RANDOM WALKS ON COMPACT SEMIGROUPS

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ABSTRACT. Let β be a regular Borel probability measure with support F on a compact semigroup S. Let $X_1, X_2, \cdots$ be a sequence of independent random variables on some probability space (Ω, Σ, P) with values in S, having identical distribution $P(X_n \in B) = \beta(B)$. The random walk $Z_n = X_1 X_2 \cdots X_n$ is studied in this paper. Let D be the closed semigroup generated by F. An element x in D is called recurrent iff $P_x(Z_n \in N_x \text{ i.o.}) = 1$ for every open set N_x containing x. This paper characterizes the recurrence of an element x in terms of divergence of the series $\sum_{n=1}^{\infty} \beta^n(N_x)$ for every open set N_x containing x. It also shows that the set of recurrent states of $\{Z_n\}$ is precisely the kernel of D.

1. Let S be a compact semigroup, i.e. a compact Hausdorff space which is algebraically a semigroup with jointly continuous multiplication. Let β be a regular probability measure defined on the Borel subsets of S. Let $X_1, X_2, \cdots$ be a sequence of independent random variables on some probability space (Ω, Σ, P) with values in S, having identical distribution $P(X_n \in B) = \beta(B)$ for Borel sets B in S. For probabilistic considerations, we will, with no loss of generality, identify the process X_n with the usual coordinate representation process in the sequence space S^∞. The sequence $Z_n = X_1 X_2 \cdots X_n$, $n = 1, 2, \cdots$, is called a right random walk on S generated by β. The sequence $W_n = X_n X_{n-1} \cdots X_1$, $n = 1, 2, \cdots$, is analogously called a left random walk on S. The set $D = \text{cl}(\bigcup_{n=1}^{\infty} F^n)$, the closed subsemigroup generated by $F = \{x \in S:$ every open set containing x has positive β-measure$\}$ = the support of β, is called the support of the random walk $\{Z_n\}$ or $\{W_n\}$. By the kernel K of D, we mean the minimal two-sided ideal (which always exists since D is a compact semigroup) of D. It is well known that K is a completely simple semigroup (i.e. simple and contains a primitive idempotent) and, for every $x \in K$, $xK = xD$ is a minimal right ideal of D and contains an idempotent. For these facts, we refer to [1].

Throughout this paper, we will use the following notations.

$$y^{-1}B = \{x \in S: yx \in B\},$$
$$A^{-1}B = \bigcup \{y^{-1}B: y \in A\} \quad \text{and}$$
$$Z_k^{-1}Z_n = X_{k+1}X_{k+2} \cdots X_n.$$

Let β^n denote $\beta * \beta * \cdots * \beta$ (n-times). Then $P(Z_k \in B) = \beta^k(B)$ and $P(Z_{n+k} \in B | Z_n = x) = \beta^k(x^{-1}B)$. [Note that implicitly we have assumed that the Z_n's are measurable in the sequence probability space. The measurability of the Z_n's is, of course, a fact when S is second countable.]

An element $x \in D$ is called a recurrent state for $\{Z_n\}$ if

$$P_x(Z_n \in N_x \text{ i.o.}) \equiv P(Z_n \in N_x \text{ infinitely often} | Z_1 = x) = 1$$

for every open set N_x containing x. A random walk having at least one recurrent state is called recurrent. Recurrent random walks were studied recently by Rosenblatt [8], Martin-Löff [6] and Larisse ([3], [4]) in the case of discrete semigroups and by the authors [10] in the case of compact abelian and completely simple semigroups. In [10], the authors showed that the set of recurrent states of the random walk $\{Z_n\}$ on a compact abelian S is precisely the kernel K of D. The methods in [10] exploited the abelian property of S and so could not be extended to the nonabelian case.

The purpose of this paper is to characterize the recurrence of an element $x \in D$ in terms of divergence of the series $\sum_{n=1}^{\infty} \beta^n(N_x)$ for every open set N_x containing x and to show that the set of recurrent states of $\{Z_n\}$ is precisely the kernel K of D, in the case of compact semigroups S.

It is relevant to mention that the same problem of recurrence was first considered by Chung and Fuchs on the Euclidean n-space in [2] and by Loynes on locally compact and compact groups in [5] with a little different definition which is equivalent to our definition on groups. Our definition of recurrence on semigroups differs from that of Chung and Fuchs or Loynes since one can easily find compact semigroups where there are random walks which are not recurrent in their sense; for instance, if $S = E \times G$, where $G = \{1, a\}$ is a group and $E = \{e_1, e_2\}$ is a left-zero semigroup and β is the normed counting measure on S, then it is easy to see that, for every x in S, $P(Z_n = x \text{ i.o.}) = \frac{1}{2} < 1$.

In what follows, S (and hence D) is always compact. We write: (a) $x \to y$ when $y \in xD$ and x, y elements of D (b) $x \to y$ i.o. when

$$P_x(Z_n \in N_y \text{ i.o.}) = 1$$

for every open set N_y containing y, for x, y elements of D.

2. Before we present our main results, we need the following lemmas.

LEMMA 2.1. *Given $x \in S$ and N_y, an open set containing y in S, we can find open sets N_1 and N_2 containing x and y respectively such that $N_1^{-1} N_2 \subseteq x^{-1} N_y$.*

PROOF. Suppose first that $x^{-1}N_y$ is empty. Then $y \notin xS$. If for every open set N' containing y and open set N'' containing x, $N' \cap N''S$ is nonempty, then we can find a net y_u converging to y and a net x_u converging to x such that $y_u = x_u s_u$, where s_u's are elements in S. Since S is compact, we can find a subnet of s_u converging to some element s in S. But then $y = xs$ and this contradicts that $x^{-1}N_y$ is empty. Hence, when $x^{-1}N_y$ is empty, there exist open sets N_1 and N_2 containing x and y respectively such that $N_2 \cap N_1 S$ is empty which means that $N_1^{-1}N_2$ is empty.

Suppose next that $x^{-1}N_y$ is nonempty. Suppose, for every open set N' containing y and contained in N_0, a compact neighbourhood of y such that $N_0 \subset N_y$, and every open set N'' containing x, we have $N''^{-1}N' \cap (x^{-1}N_y)^c$ is nonempty. Then we can find a net x_u converging to x and $z_u \in N''^{-1}N' \cap (x^{-1}N_y)^c$ such that $x_u z_u \in N_0$ and $x z_u \notin N_y$. Since S is compact, we can find a subnet of z_u converging to z such that $xz \in N_0$ and $xz \notin N_y$. This is a contradiction and the lemma follows.

LEMMA 2.2. *If $x \to y$ i.o. and $x \to z$, then $z \to y$ i.o.*

PROOF. If $z \nrightarrow y$ i.o., then there is an open set N containing y such that $P_z(Z_n \in N$ finitely often$) > 0$. By Lemma 2.1, we can find open sets N_z and N_y containing z and y respectively such that $N_z^{-1}N_y \subset z^{-1}N$. But then

$P(Z_n \in N_z^{-1}N_y$ finitely often$) > 0$ and

$$
\begin{aligned}
(1)\quad P_x(Z_n \in N_y \text{ finitely often}) &\geqq P_x(Z_k \in N_z, Z_n \in N_y \text{ finitely often}) \\
&\geqq P_x(Z_k \in N_z, Z_k^{-1}Z_n \in N_z^{-1}N_y \text{ finitely often}) \\
&= P_x(Z_k \in N_z)P(Z_n \in N_z^{-1}N_y \text{ finitely often}) \\
&\geqq P_x(Z_k \in N_z)P(Z_n \in z^{-1}N \text{ f.o.}) > 0,
\end{aligned}
$$

if Z_k is such that $P_x(Z_k \in N_z) > 0$. This is a contradiction to the fact that $x \to y$ i.o. and the lemma follows.

LEMMA 2.3. *The set R of the recurrent states of $\{Z_n\}$ is a subset of the kernel K of D.*

PROOF. Let $x \in R$. Then $x \in xD$. We claim that xD is a minimal right ideal of D. To prove this, let $a \in xD$. Suppose that $x \notin aD$. Then, by Lemma 2.1, we can find open sets N_x and N_a containing x and a respectively such that $N_a^{-1}N_x \cap D$ is empty. Since $a \in xD$, for some positive integer k, $P_x(Z_k \in N_a) > 0$. Now

$$
\begin{aligned}
P_x(Z_n \in N_x \text{ finitely often}) &\geqq P_x(Z_k \in N_a, Z_{n+k} \in N_x \text{ finitely often}) \\
&\geqq P_x(Z_k \in N_a, Z_k^{-1}Z_{n+k} \in N_a^{-1}N_x \text{ finitely often}) \\
&= P_x(Z_k \in N_a) > 0.
\end{aligned}
$$

This contradicts that $x \in R$ and therefore, $x \in aD$ so that xD is a minimal right ideal of D. Since the kernel K of D is the union of all minimal right ideals of D, $x \in K$.

THEOREM 2.4. *Let $y \in D$. Then the following are equivalent*:
(a) $y \to y$ *i.o. with respect to* $\{Z_n\}$.
(b) $\sum_{n=1}^{\infty} \beta^n(y^{-1}N_y) = \infty$ *for every open set N_y containing y*.
(c) $\sum_{n=1}^{\infty} \beta^n(N_y) = \infty$ *for every open set N_y containing y*.
(d) $y \to y$ *i.o. with respect to* $\{W_n\}$.
(e) $\sum_{n=1}^{\infty} \beta^n(N_y y^{-1}) = \infty$ *for every open set N_y containing y*.

PROOF. We will prove that (a), (b) and (c) are equivalent. Then by dual arguments, (c), (d) and (e) also will be equivalent.

First, (a) implies (b), by the Borel-Cantelli Lemma.

Next, we show that (b) implies (c). We assume (b) and that there is an open set N_y containing y such that $\sum_{n=1}^{\infty} \beta^n(N_y) < \infty$. Let N' be a compact neighbourhood of y such that $N' \subset N_y$. Then $y^{-1}N'$ is compact. Noting that for $x \in y^{-1}N'$, $yx \in N'$ and so we can find open sets $N_x(y)$ and $N(x)$ containing y and x respectively such that $N_x(y)N(x) \subset N_y$, we can find (using compactness) an open set N containing y such that $y^{-1}N \subset z^{-1}N_y$ for every z in N. Now for every positive integer k,

$$\sum_{n=1}^{\infty} \beta^n(N_y) \geqq \int \left[\sum_{n=k+1}^{\infty} \beta^{n-k}(z^{-1}N_y) \right] \beta^k(dz)$$

which means that we can find a set A such that, for every positive integer k, $\beta^k(A) = 0$ and, for $z \notin A$, $\sum_{n=1}^{\infty} \beta^n(z^{-1}N_y) < \infty$. Hence there exists $w \in N$ such that $\sum_{n=1}^{\infty} \beta^n(w^{-1}N_y) < \infty$. Since $y^{-1}N \subset w^{-1}N_y$, this contradicts the assumption of (b).

Finally, we show that (c) implies (a). Let k be an arbitrary positive integer and N_y be an open set containing y. Then if (c) holds, we have $\sum_{j=1}^{k} \sum_{i=0}^{\infty} P(Z_{j+ik} \in N_y) = \infty$. So we can find an integer m such that $1 \leqq m \leqq k$ and $\sum_{i=0}^{\infty} P(Z_{m+ik} \in N_y) = \infty$. Now

$$1 \geqq P(Z_n \in N_y \text{ finitely often})$$

$$\geqq \sum_{i=0}^{\infty} P(Z_{m+ik} \in N_y, Z_n \notin N_y \text{ for all } n \geqq m + (i+1)k)$$

$$\geqq \sum_{i=0}^{\infty} P(Z_{m+ik} \in N_y, Z_{m+ik}^{-1}Z_n \notin N_y^{-1}N_y \text{ for all } n \geqq m + (i+1)k)$$

$$= P(Z_n \notin N_y^{-1}N_y \text{ for all } n \geqq k) \sum_{i=0}^{\infty} P(Z_{m+ik} \in N_y)$$

which means that, for any open set N_y containing y and every positive

integer k,

$$P(Z_n \notin N_y^{-1}N_y \text{ for all } n \geqq k) = 0.$$

Now given an open set N containing y, by Lemma 2.1, we can find an open set N_y containing y such that $N_y^{-1}N_y \subset y^{-1}N_y$. Then

$$P_y(Z_n \in N \text{ finitely often}) \leqq P(Z_n \in N_y^{-1}N_y \text{ finitely often}) = 0.$$

Hence $y \rightarrow y$ i.o. and (a) holds.

THEOREM 2.5.   *Let $y \in D$ and $x \in K =$ the kernel of D. Then the following are equivalent*:
(a) *$x \rightarrow y$ i.o. with respect to $\{Z_n\}$.*
(b) *$\sum_{n=1}^{\infty} \beta^n(x^{-1}N_y) = \infty$ for every open set N_y containing y.*
(c) *$x \rightarrow y$ and $\sum_{n=1}^{\infty} \beta^n(N_y) = \infty$ for every open set N_y containing y.*

PROOF.   First, (a) implies (b), by the Borel-Cantelli Lemma.

Next, we show that (b) implies (c). Suppose that (b) holds. Then by following the proof of "(c) implies (a)" in Theorem 2.4 and noting that

$$P_x(Z_{m+ik} \in N_y, Z_{m+ik}^{-1}Z_n \notin N_y^{-1}N_y \text{ for all } n \geqq m + (i + 1)k)$$
$$= P_x(Z_{m+ik} \in N_y) \cdot P(Z_n \notin N_y^{-1}N_y \text{ for all } n \geqq k),$$

we see that, for every open set N_y containing y and each positive integer k, $P(Z_n \notin N_y^{-1}N_y \text{ for all } n \geqq k) = 0$. Then it follows as in Theorem 2.4 that $y \rightarrow y$ i.o. This implies (c), by Theorem 2.4.

Finally, we show that (c) implies (a). Since (c) holds, by Theorem 2.4, $y \rightarrow y$ i.o. Since $x \rightarrow y$, $y \in xD$. But xD is a minimal right ideal of D, x being an element of K. Therefore, $x \in xD = yD$ so that $y \rightarrow x$. By Lemma 2.2, $x \rightarrow y$ i.o. The proof of the theorem is complete.

REMARK.   Using Theorem 2.4, it is now easy to show that the set R of recurrent states of $\{Z_n\}$ is precisely the kernel K of D. If, for each $y \in D$, y is not recurrent, then by Theorem 2.4 we can find an open neighbourhood N_y for each y such that $\sum_{n=1}^{\infty} \beta^n(N_y)$ is finite. But then since D is compact, $\sum_{n=1}^{\infty} \beta^n(D)$ is finite, which is absurd. Hence R is non-empty. Also, R is a left ideal of D. If R' is the set of recurrent states of the left random walk $\{W_n\}$, then, by Theorem 2.4, $R = R'$. Also, R' is a right ideal of D. Therefore, R is a two-sided ideal of D and hence contains K. By Lemma 2.3, $R \subset K$. Hence $R = K$. A direct proof of this interesting fact can also be given independently of Theorem 2.4. This is shown in what follows.

THEOREM 2.6.   *The set R of recurrent states of $\{Z_n\}$ is precisely the kernel K of D.*

PROOF. Because of Lemma 2.3, suffice it to show that $K \subset R$. We will use the fact that for any regular probability measure Q on S, the function $x \to Q(x^{-1}U)$ is lower semicontinuous for open U. Now let $x \in K$. For any $y \in xD$, we have $x \in yD$. For any open neighbourhood U of x, $U \cap yD$ is nonempty. Thus for some n, $U \cap yF^n$ is nonempty. Let

$$U_n = \{y \in xD : U \cap yF^n \text{ is nonempty}\}.$$

Then U_n is relatively open in the compact set xD. So there is a finite subcover and, for some finite n,

$$g(y) \equiv \sum_{j \leq n} \beta^j(y^{-1}U) > 0$$

for all $y \in xD$. Then by the lower semicontinuity of $g(y)$, there is a $p > 0$ such that $g(y) \geq p$ for all $y \in xD$. Let $q = p/n$. For any positive integer k, the conditional probability that $xZ_m \notin U$ for $nk < m \leq n(k+1)$, given any conditions on X_j for $j \leq nk$, is at most $1 - q$. Hence

$$P(xZ_m \notin U \text{ for } nk < m \leq nN) \leq (1 - q)^{N-k}$$

which converges to zero as N approaches infinity. This means that

$$P_x(Z_m \in U \text{ finitely often}) = 0$$

so that $x \in R$. The proof is complete.

ACKNOWLEDGEMENT. We are grateful to the referee for showing us the direct proof of Theorem 2.6.

ADDED IN PROOF. Theorem 2.5 is true for every $x, y \in D$, if S is compact abelian. For inequality (1) (Proof of Lemma 2.2) (with P_x, z, N_z replaced by P, x, N_x resp.) shows that if $x \mapsto y$ i.o., then $P(Z_n \in N_y \text{ f.o.}) > 0$, a contradiction to a result of [10], since for every y in the kernel $P[Z_n \in N_y \text{ i.o.}] = 1$ for every open set N_y containing y.

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