{5}$$

It is not at all clear if function $V(x)$ is well defined. More precisely, one has to show that $v(x + S(\tau_x))$ and $\sum_{k=1}^{\tau_x - 1} f(x + S(k))$ are integrable. Furthermore, one has to show that V does not depend on choice of a and ε from the definition of G.

[2] We are grateful to Professor Ancona for pointing out the reference.

Finally, we define

$$K_+ := \{x \in K : \text{there exists } \gamma > 0 \text{ such that for every } R > 0 \\ \text{there exists } n \text{ such that } \mathbf{P}(x + S(n) \in D_{R,\gamma}, \tau_x > n) > 0\},$$

where $D_{R,\gamma} := \{x \in K : |x| \geq R, \operatorname{dist}(x, \partial K) \geq \gamma |x|\}$.

THEOREM 1. *Assume the normalization as well as the moment assumption hold. Then, for sufficiently small a, the function V is finite and harmonic for the killed random walk $\{S(n)\}$, that is,*

$$\mathbf{E}[V(x + S(1)), \tau_x > 1] = V(x). \tag{6}$$

The function $V(x)$ is strictly positive on the set K_+. Moreover, as $n \to \infty$,

$$\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-p/2}, \qquad x \in K, \tag{7}$$

where $\varkappa$ is an absolute constant.

Our moment assumption is optimal in the sense that the asymptotic behavior of $\mathbf{P}(\tau_x > n)$ is in general different if $\mathbf{E}|X|^p = \infty$. Indeed, consider a cone with $p > 2$ and let X be of the form $X = R\xi$, where R is a nonnegative random variable with

$$\mathbf{P}(R > u) \sim u^{-\alpha}, \qquad \alpha \in (2, p)$$

and ξ takes values on the unit sphere with some positive density on Σ. Clearly, $\mathbf{E}|X|^p = \infty$, that is, the moment assumption is not fulfilled. It follows from the structure of X that

$$\mathbf{P}(x + X \in D_{\sqrt{n},\gamma}) \sim n^{-\alpha/2} \mathbf{P}(\xi \in \Sigma \cap D_{0,\gamma}) \geq c n^{-\alpha/2}$$

for some positive c and all sufficiently small γ. Then

$$\mathbf{P}(\tau_x > n) \geq \mathbf{P}(x + X(1) \in D_{\sqrt{n},\gamma}) \mathbf{P}\Big(\max_{k \leq n-1} |S(k)| < \gamma\sqrt{n}\Big) \geq c n^{-\alpha/2},$$

where in the last step we used the functional central limit theorem. Therefore, the tail of τ_x is heavier than that of τ_x^{bm}. We conjecture that this lower bound is precise, that is,

$$\mathbf{P}(\tau_x > n) \sim \theta \mathbf{E}\tau_x n^{-\alpha/2}.$$

Behind this relation is the well-known principle of one big jump: in order to stay up to large moment of time n inside the cone, it is sufficient to make one big (of order $\sqrt{n}$) jump into the inner part of the cone K near time 0.

We note that assumptions $\mathbf{E}X_j^2 = 1$, $j = 1, \ldots, d$ and $\operatorname{cov}(X_i, X_j) = 0$ do not restrict the generality. More precisely, if they are not fulfilled and the covariance matrix of X is positive-definite, then there exists a matrix M such that $Y = MX$

satisfies these conditions. (If the covariance matrix is not positive-definite, then the random walk lives on a hyperplane.) Therefore, we have a random walk confined to a new cone $M(K)=\{Mx, x\in K\}$. In the following example, we show the influence of the correlation on the tail behavior of τ_x.

EXAMPLE 2. Consider a two-dimensional random walk with zero mean, $\mathbf{E}X_1^2=\mathbf{E}X_2^2=1$ and $\operatorname{cov}(X_1,X_2)=\rho\in(-1,1)$. Let K be the positive quadrant, that is, $K=\mathbb{R}_+^2$. In order to apply Theorem 1 we first need to find a matrix M such that the coordinates of the vector $Y=MX$ become uncorrelated. Let φ solve the equation $\sin 2\varphi=\rho$. Then the matrix

$$M=\frac{1}{\sqrt{1-\rho^2}}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

leads to uncorrelated coordinates. Therefore, $M(K)$ has opening $\arccos(-\rho)$. Then, as it has been shown by Burkholder [9], $p=\pi/\arccos(-\rho)$. If $\mathbf{E}|X|^{\pi/\arccos(-\rho)}$ is finite, then according to Theorem 1, we have

$$\mathbf{P}(\tau_x>n)\sim V(x)n^{-\pi/2\arccos(-\rho)}\qquad \text{as } n\to\infty.$$

It is worth mentioning that the minimal moment condition depends on the covariance between X_1 and X_2.

If V is harmonic for the random walk $MS(n)$ in the cone $M(K)$, we have also a harmonic function for $S(n)$ in K. Indeed, one can easily verify that $V(Mx)$ possesses this property.

We now turn to the asymptotic behavior of $S(n)$ conditioned to stay in K. To state our results, we introduce the limit process. For the d-dimensional Brownian motion with starting point $x\in K$, one can define a Brownian motion conditioned to stay in the cone via Doob's h-transform. For that, we make a change of measure using the harmonic function u:

$$\widehat{\mathbf{P}}_x^{(u)}\big(B(t)\in dy\big)=\mathbf{P}\big(x+B(t)\in dy, \tau_x^{\mathrm{bm}}>t\big)\frac{u(y)}{u(x)}.$$

This is possible since $u(x)>0$ inside the cone and $u(x+B(t\wedge\tau_x^{\mathrm{bm}}))$ is a martingale. Similarly, we define a random walk conditioned to stay in the cone K by

$$\widehat{\mathbf{P}}_x^{(V)}\big(S(n)\in dy\big)=\mathbf{P}\big(x+S(n)\in dy, \tau_x>n\big)\frac{V(y)}{V(x)},\qquad x\in\big\{x:V(x)>0\big\}.$$

This is possible due to Theorem 1, where harmonicity of V is proved. We note also, that if we choose the starting point in K_+ then $S(n)$ under $\widehat{\mathbf{P}}_x^{(V)}$ lives on $\{x:V(x)>0\}$, since this measure does not allow transitions to the set $\{x:V(x)=0\}$.

THEOREM 3. *Assume that the normalization as well as the moment assumption are fulfilled, then*

$$\mathbf{P}\left(\frac{x+S(n)}{\sqrt{n}}\in\cdot\,\Big|\tau_x>n\right)\to\mu \qquad \textit{weakly}, \tag{8}$$

where μ *is the probability measure on* K *with the density* $H_0u(y)e^{-|y|^2/2}$, *where* H_0 *is the normalizing constant.*

Furthermore, for every $x\in K$, *the process* $X^n(t)=\frac{S([nt])}{\sqrt{n}}$ *under the probability measure* $\widehat{\mathbf{P}}^{(V)}_{x\sqrt{n}}$ *converges weakly in the uniform topology on* $D[0,\infty)$ *to the Brownian motion under the measure* $\widehat{\mathbf{P}}^{(u)}_x$.

This is an extension of the classical theorems for one-dimensional random walks conditioned to stay positive by Iglehart [30] and Bolthausen [5]. Shimura [41] and Garbit [25] have proven similar results for two-dimensional random walks in convex cones.

REMARK 4. It is worth mentioning that one can prove Theorems 1 and 3 for more general cones if one restricts themselves to a smaller class of random walks. If the jumps of $\{S(n)\}$ are bounded, then the possibility to extend u to a bigger cone is superfluous. In this situation, one can show that V is harmonic for arbitrary starlike cone if we define v by the relation

$$v(x)=u(x_*+x)\qquad\text{for an appropriate } x_*\in K.$$

(For details, see Section 2.3.) Having constructed the harmonic function V, the proofs of all asymptotic statements from Theorems 1 and 3 do not require any change. As a result, we get an asymptotic counterpart of the results proven by Varopoulos [43, 44]. He derived upper and lower bounds for probabilities $\mathbf{P}(\tau_x>n)$ and $\mathbf{P}(x+S(n)=y,\tau_x>n)$ in terms of the harmonic function u. All his bounds have the right order in n. In order to obtain these estimates, he constructs superharmonic and subharmonic functions for $\{S(n)\}$ in terms of u. (It is equivalent to construction of super- and submartingale from $\{S(n)\}$.) And in order to obtain asymptotic results we construct a harmonic function (martingale) for the random walk. This is the main difference between our approach and that of Varopoulos.

1.4. *Local limit theorems.* In this paragraph, we are going to determine the asymptotic behavior of local probabilities for random walks conditioned to stay in a cone. As it is usual in studying local probabilities, one has to distinguish between lattice and nonlattice cases. We shall consider lattice walks only, and analogous results for nonlattice walks can be proved in the same way. The reason to choose the lattice case is an application of the local limit theorems we prove here to lattice path counting problems, which are very popular in combinatorics. Another interesting application of local limit theorems could be the study of the asymptotic behavior of the Green function for random walks in a cone. This, combined with the Martin boundary theory, will allow to find all harmonic functions.

- *Lattice assumption*: X takes values on a lattice R which is a nondegenerate linear transformation of $\mathbb{Z}^d$. Furthermore, we assume that the distribution of X is strongly aperiodic, that is, for every $x \in R$, the smallest subgroup of R which contains the set

$$\{y : y = x + z \text{ with some } z \text{ such that } \mathbf{P}(X = z) > 0\}$$

is R itself.

We first state a version of the Gnedenko local limit theorem.

THEOREM 5. *Assume that the assumptions of Theorem 1 and the lattice assumption hold. Then*

$$\sup_{y \in K} \left| n^{p/2+d/2} \mathbf{P}\big(x + S(n) = y, \tau_x > n\big) - \varkappa V(x) H_0 u\left(\frac{y}{\sqrt{n}}\right) e^{-|y|^2/2n} \right| \to 0. \tag{9}$$

To prove a local limit theorem in the one-dimensional case, that is, for random walks conditioned to stay positive, one starts usually from the Wiener–Hopf factorization; see [8, 10, 45]. Our approach is completely different, and uses the integral limit theorem for conditioned random walks (Theorem 1) and a local limit theorem for unconditioned random walks. Therefore, it works in all dimensions and all cones, where Theorem 1 holds. In particular, our method gives simple probabilistic proofs of local limit theorems for random walks conditioned to stay positive.

We next find asymptotic behavior of $\mathbf{P}(x + S(n) = y, \tau_x > n)$ for fixed y. [Note that Theorem 5 says only that this probability is $o(n^{-p/2-d/2})$.]

THEOREM 6. *Under the assumptions of the preceding theorem, for every fixed* $y \in K$,

$$\mathbf{P}\big(x + S(n) = y, \tau_x > n\big) \sim \varrho H_0^2 \frac{V(x)V'(y)}{n^{p+d/2}}, \tag{10}$$

where V' *is the harmonic function for the random walk* $\{-S(n)\}$ *and*

$$\varrho = \varkappa^2 \int_K u^2(w) e^{-|w|^2/2} \, dw.$$

Furthermore, for every $t \in (0, 1)$ *and any compact* $D \subset K$

$$\mathbf{P}\left(\frac{x + S([tn])}{\sqrt{n}} \in D \,\middle|\, \tau_x > n, x + S(n) = y\right) \to Q_t(D), \tag{11}$$

where Q_t is the measure on K with the density

$$\frac{1}{\rho(2\pi)^d}\frac{1}{(t(1-t))^{p+d/2}}u^2(z)e^{-|z|^2/2t(1-t)}\,dz.$$

In the next two subsections, we mention some interesting applications of this theorem.

1.5. *Application to lattice paths enumeration.* Starting from the classical ballot problem, counting of lattice paths confined to a certain domain, attracts a lot of attention. For lattice paths in Weyl chambers associated with reflection groups, one often uses a generalization of the classical reflection principle of Andre, which has been proved by Gessel and Zeilberger [26]. Unfortunately, the latter result can be applied only to some special random walks which are not allowed to jump over the boundary of the chamber. Additionally, the set of all possible steps should be invariant with respect to all reflections. Grabiner and Magyar [28] give the complete list of all random walks to which the reflection principle can be applied. Recently, the reflection principle of Gessel and Zeilberger has been slightly generalized by Feierl [21]. He derived a new version of the reflection principle for random walks with steps which are at most finite combinations of steps from the list of Grabiner and Magyar. Another very popular cone is the positive quadrant in $\mathbb{Z}^2$. Here, we mention papers of Bousquet-Melou [6] and of Bousquet-Melou and Mishna [7], where the authors obtained exact results for some random walks with bounded steps in the quarter plane. Raschel [40] and Kurkova and Raschel [34] also considered a two-dimensional random walk in the quarter plane, and proved some asymptotic results for the exit position. All these papers are based on the analytical approach suggested in the book of Fayolle, Iasnogorodski and Malyshev [19]. This method works for random walks on $\mathbb{Z}^2$ that can jump only to the nearest neighbors.

We next show how one can determine the asymptotic behavior of the number of walks with endpoints x and y confined to a cone from our results.

Consider lattice paths with the step set $\mathcal{S} = \{s_1, s_2, \dots, s_N\}$. We assume that the corresponding random walk on $\mathbb{Z}^d$ is strongly aperiodic.

If the vector sum of all s_i is not equal to zero, one has to perform the Cramer transformation with an appropriate parameter. For $R(h) = N^{-1}\sum_{i=1}^N e^{(h,s_i)}$, we set

$$\mathbf{P}(Y = s_i) = \frac{1}{NR(h)}e^{(h,s_i)}.$$

If there exists h_0 such that $\mathbf{E}Y = 0$, then we have the following formula for the number of walks with endpoints x and y:

$$N_n(x,y) = N^n\big(R(h_0)\big)^n e^{(h_0,x-y)}\mathbf{P}\Bigg(x + \sum_{k=1}^n Y(k) = y, \tau_x > n\Bigg).$$

It is clear that $\mathbf{E}Y = 0$ if and only if $R(h)$ attains its minimum at $h = h_0$. A necessary and sufficient condition for the existence of the global minimum for R is that the step set is not contained in a closed half-space.

There exists a linear transformation with matrix M such that $X = MY$ has uncorrelated coordinates and

$$\mathbf{P}\left(x + \sum_{k=1}^{n} Y(k) = y, \tau_x > n\right) = \mathbf{P}(Mx + S(n) = My, \tau_x > n).$$

Since the number of possible steps is finite, we have a random walk with bounded jumps. Therefore, we may use our results in any starlike cone; see Remark 4. Applying Theorem 6 to the random walk $S(n)$ and cone MK, we obtain

$$N_n(x, y) = C(x, y)\big(NR(h)\big)^n n^{-p-d/2}\big(1 + o(1)\big) \qquad \text{as } n \to \infty. \tag{12}$$

It is worth mentioning that not only $C(x, y)$ but also p may depend on h. This means that p depends not only on the cone K but also on the step set $\mathcal{S}$. An essential disadvantage of this approach is the fact that we cannot give an explicit expression for the function $C(x, y)$ and, therefore, we can only determine the rate of growth of $N_n(x, y)$. Nevertheless, for large values of x and y inside the cone one can obtain an approximation for $C(x, y)$ from the relation $V(x) \sim u(x)$.

We also note that upper bounds for $N_n(x, y)$ can be obtained from the estimates due to Varopoulos. It follows from (0.7.4) in [43] that $C(x, y)$ from (12) can be bounded from above by $Cu(x + x_0)u(y + x_0)$ with some appropriate x_0. An essential advantage of this bound consists in the fact that u is more accessible than V.

Finally, we mention that our derivation of (12) is purely probabilistic, since we use a strong approximation to prove Theorem 3. And it is not at all clear how to prove (12) by combinatorial methods. The only case known in the literature are random walks with small steps in the quarter plane: Fayolle and Raschel [20] derived a version of (12) by means of the kernel method.

1.6. *Random walks in Weyl chambers.* As it has already been mentioned, random walks in Weyl chambers have attracted a lot of attention in the recent past.

Let us first consider the chamber of type A, that is,

$$W_A := \{x \in \mathbb{R}^d : x_1 < x_2 < \cdots < x_d\}.$$

In this case, one has $u(x) = \prod_{i<j}(x_j - x_i)$ and $p = d(d-1)/2$. W_A is convex and u is harmonic on the whole space $\mathbb{R}^d$. Therefore, we may apply all our theorems to all random walks satisfying normalization and moment conditions. If one additionally assumes that the coordinates of X are exchangeable, or even independent, then $f(x) = 0$. This has been shown by König, O'Connell and Roch [32]. Therefore,

$$V(x) = u(x) - \mathbf{E}u\big(x + S(\tau_x)\big). \tag{13}$$

This form of the harmonic function has been suggested by Eichelsbacher and König [17]. It is worth mentioning that if the coordinates of X are independent, then the moment condition from the present paper is not optimal. It is shown in [16] that all the statements in Theorems 1 and 3 remain valid under the condition $\mathbf{E}|X|^{d-1} < \infty$. For two further Weyl chambers,

$$W_C := \{x \in \mathbb{R}^d : 0 < x_1 < x_2 < \cdots < x_d\}$$

and

$$W_D := \{x \in \mathbb{R}^d : |x_1| < x_2 < \cdots < x_d\}$$

and random walks with independent coordinates König and Schmid [33] have proven versions of Theorems 1 and 3 under moment conditions which are weaker than $\mathbf{E}|X|^p < \infty$. However, they have imposed an additional symmetry condition. More precisely, they have assumed that some odd moments of the distribution of coordinates are zero. This has been done in order to make $u(x + S(n))$ a martingale and $f(x) = 0$. As a result, they had the harmonic function of the form (13). One can verify that all the statements of [33] remain valid without the symmetry condition mentioned above, if one takes the harmonic function from our Theorem 1. (One has first to show that this function is well defined under the moment condition imposed by the authors of [33].)

We next note that if our random walk has independent coordinates, then Theorems 5 and 6 are valid for Weyl chamber under weaker moment assumptions. Indeed, as we have already mentioned, one needs an integral limit theorem for the conditioned random walk. Therefore, the moment conditions from [16] and [33] are sufficient for the validity of the local limit theorems. Applying (11) to the Weyl chamber of type A, we then see that the distribution of the excursion at time tn converges to the measure determined by the density

$$\frac{1}{\rho(2\pi)^d} \frac{1}{(t(1-t))^{p+d/2}} \left(\prod_{i<j} (z_j - z_i) \right)^2 e^{-|z|^2/2t(1-t)}\, dz, \qquad z \in W_A,$$

which is known to be the density of the distribution of eigenvalues in GUE. This result corresponds to Theorem 1 of Baik and Suidan [2].

1.7. *Description of our method.* In the one-dimensional case, we have only two cones: positive and negative half-axis. To determine the behavior of $\mathbf{P}(\tau_0 > n)$ in this classical case one uses the Wiener–Hopf factorization. For an arbitrary starting point x inside one of the half-axis, one has $\mathbf{P}(\tau_x > n) \sim H(x)\mathbf{P}(\tau_0 > n)$, where $H(x)$ is the renewal function based on ladder heights. And one can easily infer that $H(x)$ is harmonic. It is worth mentioning that the Wiener–Hopf method is quite powerful as one does not need to impose any moment conditions on the random walk.

Unfortunately, there are no general versions of the Wiener–Hopf factorization for the multidimensional case. We have found only two such attempts in the literature. First, Mogulskii and Pecherskii [37] proved some factorization identities for random walks on semigroups. However, it is not clear how to get asymptotics for exit times from these identities. The second one is the paper by Greenwood and Shaked [29], where a factorization over a family of two-dimensional cones is performed. As a consequence, the authors determined the asymptotic behavior of some special first passage times.

The first step in the proof of Theorem 1 consists in the construction of the harmonic function V; see Section 2. Here, we use the universality idea and construct V from the harmonic function u for the Brownian motion:

$$V(x) = \lim_{n\to\infty} \mathbf{E}\big[u\big(x + S(n)\big); \tau_x > n\big], \qquad x \in K.$$

An additional difficulty arises from the fact that although $u(x + B(t \wedge \tau_x^{\mathrm{bm}}))$ is a martingale the sequence $u(x + S(n \wedge \tau_x))$ is not. This explains correction terms $f(x+S(k))$ in Lemma 11. Another difficulty is that in general we have $x+S(\tau_x) \notin \partial K$ with positive probability. This is the reason for introducing the extendability condition on the cone K.

The second step of the proof is a coupling with the Brownian motion. Although this idea is quite natural, its naive application (starting from the beginning) gives only rough asymptotics:

$$\mathbf{P}(\tau_x > n) \sim n^{-p/2+o(1)}, \qquad n \to \infty.$$

To obtain exact asymptotics, one has to wait until the random walk moves far from the boundary of the cone. In Lemma 14, we show that this happens with a high probability. Then in Lemma 20 we couple out random walk with the Brownian motion using an extended version of Sakhanenko's coupling; see Lemma 17. This allows us to obtain the exact asymptotics when starting at y far from the boundary,

$$\mathbf{P}(\tau_y > n) \sim \varkappa u(y) n^{-p/2}.$$

Section 4 is the final step of the proof of Theorem 1. We use the Markov property at the first time ν_n when the random walk is far from the boundary and the formula we obtained from the coupling in Lemma 20. Informally, this results in

$$\begin{aligned}\mathbf{P}(\tau_x > n) &\approx \int \mathbf{P}(\tau_x > \nu_n, S_{\nu_n} \in dy)\mathbf{P}(\tau_y > n)\\ &\approx \varkappa \int \mathbf{P}(\tau_x > \nu_n, S_{\nu_n} \in dy) u(y) n^{-p/2}\\ &\approx \varkappa V(x) n^{-p/2}.\end{aligned}$$

These relations are proved in Lemmas 21, 24. Proof of Theorem 3 uses the same ideas.

It is worth mentioning that the method of constructing harmonic functions for random walks described above works also for Markov processes in discrete time. After the first version of the present paper was finished, we applied our approach to the following two problems.

First, in [15] we found asymptotics for $\mathbf{P}(\tau_{x,y} > n)$, where $\tau_{x,y} = \min\{n \ge 1 : x + ny + \sum_{k=1}^{n} S(k) \le 0\}$ and $S(k)$ is a driftless random walk with $\mathbf{E}|S(1)|^{2+\delta} < \infty$. The process $\sum_{k=1}^{n} S(k)$ is not Markovian, but one can obtain the Markov property increasing the dimension of the process. More precisely, $(\sum_{k=1}^{n} S(k), S(n))$ is a Markov chain, and consequently, $\tau_{x,y}$ becomes the exit time from the cone $\mathbb{R}_+ \times \mathbb{R}$.

Second, in [14], our joint paper with Dima Korshunov, we investigated the asymptotic behavior of the stationary distribution for a positive recurrent Markov chain on $\mathbb{R}_+$ with asymptotically zero drift. The crucial step was again a construction of harmonic functions for a chain killed at leaving an interval $[x_0, \infty)$, $x_0 > 0$.

Based on these two examples, we conjecture that our approach should work for a wide class of Markov chains, which converge, after an appropriate scaling, to diffusion processes.

2. Finiteness and positivity of V.

This section is devoted to the construction of the harmonic function V. We consider first the case $d \ge 2$. The one-dimensional case will be considered in Section 2.4.

2.1. *Finiteness*.

We first derive some properties of the functions $v(x)$ and $f(x)$.

LEMMA 7. *Let u be harmonic on $K^{4\varepsilon}$ and $|u(x)| \le c|x|^p$, $x \in K^{4\varepsilon}$. Then we have the following estimates for the derivatives*

$$
\begin{aligned}
|u_{x_i}| &\le C|x|^{p-1}, & x &\in K^{3\varepsilon},\\
|u_{x_i x_j}| &\le C|x|^{p-2}, & x &\in K^{2\varepsilon},\\
|u_{x_i x_j x_k}| &\le C|x|^{p-3}, & x &\in K^{\varepsilon}.
\end{aligned}
\tag{14}
$$

Here, and throughout the text we denote as C, c some generic constants.

PROOF OF LEMMA 7. Since u is harmonic on $K^{4\varepsilon}$ all its derivatives are harmonic as well. Let $y \in K^{3\varepsilon}$. It immediately follows from the definition of the cone $K^{4\varepsilon}$ that the ball $B(y, \eta|y|) \subset K^{4\varepsilon}$ for $\eta = \varepsilon/(1 + 3\varepsilon)$. Indeed, let x be such that $\operatorname{dist}(y, x) \le 3\varepsilon|x|$. Then, since $|y| \le (1 + 3\varepsilon)|x|$, for $z \in B(y, \eta|y|)$,

$$\operatorname{dist}(z, x) \le \operatorname{dist}(z, y) + \operatorname{dist}(y, x) < \eta|y| + 3\varepsilon|x| < 4\varepsilon|x|.$$

Hence, we can apply the mean-value formula for harmonic functions to function u_{x_i} and obtain

$$
\begin{aligned}
|u_{x_i}| &= \left| \frac{1}{\mathrm{Vol}(B(y, \eta|y|))} \int_{B(y,\eta|y|)} u_{x_i}\, dx \right| \\
&= \left| \frac{1}{|\eta|y||^d \alpha(d)} \int_{\partial B(y,\eta|y|)} u \nu_i \, ds \right| \\
&\le \frac{d\alpha(d)(\eta|y|)^{d-1}}{|\eta|y||^d \alpha(d)} \max_{x \in \partial B(y,\eta|y|)} u(x) \\
&\le c \frac{d}{\eta|y|} (1+\eta)^p |y|^p = c \frac{d(1+\eta)^p}{\eta} |y|^{p-1}.
\end{aligned}
$$

Here, $\alpha(d)$ is the volume of the unit ball and we used the Gauss–Green theorem. In the second line of the display, ν_i is the outer normal and integration takes place on the surface of the ball $B(y, \eta|y|)$.

The higher derivatives can be treated likewise. The claim of the lemma immediately follows. □

Next, we require a bound on $f(x)$.

LEMMA 8. *Let the assumptions of Lemma 7 hold and f be defined by* (4). *Let the moment and normalization assumptions hold. Then, for some $\delta > 0$,*

$$
|f(x)| \le C|x|^{p-2-\delta} \qquad \text{for all } x \in K \text{ with } |x| \ge 1.
$$

Furthermore,

$$
|f(x)| \le C \qquad \text{for all } x \in K \text{ with } |x| \le 1.
$$

PROOF. Let $x \in K$ be such that $|x| \ge 1$. Put $g(x) = |x|^{1-a}$, where we pick constant a later. Fix some $\eta \in (0, \varepsilon)$ satisfying $\eta + \eta^{1/(1-a)} \le 1$. Then, for any $y \in B(0, \eta g(x))$, the sum $x + y \in G$. By the Taylor theorem,

$$
\left| u(x+y) - u(x) - \nabla u \cdot y - \frac{1}{2} \sum_{i,j} u_{x_i x_j} y_i y_j \right| \le R_3(x)|y|^3.
$$

The remainder can be estimated by Lemma 7

$$
R_3(x) = \max_{z \in B(x, \eta g(x))} \max_{i,j,k} |u_{x_i x_j x_k}(z)| \le C(1+\eta)^{p-3}|x|^{p-3},
$$

which will give us

$$
\left| u(x+y) - u(x) - \nabla u \cdot y - \frac{1}{2} \sum_{i,j} u_{x_i x_j} y_i y_j \right| \le C|x|^{p-3}|y|^3. \tag{15}
$$

Since $v = u$ on G, we can proceed as follows:

$$\begin{aligned}|f(x)| &= \big|\mathbf{E}\big(u(x+X)-u(x)\big)1\big(|X|\le \eta g(x)\big)\big| \\ &\quad + \big|\mathbf{E}\big(v(x+X)-v(x)\big)1\big(|X|>\eta g(x)\big)\big| \\ &\le \bigg|\mathbf{E}\bigg[\bigg(\nabla u\cdot X+\frac{1}{2}\sum_{i,j}u_{x_ix_j}X_iX_j\bigg)1\big(|X|\le\eta g(x)\big)\bigg]\bigg| \\ &\quad + C|x|^{p-3}\mathbf{E}\big[|X|^3 1\big(|X|\le \eta g(x)\big)\big] \\ &\quad + C\mathbf{E}\big[|x|^p+\max\big(|X+x|^p,1\big)1\big(|X|>\eta g(x)\big)\big].\end{aligned}$$

Here we used also the bound $|v(z)|\le C\max\{1,|z|^p\}$.

After rearranging the terms, we obtain

$$\begin{aligned}\big|f(x)\big| &\le \bigg|\mathbf{E}\bigg[\nabla u\cdot X+\frac{1}{2}\sum_{i,j}u_{x_ix_j}X_iX_j\bigg]\bigg| \\ &\quad + \bigg|\mathbf{E}\bigg[\bigg(\nabla u\cdot X+\frac{1}{2}\sum_{i,j}u_{x_ix_j}X_iX_j\bigg)1\big(|X|>\eta g(x)\big)\bigg]\bigg| \\ &\quad + C|x|^{p-3}\mathbf{E}\big[|X|^3 1\big(|X|\le \eta g(x)\big)\big] \\ &\quad + C\mathbf{E}\big[\big(|x|^p+\max\big(|X+x|^p,1\big)\big)1\big(|X|>\eta g(x)\big)\big].\end{aligned}$$

Now note that the first term is 0 due to $\mathbf{E}X_i = 0$, $\operatorname{cov}(X_i, X_j) = 0$ and $\Delta u = 0$. The partial derivatives of the function v in the second term are estimated via Lemma 7. As a result,

$$\begin{aligned}\big|f(x)\big| \le C\big(&|x|^{p-1}\mathbf{E}\big[|X|;|X|>\eta g(x)\big]+|x|^{p-2}\mathbf{E}\big[|X|^2;|X|>\eta g(x)\big] \\ &+|x|^{p-3}\mathbf{E}\big[|X|^3;|X|\le\eta g(x)\big]+|x|^p\mathbf{P}\big(|X|>\eta g(x)\big) \\ &+\mathbf{E}\big[\max\big(|X|^p,1\big);|X|>\eta g(x)\big]\big).\end{aligned}$$

Hence, from the Markov inequality and

$$\mathbf{E}\big[\max\big(|X|^p,1\big);|X|>\eta g(x)\big]\le \mathbf{E}\big[|X|^p;|X|>\eta g(x)\big]+|x|^p\mathbf{P}\big(|X|>\eta g(x)\big)$$

we conclude

$$\begin{aligned}\big|f(x)\big| &\le C\frac{|x|^p}{\eta^2 g^2(x)}\mathbf{E}\big[|X|^2;|X|>\eta g(x)\big]+C|x|^{p-3}\mathbf{E}\big[|X|^3;|X|\le\eta g(x)\big] \\ &\quad + C\mathbf{E}\big[|X|^p;|X|>\eta g(x)\big].\end{aligned} \tag{16}$$

Now recall the moment assumption that $\mathbf{E}|X|^{2+2\delta}<\infty$ for some $\delta>0$. The first term is estimated via the Chebyshev inequality,

$$\frac{|x|^p}{\eta^2 g^2(x)}\mathbf{E}\big[|X|^2;|X|>\eta g(x)\big]\le\frac{|x|^p}{\eta^{2+2\delta}g^{2+2\delta}(x)}\mathbf{E}|X|^{2+2\delta}.$$

The second term can be estimated similarly,

$$|x|^{p-3}\mathbf{E}\big[|X|^3; |X| \le \eta g(x)\big] \le |x|^{p-3}\eta^{1-2\delta}g^{1-2\delta}(x)\mathbf{E}|X|^{2+2\delta}.$$

Choosing a sufficiently small, we see that the expectations in the first line of (16) are bounded by $C|x|^{p-2-\delta}$. In order to bound the last term in (16) we have to distinguish between $p \le 2$ and $p > 2$.

If $p \le 2$, then, by the Chebyshev inequality,

$$\mathbf{E}\big[|X|^p; |X| > \eta g(x)\big] \le \frac{1}{(\eta g(x))^{2+2\delta-p}}\mathbf{E}\big[|X|^{2+2\delta}\big] \le C|x|^{p-2-\delta}$$

for all a sufficiently small.

In case $p > 2$ we have, according to our moment condition, $\mathbf{E}[|X|^p] < \infty$. Consequently,

$$\mathbf{E}\big[|X|^p; |X| > \eta g(x)\big] \le C.$$

The second statement follows easily from the fact that $v(x)$ is bounded on $|x| \le 1$ and the inequality $\mathbf{E}[v(x+X)] \le C(1+\mathbf{E}[|X|^p])$. $\square$

LEMMA 9. *For any $x \notin K$,*

$$\big|v(x)\big| \le C\big(1+|x|^{p-a}\big).$$

PROOF. If $x \notin G$, then the inequality follows from the definition of v. Assume now that $x \in G \setminus K$. If $|x| \le 1$, then $|v(x)|$ is clearly bounded. But if $|x| > 1$, then $\mathrm{dist}(x, \partial K) \le |x|^{1-a}$. And it follows from the Taylor formula (recall that $v|_{\partial K} = 0$) and Lemma 7 that

$$\big|v(x)\big| \le C|x|^{p-1}\,\mathrm{dist}(x, \partial K) \le C|x|^{p-a}. \tag{17}$$

Thus, the proof is finished. $\square$

LEMMA 10. *For every $\beta < p$, we have*

$$\mathbf{E}\big[\tau_x^{\beta/2}\big] \le C\big(1+|x|^\beta\big) \tag{18}$$

and

$$\mathbf{E}\big[M^\beta(\tau_x)\big] \le C\big(1+|x|^\beta\big), \tag{19}$$

where $M(\tau_x) := \max_{k\le\tau_x}|x+S(k)|$.

This is the statement of Theorem 3.1 of [36]. One has only to notice that $e(\Gamma, R)$ in that theorem is denoted by p in our paper.

Next, we need to define an auxiliary process. Let

$$\begin{aligned} &Y_0 = v(x);\\ &Y_{n+1} = v\big(x+S(n+1)\big) - \sum_{k=0}^{n} f\big(x+S(k)\big), \qquad x \in K, n \ge 0. \end{aligned} \tag{20}$$

LEMMA 11. *The sequence Y_n defined in (20) is a martingale.*

PROOF. The integrability of the sequence Y_n is immediate from the bound $u(x) \le C|x|^p$ and from Lemmas 8 and 9. Further,

$$\begin{aligned}\mathbf{E}[Y_{n+1} - Y_n|\mathcal{F}_n] &= \mathbf{E}\big[\big(v(x+S(n+1)) - v(x+S(n)) - f(x+S(n))\big)|\mathcal{F}_n\big]\\ &= -f(x+S(n)) + \mathbf{E}\big[\big(v(x+S(n+1)) - v(x+S(n))\big)|S(n)\big]\\ &= -f(x+S(n)) + f(x+S(n)) = 0,\end{aligned}$$

where we used the definition of the function f in (4). □

LEMMA 12. *For sufficiently small $a > 0$, the function V from (5) is well defined. Furthermore,*

$$V(x) = \lim_{n\to\infty} \mathbf{E}\big[u(x+S(n)); \tau_x > n\big], \qquad x \in K. \tag{21}$$

This equality implies that V does not depend on the choice of a and ε in the definition of G.

PROOF. First, using (20) we obtain,

$$\begin{aligned}\mathbf{E}\big[v(x+S(n)); \tau_x > n\big] &= \mathbf{E}[Y_n; \tau_x > n] + \sum_{l=0}^{n-1} \mathbf{E}\big[f(x+S(l)); \tau_x > n\big]\\ &= \mathbf{E}Y_n - \mathbf{E}[Y_n; \tau_x \le n] + \sum_{l=0}^{n-1} \mathbf{E}\big[f(x+S(l)); \tau_x > n\big].\end{aligned}$$

Since Y_k is a martingale, $\mathbf{E}Y_n = \mathbf{E}Y_0 = v(x)$ and $\mathbf{E}[Y_n; \tau_x \le n] = \mathbf{E}[Y_{\tau_x}; \tau_x \le n]$. Using the definition of Y_n once again, we arrive at

$$\begin{aligned}\mathbf{E}\big[v(x+S(n)); \tau_x > n\big] &= v(x) - \mathbf{E}\big[v(x+S(\tau_x)), \tau_x \le n\big]\\ &\quad + \mathbf{E}\left[\sum_{l=0}^{\tau_x - 1} f(x+S(l)); \tau_x \le n\right]\\ &\quad + \sum_{l=0}^{n-1} \mathbf{E}\big[f(x+S(l)); \tau_x > n\big].\end{aligned}$$

Combining Lemmas 9 and 10, we obtain

$$\mathbf{E}\big|v(x+S(\tau_x))\big| \le \mathbf{E}M^{p-a}(\tau_x) \le C\big(1+|x|^{p-a}\big). \tag{22}$$

Then the dominated convergence theorem implies that

$$\mathbf{E}\big[v(x+S(\tau_x)), \tau_x \le n\big] \to \mathbf{E}v(x+S(\tau_x)). \tag{23}$$

To estimate the third and fourth terms, it is sufficient to prove that

$$\mathbf{E}\left[\sum_{l=0}^{\tau_x-1}|f(x+S(l))|\right]\le C(1+|x|^{p-\delta}). \tag{24}$$

Indeed, the dominated convergence theorem then implies that

$$\mathbf{E}\left[\sum_{l=0}^{\tau_x-1} f(x+S(l));\,\tau_x\le n\right]\to\mathbf{E}\left[\sum_{l=0}^{\tau_x-1} f(x+S(l))\right]$$

and

$$\left|\sum_{l=0}^{n-1}\mathbf{E}[f(x+S(l));\,\tau_x>n]\right|\le\mathbf{E}\left[\sum_{l=0}^{\tau_x-1}|f(x+S(l))|;\,\tau_x>n\right]\to 0$$

since τ_x is finite a.s.

Hence, it remains to prove (24). Consider first the case $p>2$. Assuming that $\delta<p-2$ and using Lemma 8, we get

$$\mathbf{E}\left[\sum_{l=0}^{\tau_x-1}|f(x+S(l))|\right]\le C\mathbf{E}[\tau_x M^{p-2-\delta}(\tau_x)].$$

Applying Hölder's inequality with $p'<p/2$ and $q'<p/(p-2-\delta)$ and Lemma 10, we obtain

$$\mathbf{E}\left[\sum_{l=0}^{\tau_x-1}|f(x+S(l))|\right]\le(\mathbf{E}\tau_x^{p'})^{1/p'}(\mathbf{E}M^{q'(p-2-\delta)}(\tau_x))^{1/q'}<C(1+|x|^{p-\delta}).$$

Such a choice of p' and q' is possible since $(p/2)^{-1}+(p/(p-2-\delta))^{-1}<1$. This proves (24) for $p>2$.

Next, consider the case $p\le 2$. We split the sum in (24) into four parts,

$$\begin{aligned}\mathbf{E}\left[\sum_{l=0}^{\tau_x-1}|f(x+S(l))|\right] &= f(x)+\sum_{l=1}^{\infty}\mathbf{E}[|f(x+S(l))|;\,\tau_x>l]\\ &= f(x)+\sum_{l=1}^{\infty}\mathbf{E}[|f(x+S(l))|;\,|x+S(l)|\le 1,\tau_x>l]\\ &\quad+\sum_{l=1}^{\infty}\mathbf{E}[|f(x+S(l))|;\,1<|x+S(l)|\le\sqrt{l},\tau_x>l]\\ &\quad+\sum_{l=1}^{\infty}\mathbf{E}[|f(x+S(l))|;\,|x+S(l)|>\sqrt{l},\tau_x>l]\\ &=: f(x)+\Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

According to Theorem 6.2 of [18],

$$\sup_{z\in\mathbb{R}^d} \mathbf{P}(|S(n)-z|\le 1)\le Cn^{-d/2}. \tag{25}$$

By Lemma 8, $|f(y)|\le C$ for $|y|\le 1$. From this bound and (25), we obtain

$$\begin{aligned}\Sigma_1 &\le C\sum_{l=1}^{\infty}\mathbf{P}(|x+S(l)|\le 1, \tau_x>l)\\ &\le C\sum_{l=1}^{\infty}\mathbf{P}(\tau_x>l/2)\sup_y \mathbf{P}(|y+S(l/2)|\le 1)\\ &\le C\sum_{l=1}^{\infty} l^{-d/2}\mathbf{P}(\tau_x>l/2)\\ &\le C\mathbf{E}[\tau_x^{(p-\delta)/2}]\sum_{l=1}^{\infty} l^{-d/2-(p-\delta)/2}\le C(1+|x|^{p-\delta}),\end{aligned}$$

where the sum is convergent due to $d\ge 2$.

Second, by Lemma 8,

$$\begin{aligned}\Sigma_2 &\le C\sum_{l=1}^{\infty}\mathbf{E}[|x+S(l)|^{p-2-\delta}; 1\le|x+S(l)|\le\sqrt{l}, \tau_x>l]\\ &\le C\sum_{l=1}^{\infty}\sum_{j=1}^{\sqrt{l}}\mathbf{E}[|x+S(l)|^{p-2-\delta}; j\le|x+S(l)|\le j+1, \tau_x>l]\\ &\le C\sum_{l=1}^{\infty}\sum_{j=1}^{\sqrt{l}} j^{p-2-\delta}\mathbf{P}(j\le|x+S(l)|\le j+1, \tau_x>l).\end{aligned}$$

Now we note that

$$\mathbf{P}(j\le|x+S(l)|\le j+1, \tau_x>l)\le \mathbf{P}(\tau_x>l/2)\sup_y\mathbf{P}(|y+S(l/2)|\in[j, j+1]).$$

Covering the region $\{z:|z|\in[j,j+1]\}$ by Cj^{d-1} unit balls and using (25), we get

$$\mathbf{P}(j\le|x+S(l)|\le j+1, \tau_x>l)\le Cj^{d-1}l^{-d/2}\mathbf{P}(\tau_x>l/2).$$

Then

$$\begin{aligned}\Sigma_2 &\le C\sum_{l=1}^{\infty}\sum_{j=1}^{\sqrt{l}} j^{p-2-\delta}j^{d-1}l^{-d/2}\mathbf{P}(\tau_x>l/2)\\ &\le C\sum_{l=1}^{\infty} l^{p/2-1-\delta/2}\mathbf{P}(\tau_x>l/2)\\ &\le C\mathbf{E}[\tau_x^{(p-\delta)/2}]\le C(1+|x|^{p-\delta}),\end{aligned}$$

by Lemma 10.

Third, by Lemma 8 and the fact that $p \le 2$,

$$\Sigma_3 \le C \sum_{l=1}^{\infty} \mathbf{E}\big[|x+S(l)|^{p-2-\delta}; |x+S(l)| > \sqrt{l}, \tau_x > l\big]$$
$$\le C \sum_{l=1}^{\infty} l^{(p-2-\delta)/2} \mathbf{P}(\tau_x > l)$$
$$\le C\mathbf{E}\big[\tau_x^{(p-\delta)/2}\big] \le C\big(1+|x|^{p-\delta}\big). \qquad \square$$

2.2. *Positivity.* In this paragraph, we show that V is strictly positive on K_+ and prove some further properties of this function.

LEMMA 13. *The function V possesses the following properties*:

(a) *For any $\gamma > 0$, $R > 0$, uniformly in $x \in D_{R,\gamma}$ we have $V(tx) \sim u(tx)$ as $t \to \infty$.*

(b) *For all $x \in K$, we have $V(x) \le C(1+|x|^p)$.*

(c) *The function V is harmonic for the killed random walk*, *that is*,

$$V(x) = \mathbf{E}\big[V\big(x+S(n_0)\big), \tau_x > n_0\big], \qquad x \in K, n_0 \ge 1.$$

(d) *The function V is strictly positive on K_+.*

(e) *If $x \in K$, then $V(x) \le V(x+x_0)$, for all x_0 such that $x_0 + K \subset K$.*

PROOF. To prove the part (a), it suffices to note that $t^{-p}u(tx) = u(x)$, $\inf_{x\in D_{R,\gamma}} u(x) > 0$, and use bounds (22), (24). These inequalities together with $|u(x)| \le C|x|^p$ give the part (b).

It suffices to prove (c) for $n_0 = 1$, since for bigger values of n_0 one can then use the Markov property of $S(n)$. It is clear that

$$\mathbf{E}\big[u\big(x+S(n+1)\big), \tau_x > n+1\big]$$
$$= \int_K \mathbf{P}\big(x+S(1) \in dy, \tau_x > 1\big)\mathbf{E}\big[u\big(y+S(n)\big), \tau_y > n\big].$$

According to Lemma 12, $\mathbf{E}[u(y+S(n)), \tau_y > n] \to V(y)$ for every $y \in K$. Furthermore, it follows from (22), (24) that $\mathbf{E}[u(y+S(n)), \tau_y > n] \le C(1+|y|^p)$. This allows one to apply the dominated convergence theorem, which gives

$$V(x) = \lim_{n\to\infty} \mathbf{E}\big[u\big(x+S(n+1)\big), \tau_x > n+1\big] = \int_K \mathbf{P}\big(x+S(1) \in dy, \tau_x > 1\big)V(y).$$

To prove the positivity of $V(x)$, assume that $x \in K_+$. Then for every $R > 0$ there exists $n_0 = n_0(R)$ such that $\mathbf{P}(x+S(n_0) \in D_{R,\gamma}, \tau_x > n_0) > 0$ with some

$\gamma = \gamma(x)$. According to the first part of the lemma, there exist $R > 0$ such that $\inf_{y\in D_{R,\gamma}} V(y) > 0$. Consequently,

$$\begin{aligned} V(x) &= \mathbf{E}\bigl[V\bigl(x+S(n_0)\bigr); \tau_x > n_0\bigr] \\ &\ge \mathbf{E}\bigl[V\bigl(x+S(n_0)\bigr), x+S(n_0)\in D_{R,\gamma}, \tau_x > n_0\bigr] > 0. \end{aligned}$$

To prove (e), we first show that the same property holds for $u(x)$. Indeed, if x_0 is such that $x_0 + K \subset K$, then

$$\{\tau_x^{\mathrm{bm}} > t\} \subset \{\tau_{x+x_0}^{\mathrm{bm}} > t\} \qquad \text{for all } x\in K, t>0.$$

Then, in view of (1),

$$\varkappa u(x) = \lim_{t\to\infty} t^{p/2}\mathbf{P}\bigl(\tau_x^{\mathrm{bm}} > t\bigr) \le \lim_{t\to\infty} t^{p/2}\mathbf{P}\bigl(\tau_{x+x_0}^{\mathrm{bm}} > t\bigr) = \varkappa u(x+x_0).$$

Applying now Lemma 12, we get

$$\begin{aligned} V(x) &= \lim_{n\to\infty} \mathbf{E}\bigl[u\bigl(x+S(n)\bigr); \tau_x > n\bigr] \\ &\le \lim_{n\to\infty} \mathbf{E}\bigl[u\bigl(x+x_0+S(n)\bigr); \tau_{x+x_0} > n\bigr] = V(x+x_0). \end{aligned}$$

Thus, the proof is finished. □

2.3. *An alternative construction of a harmonic function for random walks with bounded jumps*. In this paragraph, we show that V remains well-defined and a strictly positive harmonic function for random walks with bounded jumps if we take $v(x) = u(x+x_*)$.

Assume that $\mathbf{P}(|X|\le R) = 1$ and let x_* satisfy the condition

$$\operatorname{dist}(x,\partial K) > R \qquad \text{for every } x\in K_* := x_* + K.$$

(One can choose $x_* = t_* x_0$ with sufficiently large t_*.) Therefore, $f(x) = \mathbf{E}v(x+X) - v(x)$ is well defined and the statement of Lemma 8 is valid with $\delta = 1$. This implies, by the same arguments as in the proof of Lemma 12, that

$$\mathbf{E}\Biggl[\sum_{l=1}^{\tau_x - 1} \bigl|f\bigl(x+S(l)\bigr)\bigr|\Biggr] < C\bigl(1+|x|^{p-\delta}\bigr).$$

To show that $v(x+S(\tau_x))$ is integrable, we assume that

$$u(x) \le C|x|^{p-\delta}\operatorname{dist}(x,\partial K).$$

(If K is convex, then this inequality holds with $\delta = 1$, see [43], formula (0.2.3).) Since $\operatorname{dist}(x_* + x + S(\tau_x), \partial K)$ is bounded, then in view of Lemma 10

$$\mathbf{E}v\bigl(x+S(\tau_x)\bigr) \le C\mathbf{E}\bigl|x+S(\tau_x)\bigr|^{p-\delta} < C\bigl(1+|x|^{p-\delta}\bigr).$$

Thus, V is well defined. Repeating the derivation of (21), we obtain

$$V(x) = \lim_{n\to\infty} \mathbf{E}\bigl[u\bigl(x+x_0+S(n)\bigr); \tau_x > n\bigr].$$

This relation implies that V is harmonic. The positivity follows from Lemma 13.

Formally, V might depend on x_*. But one can show, using the coupling with the Brownian motion from the next section, that V is independent of x_*. It is sufficient to note that one can replace $u(y) \sim u(x_* + y)$ under the conditions of Lemma 20 below.

2.4. *Construction of harmonic function in the one-dimensional case*. If $d = 1$, then $K = (0, \infty)$. Random walks confined to the positive half-line are well studied in the literature. The main tool is the Wiener–Hopf factorization. This method allows one to construct the harmonic function for any oscillating random walk. It turns out that the ladder heights renewal function is harmonic for $S(n)$ killed at leaving $(0, \infty)$.

For the sake of completeness, we indicate how our method works for one-dimensional random walks.

The harmonic function for the killed Brownian motion is $u(x) = x 1_{\mathbb{R}_+}(x)$. We extend it to a harmonic function on the whole axis by putting $u(x) = x$, $x \in \mathbb{R}$. Since $u(x + S(n))$ is a martingale, the corrector function $f \equiv 0$. Therefore,

$$V(x) = u(x) - \mathbf{E}\big[u\big(x + S(\tau_x)\big)\big].$$

This function is strictly positive on K. It is well defined provided that the expectation $\mathbf{E}|u(x + S(\tau_x))|$ is finite. The latter property can be shown by constructing an appropriate positive supermartingale. Namely, put

$$h(x) = \begin{cases} (R + x)^{1-a}, & x > 0, \\ |x|, & x \le 0. \end{cases}$$

Then, after some computations, one can show that for sufficiently large R and sufficiently small a the process $h(x + S(n \wedge \tau_x))$ is a positive supermartingale provided $\mathbf{E}|X(1)|^{2+\delta} < \infty$ for some $\delta > 0$. Hence, by the optional stopping theorem

$$\mathbf{E}\big|x + S(\tau_x)\big| \le (R + x)^{1-a}.$$

This shows the finiteness of $\mathbf{E}|u(x + S(\tau_x))|$. In addition, this estimate implies that $V(x) \sim x$ as $x \to \infty$.

3. Coupling.

Let $\varepsilon > 0$ be a constant and let

$$K_{n,\varepsilon} = \big\{x \in K : \operatorname{dist}(x, \partial K) \ge n^{1/2-\varepsilon}\big\}. \tag{26}$$

Define

$$\nu_n := \min\big\{k \ge 1 : x + S(k) \in K_{n,\varepsilon}\big\}.$$

LEMMA 14. *There exists a positive constant C such that, for every $\varepsilon > 0$,*

$$\mathbf{P}\big(\nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}\big) \le \exp\{-Cn^{\varepsilon}\}.$$

PROOF. Set, for brevity, $b_n = [n^{1/2-\varepsilon}]$, where a is a positive number. Clearly,

$$\begin{aligned}
&\mathbf{P}(\nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}) \\
&\quad \le \mathbf{P}(x + S(b_n^2), x + S(2b_n^2), \dots, x + S([n^\varepsilon]b_n^2) \in K \setminus K_{n,\varepsilon}) \\
&\quad \le \Bigl(\sup_{y \in K \setminus K_{n,\varepsilon}} \mathbf{P}(y + S(b_n^2) \in K \setminus K_{n,\varepsilon})\Bigr)^{[n^\varepsilon]}.
\end{aligned}$$

It follows from the scaling property of the cone that

$$\sup_{y \in K \setminus K_{n,\varepsilon}} \mathbf{P}(y + S(b_n^2) \in K \setminus K_{n,\varepsilon}) = \sup_{y \in K \setminus K_{1,\varepsilon}} \mathbf{P}\biggl(y + \frac{S(b_n^2)}{n^{1/2-\varepsilon}} \in K \setminus K_{1,\varepsilon}\biggr).$$

Therefore, it is sufficient to show that the right-hand side is separated from 1. To this end, recall that there exists x_0 with $|x_0| = 1$ such that $x_0 + K \subset K$ and $\operatorname{dist}(x_0 + K, \partial K) > 0$. Then, by the scaling property of cones, for sufficiently large t_0, the distance $\operatorname{dist}(t_0 x_0 + K, \partial K) \ge 1$. Hence, $t_0 x_0 + K \subset K_{1,\varepsilon}$.

Let $d(x) = \operatorname{dist}(x + K, K^c)$. Since the boundary of the cone is continuous this function is continuous. We assumed that $d(x_0) > 0$. Therefore, the set $K_0 = \{x : d(x) > 0\}$ is open and nonempty. Since K is a cone, the set K_0 is a cone as well. Since $K_0 + K \subset K$, we have $y + K_0 \subset K$ for every $y \in K$. Consequently,

$$t_0 x_0 + y + K_0 \subset t_0 x_0 + K \subset K_{1,\varepsilon}$$

for all $y \in K$. This relation yields

$$\sup_{y \in K \setminus K_{1,\varepsilon}} \mathbf{P}\biggl(y + \frac{S(b_n^2)}{n^{1/2-\varepsilon}} \in K \Bigm\backslash K_{1,\varepsilon}\biggr) \le 1 - \mathbf{P}\biggl(\frac{S(b_n^2)}{n^{1/2-\varepsilon}} \in t_0 x_0 + K_0\biggr).$$

Further, by the central limit theorem,

$$\lim_{n \to \infty} \mathbf{P}\biggl(\frac{S(b_n^2)}{n^{1/2-\varepsilon}} \in t_0 x_0 + K_0\biggr) = \mathbf{P}(B(1) \in t_0 x_0 + K_0).$$

Since K_0 is open, the probability $\mathbf{P}(B(1) \in t_0 x_0 + K_0)$ is strictly positive. This completes the proof of the lemma. □

REMARK 15. We used in the proof of the last lemma that every convex cone is starlike. Now we prove this fact. Fix some $x_0 \in \Sigma$. Then, due to convexity, $x_0 + K \subset K$. Assume that there exists $y \in K$ such that $\operatorname{dist}(x_0 + y, \partial K) < \operatorname{dist}(x_0, \partial K)$. Let $x \in \partial K$ satisfy $\operatorname{dist}(x_0 + y, \partial K) = \operatorname{dist}(x_0 + y, x)$. Using the convexity of K once again, we see that there exists a hyperplane $H(x)$ such that $H(x) \cap K = \varnothing$ and $\operatorname{dist}(x_0 + y, H(x)) < \operatorname{dist}(x_0, \partial K)$. But then $\operatorname{dist}(x_0 + y, H(x)) < \operatorname{dist}(x_0, H(x))$, and this implies that the half-line $\{x_0 + ty, y > 0\}$ cuts $H(x)$ and leaves the cone K, what contradicts the fact that $x_0 + K \subset K$.

LEMMA 16. *For every* $\varepsilon > 0$ *the inequality*

$$\mathbf{E}[u(x+S(n^{1-\varepsilon})); \nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}] \le C(x)\exp\{-Cn^{c_\varepsilon}\}$$

holds.

PROOF. Since $\nu_n > n^{1-\varepsilon}$ and $\tau_x > n^{1-\varepsilon}$,

$$\operatorname{dist}(x+S(n^{1-\varepsilon}), \partial K) \le n^{1/2-\varepsilon}.$$

Therefore, applying the Taylor formula (and recalling that u vanishes on the boundary), we obtain

$$\begin{aligned} u(x+S(n^{1-\varepsilon})) &\le C|x+S(n^{1-\varepsilon})|^{p-1}\operatorname{dist}(x+S(n^{1-\varepsilon}), \partial K) \\ &\le C|x+S(n^{1-\varepsilon})|^{p-1} n^{1/2-\varepsilon}. \end{aligned}$$

Hence, by the Hölder inequality,

$$\begin{aligned} &\mathbf{E}[u(x+S(n^{1-\varepsilon})); \nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}] \\ &\quad\le Cn^{1/2-\varepsilon}\mathbf{E}[|x+S(n^{1-\varepsilon})|^{p-1}; \nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}] \\ &\quad\le Cn^{1/2}\mathbf{E}[|x+S(n^{1-\varepsilon})|^{p}]^{(p-1)/p}\mathbf{P}(\nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon})^{1/p}. \end{aligned}$$

An application of Lemma 14 and a classical martingale bound

$$\mathbf{E}|S(n^{1-\varepsilon})|^p \le Cn^{p/2}$$

gives the required exponential bound. □

We start by formulating an estimate of the quality of the normal approximation of high-dimensional random walks which follows from a result of Gëttse and Zaitsev [27]; see Theorem 4 there.

LEMMA 17. *If* $\mathbf{E}|X|^{2+\delta} < \infty$ *for some* $\delta \in (0,1)$, *then one can define a random walk with the same distribution as* $S(n)$ *and a Brownian motion* $B(t)$ *on the same probability space such that*, *for any* γ *satisfying* $0 < \gamma < \frac{\delta}{2(2+\delta)}$,

$$\mathbf{P}\Big(\sup_{u\le n}|S([u]) - B(u)| \ge n^{1/2-\gamma}\Big) \le Cn^{2\gamma+\gamma\delta-\delta/2}. \tag{27}$$

PROOF. According to Theorem 4 and (1.13) of [27], one can construct on a joint probability space a copy of $S(n)$ and a standard Gaussian random walk $W(n)$ satisfying

$$\begin{aligned} &\mathbf{P}\Big(\max_{k\le n}|S(k) - W(k)| \ge \frac{1}{2}n^{1/2-\gamma}\Big) \\ &\quad\le C\Big(\frac{1}{2}n^{1/2-\gamma}\Big)^{-(2+\delta)} n\mathbf{E}|X(1)|^{2+\delta} \\ &\quad\le Cn^{2\gamma+\gamma\delta-\delta/2}. \end{aligned}$$

But, in view of the classical Lévy construction of the Brownian motion, we may assume that there is a Brownian motion $B(t)$ on the same probability space with the property $B(k) = W(k), k \ge 0$. Therefore,

$$\mathbf{P}\Big(\max_{k\le n}|S(k)-B(k)| \ge \frac{1}{2}n^{1/2-\gamma}\Big) \le Cn^{2\gamma+\gamma\delta-\delta/2}. \tag{28}$$

Moreover,

$$\begin{aligned}\mathbf{P}\Big(\sup_{u\le n}\big|B(u)-B([u])\big| \ge \frac{1}{2}n^{1/2-\gamma}\Big) &\le n\mathbf{P}\Big(\sup_{t\le 1}|B(t)| \ge \frac{1}{2}n^{1/2-\gamma}\Big)\\ &\le dn\mathbf{P}\Big(\sup_{t\le 1}|B_1(t)| \ge \frac{1}{2\sqrt{d}}n^{1/2-\gamma}\Big)\\ &\le \frac{4dn}{\sqrt{2\pi}}\int_{n^{1/2-\gamma}/2\sqrt{d}}^{\infty} e^{-u^2/2}\,du.\end{aligned} \tag{29}$$

In the last step, we used the reflection principle and the bound

$$\mathbf{P}\Big(\sup_{t\le 1}|B_1(t)| \ge x\Big) \le 2\mathbf{P}\Big(\sup_{t\le 1}B_1(t) \ge x\Big).$$

By the triangle inequality,

$$\begin{aligned}\mathbf{P}\Big(\sup_{u\le n}\big|S([u])-B(u)\big| \ge n^{1/2-\gamma}\Big) &\le \mathbf{P}\Big(\max_{k\le n}|S(k)-B(k)| \ge \frac{1}{2}n^{1/2-\gamma}\Big)\\ &\quad + \mathbf{P}\Big(\sup_{u\le n}\big|B(u)-B([u])\big| \ge \frac{1}{2}n^{1/2-\gamma}\Big).\end{aligned}$$

Applying (28) and (29), we complete the proof. □

LEMMA 18. *There exists a finite constant C such that*

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \le C\frac{|x|^p}{t^{p/2}}, \qquad x \in K. \tag{30}$$

Moreover,

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \sim \varkappa\frac{u(x)}{t^{p/2}}, \tag{31}$$

uniformly in $x \in K$ satisfying $|x| \le \theta_t\sqrt{t}$ with some $\theta_t \to 0$. Finally, the density $b_t(x,z)$ of the probability $\mathbf{P}(\tau_x^{\mathrm{bm}} > t, x + B(t) \in dz)$ is

$$b_t(x,z) \sim \varkappa_0 t^{-d/2}e^{-|z|^2/(2t)}u(x)u(z)t^{-p} \tag{32}$$

uniformly in $x, z \in K$ satisfying $|x| \le \theta_t\sqrt{t}$ and $|z| \le \sqrt{t/\theta_t}$ with some $\theta_t \to 0$.

These statements can be derived from estimates in [3].

PROOF OF LEMMA 18. According to Theorem 1 of [3],

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) = \sum_{j=1}^{\infty} B_j \left(\frac{|x|^2}{2t}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + \frac{d}{2}, \frac{-|x|^2}{2t}\right) m_j\left(\frac{x}{|x|}\right), \tag{33}$$

where

$$a_j := \sqrt{\lambda_j + \left(\frac{d}{2} - 1\right)^2} - \frac{d}{2} + 1$$

and

$$B_j := \frac{\Gamma((a_j + d)/2)}{\Gamma(a_j + d/2)} \int_\Sigma m_j(\theta)\, d\theta.$$

By the definition,

$${}_1F_1(a, b, z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots. \tag{34}$$

Then, for all $x \in K$ with $|x|^2 \le t$, we have

$${}_1F_1\left(\frac{a_j}{2}, a_j + \frac{d}{2}, \frac{-|x|^2}{2t}\right) \le e^{|x|^2/2t} \le e^{1/2}.$$

Furthermore, in view of Lemma 5 of [3],

$$\begin{aligned} |m_j(\theta)| &\le \frac{C}{\sqrt{I_{a_j - 1 + d/2}(1)}} m_1(\theta) \\ &\le C 2^{a_j/2} \sqrt{\Gamma(a_j + d/2)}\, m_1(\theta), \qquad \theta \in \Sigma, \end{aligned} \tag{35}$$

where $I_\nu(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} (x/2)^{\nu+2m}$ is the modified Bessel function. Applying (34) and (35) to the corresponding terms in (33), we obtain

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \le C m_1\left(\frac{x}{|x|}\right) \sum_{j=1}^{\infty} B_j 2^{a_j/2} \sqrt{\Gamma(a_j + d/2)} \left(\frac{|x|^2}{2t}\right)^{a_j/2}.$$

Using the Stirling formula and (2.3) from [3], one can easily get

$$B_j 2^{a_j/2} \sqrt{\Gamma(a_j + d/2)} \le C \lambda_j^{d/4}.$$

Consequently,

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \le C m_1\left(\frac{x}{|x|}\right) \sum_{j=1}^{\infty} \lambda_j^{d/4} \left(\frac{|x|^2}{2t}\right)^{a_j/2}.$$

According to the Weyl asymptotic formula, see [11], page 172,

$$cj^{2/(d-1)} \le \lambda_j \le Cj^{2/(d-1)}.$$

This implies that

$$\sum_{j=1}^{\infty} \lambda_j^{d/4} \left(\frac{|x|^2}{2t}\right)^{a_j/2} \le C\left(\frac{|x|^2}{2t}\right)^{a_1/2}$$

for all x satisfying $|x|^2 \le t$. Therefore,

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \le Cm_1\left(\frac{x}{|x|}\right)\left(\frac{|x|^2}{2t}\right)^{a_1/2} = C\frac{u(x)}{t^{p/2}}, \qquad |x|^2 \le t. \tag{36}$$

This immediately implies that (30) holds.

The same arguments give also

$$\sum_{j=2}^{\infty} B_j \left(\frac{|x|^2}{2t}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + \frac{d}{2}, \frac{-|x|^2}{2t}\right) m_j\left(\frac{x}{|x|}\right) \le Cm_1\left(\frac{x}{|x|}\right)\left(\frac{|x|^2}{2t}\right)^{a_2/2}.$$

Since $a_2 > a_1$,

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \sim B_1\left(\frac{|x|^2}{2t}\right)^{a_1/2} {}_1F_1\left(\frac{a_1}{2}, a_1 + \frac{d}{2}, \frac{-|x|^2}{2t}\right) m_1\left(\frac{x}{|x|}\right)$$

uniformly in $|x| \le \theta_t\sqrt{t}$. Noting that ${}_1F_1(\frac{a_1}{2}, a_1 + \frac{d}{2}, \frac{-|x|^2}{2t}) \to 1$ uniformly in $|x| \le \theta_t\sqrt{t}$, we get (31).

According to Lemma 1 from [3],

$$b_t(x,z) = \frac{e^{-(|x|^2+|z|^2)/2t}}{t|x|^{d/2-1}|z|^{d/2-1}} \sum_{j=1}^{\infty} I_{a_j-1+d/2}\left(\frac{|x||z|}{t}\right) m_j\left(\frac{x}{|x|}\right) m_j\left(\frac{z}{|z|}\right).$$

From the assumptions $|x| \le \theta_t\sqrt{t}$ and $|z| \le \sqrt{t/\theta_t}$, we get uniform convergence as $\frac{|x||z|}{t} \to 0$. Recalling the definition of the Bessel functions and using (35), we obtain

$$b_t(x,z) \sim \frac{1}{\Gamma(a_1 + d/2)} \frac{e^{-(|x|^2+|z|^2)/2t}}{t|x|^{d/2-1}|z|^{d/2-1}} \left(\frac{|x||z|}{2t}\right)^{a_1-1+d/2} m_1\left(\frac{x}{|x|}\right) m_1\left(\frac{z}{|z|}\right)$$

uniformly in $|x| \le \theta_t\sqrt{t}$ and $|z| \le \sqrt{t/\theta_t}$. Simplifying this expression, and recalling the definitions of p and u, we get

$$b_t(x,z) \sim \kappa_0 u(x)u(z)e^{-(|x|^2+|z|^2)/2t} t^{-p-d/2}.$$

Noting that $e^{-|x^2|/2t} \to 1$, we obtain (32). $\square$

LEMMA 19. *If K is convex then there exists a finite constant C such that*

$$u(y) \ge C\big(\mathrm{dist}(y, \partial K)\big)^p, \qquad y \in K.$$

If K is starlike and C^2, then

$$u(y) \ge C|y|^{p-1}\,\mathrm{dist}(y, \partial K), \qquad y \in K.$$

PROOF. It is clear that

$$\{\tau_y^{\mathrm{bm}} > t\} \supset \Big\{\sup_{s\le t}|B(s)| < \mathrm{dist}(y, \partial K)\Big\}.$$

Using the scaling property, we obtain

$$\mathbf{P}(\tau_y^{\mathrm{bm}} > t) \ge \mathbf{P}\bigg(\sup_{s\le 1}|B(s)| < \frac{\mathrm{dist}(y, \partial K)}{\sqrt{t}}\bigg).$$

If K is convex, then it has been proved in [43], see Theorem 1 and (0.4.1) there, that

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \le C\frac{u(x)}{t^{p/2}}, \qquad x \in K, t > 0. \tag{37}$$

Using this bound with $t = (\mathrm{dist}(y, \partial K))^2$, we get

$$\frac{u(y)}{(\mathrm{dist}(y, \partial K))^p} \ge C\mathbf{P}\big(\tau_y^{\mathrm{bm}} > (\mathrm{dist}(y, \partial K))^2\big) \ge C\mathbf{P}\Big(\sup_{s\le 1}|B(s)| < 1\Big).$$

Thus, the first statement is proved. The second one follows easily from (0.2.1) in [43]. □

Using the coupling, we can translate the results of Lemma 18 to the random walks setting when $y \in K_{n,\varepsilon}$.

LEMMA 20. *For all sufficiently small $\varepsilon > 0$,*

$$\mathbf{P}(\tau_y > n) = \varkappa u(y)n^{-p/2}\big(1 + o(1)\big) \qquad \text{as } n \to \infty \tag{38}$$

uniformly in $y \in K_{n,\varepsilon}$ such that $|y| \le \theta_n\sqrt{n}$ for some $\theta_n \to 0$. Moreover, there exists a constant C such that

$$\mathbf{P}(\tau_y > n) \le C\frac{|y|^p}{n^{p/2}}, \tag{39}$$

uniformly in $y \in K_{n,\varepsilon}, n \ge 1$. Finally, for any compact set $D \subset K$,

$$\mathbf{P}\big(\tau_y > n, y + S(n) \in \sqrt{n}D\big) \sim \varkappa_0 u(y)n^{-p/2}\int_D dz\, e^{-|z|^2/2}u(z) \tag{40}$$

uniformly in $y \in K_{n,\varepsilon}$ such that $|y| \le \theta_n\sqrt{n}$ for some $\theta_n \to 0$.

PROOF. For every $y \in K_{n,\varepsilon}$ denote

$$y^{\pm} = y \pm R_0 x_0 n^{1/2-\gamma},$$

where x_0 is such that $|x_0| = 1$, $x_0 + K \subset K$ and R_0 is such that $\mathrm{dist}(R_0x_0 + K, \partial K) > 1$. Note also that this choice of R_0 ensures that $R_0 x_0 n^{1/2-\gamma} \subset K_{n,\gamma}$.

If we take $\gamma > \varepsilon$, then for any $\varepsilon' > \varepsilon$ there exists $n(\varepsilon')$ such that $y^{\pm} \in K_{n,\varepsilon'}$ as soon as $n \ge n(\varepsilon')$ and $y \in K_{n,\varepsilon}$.

Define

$$A_n = \Big\{\sup_{u\le n}\big|S([u]) - B(u)\big| \le n^{1/2-\gamma}\Big\},$$

where B is the Brownian motion constructed in Lemma 17. The choice of R_0 ensures that $\tau^{\mathrm{bm}}_{y^+} > n$ on the set $\{\tau_y > n\} \cap A_n$. Then, using (27), we obtain

$$\begin{aligned}\mathbf{P}(\tau_y > n) &= \mathbf{P}(\tau_y > n, A_n) + o(n^{-r}) \\ &\le \mathbf{P}(\tau^{\mathrm{bm}}_{y^+} > n) + o(n^{-r}),\end{aligned} \tag{41}$$

where $r = r(\delta, \gamma) = \delta/2 - 2\gamma - \gamma\delta$. In the same way, one can get

$$\mathbf{P}(\tau^{\mathrm{bm}}_{y^-} > n) \le \mathbf{P}(\tau_y > n) + o(n^{-r}). \tag{42}$$

If $|y| \le \theta_n\sqrt{n}$, then $|y^{\pm}| \le \theta_n\sqrt{n} + R_0x_0n^{1/2-\gamma} = \theta_n'\sqrt{n}$. Therefore, by Lemma 18,

$$\mathbf{P}(\tau^{\mathrm{bm}}_{y^\pm} > n) \sim \varkappa u(y^{\pm}) n^{-p/2}.$$

It follows from the Taylor formula and Lemma 7 that

$$\big|u(y^{\pm}) - u(y)\big| \le C|y|^{p-1}\big|y^{\pm} - y\big| \le Cn^{p/2-\gamma} \tag{43}$$

for all y with $|y| \le \sqrt{n}$. If K is convex, then, according to the first part of Lemma 19,

$$u(y)n^{-p/2} \ge C\big(\mathrm{dist}(y, \partial K)\big)^p n^{-p/2} \ge Cn^{-p\varepsilon}, \qquad y \in K_{n,\varepsilon}. \tag{44}$$

If K is not necessarily convex but C^2, then we may apply the second part of Lemma 19, which gives the same estimate $u(y) \ge Cn^{p(1/2-\varepsilon)}$.

Combining (43) and (44), we obtain for $\gamma > p\varepsilon$ an estimate

$$u(y^{\pm}) = u(y)\big(1 + o(1)\big), \qquad y \in K_{n,\varepsilon}, |y| \le \sqrt{n}.$$

Therefore, we have

$$\mathbf{P}(\tau^{\mathrm{bm}}_{y^\pm} > n) = \varkappa u(y)n^{-p/2}\big(1 + o(1)\big).$$

From this relation and bounds (41) and (42), we obtain

$$\mathbf{P}(\tau_y > n) = \varkappa u(y)n^{-p/2}\big(1 + o(1)\big) + o(n^{-r}).$$

Thus, it remains to show that

$$n^{-r} = o\big(u(y)n^{-p/2}\big) \tag{45}$$

for all sufficiently small $\varepsilon > 0$ and all $y \in K_{n,\varepsilon}$ with $|y| \le \sqrt{n}$.

Using (44), we see that (45) will be valid for all ε satisfying

$$r = \delta/2 - 2\gamma - 2\gamma\delta > p\varepsilon.$$

This proves (38). To prove (39), it is sufficient to substitute (30) in (41).

The proof of (40) is similar. Define two sets,

$$D^+ = \big\{z \in K : \mathrm{dist}(z, D) \le \big(|x_0| + 1\big)n^{-\gamma}\big\},$$
$$D^- = \big\{z \in D : \mathrm{dist}(z, \partial D) \ge \big(|x_0| + 1\big)n^{-\gamma}\big\}.$$

Clearly, $D^- \subset D \subset D^+$. Then, arguing as above, we get

$$\begin{aligned} &\mathbf{P}\big(\tau_y > n, y + S(n) \in \sqrt{n}D\big) \\ &\quad \le \mathbf{P}\big(\tau_y > n, y + S(n) \in \sqrt{n}D, A_n\big) + o\big(n^{-r}\big) \\ &\quad \le \mathbf{P}\big(\tau^{\mathrm{bm}}_{y^+} > n, y^+ + B(n) \in \sqrt{n}D^+, A_n\big) + o\big(n^{-r}\big) \\ &\quad \le \mathbf{P}\big(\tau^{\mathrm{bm}}_{y^+} > n, y^+ + B(n) \in \sqrt{n}D^+\big) + o\big(n^{-r}\big). \end{aligned} \tag{46}$$

Similarly,

$$\begin{aligned} &\mathbf{P}\big(\tau_y > n, y + S(n) \in \sqrt{n}D\big) \\ &\quad \ge \mathbf{P}\big(\tau^{\mathrm{bm}}_{y^-} > n, y^- + B(n) \in \sqrt{n}D^-\big) + o\big(n^{-r}\big). \end{aligned} \tag{47}$$

Now we apply (32) and obtain

$$\begin{aligned} \mathbf{P}\big(\tau^{\mathrm{bm}}_{y^\pm} > n, y^\pm + B(n) \in \sqrt{n}D^\pm\big) &\sim \varkappa_0 u\big(y^\pm\big)\int_{\sqrt{n}D^\pm} dz\, e^{-|z|^2/(2n)}u(z)n^{-d/2}n^{-p} \\ &= \varkappa_0 u\big(y^\pm\big)\int_{D^\pm} dz\, e^{-|z|^2/2}u(z)n^{-p/2}. \end{aligned}$$

It is sufficient to note now that

$$u\big(y^\pm\big) \sim u(y) \quad \text{and} \quad \int_{D^\pm} dz\, e^{-|z|^2/2}u(z) \to \int_D dz\, e^{-|z|^2/2}u(z)$$

as $n \to \infty$. From these relations and bounds (46) and (47), we obtain

$$\mathbf{P}\big(\tau_y > n, y + S(n) \in \sqrt{n}D\big) = \big(\varkappa_0 + o(1)\big)u(y)\int_D dz\, e^{-|z|^2/2}u(z)n^{-p/2} + o\big(n^{-r}\big).$$

Recalling (45), we arrive at the conclusion. $\square$

4. Asymptotics for $\mathbf{P}(\tau_x > n)$. We first note that, in view of Lemma 14,

$$
\begin{aligned}
\mathbf{P}(\tau_x > n) &= \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) + \mathbf{P}(\tau_x > n, \nu_n > n^{1-\varepsilon}) \\
&= \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) + O(e^{-Cn^{\varepsilon}}).
\end{aligned} \tag{48}
$$

Using the strong Markov property, we get the following estimates for the first term:

$$
\begin{aligned}
&\int_{K_{n,\varepsilon}} \mathbf{P}(x + S(\nu_n) \in dy, \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}) \mathbf{P}(\tau_y > n) \\
&\quad \le \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) \\
&\quad \le \int_{K_{n,\varepsilon}} \mathbf{P}(x + S(\nu_n) \in dy, \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}) \mathbf{P}(\tau_y > n - n^{1-\varepsilon}).
\end{aligned} \tag{49}
$$

Applying now Lemma 20, we obtain

$$
\begin{aligned}
&\mathbf{P}(\tau_x > n; \nu_n \le n^{1-\varepsilon}) \\
&\quad = \frac{\varkappa + o(1)}{n^{p/2}} \mathbf{E}[u(x + S(\nu_n)); \tau_x > \nu_n, |x + S(\nu_n)| \le \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}] \\
&\qquad + O\left(\frac{1}{n^{p/2}} \mathbf{E}[|x + S(\nu_n)|^p; \tau_x > \nu_n, |x + S(\nu_n)| > \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}]\right) \\
&\quad = \frac{\varkappa + o(1)}{n^{p/2}} \mathbf{E}[u(x + S(\nu_n)); \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}] \\
&\qquad + O\left(\frac{1}{n^{p/2}} \mathbf{E}[|x + S(\nu_n)|^p; \tau_x > \nu_n, |x + S(\nu_n)| > \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}]\right).
\end{aligned} \tag{50}
$$

We now show that the first expectation converges to $V(x)$ and that the second expectation is negligibly small.

LEMMA 21. *Under the assumptions of Theorem 1,*

$$
\lim_{n\to\infty} \mathbf{E}[u(x + S(\nu_n)); \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}] = V(x).
$$

PROOF. By the definition of Y_n,

$$
u(x + S(\nu_n)) = Y_{\nu_n} + \sum_{k=0}^{\nu_n - 1} f(x + S(k)).
$$

Consequently,

$$
\begin{aligned}
&\mathbf{E}[u(x + S(\nu_n)); \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}] \\
&\quad = \mathbf{E}[Y_{\nu_n}; \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}] \\
&\qquad + \mathbf{E}\left[\sum_{k=0}^{\nu_n - 1} f(x + S(k)); \tau_x > \nu_n, \nu_n \le n^{1-\varepsilon}\right].
\end{aligned}
$$

Recall that it was shown in Lemma 12 that

$$\mathbf{E}\sum_{k=0}^{\tau_x-1}\left|f\big(x+S(k)\big)\right|<\infty. \tag{51}$$

Then, since $\nu_n\to\infty$,

$$\left|\mathbf{E}\left[\sum_{k=0}^{\nu_n-1} f\big(x+S(k)\big);\tau_x>\nu_n,\nu_n\le n^{1-\varepsilon}\right]\right|\le\mathbf{E}\left[\sum_{k=0}^{\tau_x-1}\left|f\big(x+S(k)\big)\right|;\tau_x>\nu_n\right]$$
$$\to 0.$$

Rearranging the terms, we have

$$\begin{aligned}\mathbf{E}\big[Y_{\nu_n};\tau_x>\nu_n,\nu_n\le n^{1-\varepsilon}\big]&=\mathbf{E}\big[Y_{\nu_n\wedge n^{1-\varepsilon}};\tau_x>\nu_n\wedge n^{1-\varepsilon},\nu_n\le n^{1-\varepsilon}\big]\\&=\mathbf{E}\big[Y_{\nu_n\wedge n^{1-\varepsilon}};\tau_x>\nu_n\wedge n^{1-\varepsilon}\big]\\&\quad-\mathbf{E}\big[Y_{n^{1-\varepsilon}};\tau_x>n^{1-\varepsilon},\nu_n>n^{1-\varepsilon}\big].\end{aligned}\tag{52}$$

Recalling the definition of Y_n, we get

$$\begin{aligned}&\mathbf{E}\big[Y_{n^{1-\varepsilon}};\tau_x>n^{1-\varepsilon},\nu_n>n^{1-\varepsilon}\big]\\&\quad=\mathbf{E}\big[u\big(x+S(n^{1-\varepsilon})\big);\tau_x>n^{1-\varepsilon},\nu_n>n^{1-\varepsilon}\big]\\&\quad\quad-\mathbf{E}\left[\sum_{k=0}^{n^{1-\varepsilon}-1} f\big(x+S(k)\big);\tau_x>n^{1-\varepsilon},\nu_n>n^{1-\varepsilon}\right].\end{aligned}$$

The first term goes to zero due to Lemma 16, the second term vanishes by (51) and by the dominated convergence theorem. Therefore,

$$\mathbf{E}\big[Y_{n^{1-\varepsilon}};\tau_x>n^{1-\varepsilon},\nu_n>n^{1-\varepsilon}\big]\to 0. \tag{53}$$

Further,

$$\begin{aligned}\mathbf{E}\big[Y_{\nu_n\wedge n^{1-\varepsilon}};\tau_x>\nu_n\wedge n^{1-\varepsilon}\big]&=\mathbf{E}[Y_{\nu_n\wedge n^{1-\varepsilon}}]-\mathbf{E}\big[Y_{\nu_n\wedge n^{1-\varepsilon}};\tau_x\le\nu_n\wedge n^{1-\varepsilon}\big]\\&=\mathbf{E}Y_0-\mathbf{E}\big[Y_{\nu_n\wedge n^{1-\varepsilon}};\tau_x\le\nu_n\wedge n^{1-\varepsilon}\big]\\&=u(x)-\mathbf{E}\big[Y_{\tau_x};\tau_x\le\nu_n\wedge n^{1-\varepsilon}\big],\end{aligned}$$

where we have used the martingale property of Y_n. Noting that $\nu_n\wedge n^{1-\varepsilon}\to\infty$ almost surely, we have

$$Y_{\tau_x}\mathbf{1}\{\tau_x\le\nu_n\wedge n^{1-\varepsilon}\}\to Y_{\tau_x}.$$

Then, using the integrability of Y_{τ_x} [see (22) and (24), and the dominated convergence theorem], we obtain

$$\mathbf{E}\big[Y_{\tau_x};\tau_x\le\nu_n\wedge n^{1-\varepsilon}\big]\to\mathbf{E}Y_{\tau_x} \tag{54}$$

Combining (52)–(54), we obtain

$$\mathbf{E}\big[u\big(x+S(\nu_n)\big);\tau_x>\nu_n,\nu_n\le n^{1-\varepsilon}\big]\to u(x)-\mathbf{E}Y_{\tau_x}=V(x).$$

This proves the lemma. □

In what follows, we will use the Fuk–Nagaev inequalities several times. For the reader's convenience, we state them in the following lemma.

LEMMA 22. *Let ξ_i be independent identically distributed random variables with $\mathbf{E}[\xi_1]=0$ and $\mathbf{E}[\xi_1^2]<\infty$. Then, for all $x,y>0$,*

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i\ge x,\max_{i\le n}\xi_i\le y\right)\le e^{x/y}\left(\frac{n\mathbf{E}[\xi^2]}{xy}\right)^{x/y} \tag{55}$$

and

$$\mathbf{P}\left(\sum_{i=1}^n \xi_i\ge x\right)\le e^{x/y}\left(\frac{n\mathbf{E}[\xi^2]}{xy}\right)^{x/y}+n\mathbf{P}(\xi>y). \tag{56}$$

The second inequality is (1.56) from Corollary 1.11 of [39]. The first one is not directly stated there, but it can be found in the proof of Theorem 4 of [23]. There are no proofs in [39] and we refer the interested reader to the original paper [23].

COROLLARY 23. *For all $x,y>0$,*

$$\mathbf{P}\Big(\big|S(n)\big|>x,\max_{k\le n}\big|X(k)\big|\le y\Big)\le 2de^{x/\sqrt{d}y}\left(\frac{\sqrt{d}n}{xy}\right)^{x/\sqrt{d}y} \tag{57}$$

and

$$\mathbf{P}\big(\big|S(n)\big|>x\big)\le 2de^{x/\sqrt{d}y}\left(\frac{\sqrt{d}n}{xy}\right)^{x/\sqrt{d}y}+n\mathbf{P}\big(\big|X(1)\big|>y\big). \tag{58}$$

PROOF. It is clear that

$$\begin{aligned}\mathbf{P}\Big(\big|S(n)\big|>x,\max_{k\le n}\big|X(k)\big|\le y\Big)&\le\sum_{j=1}^d\mathbf{P}\Big(\big|S_j(n)\big|>\frac{x}{\sqrt{d}},\max_{k\le n}\big|X_j(k)\big|\le y\Big)\\&\le\sum_{j=1}^d\mathbf{P}\Big(S_j(n)>\frac{x}{\sqrt{d}},\max_{k\le n}X_j(k)\le y\Big)\\&\quad+\sum_{j=1}^d\mathbf{P}\Big(S_j(n)<-\frac{x}{\sqrt{d}},\min_{k\le n}X_j(k)\ge -y\Big).\end{aligned}$$

Applying now (55) to every summand and recalling that $\mathbf{E}[(X_j(1))^2]=1$, we get (57). The bound (58) follows from (57) and inequality

$$\mathbf{P}\Big(|S(n)|>x,\max_{k\le n}|X(k)|>y\Big)\le\mathbf{P}\Big(\max_{k\le n}|X(k)|>y\Big)\le n\mathbf{P}(|X(1)|>y).\qquad\square$$

LEMMA 24. *Under the assumptions of Theorem 1*,

$$\lim_{n\to\infty}\mathbf{E}\big[|x+S(\nu_n)|^p;\tau_x>\nu_n,|S(\nu_n)|>\theta_n\sqrt{n},\nu_n\le n^{1-\varepsilon}\big]=0.$$

PROOF. We take $\theta_n=n^{-\varepsilon/8}$. Let

$$\mu_n:=\min\{j\ge 1:|X(j)|>n^{1/2-\varepsilon/4}\}.$$

Since $|S(\nu_n)|\le n^{3/2}$ on the event $\{\mu_n>\nu_n,\nu_n\le n^{1-\varepsilon}\}$, we arrive at the following bound:

$$\begin{aligned}\mathbf{E}&\big[|x+S(\nu_n)|^p;\tau_x>\nu_n,|S(\nu_n)|>\theta_n\sqrt{n},\nu_n\le n^{1-\varepsilon},\mu_n>\nu_n\big]\\&\le Cn^{p(3/2)}\mathbf{P}\big(|S(\nu_n)|>\theta_n\sqrt{n},\nu_n\le n^{1-\varepsilon},\mu_n>\nu_n\big)\\&\le Cn^{p(3/2)}\sum_{j=1}^{n^{1-\varepsilon}}\mathbf{P}\big(|S(j)|>\theta_n\sqrt{n},\mu_n>j\big).\end{aligned}$$

Applying now (57) with $x=\theta_n\sqrt{n}=n^{1/2-\varepsilon/8}$, $y=n^{1/2-\varepsilon/4}$ to every probability term, we get

$$\sum_{j=1}^{n^{1-\varepsilon}}\mathbf{P}\big(|S(j)|>\theta_n\sqrt{n},\mu_n>j\big)\le 2d\sum_{j=1}^{n^{1-\varepsilon}}\left(\frac{(ed)j}{n^{1-3\varepsilon/8}}\right)^{n^{\varepsilon/8}/\sqrt{d}}\le\exp\{-Cn^{\varepsilon/8}\}.$$

As a result,

$$\mathbf{E}\big[|x+S(\nu_n)|^p;\tau_x>\nu_n,|S(\nu_n)|>\theta_n\sqrt{n},\nu_n\le n^{1-\varepsilon},\mu_n>\nu_n\big]\to 0.\tag{59}$$

Next,

$$\begin{aligned}\mathbf{E}&\big[|x+S(\nu_n)|^p;\tau_x>\nu_n,|S(\nu_n)|>\theta_n\sqrt{n},\nu_n\le n^{1-\varepsilon},\mu_n\le\nu_n\big]\\&\le\mathbf{E}\big[|x+S(\nu_n)|^p;\tau_x>\mu_n,\nu_n\le n^{1-\varepsilon},\mu_n\le\nu_n\big]\\&\le\sum_{j=1}^{n^{1-\varepsilon}}\mathbf{E}\big[|x+S(\nu_n)|^p;\tau_x>j,\nu_n\le n^{1-\varepsilon},j\le\nu_n,\mu_n=j\big].\end{aligned}$$

For each j, we split the sum $S(\nu_n)$ in 3 parts

$$|x+S(\nu_n)|^p\le C\big(|x+S(j-1)|^p+|X(j)|^p+|S(\nu_n)-S(j)|^p\big).$$

Then

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[\big|S(\nu_n)-S(j)\big|^p; \tau_x > j, \nu_n \le n^{1-\varepsilon}, j \le \nu_n, \mu_n = j\big]$$

$$\le \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big|M\big(n^{1-\varepsilon}\big)\big|^p \mathbf{P}(\tau_x > j-1, \mu_n = j)$$

$$\le C\sum_{j=1}^{n^{1-\varepsilon}} n^{(1-\varepsilon)p/2}\mathbf{P}(\tau_x > j-1)\mathbf{P}\big(\big|X(j)\big| > n^{1/2-\varepsilon/4}\big)$$

$$= Cn^{(1-\varepsilon)p/2}\mathbf{P}\big(\big|X(1)\big| > n^{1/2-\varepsilon/4}\big)\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{P}(\tau_x > j-1).$$

The bound $\mathbf{E}|M(n^{1-\varepsilon})|^p \le Cn^{(1-\varepsilon)p/2}$ holds due to the Doob and Rosenthal inequalities for $p \ge 2$ and additionally Hölder's inequality for $p < 2$.

There are two cases now. For $p > 2$, the sum

$$\sum_{j=1}^{\infty} \mathbf{P}(\tau_x > j) < \infty,$$

since $\mathbf{E}\tau_x < \infty$. In addition, by the Chebyshev inequality,

$$n^{(1-\varepsilon)p/2}\mathbf{P}\big(|X| > n^{1/2-\varepsilon/4}\big) \le n^{(1-\varepsilon)p/2}\frac{\mathbf{E}|X|^p}{n^{p(1/2-\varepsilon/4)}} \le n^{-p\varepsilon/4}\mathbf{E}|X|^p \to 0.$$

Next, for $p \le 2$, we use the fact that $\mathbf{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$,

$$n^{(1-\varepsilon)p/2}\mathbf{P}\big(|X| > n^{1/2-\varepsilon/4}\big) \le n^{(1-\varepsilon)p/2}\frac{\mathbf{E}|X|^{2+\delta}}{n^{(2+\delta)(1/2-\varepsilon/4)}}.$$

Since $\mathbf{E}\tau_x^{p/2-\beta} < \infty$, for any $\beta \in (0, p/2)$,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{P}(\tau_x > j) \le \mathbf{E}\tau_x^{p/2-\beta}\sum_{j=1}^{n}\frac{1}{j^{p/2-\beta}} \le Cn^{1-p/2+\beta}.$$

Then

$$\begin{aligned}&\sum_{j=1}^{n^{1-\varepsilon}} n^{(1-\varepsilon)p/2}\mathbf{P}(\tau_x > j-1)\mathbf{P}\big(|X| > n^{1/2-\varepsilon/4}\big)\\ &\qquad \le Cn^{1-p/2+\beta}n^{(1-\varepsilon)p/2}n^{-(2+\delta)(1/2-\varepsilon/4)}\\ &\qquad = Cn^{\beta-\varepsilon(p-1)/2-\delta(1/2-\varepsilon/4)} \to 0,\end{aligned} \tag{60}$$

once we pick sufficiently small $\varepsilon > 0$ and $\beta > 0$. Therefore, in each case,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[\big|S(\nu_n) - S(j)\big|^p; \tau_x > j, \nu_n \le n^{1-\varepsilon}, j \le \nu_n, \mu_n = j\big] \to 0. \tag{61}$$

Next, we analyze

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[\big|X(j)\big|^p; \tau_x > j, \nu_n \le n^{1-\varepsilon}, j \le \nu_n, \mu_n = j\big]$$
$$\le \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|X|^p; |X| > n^{1/2-\varepsilon/4}\big]\mathbf{P}(\tau_x > j-1).$$

As above, there are two cases: $p > 2$ and $p \le 2$. If $p > 2$, then we apply

$$\sum_{j=1}^{\infty} \mathbf{P}(\tau_x > j) < \infty, \qquad \mathbf{E}\big[|X|^p; |X| > n^{1/2-\varepsilon/4}\big] \to 0.$$

If $p \le 2$, then

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|X|^p; |X| > n^{1/2-\varepsilon/4}\big]\mathbf{P}(\tau_x > j)$$
$$\le Cn^{-p/2+\beta+1}\mathbf{E}|X|^{2+\delta}n^{-(2-p+\delta)(1/2-\varepsilon/4)}$$
$$\le Cn^{\beta-\delta(1/2-\varepsilon/4)+(2-p)\varepsilon/4} \to 0$$

once we pick sufficiently small $\varepsilon > 0$ and $\beta > 0$. Therefore,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[\big|X(j)\big|^p; \tau_x > j, \nu_n \le n^{1-\varepsilon}, j \le \nu_n, \mu_n = j\big] \to 0. \tag{62}$$

Further,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[\big|x + S(j-1)\big|^p; \tau_x > j, \nu_n \le n^{1-\varepsilon}, j \le \nu_n, \mu_n = j\big]$$
$$\le 2^p|x|^p\mathbf{P}\big(\mu_n \le n^{1-\varepsilon}\big)$$
$$+ 2^p \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[\big|S(j-1)\big|^p; \tau_x > j, \mu_n = j\big]$$
$$\le 2^p|x|^p\mathbf{P}\big(\mu_n \le n^{1-\varepsilon}\big) \tag{63}$$

$$+ 2^p \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|S(j-1)|^p; |S(j-1)| > n^{1/2-\varepsilon/8}, \mu_n = j\big]$$

$$+ 2^p \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|S(j-1)|^p; |S(j-1)| \le n^{1/2-\varepsilon/8}, \tau_x > j, \mu_n = j\big].$$

Using the Chebyshev inequality, we obtain

$$\mathbf{P}(\mu_n \le n^{1-\varepsilon}) \le n^{1-\varepsilon}\mathbf{P}(|X| > n^{1/2-\varepsilon/4}) \le dn^{-\varepsilon/2}.$$

For the second term in (63) we note that on $\mu_n = j$ the sum $|S(j-1)| \le n^{3/2}$. Hence,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|S(j-1)|^p; |S(j-1)| > n^{1/2-\varepsilon/8}, \mu_n = j\big]$$

$$\le n^{3p/2} \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{P}\big(|S(j-1)| > n^{1/2-\varepsilon/8}, \mu_n = j\big)$$

$$\le Cn^{3p/2+1} \exp\{-Cn^{\varepsilon/8}\},$$

by the Fuk–Nagaev inequality (57). The third term,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|S(j-1)|^p; |S(j-1)| \le n^{1/2-\varepsilon/8}, \tau_x > j, \mu_n = j\big]$$

$$\le \sum_{j=1}^{n^{1-\varepsilon}} n^{p(1/2-\varepsilon/8)}\mathbf{P}(\tau_x > j, \mu_n = j)$$

$$\le n^{p(1/2-\varepsilon/8)}\mathbf{P}(|X| > n^{1/2-\varepsilon/4}) \sum_{j=1}^{n^{1-\varepsilon}} \mathbf{P}(\tau_x > j-1) \to 0$$

as has already been shown in (60). Hence,

$$\sum_{j=1}^{n^{1-\varepsilon}} \mathbf{E}\big[|x + S(j-1)|^p; \tau_x > j, \nu_n \le n^{1-\varepsilon}, j \le \nu_n, \mu_n = j\big] \to 0. \tag{64}$$

Now the claim follows from equations (59), (61), (62) and (64). □

Now we are in position to complete the proof of Theorem 1. It follows from the lemmas and (48) and (50) that

$$\mathbf{P}(\tau_x > n) = \frac{\varkappa V(x)}{n^{p/2}}\big(1 + o(1)\big).$$

5. Weak convergence results.

LEMMA 25. *For any $x \in K$, the distribution $\mathbf{P}(\frac{x+S(n)}{\sqrt{n}} \in \cdot | \tau_x > n)$ weakly converges to the distribution with the density $H_0 e^{-|y|^2/2} u(y)$, where H_0 is the normalizing constant.*

PROOF. It suffices to show that, for any compact $A \subset K$,

$$\frac{\mathbf{P}(x + S(n) \in \sqrt{n}A, \tau_x > n)}{\mathbf{P}(\tau_x > n)} \to H_0 \int_A e^{-|y|^2/2} u(y)\, dy. \tag{65}$$

Take θ_n which goes to zero slower than any power function. First note that, as in (48) and (50),

$$\begin{aligned}
&\mathbf{P}\bigl(x + S(n) \in \sqrt{n}A, \tau_x > n\bigr) \\
&\quad = \mathbf{P}\bigl(\tau_x > n, x + S(n) \in \sqrt{n}A, \nu_n \le n^{1-\varepsilon}\bigr) + O\bigl(e^{-Cn^{\varepsilon}}\bigr) \\
&\quad = \mathbf{P}\bigl(\tau_x > n, x + S(n) \in \sqrt{n}A, \bigl|S(\nu_n)\bigr| \le \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\bigr) + o\bigl(\mathbf{P}(\tau_x > n)\bigr).
\end{aligned}$$

In the last line, we used the following estimates which hold by the Markov property, Lemmas 20 and 24,

$$\begin{aligned}
&\mathbf{P}\bigl(\tau_x > n, \bigl|S(\nu_n)\bigr| > \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\bigr) \\
&\quad \le \frac{C}{n^{p/2}} \mathbf{E}\bigl[\bigl|x + S(\nu_n)\bigr|^p; \tau_x > \nu_n, \bigl|S(\nu_n)\bigr| > \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\bigr] \\
&\quad = o\bigl(n^{-p/2}\bigr) = o\bigl(\mathbf{P}(\tau_x > n)\bigr).
\end{aligned}$$

Next,

$$\begin{aligned}
&\mathbf{P}\bigl(\tau_x > n, x + S(n) \in \sqrt{n}A, \bigl|S(\nu_n)\bigr| \le \theta_n \sqrt{n}, \nu_n \le n^{1-\varepsilon}\bigr) \\
&\quad = \sum_{k=1}^{n^{1-\varepsilon}} \int_{K_{n,\varepsilon} \cap \{|y-x| \le \theta_n \sqrt{n}\}} \mathbf{P}\bigl(\tau_x > k, x + S(k) \in dy, \nu_n = k\bigr) \\
&\qquad\qquad \times \mathbf{P}\bigl(\tau_y > n - k, y + S(n-k) \in \sqrt{n}A\bigr).
\end{aligned}$$

Using the coupling and arguing as in Lemma 20, one can show that

$$\mathbf{P}\bigl(\tau_y > n - k, y + S(n-k) \in \sqrt{n}A\bigr) \sim \mathbf{P}\bigl(\tau_y^{\mathrm{bm}} > n, y + B(n) \in \sqrt{n}A\bigr)$$

uniformly in $k \le n^{1-\varepsilon}$ and $y \in K_{n,\varepsilon}$. Next, we apply asymptotics (32) and obtain that

$$\mathbf{P}\bigl(\tau_y > n - k, y + S(n-k) \in \sqrt{n}A\bigr) \sim \varkappa_0 \int_A dz\, e^{-|z|^2/2} u(y) u(z) n^{-p/2}$$

uniformly in $y \in K_{n,\varepsilon}$, $|y| \le \theta_n\sqrt{n}$. As a result, we obtain

$$\begin{aligned}
&\mathbf{P}\big(x+S(n)\in\sqrt{n}A,\tau_x>n\big)\\
&\quad\sim\int_A dz\,e^{-|z|^2/2}u(z)n^{-p/2}\\
&\qquad\times\varkappa_0\mathbf{E}\big[u\big(x+S(\nu_n)\big),\tau_x>\nu_n,\big|S(\nu_n)\big|\le\theta_n\sqrt{n},\nu_n\le n^{1-\varepsilon}\big]\\
&\quad\sim\varkappa_0\int_A dz\,e^{-|z|^2/2}u(z)n^{-p/2}V(x),
\end{aligned}$$

where the latter equivalence holds due to Lemmas 21 and 24. Substituting the latter equivalence in (65) and using the asymptotics for $\mathbf{P}(\tau_x > n)$, we arrive at the conclusion. □

Now we change the notation slightly. Let

$$\mathbf{P}_x\big(S(n)\in A\big)=\mathbf{P}\big(x+S(n)\in A\big).$$

LEMMA 26. *Let $X^n(t)=\frac{S([nt])}{\sqrt{n}}$ be the family of processes with the probability measure $\widehat{\mathbf{P}}^{(V)}_{x\sqrt{n}}$, $x\in K$. Then X^n converges weakly in the uniform topology on $D[0,\infty)$ to the Brownian motion conditioned to stay in K with the probability measure $\widehat{\mathbf{P}}^{(u)}_x$.*

PROOF. To prove the claim, we need to show that the convergence takes place in $D[0,l]$ for every l. The proof is identical for each l, so we let $l=1$ to simplify notation. Thus, it is sufficient to show that for every functional $f: 0\le f\le 1$ uniformly continuous on $D[0, 1]$ with respect to the uniform topology,

$$\widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}}f\big(X^n\big)\to\widehat{\mathbf{E}}^{(u)}_x f(B)\qquad\text{as } n\to\infty.$$

We first show that

$$\frac{1}{V(x\sqrt{n})}\mathbf{E}\big[V\big(x\sqrt{n}+S(n)\big),\big|S(n)\big|>R\sqrt{n}\big]\le g(R),\tag{66}$$

where $g(R)\to 0$ as $R\to\infty$. Using Lemma 13(a) and (b), we have, for all $R>1$,

$$\begin{aligned}
&\frac{1}{V(x\sqrt{n})}\mathbf{E}\big[V\big(x\sqrt{n}+S(n)\big),\big|S(n)\big|>R\sqrt{n}\big]\\
&\quad\le\frac{C}{n^{p/2}}\big(n^{p/2}\mathbf{P}\big(\big|S(n)\big|>R\sqrt{n}\big)+\mathbf{E}\big[\big|S(n)\big|^p,\big|S(n)\big|>R\sqrt{n}\big]\big)\\
&\quad\le\frac{C}{n^{p/2}}\mathbf{E}\big[\big|S(n)\big|^p,\big|S(n)\big|>R\sqrt{n}\big].
\end{aligned}$$

If $p > 2$, then

$$\begin{aligned}\mathbf{E}\big[|S(n)|^p; |S(n)| > R\sqrt{n}\big] \\ = p\int_{R\sqrt{n}}^{\infty} z^{p-1}\mathbf{P}\big(|S(n)| > z\big)\,dz + R^p n^{p/2}\mathbf{P}\big(|S(n)| > R\sqrt{n}\big).\end{aligned}$$

Choosing $y = z/r$ in the inequality (58), we have

$$\mathbf{P}\big(|S(n)| > z\big) \le C(r)\Big(\frac{n}{z^2}\Big)^r + n\mathbf{P}\big(|X| > z/r\big).$$

Using the latter bound with $r > p/2$, we have

$$\begin{aligned}\mathbf{P}\big(|S(n)| > R\sqrt{n}\big) &\le C(r)R^{-2r} + n\mathbf{P}\big(|X| > R\sqrt{n}/r\big) \\ &\le C(r)R^{-2r} + \frac{r^2}{R^2}\mathbf{E}\big[|X|^2, |X| > R\big]\end{aligned} \tag{67}$$

and

$$\begin{aligned}&\int_{R\sqrt{n}}^{\infty} z^{p-1}\mathbf{P}\big(|S(n)| > z\big)\,dz \\ &\le C(r)pn^r\int_{R\sqrt{n}}^{\infty} z^{p-1-2r}\,dz + np\int_{R\sqrt{n}}^{\infty} z^{p-1}\mathbf{P}\big(|X| > z/r\big)\,dz \\ &\le C(r)\frac{p}{2r-p}n^{p/2}R^{p-2r} + r^p n\mathbf{E}\big[|X|^p, |X| > R\sqrt{n}/r\big] \\ &\le C(p,r)n^{p/2}\big(R^{p-2r} + \mathbf{E}\big[|X|^p, |X| > R\big]\big)\end{aligned} \tag{68}$$

for all sufficiently large n. This implies that (66) holds for $p > 2$.

If $p \le 2$ then, combining the Markov inequality and (68), we get for any $r > 1 + \delta/2$,

$$\begin{aligned}\mathbf{E}\big[|S(n)|^p; |S(n)| > R\sqrt{n}\big] &\le (R\sqrt{n})^{p-2-\delta}\mathbf{E}\big[|S(n)|^{2+\delta}, |S(n)| > R\sqrt{n}\big] \\ &\le C(2+\delta, r)n^{p/2}R^{2+\delta-2r}.\end{aligned}$$

Thus, the bound (66) is valid for all p.

Fix also some $\varepsilon > 0$. It follows easily from Lemma 13(a), (b) and the central limit theorem that

$$\begin{aligned}&\frac{1}{V(x\sqrt{n})}\mathbf{E}\big[V\big(x\sqrt{n} + S(n)\big), \tau_{x\sqrt{n}} > n, |S(n)| \le R\sqrt{n}, \\ &\qquad\qquad \operatorname{dist}\big(x\sqrt{n} + S(n), \partial K\big) \le \varepsilon\sqrt{n}\big] \\ &\le C\mathbf{P}\big(\operatorname{dist}\big(x\sqrt{n} + S(n), \partial K\big) \le \varepsilon\sqrt{n}\big) \\ &\le C\mathbf{P}\big(\operatorname{dist}\big(x + B(1), \partial K\big) \le \varepsilon\big).\end{aligned}$$

Since the distribution of $B(1)$ is isotropic,

$$\mathbf{P}\big(\mathrm{dist}\big(x+B(1),\partial K\big)\le\varepsilon\big)\le C\varepsilon.$$

Therefore,

$$\begin{aligned}\frac{1}{V(x\sqrt{n})}\mathbf{E}\big[V\big(x\sqrt{n}+S(n)\big),\tau_{x\sqrt{n}}>n,\big|S(n)\big|\le R\sqrt{n},\\ \mathrm{dist}\big(x\sqrt{n}+S(n),\partial K\big)\le\varepsilon\sqrt{n}\big]\\ \le C\varepsilon.\end{aligned}\tag{69}$$

It is clear that similar bounds are valid for the Brownian motion. More precisely,

$$\frac{1}{u(x)}\mathbf{E}\big[u\big(x+B(1)\big),\big|B(1)\big|>R\big]\le g(R)\tag{70}$$

and

$$\begin{aligned}&\frac{1}{u(x)}\mathbf{E}\big[u\big(x+B(1)\big),\tau_x^{\mathrm{bm}}>1,\big|B(1)\big|\le R,\mathrm{dist}\big(x+B(1),\partial K\big)\le\varepsilon\big]\\ &\quad\le C\big(|x|+R\big)^{d-1}\varepsilon.\end{aligned}\tag{71}$$

Define

$$D_n:=\big\{\big|S(n)\big|\le R\sqrt{n},\mathrm{dist}\big(x\sqrt{n}+S(n),\partial K\big)\ge\varepsilon\sqrt{n}\big\}$$

and

$$D^{\mathrm{bm}}:=\big\{\big|B(1)\big|\le R,\mathrm{dist}\big(x+B(1),\partial K\big)\ge\varepsilon\big\}.$$

Using Lemma 13(a), one can easily get

$$\begin{aligned}&\frac{1}{V(x\sqrt{n})}\mathbf{E}\big[f(X^n)V\big(x\sqrt{n}+S(n)\big)1_{D_n},\tau_{x\sqrt{n}}>n\big]\\ &\quad=\big(1+o(1)\big)\frac{1}{u(x\sqrt{n})}\mathbf{E}\big[f(X^n)u\big(x\sqrt{n}+S(n)\big)1_{D_n},\tau_{x\sqrt{n}}>n\big]\\ &\quad=\big(1+o(1)\big)\frac{1}{u(x)}\mathbf{E}\bigg[f(X^n)u\bigg(x+\frac{S(n)}{\sqrt{n}}\bigg)1_{D_n},\tau_{x\sqrt{n}}>n\bigg].\end{aligned}$$

We next note that $u(x+\cdot)f(\cdot)1_{D^{\mathrm{bm}}\cap\{\tau_x^{\mathrm{bm}}>1\}}$ is bounded and its discontinuities are a null-set with respect to the Wiener measure on $D[0,1]$ equipped with the Borel σ-algebra induced by the uniform topology. Thus, due to the Donsker invariance principle on $D[0,1]$ with the uniform topology,

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{V(x\sqrt{n})}\mathbf{E}\big[f(X^n)V\big(x\sqrt{n}+S(n)\big)1_{D_n},\tau_{x\sqrt{n}}>n\big]\\ &\quad=\frac{1}{u(x)}\mathbf{E}\big[f(B)u\big(x+B(1)\big)1_{D^{\mathrm{bm}}},\tau_x^{\mathrm{bm}}>1\big].\end{aligned}$$

For details on the invariance principle on $D[0,1]$ with the uniform topology and on the Wiener measure on this space, we refer to Billingsley's book [4], Section 18.

From this convergence and bounds (66)–(71), we conclude that

$$\limsup_{n\to\infty}\bigl|\widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}} f(X^n) - \widehat{\mathbf{E}}^{(u)}_x f(B)\bigr| \le 2g(R) + C\bigl(|x|+R\bigr)^{d-1}\varepsilon.$$

Letting first $\varepsilon \to 0$ and then $R \to \infty$, we get

$$\limsup_{n\to\infty}\bigl|\widehat{\mathbf{E}}^{(V)}_{x\sqrt{n}} f(X^n) - \widehat{\mathbf{E}}^{(u)}_x f(B)\bigr| = 0.$$

Thus, the lemma is proved. □

6. Proof of local limit theorems.

6.1. *Preliminary estimates.*

LEMMA 27. *For all $y \in K$ and all $n \ge 1$,*

$$\mathbf{P}\bigl(x+S(n)=y, \tau_x > n\bigr) \le \frac{C}{n^{d/2}}\mathbf{P}(\tau_x > n/2) \le C(x) n^{-p/2-d/2}. \tag{72}$$

PROOF. It follows easily from (25) that

$$\mathbf{P}\bigl(S(j)=z\bigr) \le Cj^{-d/2}, \qquad z \in \mathbb{Z}^d. \tag{73}$$

Therefore, for $m = [n/2]$ we have

$$\begin{aligned}
&\mathbf{P}\bigl(x+S(n)=y, \tau_x > n\bigr)\\
&\quad= \sum_{z\in K}\mathbf{P}\bigl(x+S(m)=z, \tau_x > m\bigr)\mathbf{P}\bigl(z+S(n-m)=y, \tau_z > n-m\bigr)\\
&\quad\le \sum_{z\in K}\mathbf{P}\bigl(x+S(m)=z, \tau_x > m\bigr)\mathbf{P}\bigl(z+S(n-m)=y\bigr)\\
&\quad\le Cn^{-d/2}\mathbf{P}(\tau_x > m).
\end{aligned}$$

But we know that $\mathbf{P}(\tau_x > m) \le C(x)m^{-p/2}$. This completes the proof of the lemma. □

Comparing (72) with the claim in Theorem 5, we see that (72) has the right order for typical values of y, that is, for y of order $n^{1/2}$. But for smaller values of y that bound is too rough.

LEMMA 28. *For all $x, y \in K$ and all $n \ge 1$,*

$$\mathbf{P}\bigl(x+S(n)=y, \tau_x > n\bigr) \le C(x,y) n^{-p-d/2}. \tag{74}$$

PROOF. We first split the trajectory $S(1), S(2), \dots, S(n)$ into two parts

$$\mathbf{P}\big(x+S(n)=y, \tau_x>n\big) = \sum_{z\in K} \mathbf{P}\big(x+S(m)=z, \tau_x>m\big)\mathbf{P}\big(z+S(n-m)=y, \tau_z>n-m\big),$$

where $m=[n/2]$. Then we reverse the time in the second part:

$$\begin{aligned}
&\mathbf{P}\big(z+S(n-m)=y, \tau_z>n-m\big)\\
&\quad=\mathbf{P}\big(z+S(k)\in K, k=1,2,\dots,n-m-1, z+S(n-m)=y\big)\\
&\quad=\mathbf{P}\Bigg(z+S(n-m)-\sum_{j=k+1}^{n-m} X(j)\in K, k=1,2,\dots,n-m-1,\\
&\qquad\qquad z+S(n-m)=y\Bigg)\\
&\quad=\mathbf{P}\big(y-S(k)\in K, k=1,2,\dots,n-m-1, y-S(n-m)=z\big)\\
&\quad=\mathbf{P}\big(y-S(n-m)=z, \tau'_y>n-m\big),
\end{aligned}$$

where $\tau'_y=\min\{k\ge 1: y-S(k)\notin K\}$. Applying Lemma 27 to the random walk $\{-S(n)\}$, we obtain

$$\mathbf{P}\big(z+S(n-m)=y, \tau_z>n-m\big)\le C(y)n^{-p/2-d/2}.$$

Consequently,

$$\begin{aligned}
\mathbf{P}\big(x+S(n)=y, \tau_x>n\big)&\le C(y)n^{-p/2-d/2}\sum_{z\in K}\mathbf{P}\big(x+S(m)=z, \tau_x>m\big)\\
&\le C(y)n^{-p/2-d/2}C(x)n^{-p/2}.
\end{aligned}$$

Thus, the proof is finished. □

LEMMA 29. *There exist constants a and C such that, for every $u>0$,*

$$\limsup_{n\to\infty}\sup_{|x-z|\ge u\sqrt{n}} n^{d/2}\mathbf{P}\big(x+S(n)=z\big)\le C\exp\{-au^2\} \tag{75}$$

and

$$\limsup_{n\to\infty}\sup_{x,z\in M_{n,u}} n^{d/2}\mathbf{P}\big(x+S(n)=z, \tau_x\le n\big)\le C\exp\{-au^2\}, \tag{76}$$

where $M_{n,u}:=\{z:\operatorname{dist}(z,\partial K)\ge u\sqrt{n}\}$.

PROOF. Put again $m=[n/2]$. For x and z with $|x-z|\ge u\sqrt{n}$, we have

$$\begin{aligned}
\mathbf{P}\big(x+S(n)=z\big)&\le\mathbf{P}\big(x+S(n)=z, |S(m)|\ge u\sqrt{n}/2\big)\\
&\quad+\mathbf{P}\big(x+S(n)=z, |S(n)-S(m)|\ge u\sqrt{n}/2\big).
\end{aligned}$$

We first note that from the Markov property and (73) follows

$$\mathbf{P}(x+S(n)=z, |S(m)|\ge u\sqrt{n}/2)\le Cn^{-d/2}\mathbf{P}(|S(m)|\ge u\sqrt{n}/2).$$

Reversing the time, as it was done in the previous lemma, we infer that

$$\mathbf{P}(x+S(n)=z, |S(n)-S(m)|\ge u\sqrt{n}/2)\le Cn^{-d/2}\mathbf{P}(|S(n-m)|\ge u\sqrt{n}/2).$$

As a result we have

$$\mathbf{P}(x+S(n)=z)\le Cn^{-d/2}(\mathbf{P}(|S(m)|\ge u\sqrt{n}/2)+\mathbf{P}(|S(n-m)|\ge u\sqrt{n}/2)).$$

The first estimate in the lemma follows now from the central limit theorem.

To prove the second estimate we note that if $\mathrm{dist}(z,\partial K)\ge u\sqrt{n}$ then, using the Markov property, we obtain

$$\mathbf{P}(x+S(n)=z,\tau_x\le n/2)\le \max_{n/2\le k\le n}\sup_{|y-z|\ge u\sqrt{n}} n^{d/2}\mathbf{P}(y+S(k)=z).$$

Furthermore, if $\mathrm{dist}(x,\partial K)\ge u\sqrt{n}$ then, reversing additionally the time, we get

$$\mathbf{P}(x+S(n)=z,n/2<\tau_x\le n)\le \max_{n/2\le k\le n}\sup_{|y-x|\ge u\sqrt{n}} n^{d/2}\mathbf{P}(y+S(k)=x).$$

Applying (75), we complete the proof. □

6.2. *Proof of Theorem* 5. For simplicity we assume that X takes values on $\mathbb{Z}^d$. We split the cone into three parts:

$$K^{(1)}:=\{y\in K:|y|>A\sqrt{n}\},$$
$$K^{(2)}:=\{y\in K:|y|\le A\sqrt{n},\mathrm{dist}(y,\partial K)\le 2\varepsilon\sqrt{n}\},$$
$$K^{(3)}:=\{y\in K:|y|\le A\sqrt{n},\mathrm{dist}(y,\partial K)> 2\varepsilon\sqrt{n}\}$$

with some $A>0$ and $\varepsilon>0$. Noting that

$$\lim_{A\to\infty}\sup_{y\in K^{(1)}} u(y/\sqrt{n})e^{-|y|^2/2n}=0$$

and

$$\lim_{A\to\infty}\lim_{\varepsilon\to 0}\sup_{y\in K^{(2)}} u(y/\sqrt{n})e^{-|y|^2/2n}=0,$$

one can easily see that the theorem will be proved if we show that

$$\lim_{A\to\infty}\limsup_{n\to\infty} n^{p/2+d/2}\sup_{y\in K^{(1)}}\mathbf{P}(x+S(n)=y,\tau_x>n)=0,$$
$$\lim_{A\to\infty}\lim_{\varepsilon\to 0}\limsup_{n\to\infty} n^{p/2+d/2}\sup_{y\in K^{(2)}}\mathbf{P}(x+S(n)=y,\tau_x>n)=0$$

and

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sup_{y\in K^{(3)}}\Big|n^{p/2+d/2}\mathbf{P}\big(x+S(n)=y,\tau_x>n\big)$$
$$-\varkappa V(x)H_0u\Big(\frac{y}{\sqrt{n}}\Big)e^{-|y|^2/2n}\Big|=0.$$

This is done in (77), (81) and (87), respectively.

We have

$$\mathbf{P}\big(x+S(n)=y,\tau_x>n\big)=\mathbf{P}\big(x+S(n)=y,\tau_x>n,\big|S(n/2)\big|\le A\sqrt{n}/2\big)$$
$$+\mathbf{P}\big(x+S(n)=y,\tau_x>n,\big|S(n/2)\big|>A\sqrt{n}/2\big).$$

Using the Markov property and (73), we get, for all $y\in K^{(1)}$,

$$\mathbf{P}\big(x+S(n)=y,\tau_x>n,\big|S(n/2)\big|>A\sqrt{n}/2\big)$$
$$\le C(x)n^{-d/2-p/2}\mathbf{P}\big(\big|x+S(n/2)\big|>A\sqrt{n}/2-|x|\,|\tau_x>n/2\big).$$

Applying now (8) in Theorem 3, we obtain, uniformly in $y\in K^{(1)}$,

$$\lim_{A\to\infty}\limsup_{n\to\infty}n^{p/2+d/2}\mathbf{P}\big(x+S(n)=y,\tau_x>n,\big|S(n/2)\big|>A\sqrt{n}/2\big)$$
$$\le C(x)\lim_{A\to\infty}\mu\big(\{z\in K:|z|>A/\sqrt{2}\}\big)=0.$$

Furthermore, applying Theorem 1 and (75), we get, for $|y|>A\sqrt{n}$,

$$\mathbf{P}\big(x+S(n)=y,\tau_x>n,\big|S(n/2)\big|\le A\sqrt{n}/2\big)$$
$$\le\mathbf{P}(\tau_x>n/2)\sup_{|y-z|>A\sqrt{n}/2}\mathbf{P}\big(x+z+S(n/2)=y\big)$$
$$\le C(x)n^{-d/2-p/2}\exp\{-aA^2/4\}.$$

As a result, we have

$$\lim_{A\to\infty}\limsup_{n\to\infty}n^{d/2+p/2}\sup_{y\in K^{(1)}}\mathbf{P}\big(x+S(n)=y,\tau_x>n\big)=0. \tag{77}$$

We next consider $y\in K^{(2)}$. Set $m=[n/2]$. Using the time reversion from Lemma 28 and the bound (72), we obtain

$$\mathbf{P}\big(x+S(n)=y,\tau_x>n\big)$$
$$=\sum_{z\in K}\mathbf{P}\big(x+S(m)=z,\tau_x>m\big)\mathbf{P}\big(y-S(n-m)=z,\tau'_y>n-m\big)$$
$$\le C(x)m^{-p/2-d/2}\sum_{z\in K}\mathbf{P}\big(y-S(n-m)=z,\tau'_y>n-m\big)$$
$$\le C(x)n^{-p/2-d/2}\mathbf{P}\big(\tau'_y>n-m\big).$$

We want to show that

$$\limsup_{n\to\infty} \sup_{y\in K^{(2)}} \mathbf{P}\big(\tau'_y > n-m\big) \le g(\varepsilon) \tag{78}$$

with some $g(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using the same arguments as in (41), we have

$$\mathbf{P}\big(\tau'_y > n-m\big) \le \mathbf{P}\big(\tau^{\mathrm{bm}}_{y+\varepsilon\sqrt{n}x_0} > n-m\big) + o\big(n^{-r}\big),$$

and $o(n^{-r})$ is uniform in y. Consequently, by the scaling property of the Brownian motion,

$$\sup_{y\in K^{(2)}} \mathbf{P}\big(\tau'_y > n-m\big) \le \sup_{z\in K:\,|z|\le A,\mathrm{dist}(z,\partial K)\le 2\varepsilon} \mathbf{P}\big(\tau^{\mathrm{bm}}_{z+\varepsilon x_0} > 1/2\big) + o\big(n^{-r}\big). \tag{79}$$

Note that if $\mathrm{dist}(z,\partial K) \le 2\varepsilon$ then $\mathrm{dist}(z+\varepsilon x_0, \partial K) \le C_*\varepsilon$.

The most standard way of bounding $\mathbf{P}(\tau^{\mathrm{bm}}_x > 1/2)$ is the use of the parabolic boundary Harnack principle which gives

$$\mathbf{P}\big(\tau^{\mathrm{bm}}_x > 1/2\big) \le Cu(x), \tag{80}$$

see [43], page 336, and references there. If $|x|$ is bounded and $\mathrm{dist}(x,\partial K) \le C_*\varepsilon$, then (78) is immediate from the definition of u.

But for convex cones there exists an elementary way of deriving (78) from (79), which we present below.

If K is convex, then there exists a hyperplane $H = H(z)$ such that $\mathrm{dist}(z + \varepsilon x_0, H) \le 2C_*\varepsilon$ and $K \cap H = \varnothing$. If we set $T_z := \inf\{t > 0 : z + B(t) \in H\}$ then, obviously,

$$\mathbf{P}\big(\tau^{\mathrm{bm}}_{z+\varepsilon x_0} > 1/2\big) \le \mathbf{P}(T_{z+\varepsilon x_0} > 1/2).$$

Due to the rotational invariance of the Brownian motion, the normal to H component of B is a one-dimensional Brownian motion. As a result, we have

$$\mathbf{P}(T_{z+\varepsilon x_0} > 1/2) \le \mathbf{P}\Big(2C_*\varepsilon + \inf_{t\le 1/2} B_1(t) > 0\Big)$$

uniformly in z satisfying $\mathrm{dist}(z,\partial K) \le 2\varepsilon$. Applying finally the reflection principle, we conclude from (79) that

$$\mathbf{P}\big(\tau'_y > n-m\big) \le C\varepsilon + o\big(n^{-r}\big)$$

uniformly in y satisfying $\mathrm{dist}(y,\partial K) \le 2\varepsilon\sqrt{n}$.

Summarizing,

$$\lim_{A\to\infty}\lim_{\varepsilon\to 0}\limsup_{n\to\infty} n^{d/2+p/2} \sup_{y\in K^{(2)}} \mathbf{P}\big(x+S(n)=y, \tau_x > n\big) = 0. \tag{81}$$

It remains to consider "typical" values of y, that is, $y \in K^{(3)}$. Set $m = [\varepsilon^3 n]$. We start with the representation

$$\begin{aligned}&\mathbf{P}\big(x+S(n)=y, \tau_x > n\big)\\ &\quad= \sum_{z\in K} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)\mathbf{P}\big(z+S(m)=y, \tau_z > m\big).\end{aligned} \tag{82}$$

Let $K_1(y) := \{z \in K : |z-y| < \varepsilon\sqrt{n}\}$. Applying (75), we have

$$\begin{aligned}
&\sum_{z\in K\setminus K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)\mathbf{P}\big(z+S(m)=y, \tau_z > m\big)\\
&\quad\le \sum_{z\in K\setminus K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)\mathbf{P}\big(z+S(m)=y\big)\\
&\quad\le \sum_{z\in K\setminus K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)Cn^{-d/2}\varepsilon^{-3d/2}\exp\{-a/\varepsilon\} \qquad (83)\\
&\quad\le C\mathbf{P}(\tau_x > n-m)n^{-d/2}\varepsilon^{-3d/2}\exp\{-a/\varepsilon\}\\
&\quad\le CV(x)n^{-d/2-p/2}\varepsilon^{-3d/2}\exp\{-a/\varepsilon\}
\end{aligned}$$

uniformly in y satisfying $\operatorname{dist}(y,\partial K) > 2\varepsilon\sqrt{n}$.

If $\operatorname{dist}(y,\partial K) > 2\varepsilon\sqrt{n}$ and $z \in K_1(y)$, then $\operatorname{dist}(z,\partial K) > \varepsilon\sqrt{n}$. Using (76), we have

$$\begin{aligned}
&\sum_{z\in K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)\mathbf{P}\big(z+S(m)=y, \tau_z \le m\big)\\
&\quad\le \sum_{z\in K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)Cn^{-d/2}\varepsilon^{-3d/2}\exp\{-a/\varepsilon\}\\
&\quad\le C\mathbf{P}(\tau_x > n-m)n^{-d/2}\varepsilon^{-3d/2}\exp\{-a/\varepsilon\} \qquad (84)\\
&\quad\le CV(x)n^{-d/2-p/2}\varepsilon^{-3d/2}\exp\{-a/\varepsilon\}
\end{aligned}$$

uniformly in y satisfying $\operatorname{dist}(y,\partial K) > 2\varepsilon\sqrt{n}$.

Using the local limit theorem for unconditioned random walks (see Proposition 7.9 in Spitzer's book [42]), we have, uniformly in y,

$$\begin{aligned}
\Sigma(y) &:= \sum_{z\in K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)\mathbf{P}\big(z+S(m)=y\big)\\
&= \sum_{z\in K_1(y)} \mathbf{P}\big(x+S(n-m)=z, \tau_x > n-m\big)\big(2\pi n\varepsilon^3\big)^{-d/2} \qquad (85)\\
&\qquad\times \exp\{-|y-z|^2/2\varepsilon^3 n\}\\
&\quad+ O\big(n^{-d/2-p/2}\varepsilon^{-3d/2}e^{-a/\varepsilon}\big).
\end{aligned}$$

It follows from compactness argument and the integral limit theorem for $\{S(n)\}$ conditioned to stay in K that

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{y\in K^{(3)}} \bigg| \sum_{z\in K_1(y)} \mathbf{P}\big(x+S(n-m)=z \mid \tau_x > n-m\big)\exp\{-|y-z|^2/2\varepsilon^3 n\}\\
&\qquad - H_0 \int_{|(1-\varepsilon^3)^{1/2}r - y/\sqrt{n}|<\varepsilon} u(r)e^{-|r|^2/2}e^{-|(1-\varepsilon^3)^{1/2}r-y/\sqrt{n}|^2/2\varepsilon^3}\,dr\bigg|\\
&\quad= 0
\end{aligned}$$

for every fixed ε. Set, for brevity,

$$I_1(y,n,\varepsilon) := \int_{|(1-\varepsilon^3)^{1/2}r-y/\sqrt{n}|<\varepsilon} u(r)e^{-|r|^2/2}e^{-|(1-\varepsilon^3)^{1/2}r-y/\sqrt{n}|^2/2\varepsilon^3}\,dr$$

and

$$I_2(y,n,\varepsilon) := \int_{|(1-\varepsilon^3)^{1/2}r-y/\sqrt{n}|<\varepsilon} e^{-|(1-\varepsilon^3)^{1/2}r-y/\sqrt{n}|^2/2\varepsilon^3}\,dr.$$

Since $u(r)e^{-|r|^2/2}$ is uniformly continuous, we have

$$\limsup_{\varepsilon\to 0}\sup_{n\ge 1}\sup_{y\in K^{(3)}} \frac{|I_1(y,n,\varepsilon)-u(y/\sqrt{n})e^{-|y|^2/2n}I_2(y,n,\varepsilon)|}{I_2(y,n,\varepsilon)} = 0.$$

Noting that

$$\begin{aligned} I_2(n,y,\varepsilon) &= \bigl(1-\varepsilon^3\bigr)^{-d/2}\varepsilon^{3d/2}\int_{|r'|<\varepsilon^{-1/2}} e^{-|r'|^2/2}\,dr' \\ &\sim \varepsilon^{3d/2}\int_{\mathbb{R}^d} e^{-|r'|^2/2}\,dr' = \bigl(2\pi\varepsilon^3\bigr)^{d/2}, \end{aligned}$$

we conclude that

$$\limsup_{\varepsilon\to 0}\sup_{n\ge 1}\sup_{y\in K^{(3)}} \frac{|I_1(y,n,\varepsilon)-u(y/\sqrt{n})e^{-|y|^2/2n}(2\pi\varepsilon^3)^{d/2}|}{(2\pi\varepsilon^3)^{d/2}} = 0.$$

Consequently,

$$\begin{aligned} \limsup_{n\to\infty}\sup_{y\in K^{(3)}} \Bigg| &\sum_{z\in K_1(y)} \mathbf{P}\bigl(x+S(n-m)=z|\tau_x>n-m\bigr)\exp\{-|y-z|^2/2\varepsilon^3 n\} \\ &\qquad - H_0u(y/\sqrt{n})e^{-|y|^2/2n}\bigl(2\pi\varepsilon^3\bigr)^{d/2}\Bigg| \\ &= o\bigl(\varepsilon^{3d/2}\bigr). \end{aligned}$$

From this relation and (85), we infer

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sup_{y\in K^{(3)}} \bigl|n^{d/2+p/2}\Sigma(y) - \varkappa V(x)H_0u(y/\sqrt{n})e^{-|y|^2/2n}\bigr| = 0. \tag{86}$$

Combining (82), (83), (84) and (86), we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\sup_{y\in K^{(3)}} \Bigg| &n^{p/2+d/2}\mathbf{P}\bigl(x+S(n)=y,\tau_x>n\bigr) \\ &- \varkappa V(x)H_0u\biggl(\frac{y}{\sqrt{n}}\biggr)e^{-|y|^2/2n}\Bigg| = 0. \end{aligned} \tag{87}$$

6.3. *Proof of Theorem 6*. Set $m = [(1-t)n]$ and write

$$\begin{aligned}
&\mathbf{P}\big(x+S(n)=y,\tau_x>n\big)\\
&\quad=\sum_{z\in K}\mathbf{P}\big(x+S(n-m)=z,\tau_x>n-m\big)\mathbf{P}\big(z+S(m)=y,\tau_z>m\big) \qquad (88)\\
&\quad=\sum_{z\in K}\mathbf{P}\big(x+S(n-m)=z,\tau_x>n-m\big)\mathbf{P}\big(y+S'(m)=z,\tau'_y>m\big),
\end{aligned}$$

where S' is distributed as $-S$.

We first note that, according to Theorem 1 and Lemma 27,

$$\begin{aligned}
\Sigma_1(A,n) &:= \sum_{z\in K:|z|>A\sqrt{n}}\mathbf{P}\big(x+S(n-m)=z,\tau_x>n-m\big)\\
&\qquad\qquad\times\mathbf{P}\big(y+S'(m)=z,\tau'_y>m\big)\\
&\le C(x,y)n^{-p-d/2}\mathbf{P}\big(|S'(m)|>A\sqrt{n}-|y|\,|\,\tau'_y>m\big).
\end{aligned}$$

Therefore, in view of Theorem 3,

$$\lim_{A\to\infty}\lim_{n\to\infty}n^{p+d/2}\Sigma_1(A,n)=0. \qquad (89)$$

Applying (9) first to $\{S'(n)\}$ and then to $\{S(n)\}$, we get

$$\begin{aligned}
\Sigma_2(A,n) &:= \sum_{z\in K:|z|\le A\sqrt{n}}\mathbf{P}\big(x+S(n-m)=z,\tau_x>n-m\big)\\
&\qquad\qquad\times\mathbf{P}\big(y+S'(m)=z,\tau'_y>m\big)\\
&= \varkappa^2\frac{V(x)V'(y)H_0^2}{(t(1-t))^{p/2+d/2}}n^{-p-d}\\
&\quad\times\sum_{z\in K:|z|\le A\sqrt{n}}u\bigg(\frac{z}{\sqrt{tn}}\bigg)u\bigg(\frac{z}{\sqrt{(1-t)n}}\bigg)\\
&\qquad\qquad\times\exp\bigg\{-\frac{|z|^2}{2tn}-\frac{|z|^2}{2(1-t)n}\bigg\}\\
&\quad+o(R_n),
\end{aligned}$$

where

$$\begin{aligned}
R_n &= \mathbf{P}(\tau_x>n-m)m^{-p/2-d/2}\\
&\quad+\big(n(n-m)\big)^{-p/2-d/2}\sum_{|z|\le A\sqrt{n}}u\bigg(\frac{z}{\sqrt{tn}}\bigg)\exp\bigg\{-\frac{|z|^2}{2tn}\bigg\}.
\end{aligned}$$

Using Theorem 1 and noting that the sum is of order $n^{d/2}$, we conclude that $R_n \le Cn^{-p-d/2}$. Therefore,

$$\Sigma_2(A,n) = \varkappa^2 \frac{V(x)V'(y)H_0^2}{(t(1-t))^{p/2+d/2}} n^{-p-d} \times \sum_{z\in K:|z|\le A\sqrt{n}} u\Big(\frac{z}{\sqrt{tn}}\Big) u\Big(\frac{z}{\sqrt{(1-t)n}}\Big) \exp\Big\{-\frac{|z|^2}{2tn} - \frac{|z|^2}{2(1-t)n}\Big\} + o\big(n^{-p-d/2}\big).$$

Thus, it remains to compute the limiting value of the sum in the latter formula. Using the homogeneity of u, we get

$$\lim_{n\to\infty} n^{-d/2} \sum_{z\in K:|z|\le A\sqrt{n}} u\Big(\frac{z}{\sqrt{tn}}\Big) u\Big(\frac{z}{\sqrt{(1-t)n}}\Big) \exp\Big\{-\frac{|z|^2}{2t(1-t)n}\Big\} = \frac{1}{(t(1-t))^{p/2}} \int_{w\in K:|w|\le A} u^2(w) e^{-|w|^2/2t(1-t)}\, dw. \tag{90}$$

Consequently,

$$\lim_{A\to\infty}\lim_{n\to\infty} n^{p+d/2}\Sigma_2(A,n) = \varkappa^2 \frac{V(x)V'(y)H_0^2}{(t(1-t))^{p+d/2}} \int_K u^2(w) e^{-|w|^2/2t(1-t)}\, dw. \tag{91}$$

Combining (88), (89) and (91), we obtain

$$\lim_{n\to\infty} n^{p+d/2}\mathbf{P}\big(x+S(n)=y, \tau_x>n\big) = \varkappa^2 \frac{V(x)V'(y)H_0^2}{(t(1-t))^{p+d/2}} \int_K u^2(w) e^{-|w|^2/2t(1-t)}\, dw. \tag{92}$$

Substituting $v = w/\sqrt{t(1-t)}$ we see that (10) holds with $\rho = \varkappa^2 \int_K u^2(v) \times e^{-|v|^2/2}\, dv$.

Repeating the derivation of (90), we obtain

$$\lim_{n\to\infty} n^{p+d/2}\mathbf{P}\Big(\frac{x+S([tn])}{\sqrt{n}} \in D, x+S(n)=y, \tau_x>n\Big) = \frac{V(x)V'(y)H_0^2}{(2\pi)^d (t(1-t))^{p+d/2}} \int_D u^2(w) e^{-|w|^2/2t(1-t)}\, dw.$$

Combining this with (92), we get (11). Thus, the proof is finished.

Acknowledgements. The authors are grateful to the referees for a number of comments and suggestions that helped us to improve the exposition.

REFERENCES

[1] AFANASYEV, V. I., GEIGER, J., KERSTING, G. and VATUTIN, V. A. (2005). Criticality for branching processes in random environment. *Ann. Probab.* **33** 645–673. MR2123206

[2] BAIK, J. and SUIDAN, T. M. (2007). Random matrix central limit theorems for nonintersecting random walks. *Ann. Probab.* **35** 1807–1834. MR2349576

[3] BAÑUELOS, R. and SMITS, R. G. (1997). Brownian motion in cones. *Probab. Theory Related Fields* **108** 299–319. MR1465162

[4] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[5] BOLTHAUSEN, E. (1976). On a functional central limit theorem for random walks conditioned to stay positive. *Ann. Probab.* **4** 480–485. MR0415702

[6] BOUSQUET-MÉLOU, M. (2005). Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Probab.* **15** 1451–1491. MR2134111

[7] BOUSQUET-MÉLOU, M. and MISHNA, M. (2010). Walks with small steps in the quarter plane. In *Algorithmic Probability and Combinatorics*. *Contemp. Math.* **520** 1–39. Amer. Math. Soc., Providence, RI. MR2681853

[8] BRYN-JONES, A. and DONEY, R. A. (2006). A functional limit theorem for random walk conditioned to stay non-negative. *J. Lond. Math. Soc.* (2) **74** 244–258. MR2254563

[9] BURKHOLDER, D. L. (1977). Exit times of Brownian motion, harmonic majorization, and Hardy spaces. *Adv. Math.* **26** 182–205. MR0474525

[10] CARAVENNA, F. (2005). A local limit theorem for random walks conditioned to stay positive. *Probab. Theory Related Fields* **133** 508–530. MR2197112

[11] CHAVEL, I. (1984). *Eigenvalues in Riemannian Geometry*. *Pure and Applied Mathematics* **115**. Academic Press, Orlando, FL. MR0768584

[12] COHEN, J. W. (1992). *Analysis of Random Walks*. *Studies in Probability, Optimization and Statistics* **2**. IOS Press, Amsterdam. MR1196229

[13] DEBLASSIE, R. D. (1987). Exit times from cones in $\mathbf{R}^n$ of Brownian motion. *Probab. Theory Related Fields* **74** 1–29. MR0863716

[14] DENISOV, D., KORSHUNOV, D. and WACHTEL, V. (2013). Potential analysis for positive recurrent Markov chains with asymptotically zero drift: Power-type asymptotics. *Stochastic Process. Appl.* **123** 3027–3051. MR3062435

[15] DENISOV, D. and WACHTEL, V. Exit times for integrated random walks. Available at arXiv:1207.2270.

[16] DENISOV, D. and WACHTEL, V. (2010). Conditional limit theorems for ordered random walks. *Electron. J. Probab.* **15** 292–322. MR2609589

[17] EICHELSBACHER, P. and KÖNIG, W. (2008). Ordered random walks. *Electron. J. Probab.* **13** 1307–1336. MR2430709

[18] ESSEEN, C. G. (1968). On the concentration function of a sum of independent random variables. *Z. Wahrsch. Verw. Gebiete* **9** 290–308. MR0231419

[19] FAYOLLE, G., IASNOGORODSKI, R. and MALYSHEV, V. (1999). *Random Walks in the Quarter-Plane: Algebraic Methods, Boundary Value Problems and Applications*. *Applications of Mathematics (New York)* **40**. Springer, Berlin. MR1691900

[20] FAYOLLE, G. and RASCHEL, K. (2012). Some exact asymptotics in the counting of walks in the quarter plane. In *23rd Intern. Meeting on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA'12)*. *Discrete Math. Theor. Comput. Sci. Proc., AQ* 109–124. Assoc. Discrete Math. Theor. Comput. Sci., Nancy. MR2957325

[21] FEIERL, T. (2014). Asymptotics for the number of walks in a Weyl chamber of type B. *Random Structures Algorithms* **45** 261–305. MR3245292

[22] FISHER, M. E. (1984). Walks, walls, wetting, and melting. *J. Stat. Phys.* **34** 667–729. MR0751710

[23] FUK, D. X. and NAGAEV, S. V. (1971). Probability inequalities from sums of independent random variables. *Theory Probab. Appl.* **16** 643–660.

[24] GARBIT, R. (2009). Brownian motion conditioned to stay in a cone. *J. Math. Kyoto Univ.* **49** 573–592. MR2583602

[25] GARBIT, R. (2011). A central limit theorem for two-dimensional random walks in a cone. *Bull. Soc. Math. France* **139** 271–286. MR2828570

[26] GESSEL, I. M. and ZEILBERGER, D. (1992). Random walk in a Weyl chamber. *Proc. Amer. Math. Soc.* **115** 27–31. MR1092920

[27] GËTTSE, F. and ZAĬTSEV, A. YU. (2009). The accuracy of approximation in the multidimensional invariance principle for sums of independent identically distributed random vectors with finite moments. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **368** 110–121, 283–284. MR2749186

[28] GRABINER, D. J. and MAGYAR, P. (1993). Random walks in Weyl chambers and the decomposition of tensor powers. *J. Algebraic Combin.* **2** 239–260. MR1235279

[29] GREENWOOD, P. and SHAKED, M. (1977). Fluctuations of random walk in R^d and storage systems. *Adv. in Appl. Probab.* **9** 566–587. MR0464406

[30] IGLEHART, D. L. (1974). Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probab.* **2** 608–619. MR0362499

[31] KÖNIG, W. (2005). Orthogonal polynomial ensembles in probability theory. *Probab. Surv.* **2** 385–447. MR2203677

[32] KÖNIG, W., O'CONNELL, N. and ROCH, S. (2002). Non-colliding random walks, tandem queues, and discrete orthogonal polynomial ensembles. *Electron. J. Probab.* **7** 24 pp. (electronic). MR1887625

[33] KÖNIG, W. and SCHMID, P. (2010). Random walks conditioned to stay in Weyl chambers of type C and D. *Electron. Commun. Probab.* **15** 286–296. MR2670195

[34] KURKOVA, I. and RASCHEL, K. (2012). On the functions counting walks with small steps in the quarter plane. *Publ. Math. Inst. Hautes Études Sci.* **116** 69–114. MR3090255

[35] MACPHEE, I. M., MENSHIKOV, M. V. and WADE, A. R. (2013). Moments of exit times from wedges for non-homogeneous random walks with asymptotically zero drifts. *J. Theoret. Probab.* **26** 1–30. MR3023832

[36] MCCONNELL, T. R. (1984). Exit times of N-dimensional random walks. *Z. Wahrsch. Verw. Gebiete* **67** 213–233. MR0758074

[37] MOGULSKII, A. A. and PECHERSKII, E. A. (1977). On the first exit time from a semigroup for a random walk. *Theory Probab. Appl.* **22** 818–825.

[38] MORREY, C. B. JR. and NIRENBERG, L. (1957). On the analyticity of the solutions of linear elliptic systems of partial differential equations. *Comm. Pure Appl. Math.* **10** 271–290. MR0089334

[39] NAGAEV, S. V. (1979). Large deviations of sums of independent random variables. *Ann. Probab.* **7** 745–789. MR0542129

[40] RASCHEL, K. (2012). Counting walks in a quadrant: A unified approach via boundary value problems. *J. Eur. Math. Soc.* (*JEMS*) **14** 749–777. MR2911883

[41] SHIMURA, M. (1991). A limit theorem for two-dimensional random walk conditioned to stay in a cone. *Yokohama Math. J.* **39** 21–36. MR1137264

[42] SPITZER, F. (1976). *Principles of Random Walk*, 2nd ed. Springer, New York. MR0388547

[43] VAROPOULOS, N. T. (1999). Potential theory in conical domains. *Math. Proc. Cambridge Philos. Soc.* **125** 335–384. MR1643806

[44] VAROPOULOS, N. T. (2000). Potential theory in conical domains. II. *Math. Proc. Cambridge Philos. Soc.* **129** 301–319. MR1765917

[45] VATUTIN, V. A. and WACHTEL, V. (2009). Local probabilities for random walks conditioned to stay positive. *Probab. Theory Related Fields* **143** 177–217. MR2449127

SCHOOL OF MATHEMATICS
UNIVERSITY OF MANCHESTER
OXFORD ROAD
MANCHESTER M13 9PL
UNITED KINGDOM
E-MAIL: denis.denisov@manchester.ac.uk

MATHEMATICAL INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTRASSE 39
D–80333 MUNICH
GERMANY
E-MAIL: wachtel@mathematik.uni-muenchen.de