RANDOM WALKS WITH SPHERICAL SYMMETRY

BY

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1. Introduction

Let $\mathbf{X}=(X_1, X_2, \ldots, X_n)$ be a random n-vector with spherical symmetry, that is, a random variable taking values in Euclidean n-space R^n with the property that, if A is any measurable subset of R^n, and A' is obtained from A by rotation about the origin, then

$$P(\mathbf{X}\in A)=P(\mathbf{X}\in A').$$

Then the distribution of $\mathbf{X}$ is determined by that of its length

$$X=|\mathbf{X}|=\left\{\sum_{i=1}^{n} X_i^2\right\}^{\frac{1}{2}},$$

and in particular the characteristic function of $\mathbf{X}$ is given by

$$\Phi(\mathbf{t})=E(e^{i\mathbf{t}\cdot\mathbf{X}})=E(e^{itX\cos\theta}), \tag{1}$$

where $t=|\mathbf{t}|$, and θ is the angle between the vectors $\mathbf{t}$ and $\mathbf{X}$. Clearly θ and X are independent, θ having the distribution which a uniformly distributed unit vector makes with a fixed axis.

It is readily shown that, when $n\geqslant 2$, $\lambda=\cos\theta$ has a probability density

$$f_n(\lambda)=\frac{(\frac{1}{2}n-1)!}{\pi^{\frac{1}{2}}(\frac{1}{2}n-\frac{3}{2})!}(1-\lambda^2)^{\frac{1}{2}n-\frac{3}{2}} \qquad (-1\leqslant\lambda\leqslant 1). \tag{2}$$

Hence, for any complex u,

$$\begin{aligned} E(e^{iu\cos\theta}) &= \frac{(\frac{1}{2}n-1)!}{\pi^{\frac{1}{2}}(\frac{1}{2}n-\frac{3}{2})!}\int_{-1}^{1} e^{iu\lambda}(1-\lambda^2)^{\frac{1}{2}n-\frac{3}{2}}\,d\lambda \\ &= J_{\frac{1}{2}n-1}(u)\,(\tfrac{1}{2}u)^{-\frac{1}{2}n+1}(\tfrac{1}{2}n-1)! \end{aligned}$$

by the Poisson integral ([16], 48) for the Bessel function $J\cdot(\cdot)$.

It will be convenient to write

$$n = 2(1+s), \tag{3}$$

and to recall the Bessel function $\Lambda_s(\cdot)$ defined by

$$\Lambda_s(u) = J_s(u)\ s!\ (\tfrac{1}{2}u)^{-s} = \sum_{r=0}^{\infty} (-1)^r (\tfrac{1}{2}u)^{2r} s!/r!\,(s+r)! \tag{4}$$

(cf. [10]). Then equation (1) takes the form

$$\Phi(\mathbf{t}) = E\{\Lambda_s(tX)\}. \tag{5}$$

When $n=1$, $\mathbf{X}$ is simply a real random variable with a symmetric distribution, and so

$$\Phi(\mathbf{t}) = E(\tfrac{1}{2}e^{itX} + \tfrac{1}{2}e^{-itX}) = E(\cos tX),$$

which is of the form (5) since

$$\Lambda_{-\frac{1}{2}}(u) = \cos u. \tag{6}$$

Now let $\mathbf{X}, \mathbf{Y}$ be two independent random n-vectors with spherical symmetry, and let $\mathbf{Z} = \mathbf{X}+\mathbf{Y}$. Then $\mathbf{Z}$ clearly also has spherical symmetry, and since

$$E(e^{i\mathbf{t}\cdot\mathbf{Z}}) = E(e^{i\mathbf{t}\cdot\mathbf{X}})\ E(e^{i\mathbf{t}\cdot\mathbf{Y}}),$$

it follows from (5) that

$$E\{\Lambda_s(tZ)\} = E\{\Lambda_s(tX)\}\, E\{\Lambda_s(tY)\} \tag{7}$$

for all $t \geqslant 0$. Thus, if we are considering problems involving the addition of independent, spherically symmetric random vectors, we can work wholly in terms of the distributions of length, using instead of the usual characteristic function a *radial characteristic function* defined by

$$\Psi(t) = E\{\Lambda_s(tX)\}. \tag{8}$$

This device has been used by a number of writers; see for instance [12]. We shall see that the radial characteristic function possesses many of the properties usually associated with the familiar univariate characteristic function.

It has been pointed out by Haldane [9] that it is possible to work from the beginning with the lengths of the random vectors by using the following procedure. If $\mathbf{Z} = \mathbf{X}+\mathbf{Y}$, then the length of $\mathbf{Z}$ is given by

$$Z^2 = X^2 + Y^2 + 2XY\cos\theta, \tag{9}$$

where θ is now the angle between $\mathbf{X}$ and $\mathbf{Y}$. By spherical symmetry, θ is independent of (X, Y), and $\lambda = \cos\theta$ has (for $n \geqslant 2$) the probability density (2). Thus, if we know the distributions of X and Y we can use (9) to compute that of Z.

More generally, let F be any distribution function on $[-1, 1]$. Then, if X, Y are independent, non-negative random variables, we can construct a new random variable Z by

$$Z = (X^2 + Y^2 + 2\lambda XY)^{\frac{1}{2}}, \tag{10}$$

where λ, independent of (X, Y), has the distribution function F. This does not determine Z as a random variable (because of the arbitrary choice of λ) but it does determine the distribution of Z, and indeed the joint distribution of X, Y, Z. A random variable constructed in this way will be denoted by

$$Z = X \circ Y, \tag{11}$$

where it must be emphasised that the symbol $\circ$ depends on the choice of the distribution F.

The lack of definiteness in the definition of the symbol $\circ$ forces us to make a convention about the interpretation of an expression containing the symbol more than once. We make the natural convention that the different λ used in defining these symbols shall be *independent* random variables.

With this procedure a generalisation has been given of the addition of spherically symmetric random vectors, since if F has the density (2), then $X \circ Y$ can be interpreted as $|\mathbf{X}+\mathbf{Y}|$, where $\mathbf{X}$ and $\mathbf{Y}$ are independent and spherically symmetric with $X = |\mathbf{X}|$, $Y = |\mathbf{Y}|$.

With the convention established above, we can construct from three independent non-negative random variables X, Y, Z the two expressions

$$X \circ (Y \circ Z) \quad \text{and} \quad (X \circ Y) \circ Z.$$

In the random walk case, these are both representations of $|\mathbf{X}+\mathbf{Y}+\mathbf{Z}|$, and therefore have the same distribution, and indeed the same joint distribution with X, Y, Z. Haldane has observed that this associativity property does not hold for general F, and has shown that the assumption that the two expressions have the same distribution gives relations between the cumulants of F such that exactly one of these can be specified arbitrarily. (For a rather different, but related, associativity property we refer to [15]).

The object of the present paper is to study the operation $\circ$ in those cases for which the associativity property holds. These depend on a single real parameter n, and may naturally be interpreted as describing random walks in a space whose dimension n can take any real value in $1 \leqslant n \leqslant \infty$. Thus, for instance, we can give a meaning to a random walk with spherical symmetry in $2\frac{1}{2}$ dimensions, and the resulting process will be in a sense intermediate between those in 2 and 3 dimensions.

It will prove to be possible to construct for these processes analogues of many of the results known for ordinary random walks. Thus we can consider analogues of the central limit theorem, of stable and infinitely divisible distributions, of Pólya's theorem, and of Brownian processes. Walks in different numbers of dimensions will be connected by a projection procedure. As well as being relevant to probability theory, the results obtained give an interpretation for some well-known formulae involving Bessel functions of general order.

2. The associativity condition

We shall say that the distribution function F on $[-1, 1]$ is associative if, whenever X, Y, Z are independent non-negative random variables, we have

$$X \circ (Y \circ Z) \doteq (X \circ Y) \circ Z, \tag{12}$$

where the sign $\doteq$ denotes equality of distribution. The object of this section is to identify those F which are associative.

LEMMA 1. *If F is associative then, whatever the constants $x, y \geqslant 0$,*

$$\lambda y + \mu x \doteq \mu(x^2 + y^2 + 2\lambda xy)^{\frac{1}{2}}, \tag{13}$$

where λ, μ are independent with distribution function F.

If

$$G(u) = E(e^{iu\lambda}) = \int_{-1}^{1} e^{iu\lambda}\, dF(\lambda), \tag{14}$$

then

$$G(x)\, G(y) = \int_{-1}^{1} G\{(x^2 + y^2 + 2\lambda xy)^{\frac{1}{2}}\}\, dF(\lambda), \tag{15}$$

and, whenever $t \geqslant 0$ and X, Y are independent,

$$E\{G(tX)\}\, E\{G(tY)\} = E\{G[t(X \circ Y)]\}. \tag{16}$$

Proof. In equation (12), take X, Y, Z to be the sure random variables $X = x$, $Y = y$, $Z = z$. Then

$$\begin{aligned} x^2 + (y^2 + z^2 + 2\lambda yz) + 2\mu x(y^2 + z^2 + 2\lambda yz)^{\frac{1}{2}} \\ \doteq (x^2 + y^2 + 2\lambda xy) + z^2 + 2\mu z(x^2 + y^2 + 2\lambda xy)^{\frac{1}{2}}, \end{aligned}$$

where λ, μ are independent with distribution function F. Subtracting the constant $x^2 + y^2 + z^2$ and dividing by $2z$, we have

$$\lambda y + \mu x(1 + 2\lambda y/z + y^2/z^2)^{\frac{1}{2}} \doteq \lambda xy/z + \mu(x^2 + y^2 + 2\lambda xy)^{\frac{1}{2}}.$$

As $z \to \infty$, both sides converge in probability, and hence in distribution. Hence

$$\lambda y + \mu x \doteq \mu(x^2 + y^2 + 2\lambda xy)^{\frac{1}{2}},$$

proving (13). Taking characteristic functions,

$$E(e^{i\lambda y})\,E(e^{i\mu x}) = E\{\exp[i\mu(x^2+y^2+2\lambda xy)^{\frac{1}{2}}]\},$$

which is equivalent to (15), and (16) follows at once.

The significance of (16) is that it shows that $E\{G(tX)\}$ plays the role of a characteristic function for $\circ$, (16) being the exact analogue of the formula

$$E(e^{itX})\,E(e^{itY}) = E(e^{it(X+Y)}).$$

Before using Lemma 1 to identify all the associative F, we note two special cases for which (13) holds.

Case 1. If the distribution of λ is concentrated at $\lambda=1$, then both sides of (13) are concentrated at $x+y$. We have

$$G(u) = e^{iu}.$$

In this case

$$X\circ Y = X+Y,$$

and F is obviously associative.

Case 2. If the distribution of λ is concentrated at $\lambda=0$, then both sides of (13) are concentrated at 0. Then

$$G(u) = 1,$$

and

$$(X\circ Y)^2 = X^2+Y^2.$$

Again F is clearly associative.

No other degenerate F are associative, since if F concentrates probability 1 at $\lambda=a$, (13) implies that

$$a(x+y) = a(x^2+y^2+2axy)^{\frac{1}{2}},$$

i.e.

$$a(1-a)\,xy = 0,$$

so that $a=0$ or 1.

In the following analysis, we shall assume that Cases 1 and 2 have been excluded. First consider the equation (15) for fixed positive x. From (14), the left hand side is the restriction to $y\geqslant 0$ of an entire function of the complex variable y. The right-hand side can be continued to a regular function of y in $|y|<x$, by taking the branch of

$$(x^2+y^2+2\lambda xy)^{\frac{1}{2}}$$

which is positive on $0<y<x$ and single-valued in $|y|<x$; this is possible because all the zeros of

$$x^2+y^2+2\lambda xy$$

lie on $|y|=x$. It follows that (15) holds for all y in $|y|<x$.

In particular, if we let $y\to -x$ through points in $|y|<x$, we get

$$G(x)\,G(-x) = \lim \int_{-1}^{1} G\{(x^2+y^2+2\lambda xy)^{\frac{1}{2}}\}\,dF(\lambda) = \int_{-1}^{1} G\{x(1-\lambda)\sqrt{2}\}\,dF(\lambda)$$

because of our choice of the branch of $(x^2+y^2+2\lambda xy)^{\frac{1}{2}}$. This last equation holds for all positive x, and since both sides can be continued to entire functions of x, it follows that for all complex x,

$$G(x)\,G(-x) = \int_{-1}^{1} G\{x(1-\lambda)\sqrt{2}\}\,dF(\lambda). \tag{17}$$

Hence the right hand side must be an even function of x, and so, for each odd integer n, its nth derivative vanishes at $x=0$. Thus

$$\int_{-1}^{1} 2^{\frac{1}{2}n}(1-\lambda)^n G^{(n)}(0)\,dF(\lambda) = 0,$$

and so, since if we exclude Case 1

$$\int_{-1}^{1} (1-\lambda)^n dF(\lambda) > 0,$$

we have $G^{(n)}(0)=0$ for each odd n. Therefore G is an even function.

It follows that, excluding Case 1, all the associative distribution functions F are symmetric. Notice that all the moments of F are finite, and write

$$M_n = E(\lambda^{2n}).$$

Since we are excluding Case 2, we have

$$0 < M_n \leqslant 1 \tag{18}$$

for all n. From (13),

$$E(\lambda y + \mu x)^{2n} = E\mu^{2n} E(x^2+y^2+2\lambda xy)^n.$$

This holds for all $x, y \geqslant 0$, and so is a polynomial identity in x, y. Equating coefficients of x^2y^{2n-2},

$$E(\tfrac{1}{2}\cdot 2n(2n-1)\lambda^{2n-2}\mu^2) = M_n E(n + \tfrac{1}{2}n(n-1)(2\lambda)^2),$$

which simplifies to

$$(2n-1)M_{n-1} = 2M_n(\tfrac{1}{2}M_1^{-1} + n - 1).$$

Because of (18) we can define s by

$$\tfrac{1}{2}M_1^{-1} = s+1,$$

and then

$$-\tfrac{1}{2} \leqslant s < \infty. \tag{19}$$

Then

$$M_n = M_{n-1}\,\tfrac{1}{2}(2n-1)/(n+s),$$

and so
$$M_n = 2^{-n}(2n-1)(2n-3)\dots 1/(n+s)(n+s-1)\dots(s+1)$$
$$= 2^{-2n}(2n)!\,s!/n!\,(n+s)!.$$

Hence
$$G(u) = \sum_{n=0}^{\infty} M_n (iu)^{2n}/(2n)! = \sum_{n=0}^{\infty} \frac{s!}{n!(n+s)!}(-\tfrac{1}{4}u^2)^n = \Lambda_s(u),$$

where $\Lambda_s(u)$ is defined by (4). The Poisson integral

$$\Lambda_s(u) = \frac{s!}{\pi^{\frac{1}{2}}(s-\frac{1}{2})!}\int_{-1}^{1} e^{iu\lambda}(1-\lambda^2)^{s-\frac{1}{2}}\,d\lambda \quad (s > -\tfrac{1}{2})$$

([16], 48) shows that, when $s > -\frac{1}{2}$, F is given by

$$dF(\lambda) = \frac{s!}{\pi^{\frac{1}{2}}(s-\frac{1}{2})!}(1-\lambda^2)^{s-\frac{1}{2}}\,d\lambda \quad (-1 \leqslant \lambda \leqslant 1). \tag{20}$$

When $s = -\frac{1}{2}$,
$$\Lambda_s(u) = \cos u = \tfrac{1}{2}e^{iu} + \tfrac{1}{2}e^{-iu},$$

so that F assigns probability $\frac{1}{2}$ to each of the values ± 1.

We have now shown that the characteristic function $G(u)$ of any associative F must have one of the forms

$$e^{iu},\ 1,\ \Lambda_s(u)\,(-\tfrac{1}{2} \leqslant s < \infty).$$

To complete this section we show that all such F are associative, and to do this we need the following uniqueness result, which will be used continually throughout this paper.

LEMMA 2. *If $-\frac{1}{2} \leqslant s < \infty$, and if H_1, H_2 are two (right-continuous) distribution functions on $[0, \infty)$ satisfying*

$$\int_0^{\infty} \Lambda_s(tx)\,dH_1(x) = \int_0^{\infty} \Lambda_s(tx)\,dH_2(x) \tag{21}$$

for all $t > 0$, then $H_1 = H_2$.

Proof. According to a theorem of Weber ([16], 394), we have, for all $s \geqslant -\frac{1}{2}$, $a \geqslant 0$, $p > 0$,

$$\int_0^{\infty} t^{s+1} J_s(at)\,e^{-p^2t^2}\,dt = a^s(2p^2)^{-s-1}\,e^{-a^2/4p^2},$$

which may be written as

$$\int_0^{\infty} t^{2s+1}\Lambda_s(at)\,e^{-p^2t^2}\,dt = \tfrac{1}{2}s!\,p^{-2(s+1)}\,e^{-a^2/4p^2}. \tag{22}$$

If we integrate the left hand side with respect to $dH_j(a)$ $(j = 1, 2)$ and use Fubini's theorem, we get, by (21), an expression independent of j. Hence

$$\int_0^\infty e^{-a^2/4p^2}\,dH_1(a) = \int_0^\infty e^{-a^2/4p^2}\,dH_2(a)$$

for all $p>0$, and hence, by the uniqueness theorem for Laplace transforms, ([17], 63), $H_1 = H_2$.

We are now in a position to prove the basic theorem giving necessary and sufficient conditions for a distribution function F to be associative.

THEOREM 1. *F is associative if and only if it is concentrated at 0 or at 1, or if, for some s in $-\frac{1}{2} \leqslant s < \infty$,*

$$G(u) = \int_{-1}^{1} e^{iu\lambda}\,dF(\lambda) = \Lambda_s(u). \tag{23}$$

In the latter case, if $s = -\frac{1}{2}$, F assigns probability $\frac{1}{2}$ to each of the values ± 1, and if $s > -\frac{1}{2}$, F is given by (20).

Proof. We have already proved that, if F is associative, then one of the above alternatives holds. We have also proved that if F is concentrated at 0 or at 1, it is associative. Hence it suffices to prove that, if F is given by (23), it is associative.

According to a theorem of Gegenbauer ([16], 367) we have, for $s > -\frac{1}{2}$,

$$2^s(s-\tfrac{1}{2})!\,\pi^{\frac{1}{2}} x^{-s} J_s(x)\, y^{-s} J_s(y)$$
$$= \int_0^\pi (x^2+y^2-2xy\cos\theta)^{-\frac{1}{2}s} J_s\{(x^2+y^2-2xy\cos\theta)^{\frac{1}{2}}\}\sin^{2s}\theta\,d\theta.$$

Putting $\lambda = -\cos\theta$, this may be reduced to

$$\Lambda_s(x)\,\Lambda_s(y) = \int_{-1}^{1} \Lambda_s\{(x^2+y^2+2\lambda xy)^{\frac{1}{2}}\}\,dF(\lambda), \tag{24}$$

where $dF(\lambda)$ is given by (20). If $s = -\frac{1}{2}$ this still holds, following now from

$$\cos x \cos y = \tfrac{1}{2}\cos(x+y) + \tfrac{1}{2}\cos(x-y).$$

As in Lemma 1, (24) shows that, for any independent non-negative random variables X, Y,

$$E\{\Lambda_s[t(X \circ Y)]\} = E\{\Lambda_s(tX)\}\,E\{\Lambda_s(tY)\}.$$

Hence, if X, Y, Z are independent, and if

$$W_1 = X \circ (Y \circ Z), \quad W_2 = (X \circ Y) \circ Z,$$

it follows that

$$E\{\Lambda_s(tW_1)\} = E\{\Lambda_s(tX)\}\,E\{\Lambda_s(tY)\}\,E\{\Lambda_s(tZ)\} = E\{\Lambda_s(tW_2)\}.$$

Hence, by Lemma 2, $W_1 \doteq W_2$, and the theorem is proved.

3. Symmetric random walks

It has now been shown that the only operations of the form $\circ$ which are associative are those for which F has one of the forms given in Theorem 1. If we exclude for the moment the two cases in which F is degenerate, then the possible F depend on a single parameter s taking all real values in $-\frac{1}{2} \leqslant s < \infty$. Equivalently, if we write

$$n = 2(1+s),$$

they depend on the parameter n taking values in $1 \leqslant n < \infty$. Notice that if n is a positive integer ($\neq 1$) the distribution function F has the density f_n defined by (2).

When F is such that $$\int_{-1}^{1} e^{iu\lambda}\, dF(\lambda) = \Lambda_s(u),$$

we shall denote the operation $\circ$ by $\overset{n}{\oplus}$, and where no confusion arises, we shall sometimes drop the n. (Whenever n and s are used together, they will always be connected by $n = 2(1+s)$.) When n is an integer, the random variable

$$X \overset{n}{\oplus} Y$$

can be interpreted as the length of the sum of two independent spherically symmetric n-vectors whose lengths are given by the random variables X, Y. Thus the operation $\overset{n}{\oplus}$ is a generalisation of this procedure to the case when n is no longer an integer. This may be regarded as a justification for the following terminology.

Any one of the equivalent random variables

$$X \overset{n}{\oplus} Y$$

will be called a *radial sum in n dimensions* of the two (independent non-negative) random variables X, Y. Since the operation $\overset{n}{\oplus}$ is associative, we may define unambiguously (up to distribution conditional on X, Y, Z) sums like

$$X \overset{n}{\oplus} Y \overset{n}{\oplus} Z$$

(but not sums like $$X \overset{m}{\oplus} Y \overset{n}{\oplus} Z$$
unless $m = n$).

In particular, let $X_1, X_2, \ldots$ be independent non-negative random variables with the same distribution, and let

$$S_m = X_1 \overset{n}{\oplus} X_2 \overset{n}{\oplus} \ldots \overset{n}{\oplus} X_m. \tag{25}$$

Then the process

$$\{S_0 = 0, S_1 S_2, \ldots.\}$$

will be called a *symmetric random walk in n dimensions*. When n is an integer, this accords with the usual definition, so long as "symmetric" is taken to mean "spherically symmetric" and S_m is regarded as the distance from the origin after m steps.

A little care is needed in the formal definition of the process $\{S_m\}$ because of the random elements implicit in the definition of $\overset{n}{\oplus}$. We take as sample space Ω the Cartesian product of countably many intervals $[0, \infty)$ with countably many intervals $[-1, 1]$, the probability measure P being defined on Ω as the product of the distribution of the X_i on each of the first set of intervals together with the distribution F on each of the second set. If the typical point ω of Ω has components

$$X_1(\omega),\ X_2(\omega), \ldots;\quad \lambda_1(\omega),\ \lambda_2(\omega), \ldots,$$

then $S_m(\omega)$ is defined inductively by

$$S_0(\omega) = 0,$$

$$S_{m+1}(\omega) = \{S_m^2(\omega) + X_{m+1}^2(\omega) + 2\lambda_m(\omega)\, S_m(\omega)\, X_{m+1}(\omega)\}^{\frac{1}{2}}. \tag{26}$$

From this it is evident that $\{S_m(\omega)\}$ is a Markov process with stationary transition probabilities.

It is a consequence of the associativity of $\overset{n}{\oplus}$ that we can write

$$S_{m+r} \doteq S_m \overset{n}{\oplus} S_r^{(m)},$$

where $S_r^{(m)}$ is defined inductively on r by

$$S_0^{(m)} = 0,$$

$$S_{r+1}^{(m)} = \{S_r^{(m)2} + X_{m+r+1}^2 + 2\lambda_{m+r} S_r^{(m)} X_{m+r+1}\}^{\frac{1}{2}}.$$

Notice that S_m and $S_r^{(m)}$ are independent. In fact, we can go further and prove that

$$S_{m+r} = \{S_m^2 + S_r^{(m)2} + 2\mu_{m,r} S_m S_r^{(m)}\}^{\frac{1}{2}}, \tag{27}$$

where $\mu_{m,r} = \mu_{m,r}(\omega)$ is independent of $S_m, S_r^{(m)}$ and has the distribution function F. The proof is elementary but tedious, and will be omitted. If n is an integer,

and
$$S_m = |\mathbf{X}_1 + \ldots + \mathbf{X}_m|,$$

then the interpretation of $\mu_{m,r}$ is as the cosine of the angle between the vectors

$$\mathbf{X}_1 + \ldots + \mathbf{X}_m,\quad \mathbf{X}_{m+1} + \ldots + \mathbf{X}_{m+r}.$$

Before going on to study in detail the properties of symmetric random walks we must consider the two degenerate cases which have been excluded. Consider first the distribution function F which concentrates all probability at $\lambda = 0$. If we look at the probability distribution (20), we see that, as s increases, it becomes more and more peaked, and so λ becomes more and more concentrated at zero. The characteristic function $\Lambda_s(u)$ does in fact tend to 1 as $s\to\infty$, as can easily be seen from its power series expansion (4). Thus the degenerate case $\lambda=0$ can be regarded as a limiting situation as the dimension n tends to infinity. We shall, however, continue to exclude this case from consideration, because many of the results which we shall obtain do not apply to it. This is basically because the corresponding function $G(u)$, being identically equal to 1, is essentially trivial, and the expression $E\{G(tX)\}$ contains no information about the random variable X.

The other degenerate F, which concentrates probability at $\lambda=1$, stands apart from the other associative F in being asymmetric. This corresponds to the fact that, in more than one dimension, there is no satisfactory way of defining a random walk with "positive" steps.

From now on we shall exclude these two degenerate cases, and consider only the operation $\overset{n}{\oplus}$ for $1\leqslant n<\infty$. The remainder of this paper is devoted to a study of the main properties of symmetric random walks in a possibly non-integral number of dimensions. As might be expected, many of the results are very similar to those that hold in an integral number of dimensions, but the familiar methods of proof are not always open to us. The analysis leans heavily on the idea of the radial characteristic function, and the next section is devoted to its most important properties.

4. The radial characteristic function

If X is a non-negative random variable, and $n\geqslant 1$ a finite real number, we define the *radial characteristic function* of X in n dimensions by

$$^n\Psi_X(t) = E\{\Lambda_s(tX)\}, \tag{28}$$

where, as always, n and s are connected by $n=2(1+s)$. This expectation always exists when t is real, since $\Lambda_s(u)$ is a characteristic function, and so

$$|\Lambda_s(u)|\leqslant 1 \quad (u \text{ real}). \tag{29}$$

In fact,

$$^n\Psi_X(t) = E(e^{i\lambda tX}), \tag{30}$$

where λ is independent of X and has the probability density (20) (unless $n=1$, when $\lambda=\pm 1$ with equal probabilities). Since $^n\Psi_X(t)$ is an even function of t, we shall consider it as a function of $t\geqslant 0$; we shall not have occasion to consider complex t, when the expectation (28) may or may not exist. Where appropriate, the affixes n and X will be dropped from the symbol Ψ.

According to Lemma 2, a knowledge of $^n\Psi_X(t)$ for all $t>0$ determines the distribution of X, and so (28) defines a $(1-1)$ correspondence between the set of all distributions on

$[0, \infty)$ and a certain class of functions. We shall now see that, in a sense, this correspondence is continuous in both directions.

Before doing this, we must make precise the terminology to be used for distribution functions. The distribution function $H(x)$ of the random variable X will be defined as

$$H(x) = P(X \leqslant x),$$

so that it is continuous from the right, and zero for $x<0$. If $f_r(x)$, $f(x)$ are functions of totally bounded variation, we shall follow Loève [11] in writing

$$f_r \xrightarrow{w} f$$

if $f_r(x) \to f(x)$ whenever x is a point of continuity of f; and

$$f_r \xrightarrow{c} f$$

if $f_r \xrightarrow{w} f$ and if the total variation of f_r tends to that of f. If H_r, H are distribution functions, then

$$H_r \to H$$

will always mean

$$H_r \xrightarrow{c} H.$$

THEOREM 2. *Let H_r $(r=1,2,\ldots)$ be distribution functions on $[0, \infty)$, and let*

$$\psi_r(t) = \int_0^\infty \Lambda_s(tx)\, dH_r(x),$$

where $-\frac{1}{2} \leqslant s < \infty$. If, as $r\to\infty$,

$$\psi_r(t) \to \psi(t)$$

for all $t \geqslant 0$, and if ψ is continuous at $t=0$, then there exists a (unique) distribution function $H(x)$ such that

$$H_r \to H,$$

and

$$\psi(t) = \int_0^\infty \Lambda_s(tx)\, dH(x).$$

Conversely, if $H_r \to H$, then

$$\psi_r(t) \to \psi(t)$$

uniformly in every finite interval.

Proof. The result is well-known for $s=-\frac{1}{2}$ $(\Lambda_s(u) = \cos u)$, and we therefore restrict ourselves to the case $s > -\frac{1}{2}$. Then Λ_s is the characteristic function of an absolutely continuous distribution, and so, by the Riemann-Lebesgue lemma,

$$\Lambda_s(u) \to 0 \quad \text{as} \quad u \to \infty.$$

According to the "weak compactness theorem" ([11], 179), if Σ is any subsequence of the natural numbers, then there exists a sub-subsequence $\sigma \subseteq \Sigma$ and a non-decreasing function f^σ of totally bounded variation such that, as $r \to \infty$ through σ,

$$H_r \xrightarrow{w} f^\sigma.$$

By the Helly-Bray lemma ([11], 181) and since $\Lambda_s(u) \to 0$ as $u \to \infty$,

$$\int_0^\infty \Lambda_s(tx)\, dH_r(x) \to \int_0^\infty \Lambda_s(tx)\, df^\sigma(x) \quad (t > 0),$$

so that

$$\int_0^\infty \Lambda_s(tx)\, df^\sigma(x) = \psi(t) \quad (t > 0).$$

Now let $t \to 0$, and use bounded convergence and the fact that $\psi(t) \to \psi(0) = 1$ as $t \to 0$, to give

$$\int_0^\infty df^\sigma(x) = 1.$$

We are certainly at liberty to redefine f^σ on its discontinuity set to obtain a right-continuous function H^σ, which is thus a distribution function. Hence

$$\psi(t) = \int_0^\infty \Lambda_s(tx)\, dH^\sigma(x)$$

for all $t \geq 0$. By Lemma 2, all the H^σ (for different σ) are equal (to H, say), and so

$$H_r \to H.$$

To prove the converse, suppose that $H_r \to H$. Let F be given by (20), so that

$$\Lambda_s(u) = \int_{-1}^1 e^{i\lambda u}\, dF(\lambda).$$

Then, if we write

$$\phi_r(u) = \int_0^\infty e^{iux}\, dH_r(x), \quad \phi(u) = \int_0^\infty e^{iux}\, dH(x),$$

it is well-known ([11], 191) that

$$\phi_r(u) \to \phi(u)$$

uniformly in $|u| \leq A$, for any A. Hence, since

$$\psi_r(t) = \int_0^\infty dH_r(x) \int_{-1}^1 e^{i\lambda tx}\, dF(\lambda) = \int_{-1}^1 \phi_r(\lambda t)\, dF(\lambda),$$

and since $\phi_r(\lambda t) \to \phi(\lambda t)$ uniformly in $|t| \leq A$, $|\lambda| \leq 1$, it follows that

$$\psi_r(t) \to \int_{-1}^1 \phi(\lambda t)\, dF(\lambda) = \psi(t)$$

uniformly in $|t| \leq A$. Hence the theorem is proved.

Let $\mathcal{M}$ be the class of all probability distributions on $[0, \infty)$, and let $\mathcal{J}_n$ be the class of all n-dimensional radial characteristic functions. Then ${}^n\Psi_X(t)$ determines a $(1-1)$ correspondence ${}^n\Psi : \mathcal{M} \rightarrow \mathcal{J}_n$, which if distributions of $\mathcal{M}$ are identified with their distribution functions is given explicitly by

$$({}^n\Psi \cdot H)(t) = \int_0^\infty \Lambda_s(tx)\, dH(x). \tag{31}$$

Notice that we can give $\mathcal{M}$ and $\mathcal{J}_n$ semigroup structures by the operation $\overset{n}{\otimes}$ on $\mathcal{M}$ defined by

$$(\text{distribution of } X) \overset{n}{\otimes} (\text{distribution of } Y) = (\text{distribution of } X \overset{n}{\oplus} Y), \tag{32}$$

and by pointwise multiplication in $\mathcal{J}_n$. Then ${}^n\Psi$ is a semigroup isomorphism having the continuity properties given in Theorem 2.

Since the radial characteristic function ${}^n\Psi_X(t)$ determines the distribution of X, it is important to have a method of inverting this transform. If X has a probability density $h(x)$, then its radial characteristic function is

$$\Psi_X(t) = \int_0^\infty \Lambda_s(tx)\, h(x)\, dx = s!(\tfrac{1}{2}t)^{-s} \int_0^\infty x^{-s} J_s(tx)\, h(x)\, dx.$$

This is a Hankel transform, and may be inverted under suitable conditions (cf. [12]). For instance, if $h(x)$ is of locally bounded variation, and if

$$\int_0^\infty x^{-s-\frac{1}{2}} h(x)\, dx < \infty, \tag{33}$$

then ([16], 456)

$$\tfrac{1}{2}\{h(x+0) + h(x-0)\} = \frac{x^{2s+1}}{2^{2s}(s!)^2} \int_0^\infty t^{2s+1} \Lambda_s(tx)\, \Psi(t)\, dt. \tag{34}$$

Now it is easily verified from (4) that

$$\frac{d}{dz}\{z^{2s+1}\Lambda_{s+1}(z)\} = 2(s+1)\, z^{2s+1} \Lambda_s(z),$$

and so, integrating (34) *formally* with respect to x, we get

$$H(x) = P(X \leqslant x) = \frac{x^{2s+2}}{2^{2s+1} s!(s+1)!} \int_0^\infty t^{2s+1} \Lambda_{s+1}(tx)\, \Psi(t)\, dt.$$

This formula is closely analogous to the Lévy inversion formula for ordinary characteristic functions, and it might thus be expected to hold for all random variables X (with a suitable convention for points of discontinuity of H). This, however, is not so, as may be seen by considering the random variable concentrated at zero, for which $\Psi(t)\equiv 1$. In this case the above formula for H contains the integral

$$\int_0^\infty t^{2s+1}\Lambda_{s+1}(tx)\,dt,$$

which diverges when $s \geq \frac{1}{2}$.

Thus, in order that an inversion formula be valid, it is necessary that not too much probability be concentrated at or near the origin. We can expect, however, that it will be unnecessary to impose any conditions of an analytic nature on H. A sufficient condition for an inversion formula to exist is given by the following theorem.

THEOREM 3. *If X is a strictly positive random variable with*

$$E(X^{-\frac{1}{2}(n-1)}) < \infty, \tag{35}$$

then, for all $a > 0$,

$$P(X < a) + \tfrac{1}{2}P(X = a) = \frac{a^{2s+2}}{2^{2s+1}s!(s+1)!}\int_0^{\to\infty} t^{2s+1}\Lambda_{s+1}(at)\,{}^n\Psi_X(t)\,dt. \tag{36}$$

Proof. When $s = -\frac{1}{2}$ $(n=1)$ (35) is trivially satisfied, and the theorem reduces to the classical result of Lévy (as applied to symmetric random variables). We thus confine this proof to the case $s > -\frac{1}{2}$. Then, for $A > 0$, if H is the distribution function of X,

$$\int_0^A t^{2s+1}\Lambda_{s+1}(tx)\,{}^n\Psi_X(t)\,dt = \int_0^\infty dH(y)\,\frac{2^{2s+1}s!(s+1)!}{x^{s+1}y^s}\int_0^A J_{s+1}(tx)\,J_s(ty)\,dt.$$

Hence the right hand side of (36) is equal to

$$\lim_{A\to\infty}\int_0^\infty \chi(A, a, y)\,dH(y),$$

if the limit exists, where

$$\chi(A, a, y) = \int_0^A a^{s+1}y^{-s}\,J_{s+1}(at)\,J_s(ty)\,dt.$$

But it is a classical result ([16], 454) that $\chi(A, a, y)$ tends, as $A\to\infty$, to 0, $\frac{1}{2}$ or 1 according as $y < a$, $y = a$ or $y > a$. Moreover using the estimates

$$J_p(u) = O(u^p) \quad (u\to 0)$$

and
$$J_p(u) = (\tfrac{1}{2}\pi u)^{-\frac{1}{2}}\{\cos(u - \tfrac{1}{2}p\pi - \tfrac{1}{4}\pi) + O(u^{-1})\} \quad (u\to\infty),$$

it is not difficult to show that

$$y^{s+\frac{1}{2}}\chi(A, a, y)$$

is uniformly bounded in $A > 1$, $y > 0$ for fixed a. Hence the result follows from the dominated convergence theorem.

To conclude this section, we prove a theorem giving some analytic properties of the radial characteristic function. These will be deduced from the corresponding (and well-known) results for the ordinary characteristic function.

THEOREM 4. *Let $\Psi(t)$ be the radial characteristic function ${}^n\Psi_X(t)$ of the random variable X. Then $\Psi(t)$ is uniformly continuous in $0 \leq t < \infty$. If, for some integer m, $E(X^m) < \infty$, then, as $t \to 0$,*

$$\Psi(t) = \sum_{0 \leq 2r \leq m} \frac{(-1)^r 2^{-2r} s!}{r!(r+s)!} t^{2r} E(X^{2r}) + o(t^m). \tag{37}$$

Proof. Let λ be a random variable independent of X such that

$$E(e^{iu\lambda}) = \Lambda_s(u),$$

(cf. (30)), and set $Z = \lambda X$, so that

$$\Psi(t) = E(e^{itZ}).$$

Thus $\Psi(t)$ is a characteristic function, and hence ([13], 22) is uniformly continuous. If $E(X^m) < \infty$, then

$$E(|Z|^m) \leq E(X^m) < \infty,$$

and so ([13], 30)

$$\Psi(t) = \sum_{\nu=0}^{m} \frac{(it)^\nu}{\nu!} E(Z^\nu) + o(t^m)$$

as $t \to 0$. But

$$E(Z^\nu) = E(\lambda^\nu)\, E(X^\nu),$$

and since λ has characteristic function $\Lambda_s(u)$, (4) shows that

$$E(\lambda^\nu) = \begin{cases} 0 & (\nu \text{ odd}) \\ \dfrac{(2r)!\,s!}{r!(r+s)!} 2^{-2r} & (\nu \text{ even}, = 2r). \end{cases}$$

This proves (37), and the theorem is established.

Notice that, if $E(X^m) < \infty$, we can expand $\log \Psi(t)$ in a power series

$$\log \Psi(t) = \sum_{0 \leq 2r \leq m} \frac{(-1)^r 2^{-2r} s!}{r!(r+s)!} t^{2r} \theta_r + o(t^m), \tag{38}$$

where the θ_r are functions of $\mu_j = E(X^{2j})$ for $j \leq r$. The exact functional dependence of the θ_r on the μ_j is given by the formal power series identity

$$\sum_{r=0}^{\infty} \frac{s!}{r!(r+s)!} z^r \mu_r = \exp \sum_{r=0}^{\infty} \frac{s!}{r!(r+s)!} z^r \theta_r. \tag{39}$$

Notice that the definition of the θ_r depends on the dimension n. The importance of these quantities lies in the fact that they are additive in the sense that, if

$$E(X^{2r}) < \infty, \quad E(Y^{2r}) < \infty,$$

then, with the obvious notation,

$$\theta_r(X \overset{n}{\oplus} Y) = \theta_r(X) + \theta_r(Y). \tag{40}$$

This is, of course, a reflection of the multiplicative property of the radial characteristic function.

The quantities θ_r were originally introduced by Lord [12] under the name of "polar cumulants" and re-introduced by Haldane [9] as "vectorial cumulants". The latter author made a great deal of use of them, and they lie at the root of his discussion of the associativity of operations of the type $\circ$.

Notice that θ_1 is just $E(X^2)$, and that the relation (40) reduces to the obvious, but useful, identity

$$E(X \oplus Y)^2 = E(X^2) + E(Y^2). \tag{41}$$

5. The law of large numbers and the central limit theorem

We now return to the random walk in n dimensions defined in § 3. Throughout this section the number of dimensions will be kept fixed. Let X be any non-negative random variable, and write $\psi(t)$ for its radial characteristic function in n dimensions. Let $\{S_m(\omega)\}$ be a symmetric random walk in n dimensions whose steps X_i have the same distribution as X. Then the radial characteristic function of S_m is given by

$$\Psi_m(t) = \{\psi(t)\}^m. \tag{42}$$

Now suppose that $E(X) < \infty$. Then, according to Theorem 4,

$$\psi(t) = 1 + o(t)$$

as $t \to 0$. Hence
$$\Psi_m(t/m) = \{1 + o(t/m)\}^m \to 1$$

as $m \to \infty$. It follows (using Theorem 2) that

$$S_m/m \to 0$$

in probability as $m \to \infty$. This is the analogue of the weak law of large numbers; the strong law has an analogue given by the following theorem.

THEOREM 5. *If $E(X) < \infty$, then there is probability one that, as $m \to \infty$,*

$$S_m/m \to 0.$$

Proof.

$$\sum_{m=1}^{\infty} P(X_m \geqslant m) = \sum_{m=1}^{\infty} P(X \geqslant m) = \sum_{m=1}^{\infty} mP(m \leqslant X < m+1)$$
$$\leqslant \sum_{m=1}^{\infty} E(X;\ m \leqslant X < m+1) \leqslant E(X) < \infty.$$

Hence, with probability one, $X_m < m$ for all but finitely many m. Now define

$$\bar{X}_m(\omega) = \min(X_m(\omega), m),$$

and define $\bar{S}_m(\omega)$ inductively by

$$\bar{S}_0(\omega) = 0,$$
$$\bar{S}_{m+1}(\omega) = \{\bar{S}_m^2(\omega) + \bar{X}_{m+1}^2(\omega) + 2\lambda_m(\omega)\,\bar{S}_m(\omega)\,\bar{X}_{m+1}(\omega)\}^{\frac{1}{2}},$$

so that $\{\bar{S}_m\}$ is obtained from $\{X_m\}$ in the same way, *and with the same random variables* λ_m, that $\{S_m\}$ is obtained from $\{X_m\}$. It is easy to prove by induction that

$$|\bar{S}_m - S_m| \leqslant \sum_{r=1}^{m} |\bar{X}_r - X_r|,$$

and since $\bar{X}_r = X_r$ for all sufficiently large r (with probability one) it follows that $\bar{S}_m - S_m$ remains bounded with probability one as $m \to \infty$. Hence it suffices to prove that, with probability one,

$$\bar{S}_m/m \to 0$$

as $m \to \infty$.

Now

$$E(\bar{S}_{m+1}^2 | \bar{S}_m^2, \bar{S}_{m-1}^2, \ldots, \bar{S}_0^2) = E(\bar{S}_m^2 + \bar{X}_{m+1}^2 + 2\lambda_m \bar{S}_m \bar{X}_{m+1} | \bar{S}_m^2, \ldots, \bar{S}_0^2)$$
$$= \bar{S}_m^2 + E(\bar{X}_{m+1}^2) \geqslant \bar{S}_m^2,$$

and so $\{\bar{S}_m^2\}$ is a semi-martingale. Hence, by an inequality of Doob ([5], 314), for any $a > 0$,

$$P(\max_{1 \leqslant k \leqslant m} \bar{S}_k \geqslant a) = P(\max_{1 \leqslant k \leqslant m} \bar{S}_k^2 \geqslant a^2) \leqslant a^{-2} E(\bar{S}_m^2) = a^{-2} \sum_{r=1}^{m} E(\bar{X}_r^2).$$

Fix $\varepsilon > 0$, and let A_ν be the event that, for some r in $2^{\nu-1} \leqslant r < 2^\nu$, $\bar{S}_r > \varepsilon r$.

Then
$$P(A_\nu) \leqslant P(\max_{1 \leqslant k \leqslant 2^\nu} \bar{S}_k > \varepsilon 2^{\nu-1}) \leqslant 4\varepsilon^{-2} 2^{-2\nu} \sum_{r=1}^{2^\nu} E(\bar{X}_r^2),$$

and so
$$\sum_{\nu=1}^{\infty} P(A_\nu) \leqslant 4\varepsilon^{-2} \sum_{\nu=1}^{\infty} 2^{-2\nu} \sum_{r=1}^{2^\nu} E(\bar{X}_r^2)$$
$$= 4\varepsilon^{-2} \sum_{r=1}^{\infty} E(\bar{X}_r^2) \sum_{2^\nu \geqslant r} 2^{-2\nu}$$

$$\leqslant 6\varepsilon^{-2}\sum_{r=1}^{\infty} r^{-2}E(\bar{X}_r^2)$$

$$= 6\varepsilon^{-2}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} r^{-2}E(\bar{X}_r^2\colon m-1\leqslant X_r<m)$$

$$\leqslant 6\varepsilon^{-2}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} r^{-2}\min(m^2, r^2)\,P(m-1\leqslant X<m)$$

$$= 6\varepsilon^{-2}\sum_{m=1}^{\infty} P(m-1\leqslant X<m)\left\{\sum_{r=1}^{m}1+\sum_{r=m+1}^{\infty} m^2/r^2\right\}$$

$$\leqslant 6\varepsilon^{-2}\sum_{m=1}^{\infty} P(m-1\leqslant X<m)\,2m\leqslant 12\varepsilon^{-2}E(X+1)<\infty.$$

Hence, by the Borel-Cantelli lemma, there is probability one that only finitely many of the events A_v occur, and so $\bar{S}_r\leqslant\varepsilon r$ for all sufficiently large r. Since ε is arbitrary, it follows that

$$\bar{S}_m/m\to 0$$

with probability one. This completes the proof.

The next theorem is an analogue of the second fundamental result in the theory of sums of random variables, the central limit theorem. This follows very simply from the properties of the radial characteristic function established in the previous section.

THEOREM 6. *Suppose that $E(X^2)<\infty$, and define*

$$Z_m = S_m/\{mE(X^2)\}^{\frac{1}{2}}$$

Then the distribution of Z_m converges, as $m\to\infty$, to that of a random variable Z with probability density

$$g_s(z)=\frac{2(s+1)^{s+1}}{s!}\,z^{2s+1}\exp\{-(s+1)\,z^2\},\quad (0<z<\infty). \tag{43}$$

Proof. We have

$$\begin{aligned} E\{\Lambda_s(tZ_m)\} &= \Psi_m[t\{mE(X^2)\}^{-\frac{1}{2}}] \\ &= \{\psi[t\{mE(X^2)\}^{-\frac{1}{2}}]\}^m \\ &= \{1-t^2/4(s+1)\,m+o(t^2/m)\}^m \qquad \text{(Theorem 4)} \\ &\to \exp\{-t^2/4(s+1)\} \quad \text{as} \quad m\to\infty. \end{aligned}$$

This function is continuous at $t=0$, and so, by Theorem 2, the distribution of Z_m converges to that of a random variable Z with

$$\Psi_Z(t)=\exp\{-t^2/4(s+1)\}.$$

But, by the Weber integral (22),

$$\int_0^\infty g_s(z)\,\Lambda_s(tz)\,dz = \frac{2(s+1)^{s+1}}{s!}\int_0^\infty z^{2s+1}\Lambda_s(tz)\,e^{-(s+1)z^2}dz$$

$$= \frac{2(s+1)^{s+1}}{s!}\,\tfrac{1}{2}s!(s+1)^{-s-1}e^{-t^2/4(s+1)}$$

$$= \Psi_Z(t).$$

Now $g_s(z) \geqslant 0$, and the above integral with $t=0$ shows that

$$\int_0^\infty g_s(z)\,dz = 1.$$

Hence $g_s(z)$ is a probability density on $z \geqslant 0$, and hence, by Lemma 2, is the probability density of Z. Hence the theorem is proved.

The density $g_s(z)$ defines the n-dimensional *Rayleigh distribution*, which plays the role of the normal distribution for the operation $\overset{n}{\oplus}$. It is closely related to the Pearson Type III and the χ^2 distributions, (cf. [9]). Notice that the distribution is *stable*, in the sense that, if Z_1 and Z_2 are independent and have the probability density $g_s(z)$, and if a, b are positive, then

$$c^{-1}(aZ_1 \overset{n}{\oplus} bZ_2)$$

also has this probability density, where $c = (a^2+b^2)^{\frac{1}{2}}$. In the next section we shall characterise distributions having this property.

6. Infinitely divisible and stable distributions

A probability distribution on $[0, \infty)$ will be called *infinitely divisible in n dimensions* if, for any integer k, there exist independent, non-negative random variables $Y_1, Y_2, \ldots, Y_k$ such that

$$Y_1 \overset{n}{\oplus} Y_2 \overset{n}{\oplus} \ldots \overset{n}{\oplus} Y_k$$

has the given distribution. If this distribution has (n-dimensional) radial characteristic function $\psi(t)$, then it is infinitely divisible if and only if, for each integer k, there exists an n-dimensional radial characteristic function $\psi_k(t)$ such that

$$\psi(t) = \{\psi_k(t)\}^k. \tag{44}$$

More concisely, the distribution with n-dimensional radial characteristic function ψ is infinitely divisible in n dimensions if and only if, for each integer k,

$$\psi^{1/k} \in \mathfrak{I}_n. \tag{45}$$

It turns out that much of the classical theory of infinitely divisible distributions in one dimension (for which see, for instance, [8]) can be carried over to the present situation, and in particular there is a natural analogue of the Lévy-Khinchin representation.

We denote the class of radial characteristic functions of infinitely divisible distributions (in n dimensions) by $\mathfrak{I}_n$, so that

$$\mathfrak{I}_n = \bigcap_{k=1}^{\infty}\{\psi;\ \psi^{1/k} \in \mathfrak{F}_n\}.$$

Then we have the following theorem.

THEOREM 7. *The function $\psi(t)$ belongs to $\mathfrak{I}_n$ if and only if there exists a non-decreasing right-continuous function $G(x)$ of totally bounded variation in $0 \leqslant x < \infty$ such that*

$$\log \psi(t) = \int_0^\infty \{\Lambda_s(tx) - 1\} \frac{1+x^2}{x^2}\, dG(x). \tag{46}$$

This representation of ψ is unique.

Proof.

(a) We first prove that any $\psi \in \mathfrak{I}_n$ can be expressed in the form (46). If, for each k, $\psi = (\psi_k)^k$, where $\psi_k \in \mathfrak{F}_n$, then since, by (30), $\mathfrak{F}_n \subseteq \mathfrak{F}_1$, it follows that $\psi_k \in \mathfrak{F}_1$, and so $\psi \in \mathfrak{I}_1$. Hence ([8], § 17, Theorem 1) $\psi(t) \neq 0$ for all t, and so $\psi(t) > 0$ for all t. Thus

$$\log \psi(t) = \lim_{k \to \infty} k\{\psi_k(t) - 1\}.$$

But, for some $H_k \in \mathfrak{M}$ (i.e. some distribution function on $[0, \infty)$),

$$\psi_k(t) = \int_0^\infty \Lambda_s(tx)\, dH_k(x),$$

and so, writing
$$G_k(x) = k \int_0^x \frac{y^2}{1+y^2}\, dH_k(y),$$

we have
$$\log \psi(t) = \lim_{k \to \infty} \int_0^\infty \{\Lambda_s(tx) - 1\} \frac{1+x^2}{x^2}\, dG_k(x). \tag{47}$$

Hence, for all $k \geqslant k_0(t)$, and all t,

$$0 \leqslant \int_0^\infty \{1 - \Lambda_s(tx)\} \frac{1+x^2}{x^2}\, dG_k(x) \leqslant -\log \psi(t) + 1.$$

Now the function $v(z)$ defined by

$$v(z) = \frac{1 - \cos z}{1 - \Lambda_s(z)} \quad (z \neq 0),$$

$$v(0) = 2(s+1),$$

is finite, positive and continuous for all z, and

$$v(z) \sim 1 - \cos z \quad (z \to \infty)$$

(unless $s = -\frac{1}{2}$, in which case $v(z) \equiv 1$). Hence $v(z)$ is bounded, and so there exists C, depending only on s, such that

$$1 - \cos z \leqslant C(1 - \Lambda_s(z))$$

for all z. Then

$$0 \leqslant \int_0^\infty (1 - \cos tx) \frac{1+x^2}{x^2} dG_k(x) \leqslant C(-\log \psi(t) + 1).$$

An argument used by Gnedenko and Kolmogorov ([8], § 18, proof of Theorem 1) now shows that the total variation of G_k is bounded as k varies, and that

$$\int_T^\infty dG_k(x) \to 0 \quad (T \to \infty)$$

uniformly in k. Hence $\{G_k\}$ has a subsequence $\{G_{k_i}\}$ with a limit G in the sense that

$$G_{k_i} \xrightarrow{c} G \quad (i \to \infty).$$

Hence, from (47),

$$\log \psi(t) = \int_0^\infty \{\Lambda_s(tx) - 1\} \frac{1+x^2}{x^2} dG(x).$$

(b) Conversely, suppose that $\psi(t)$ has the form (46); then we prove that $\psi \in \mathfrak{I}_n$. Consider first the characteristic function $\Lambda_s(ct)$ of the sure random variable concentrated at $c \geqslant 0$ The radial sum of r such variables has radial characteristic function

$$\{\Lambda_s(ct)\}^r \in \mathfrak{I}_n.$$

Now it is clear that $\mathfrak{I}_n$ is closed under the formation of mixtures, and so $\mathfrak{I}_n$ contains

$$\sum_{r=0}^\infty \frac{e^{-a} a^r}{r!} \{\Lambda_s(ct)\}^r = \exp\,[a\{\Lambda_s(ct) - 1\}],$$

for each $a \geqslant 0$. If $a_1, \ldots, a_q, c_1, \ldots, c_q$ are any positive numbers, it follows that

$$\prod_i \exp\,[a_i\{\Lambda_s(c_i t) - 1\}] = \exp \sum_i a_i\{\Lambda_s(c_i t) - 1\} \in \mathfrak{I}_n.$$

By Theorem 2, a pointwise limit of functions in $\mathfrak{I}_n$ lies in $\mathfrak{I}_n$ so long as it is continuous at the origin. Now any function of the form (46) is a pointwise limit of functions in $\mathfrak{I}_n$, since the integral is a limit of approximating Riemann sums which are of the form

$$\sum a_i\{\Lambda_s(c_i t) - 1\},$$

with $a_i \geqslant 0$, $c_i \geqslant 0$. Moreover, any function of this form is continuous at the origin, since

$$0 \leqslant \{1-\Lambda_s(tx)\}(1+x^2)/x^2 \leqslant \min(2, t^2x^2/4(1+s))\,(1+x^2)/x^2 \leqslant 2+t^2/2(1+s),$$

which is uniformly bounded in $0 \leqslant t \leqslant 1$, $x \geqslant 0$, and so, by bounded convergence,

$$\log \psi(t) \to 0 \quad (t \to 0).$$

Hence any function of the form (46) belongs to $\mathfrak{J}_n$. But

$$\log \psi^{1/k}(t) = \int_0^\infty \{\Lambda_s(tx) - 1\} \frac{1+x^2}{x^2}\, d\, \frac{G(x)}{k},$$

so that $\psi^{1/k} \in \mathfrak{J}_n$, and so $\psi(t) \in \mathfrak{J}_n$.

(c) Finally, we prove that the representation (46) is unique. Suppose that G_1, G_2 are non-decreasing, right-continuous, and of bounded variation in $x \geqslant 0$, and that

$$\int_0^\infty \{1-\Lambda_s(tx)\} \frac{1+x^2}{x^2}\, dG_j(x)$$

is independent of $j = 1, 2$. Using (22) and the fact that the integrand is non-negative, we see that

$$\mathbf{I}_j'(\alpha) = \int_0^\infty (1-e^{-\alpha x^2}) \frac{1+x^2}{x^2}\, dG_j(x) \quad (\alpha > 0)$$

is independent of j. Hence so is

$$\begin{aligned} \mathbf{I}_j(\beta) &= -\tfrac{1}{2}\int_{\beta+1}^{\beta+3} \mathbf{I}_j'(\alpha)\, d\alpha + \mathbf{I}_j'(\beta+2) \\ &= \int_0^\infty e^{-\beta x^2} e^{-2x^2} (\sinh x^2/x^2 - 1)\,(1+x^2)/x^2\, dG_j(x) \\ &= \int_0^\infty e^{-\beta x^2} dG_j^*(x), \text{ (say)}, \end{aligned}$$

where G_j^* is non-decreasing, right-continuous, and of bounded variation in $x \geqslant 0$. Since $\mathbf{I}_1(\beta) = \mathbf{I}_2(\beta)$ for all $\beta \geqslant 0$, it follows that $G_1^* = G_2^*$, and so $G_1 = G_2$.

Thus the proof of the theorem is complete.

The units out of which the infinitely divisible distributions are built are not (as in the one-dimensional case) Poisson distributions, but are those with radial characteristic function

$$\exp a\{\Lambda_s(ct) - 1\} \quad (a, c > 0). \tag{48}$$

These seem to be the natural generalisation of the Poisson distributions. The expression (48) is the radial characteristic function of

$$X_1 \overset{n}{\oplus} X_2 \overset{n}{\oplus} \dots \overset{n}{\oplus} X_N,$$

where all the X_i are equal to c with probability one, and N has a Poisson distribution with mean a. The distribution has atoms at 0 and at c, together with an absolutely continuous component in $x > 0$. It would be interesting to prove for this distribution analogues of some of the well-known results (such as Raikov's theorem, [13], 174) about the Poisson distribution.

It is now possible to develop the theory of infinitely divisible distributions exactly as in [8]. This will not be done in detail here. Instead we go on to study a subclass of $\mathfrak{J}_n$, that of the stable distributions. Because of the intrinsic symmetry of our problem, our definition of stability is not quite analogous to the usual one.

We shall say that a distribution on $[0, \infty)$ is *stable in n dimensions* if any positive linear combination under $\overset{n}{\oplus}$ of independent variables with this distribution is a multiple of a random variable having this same distribution. Thus a distribution with n-dimensional radial characteristic function $\psi(t)$ is stable in n dimensions if and only if, for each $a, b > 0$, there exists $c > 0$ such that

$$\psi(t/c) = \psi(t/a)\psi(t/b). \tag{49}$$

We write S_n for the class of such functions $\psi(t)$.

If $\psi \in S_n$, then ψ is a radial characteristic function, and so by (30) is the ordinary characteristic function of a symmetric distribution on $(-\infty, \infty)$. Then (49) is equivalent to the statement that this latter distribution is stable in the ordinary sense. Hence $\psi(t)$ is the characteristic function of a symmetric stable distribution, and so

$$\psi(t) = \exp\{-c|t|^\alpha\},$$

for some $c \geq 0$, $0 < \alpha \leq 2$.

Now in (46) take $G(x)$ to be defined by

$$dG(x) = x^{1-\alpha}dx/(1+x^2),$$

where $0 < \alpha < 2$. Then Theorem 7 shows that $\mathfrak{J}_n$ (and a fortiori $\mathfrak{F}_n$) contains the function $\psi(t)$ given by

$$\begin{aligned}\log \psi(t) &= \int_0^\infty \{\Lambda_s(tx) - 1\}\, x^{-1-\alpha}dx \\ &= |t|^\alpha \int_0^\infty \{\Lambda_s(y) - 1\}\, y^{-1-\alpha}dy \\ &= -A|t|^\alpha,\end{aligned}$$

say, where $A > 0$.

By taking
$$dG(x) = cA^{-1}x^{1-\alpha}dx/(1+x^2),$$
we see similarly that
$$\exp\{-c|t|^{\alpha}\} \in \mathfrak{I}_n \subseteq \mathfrak{F}_n,$$
for all $c \geqslant 0$, $0 < \alpha < 2$. By taking $G(x)$ to be equal to 0 for $x < 0$ and equal to a constant > 0 when $x \geqslant 0$, we see that
$$\exp\{-ct^2\} \in \mathfrak{I}_n \subseteq \mathfrak{F}_n,$$
for all $c \geqslant 0$. But a function ψ belongs to $\mathcal{S}_n$ if and only if it satisfies (49) and belongs to $\mathfrak{F}_n$, and hence we obtain the following theorem.

THEOREM 8. *A distribution is stable in n dimensions if and only if its n-dimensional radial characteristic function has the form*
$$\exp(-ct^{\alpha}) \quad (t \geqslant 0), \tag{50}$$
where $c \geqslant 0$, $0 < \alpha < 2$.

The parameter c is only a scale parameter, and the essentially different stable distributions depend on the value of the number α. When $c > 0$, we call α the parameter of the stable distribution. Notice that although (50) is apparently independent of the number of dimensions, this number n is implicit in the radial characteristic function. Thus the distribution function $H(x)$ whose radial characteristic function in n dimensions is given by (50) satisfies
$$\int_0^{\infty} \Lambda_s(tx)\, dH(x) = \exp(-ct^{\alpha}),$$
which involves s and thus n. However, the theorem does show that the class $\mathcal{S}_n$ is independent of n; it will therefore be denoted simply by $\mathcal{S}$.

It is clear that the stable distribution in n dimensions with parameter α is an analogue of the ordinary symmetric stable distribution with the same parameter. Thus the Rayleigh distribution, which is stable with parameter $\alpha = 2$, is analogous to the normal distribution. The distribution with radial characteristic function e^{-t}, which is stable with $\alpha = 1$, is a generalisation of the Cauchy distribution. For this case it can be verified that the use of the inversion formula (34) is justified, so that this distribution has a probability density
$$k_s(x) = \frac{x^{2s+1}}{2^{2s}(s!)^2} \int_0^{\infty} t^{2s+1} \Lambda_s(tx)\, e^{-t} dt$$
$$= \frac{x^{s+1}}{2^s s!} \int_0^{\infty} t^{s+1} J_s(tx)\, e^{-t} dt$$

$$= \frac{x^{s+1}}{2^s s!} 2^{s+1} \pi^{-\frac{1}{2}} (s+\tfrac{1}{2})! \, x^s (1+x^2)^{-s-\frac{3}{2}}$$

([6], 182).

Hence the *n-dimensional Cauchy distribution* has probability density

$$k_s(x) = \frac{2(s+\frac{1}{2})!}{s! \sqrt{\pi}} x^{n-1} (1+x^2)^{-\frac{1}{2}(n+1)} \quad (x \geq 0). \tag{51}$$

When $n=1$ this reduces to the usual Cauchy distribution, or more precisely, to the distribution of $|Y|$ where Y has a Cauchy distribution.

For other values of α it does not seem to be possible to express the distribution in closed form. Notice that, in accordance with the central limit theorem, the only case in which the stable distribution has finite variance is that in which $\alpha=2$ (the Rayleigh distribution).

7. Recurrence properties of symmetric random walks

We now return to the process $\{S_m\}$ defined by (25), which we have called a symmetric random walk in n dimensions. This is a Markov process whose state space is the non-negative real line, and the present section is devoted to its recurrence properties. Thus, in geometrical terms, we are interested in the return of the process to a spherical neighbourhood of the origin.

More precisely, we consider those values of m for S_m is less than some fixed positive h. There may or may not be infinitely many such values. We say that the random walk $\{S_m\}$ is *recurrent* if, for *all* $h>0$,

$$P(S_m < h \text{ for infinitely many } m) = 1. \tag{52}$$

We shall see later that the only other alternative is that, for all $h>0$, the probability (52) is zero.

LEMMA 3. *The random walk $\{S_m\}$ is recurrent if and only if, for all $h>0$,*

$$\sum_{m=1}^{\infty} P(S_m < h) = \infty. \tag{53}$$

Proof (cf. [3]). If, for some $h>0$, the series (53) converges, then the Borel-Cantelli lemma shows that, for that value of h, the probability in (52) is zero, and so the walk is not recurrent. Conversely, suppose that (53) is true. Set

$$q(h) = P(S_m \geq h \quad \text{for all } m \geq 1),$$

$$r(h) = P(S_m < h \quad \text{for only finitely many } m).$$

Then, since $Y \oplus Z \geqslant |Y - Z|$, and using the notation (27),

$$1 \geqslant r(h) \geqslant \sum_{k=1}^{\infty} P(S_k < h,\ S_m^{(k)} \geqslant 2h \quad \text{for all } m \geqslant 1)$$

$$= \sum_{k=1}^{\infty} P(S_k < h)\, q(2h).$$

Hence $q(2h) = 0$ for all $h > 0$. Thus

$$r(h) \leqslant \sum_{N > 1/h} \sum_{k=1}^{\infty} P(S_k < h - 1/N,\ S_{k+m} \geqslant h \quad \text{for all } m \geqslant 1) + q(h)$$

$$\leqslant \sum_{N > 1/h} \sum_{k=1}^{\infty} P(S_k < h - 1/N)\, P(S_m^{(k)} \geqslant 1/N \quad \text{for all } m \geqslant 1) + q(h)$$

$$= \sum_{N > 1/h} \sum_{k=1}^{\infty} P(S_k < h - 1/N)\, q(1/N) + q(h) = 0.$$

Therefore

$$P(S_m < h \quad \text{for infinitely many } m) = 1 - r(h) = 1.$$

Hence the walk is recurrent and the lemma is proved.

We now seek to use this lemma to establish a criterion for recurrence in terms of the distribution of the length X_j of the jth step (this distribution being, by hypothesis, independent of j). As before, we denote by $\psi(t)$ the radial characteristic function of X_j.

THEOREM 9. *The random walk $\{S_m\}$ is recurrent if and only if the integral*

$$\int_0^{\infty} \frac{1}{1-\psi(t)} \frac{t^{n-1}dt}{1+t^{n+1}} \tag{54}$$

diverges.

Proof. Let $\mu(\varrho)$ ($\varrho < 1$) be a random variable independent of $\{S_m\}$ and having a geometric distribution

$$P(\mu(\varrho) = k) = (1-\varrho)\varrho^k \quad (k = 0, 1, 2, \ldots),$$

and define

$$Z_\varrho = S_{\mu(\varrho)}.$$

Then, if Φ_ϱ is the radial characteristic function of Z_ϱ, we have

$$P(Z_\varrho < h) = (1-\varrho) \sum_{m=0}^{\infty} \varrho^m P(S_m < h),$$

and

$$\Phi_\varrho(t) = (1-\varrho) \sum_{m=0}^{\infty} \varrho^m E\{\Lambda_s(tS_m)\} = \frac{1-\varrho}{1-\varrho\psi(t)} > 0.$$

Hence the walk is recurrent if and only if, for each $h > 0$,

$$P(Z_\varrho < h)/(1-\varrho) \to \infty \tag{55}$$

as $\varrho \uparrow 1$.

Now consider, for $\beta > 0$, the integral

$$\mathbf{I}_\varrho(\beta) = \int_0^\infty t^{2s+1}\Phi_\varrho(t)\,\{\Lambda_{s+1}(\beta t)\}^2\,dt.$$

By Fubini's theorem

$$\mathbf{I}_\varrho(\beta) = E\int_0^\infty t^{2s+1}\Lambda_s(tZ_\varrho)\,\{\Lambda_{s+1}(\beta t)\}^2\,dt$$

$$= A(\beta)\,\beta^{-2s-2}E\int_{\substack{0\leqslant\theta\leqslant\pi/2;\\ \sin\theta\geqslant Z_\varrho/2\beta}} \cos^{2s+2}\theta\,d\theta$$

(cf. 16, § 13.46), where $A(\beta)$ is a positive function of β and s alone. Because the last integral is zero when $Z_\varrho > 2\beta$, it follows that there exist positive numbers $C_1(h)$, $C_2(h)$ independent of ϱ such that

$$C_1(h)\,\mathbf{I}_\varrho(h) < P(Z_\varrho < h) < C_2(h)\,\mathbf{I}_\varrho(2h).$$

Hence, from (55), the walk is recurrent if and only if

$$\mathbf{I}_\varrho(\beta)/(1-\varrho) \to \infty$$

as $\varrho \uparrow 1$ for all $\beta > 0$, i.e.

$$\int_0^\infty (1-\varrho\psi(t))^{-1}\,t^{2s+1}\{\Lambda_{s+1}(\beta t)\}^2\,dt \to \infty$$

as $\varrho \uparrow 1$. If this integral is written in the form

$$\int_{\psi(t)>0} + \int_{\psi(t)<0} = \mathbf{I}' + \mathbf{I}'',$$

then the integrand of $\mathbf{I}'$ is monotone in ϱ, and that of $\mathbf{I}''$ is, for all ϱ $(0<\varrho<1)$ less than the integrable function $t^{2s+1}\{\Lambda_{s+1}(\beta t)\}^2$, and so the limit as $\varrho \uparrow 1$ can be taken inside the integral, showing that a necessary and sufficient condition for recurrence is that

$$\int_0^\infty (1-\psi(t))^{-1}t^{n-1}\{\Lambda_{s+1}(\beta t)\}^2\,dt = \infty$$

for all $\beta > 0$. Using the fact that, if $n \geqslant 1$, either $\psi(t) \equiv 1$, or $\psi(t)$ is bounded away from 1 in $t \geqslant 1$, this condition is easily seen to be equivalent to the divergence of (54), and the theorem is proved. Notice that (54) is only a condition on $\psi(t)$ in the neighbourhood of $t=0$, since, unless $\psi(t)\equiv 1$,

$$\int_\delta^\infty \frac{1}{1-\psi(t)}\,\frac{t^{n-1}dt}{1+t^{n+1}}<\infty$$

for all $\delta>0$.

Corollary 1. *If $E(X^2)<\infty$, and if $X>0$ with positive probability, then $\{S_m\}$ is recurrent in n dimensions if and only if*

$$n\leqslant 2. \tag{56}$$

Proof. According to Theorem 4, if $E(X^2)<\infty$, then

$$\psi(t)=1-at^2+o(t^2)$$

as $t\to 0$, where $a=E(X^2)/2n>0$. Hence there exists $b>0$ such that $t^{-2}(1-\psi(t))$ is bounded away from zero and infinity in $0<t<b$. Thus (54) diverges if and only if

$$\int_0^b \frac{t^{n-1}dt}{t^2(1+t^{n+1})}=\infty,$$

i.e. if and only if $n\leqslant 2$.

Corollary 2. *If $n>2$, and if $X>0$ with positive probability, then $\{S_m\}$ is not recurrent.*

Proof. If $E(X^2)<\infty$ this follows from Corollary 1. If $E(X^2)=\infty$, then, since $\psi(t)=E\{\Lambda_s(tX)\}=E(e^{i\lambda tX})$ is the characteristic function of $Z=\lambda X$, and since

$$E(Z^2)=E(\lambda^2)\,E(X^2)=\infty,$$

it follows ([13], Theorem 2.3.1) that

$$(1-\psi(t))/t^2\to\infty,$$

as $t\to 0$. It follows at once that, for b sufficiently small,

$$\int_0^b \frac{1}{1-\psi(t)}\,\frac{t^{n-1}dt}{1+t^{n+1}}\leqslant\int_0^b\frac{1}{t^2}\,\frac{t^{n-1}dt}{1+t^{n+1}}<\infty$$

if $n>2$, and so the walk is not recurrent.

Corollary 3. *If $\{S_m\}$ is not recurrent, then, for any $h>0$,*

$$P(S_m<h\quad\text{for infinitely many}\quad m)=0, \tag{57}$$

and there exists $m_0=m_0(h)$ such that

$$P(S_m<h \text{ for some } m\geqslant m_0)<1. \tag{58}$$

Proof. If $\{S_m\}$ is not recurrent, then, for any $\beta > 0$

$$\mathrm{I}_\varrho(\beta) = O(1-\varrho),$$

and hence
$$P(Z_\varrho < h) < C_2(h)\, \mathrm{I}_\varrho(2h) = O(1-\varrho).$$

Thus, for any $h > 0$,
$$\sum_{m=1}^{\infty} P(S_m < h) < \infty.$$

The Borel-Cantelli lemma establishes (57). Let m_0 be such that

$$\sum_{m=m_0}^{\infty} P(S_m < h) < 1.$$

Then
$$P(S_m < h \quad \text{for some} \quad m \geqslant m_0(h)) \leqslant \sum_{m=m_0}^{\infty} P(S_m < h) < 1,$$

and the result is proved.

Corollary 1 is the analogue for symmetric random walks of the celebrated theorem of Pólya [14]. We see that there is a discontinuity in the behaviour of symmetric random walks with finite mean square as the dimension varies. Thus the walk is recurrent in 2 dimensions, but not in $(2+\varepsilon)$ dimensions for any $\varepsilon > 0$. Corollary 2 shows that the removal of the condition $E(X^2) < \infty$ cannot result in an increase in the critical dimension. We shall see in the next section that, in suitable circumstances, the critical dimension actually decreases.

The meaning of Corollary 3 is that the walk either returns infinitely often to every neighbourhood of the origin with probability one, or it returns to any finite neighbourhood only finitely many times with probability one. The results of this section are very similar to those obtained by Chung and Fuchs [3] for ordinary random walks, but in their problem the origin does not have the special place that is implicit in the formulation adopted in the present work.

To conclude this section, we discuss the problem of finding an invariant measure for the Markov process $\{S_m\}$. Let $p(x, A)$ $(x \in [0, \infty),\ A \subseteq [0, \infty))$ be the transition probability

$$\begin{aligned} p(x, A) &= P(S_{m+1} \in A \mid S_m = x) \\ &= P(\{x^2 + X_{m+1}^2 + 2\lambda_m\, x X_{m+1}\}^{\frac{1}{2}} \in A) \\ &= P(\{x^2 + X^2 + 2\lambda x X\}^{\frac{1}{2}} \in A). \end{aligned}$$

We say that a σ-finite measure μ on the state space $[0, \infty)$ is invariant for the process considered if, for every measurable subset A of the state space,

$$\mu(A) = \int \mu(dx)\, p(x, A). \tag{59}$$

We prove that the measure μ defined by

$$\mu(dx) = x^{n-1}dx \quad (0 \leqslant x < \infty) \tag{60}$$

is invariant for the process $\{S_m\}$.

Consider first the case $n>1$. It suffices to prove (59) for sets A of the form $[0,a)$ for $a>0$. But, if $A=[0,a)$, then

$$\int \mu(dx)\, p(x, A) = \int_0^\infty x^{n-1}dx \iint_{x^2+y^2+2\lambda xy \leqslant a^2} dH(y)\, dF(\lambda),$$

where H is the distribution function of X, and F is given by (20). Hence

$$\int \mu(dx)\, p(x, A) = \int_0^\infty dH(y)\, m(a, y),$$

where
$$m(a, y) = \iint_{x^2+y^2+2\lambda xy \leqslant a^2} x^{n-1}dx\, dF(\lambda)$$

$$= \frac{s!}{(s-\frac{1}{2})!\sqrt{\pi}} \iint_{\substack{x^2+y^2+2\lambda xy \leqslant a^2 \\ x \geqslant 0,\ |\lambda| \leqslant 1}} x^{n-1}(1-\lambda^2)^{s-\frac{1}{2}}dx\, d\lambda$$

$$= \frac{s!}{(s-\frac{1}{2})!\sqrt{\pi}} \iint_{\substack{x^2+y^2+2xy\cos\theta \leqslant a^2 \\ x \leqslant 0,\ 0 \leqslant \theta \leqslant \pi}} x^{n-1}\sin^{n-2}\theta\, dx\, d\theta.$$

Now write $\zeta = \xi + i\eta = xe^{i\theta}$. Then

$$m(a, y) = \frac{s!}{(s-\frac{1}{2})!\sqrt{\pi}} \iint_{\substack{|\zeta - y| \leqslant a \\ \eta \geqslant 0}} \eta^{n-2}d\xi\, d\eta$$

$$= \frac{s!}{(s-\frac{1}{2})!\sqrt{\pi}} \int_0^a \eta^{n-2}2(a^2-\eta^2)^{\frac{1}{2}}\, d\eta$$

$$= a^n/n$$

on evaluating the Beta function integral. Hence whatever the distribution function H of the step length X, we have

$$\int \mu(dx)\, p(x, A) = \int_0^\infty m(a, y)\, dH(y) = \int_0^\infty a^n/n\, dH(y) = a^n/n = \mu(A).$$

A similar, but simpler, argument suffices for the case $n=1$. Hence we have proved the stated result that the measure μ defined in (60) is invariant for the Markov process $\{S_m\}$.

8. Symmetric Brownian processes

In the preceding sections we considered symmetric random walks with a discrete time parameter; we now turn to the corresponding problem in continuous time. Let us call a real-valued continuous time stochastic process $Z(\tau)$ $(\tau \geqslant 0)$ with $Z(0)=0$ a *symmetric Brownian process* in n dimensions if, whenever

$$0=\tau_0<\tau_1<\ldots<\tau_k,$$

the joint distribution of $Z(\tau_1),\ldots,Z(\tau_k)$ is the same as that of

$$Y_1,\ Y_1 \overset{n}{\oplus} Y_2,\ \ldots,\ Y_1 \overset{n}{\oplus} Y_2 \overset{n}{\oplus} \ldots \overset{n}{\oplus} Y_k,$$

where the Y_i are independent, and where Y_i has a distribution $\mathcal{D}(\tau_i-\tau_{i-1})$ depending only on $\tau_i-\tau_{i-1}$.

For any given family of distributions $\{\mathcal{D}(\tau);\tau>0\}$ there will be a stochastic process $Z(\tau)$ with the assigned joint distributions if and only if the Kolmogorov consistency conditions ([5], 10) are satisfied. These conditions can easily be seen to be equivalent to the condition that, for all $\sigma,\tau>0$,

$$\mathcal{D}(\sigma)\overset{n}{\otimes}\mathcal{D}(\tau)=\mathcal{D}(\sigma+\tau), \tag{61}$$

where $\overset{n}{\otimes}$ is the semigroup operation on $\mathfrak{M}$ defined in § 4. If $\psi_\tau(\cdot)$ is the n-dimensional radial characteristic function of $\mathcal{D}(\tau)$, (61) is equivalent to

$$\psi_\sigma\psi_\tau=\psi_{\sigma+\tau}. \tag{62}$$

Thus each ψ_τ belongs to $\mathfrak{I}_n$, and in fact

$$\psi_\tau=\exp(-\tau\xi), \tag{63}$$

where
$$\xi(t)=\int_0^\infty\{1-\Lambda_s(tx)\}\frac{1+x^2}{x^2}\,dG(x), \tag{64}$$

and $G(x)$ is a non-decreasing function of bounded variation (Theorem 7).

Thus every symmetric Brownian process has joint distributions determined by (63), (64). Conversely, if G is any non-decreasing function of bounded variation, there is a symmetric Brownian process having joint distributions given by (63), (64). There will be many such processes, but they will all have the same finite-dimensional distributions.

Of particular interest are those processes for which all the distributions $\mathcal{D}(\tau)$ are of the same type, in the sense that $\mathcal{D}(\sigma)$ is obtained from $\mathcal{D}(\tau)$ by a change of scale. In this case (61) shows that $\mathcal{D}(\tau)$ must be a stable distribution, so that, for some $a\geqslant 0$, $0<\alpha\leqslant 2$,

$$\psi_\tau(t)=\exp(-a\tau t^\alpha). \tag{65}$$

Any process for which ψ_τ has this form will be called a *symmetric stable process with parameter* α. In particular, when $\alpha=2$, so that $Z(\tau)$ has a Rayleigh distribution, we call the process a *symmetric Wiener process*.

The recurrence theorem of the preceding section has an analogue for symmetric Brownian processes which will now be discussed. A process $Z(\tau)$ will be called *recurrent* if, for each $h>0$, the set

$$\mathcal{T}(h) = \{\tau; Z(\tau) < h\}$$

is unbounded with probability one.

Observe that, for any $\delta>0$, the discrete time process

$$S_m(\delta) = Z(m\delta) \quad (m = 0,1,2,\ldots) \tag{66}$$

is a symmetric random walk, the radial characteristic function of each step being ψ_δ. If, for some $\delta>0$, the random walk $\{S_m(\delta)\}$ is recurrent, then so is $Z(\tau)$, since $\mathcal{T}(h)$ then contains infinitely many of the points m_δ.

The converse result is much less trivial, and is, in fact, generally false. Whatever the joint distributions determined by the $\mathcal{D}(\tau)$, we can find a process $Z(\tau)$ with these joint distributions which is recurrent. For, if $\bar{Z}(\tau)$ is a symmetric Brownian process with the given finite-dimensional distributions, then so is the process $Z(\tau)$ defined by

$$Z(\tau) = \bar{Z}(\tau) \quad (\tau \neq \zeta_1, \zeta_2, \ldots)$$

$$Z(\zeta_j) = 0 \quad (j = 1,2,\ldots),$$

where the ζ_j are the instants of a Poisson process independent of $\bar{Z}(\tau)$. But $Z(\tau)$ is recurrent, because $\mathcal{T}(h)$ contains the points ζ_j.

Let $\xi(t)$ be any function of the form (64), and write $\mathcal{Z}(\xi)$ for the class of symmetric Brownian processes whose finite-dimensional distributions are determined by the radial characteristic functions

$$\psi_\tau = e^{-\tau\xi}.$$

Then $\mathcal{Z}(\xi)$ will certainly contain some recurrent processes. However, if we are seeking properties which depend only on the finite-dimensional distributions of the processes studied, what is of interest is whether or not *all* the processes in $\mathcal{Z}(\xi)$ are recurrent.

With this in mind, we say that $\xi(t)$ is *of recurrent type* if all the processes in $\mathcal{Z}(\xi)$ are recurrent. Then we have seen that if, for some $\delta>0$, the symmetric random walk $\{S_m(\delta)\}$ whose steps have radial characteristic function $e^{-\delta\xi}$ is recurrent, then $\xi(t)$ is of recurrent type. The next lemma establishes the converse. The proof follows closely that of Chung ([2], 179) for a similar result in the theory of Markov chains.

LEMMA 4. *If $\xi(t)$ is of recurrent type then, for some $\delta>0$, $\{S_m(\delta)\}$ is recurrent.*

Proof. First observe that, as $\tau \to 0$, $\mathcal{D}(\tau)$ approaches the distribution concentrated at zero, and so every process in $\mathcal{Z}(\xi)$ is continuous in probability. It follows from theorems of Doob ([5], § II.2) that $\mathcal{Z}(\xi)$ contains a process $Z(\tau)$ which is measurable and which is separable (relative to closed sets), any dense subset of the real line being admissible as a separability set. Then, since $\xi(t)$ is of recurrent type, $Z(\tau)$ is recurrent.

Let $\|A\|$ denote the Lebesgue measure of the set A, and let h, τ, ε be positive numbers. For any random variable w such that $0 \leqslant w \leqslant 1$,

$$E(w) \leqslant \tfrac{1}{2} \cdot P(w \leqslant \tfrac{1}{2}) + 1 \cdot P(w > \tfrac{1}{2}) = \tfrac{1}{2} + \tfrac{1}{2} P(w > \tfrac{1}{2}),$$

so that
$$P(w > \tfrac{1}{2}) \geqslant 2E(w) - 1.$$

Applying this to
$$w = \frac{1}{2\varepsilon} \|\mathcal{J}(2h) \cap (\tau, \tau + 2\varepsilon)\|,$$

and using the measurability of $Z(\tau)$, we get

$$\begin{aligned} &P(\|\mathcal{J}(2h) \cap (\tau, \tau + 2\varepsilon)\| > \varepsilon \mid Z(\tau) < h) \\ &\geqslant \varepsilon^{-1} E(\|\mathcal{J}(2h) \cap (\tau, \tau + 2\varepsilon)\| \mid Z(\tau) < h) - 1 \\ &= \varepsilon^{-1} \int_0^{2\varepsilon} P(Z(\tau + \sigma) < 2h \mid Z(\tau) < h)\, d\sigma - 1 \\ &\geqslant \varepsilon^{-1} \int_0^{2\varepsilon} P(Z(\sigma) < h)\, d\sigma - 1 = \gamma(\varepsilon), \quad \text{say.} \end{aligned}$$

Then $\gamma(\varepsilon) \to 1$ as $\varepsilon \to 0$, and so, if ε is sufficiently small $\gamma(\varepsilon) > 0$. Now, for any $\eta > 0$,

$$\begin{aligned} &P(\|\mathcal{J}(2h) \cap (\tau, \infty)\| > \varepsilon) \\ &\geqslant \sum_{m=0}^{\infty} P\{Z(\tau + r\eta) \geqslant h (0 < r < m),\ Z(\tau + m\eta) < h, \|\mathcal{J}(h) \cap (\tau + m\eta, \infty)\| > \varepsilon\} \\ &\geqslant \gamma(\varepsilon) \sum_{m=0}^{\infty} P\{Z(\tau + r\eta) \geqslant h (0 < r < m),\ Z(\tau + m\eta) < h\} \\ &= \gamma(\varepsilon)\, P\{Z(\tau + r\eta) < h \text{ for some } r \geqslant 0\}. \end{aligned}$$

Now put $\eta = 2^{-k}$, and let $k \to \infty$ through integral values. Using the separability and the recurrence of $Z(\cdot)$,

$$P(\|\mathcal{J}(2h) \cap (\tau, \infty)\| > \varepsilon) \geqslant \gamma(\varepsilon)\, P\{Z(\tau + \sigma) < h \text{ for some } \sigma \geqslant 0\} = \gamma(\varepsilon).$$

Hence, letting $\tau \to \infty$,
$$P(\|\mathcal{J}(2h)\| = \infty) \geqslant \gamma(\varepsilon),$$

and so, letting $\varepsilon \to 0$,
$$P(\|\mathcal{J}(2h)\| = \infty) = 1.$$

Hence
$$\int_0^\infty P(Z(\tau) \leqslant 2h)\, d\tau = \infty.$$

It follows from a remark of Chung ([2], 182) and from the evident continuity of $P(Z(\tau) \leqslant 2h)$ that, for all positive δ outside a set $\Delta(h)$ of the first category,

$$\sum_{m=0}^{\infty} P\{Z(m\delta) \leqslant 2h\} = \infty. \tag{67}$$

Choose δ to lie outside the set

$$\bigcup_{k=1}^{\infty} \Delta(1/k),$$

which is possible since this set is of the first category. Then (67) holds when $h = 1/k$ (k integral), and so, since the left-hand side of (67) is monotone non-decreasing in h, it holds for all $h > 0$. Hence, with this choice of δ, $\{S_m(\delta)\}$ is recurrent, and the lemma is proved.

THEOREM 10. *A function $\xi(t)$ of the form* (64) *is of recurrent type in n dimensions if and only if the integral*

$$\int_0^\infty \frac{1}{\xi(t)} \frac{t^{n-1} dt}{1 + t^{n+1}} \tag{68}$$

diverges. The function $\xi(t) = at^\alpha$ corresponding to the symmetric stable process with parameter α is of recurrent type if and only if

$$n \leqslant \alpha. \tag{69}$$

Proof. Lemma 4 and Theorem 9 show that $\xi(t)$ is of recurrent type if and only if, for some $\delta > 0$,

$$\int_0^\infty \frac{1}{1 - \psi_\delta(t)} \frac{t^{n-1} dt}{1 + t^{n+1}} = \infty$$

But, since $\psi_\delta(t) = \exp\{-\delta\xi(t)\}$ this integral diverges if and only if (68) does. In particular, if $\xi(t) = at^\alpha$, (68) diverges if and only if

$$\int_0^\infty \frac{t^{n-\alpha-1} dt}{1 + t^{n+1}} = \infty,$$

i.e. if and only if $n \leqslant \alpha$. Hence the theorem is proved.

According to (69), the critical dimension for recurrence of symmetric stable processes is no longer 2 (as in Pólya's theorem), but is now α, which can be strictly less than 2. The critical dimension need not of course be an integer.

9. Projections

If we have a random walk in an integral number of dimensions, then the projection of this walk onto any linear subspace is again a random walk in the appropriate number of dimensions. Moreover, if the original process is spherically symmetric, so is the projected process. If a typical step in the original walk has length X, then the corresponding step in the projected walk has length $X\cos\theta$, where θ is independent of X, and has the distribution that a uniformly distributed unit vector makes with the subspace. If the dimension of the original space is n, and that of the subspace $m<n$, then it is elementary to show that $\mu=\cos\theta$ has a probability density

$$\frac{2s!}{r!(s-r-1)!}\mu^{2r+1}(1-\mu^2)^{s-r-1} \quad (0\leqslant\mu\leqslant 1) \tag{70}$$

where we have written

$$m=2(1+r), \quad n=2(1+s). \tag{71}$$

This procedure may be generalised to the symmetric random walks considered in this paper as follows. If m,n are any two real numbers with $1\leqslant m<n<\infty$, and if X is any non-negative random variable, then we define a new random variable $\prod_m^n X$ by

$$\textstyle\prod_m^n X=\mu X, \tag{72}$$

where μ is independent of X and has the probability density (70). The interpretation of $\prod_m^n X$ is as the length of the projection onto an m-dimensional linear subspace of a random vector with length X which has spherical symmetry in n dimensions.

Notice that there is the same ambiguity in the definition of $\prod_m^n$ as there was in the definition of $\overset{n}{\oplus}$, due to the arbitrary choice of μ. We therefore extend the convention made in § 1 by saying that, in any expression containing several $\prod$ and $\oplus$ symbols, the random variables μ and λ implicit in these symbols are all independent. This corresponds to the fact, that, in a spherically symmetric random walk, the angles which successive steps make with any linear subspace are independent.

The projection operators $\prod_m^n$ tie up in a very natural way with the operation of radial addition and with the radial characteristic function. The connections are made explicit in the next theorem. It will be convenient to write the radial characteristic function ${}^n\Psi_X(t)$ in the alternative form ${}^n\Psi_t[X]$, so that

$${}^n\Psi_t[X]=E\{\Lambda_s(tX)\}.$$

THEOREM 11. *For any non-negative random variable* X,

$${}^n\Psi_t[X]={}^m\Psi_t[\textstyle\prod_m^n X]. \tag{73}$$

If X and Y are independent and non-negative, then

$$\prod\nolimits_m^n (X \overset{n}{\oplus} Y) \doteq (\prod\nolimits_m^n X) \overset{m}{\oplus} (\prod\nolimits_m^n Y). \tag{74}$$

Proof. Sonine's first finite integral ([16], 373) can be written

$$\Lambda_s(z) = \frac{2s!}{r!(s-r-1)!} \int_0^1 \Lambda_r(\mu z)\, \mu^{2r+1}(1-\mu^2)^{s-r-1} d\mu,$$

so that, if μ has the probability density (70),

$$\Lambda_s(z) = E\{\Lambda_r(\mu z)\}.$$

Hence

$$E\{\Lambda_s(tX)\} = E\{\Lambda_r(t\mu X)\},$$

proving (73). Therefore

$$\begin{aligned} {}^m\Psi_t[\prod\nolimits_m^n (X \overset{n}{\oplus} Y)] &= {}^n\Psi_t[X \overset{n}{\oplus} Y] = {}^n\Psi_t[X]\, {}^n\Psi_t[Y] \\ &= {}^m\Psi_t[\prod\nolimits_m^n X]\, {}^m\Psi_t[\prod\nolimits_m^n Y] = {}^m\Psi_t[(\prod\nolimits_m^n X) \overset{m}{\oplus} (\prod\nolimits_m^n Y)]. \end{aligned}$$

An application of Lemma 2 completes the proof of the theorem.

Now let $X_1, X_2, \ldots$ be independent and identically distributed, and put $\tilde{X}_j = \prod_m^n X_j$. Then, if

$$S_N = X_1 \overset{n}{\oplus} \ldots \overset{n}{\oplus} X_N, \quad \tilde{S}_N = \tilde{X}_1 \overset{m}{\oplus} \ldots \overset{m}{\oplus} \tilde{X}_N,$$

the theorem shows that

$$\prod\nolimits_m^n S_N \doteq \tilde{S}_N.$$

Thus, as far as the one-point distributions are concerned, the process $\{\tilde{S}_N\}$ is the projection of $\{S_N\}$. This does not, however, extend to the joint distributions. To see this, consider the case where m and n are integers, and let $X_j = |\mathbf{X}_j|$, where the $\mathbf{X}_j$ are independent identically distributed spherically symmetric n-vectors. Then $\tilde{S}_N$ is the length of the projection of $\mathbf{S}_N$ onto the linear subspace, but the angles which the different $\mathbf{S}_N$ make with the subspace are correlated with one another.

It is a consequence of Theorem 11 that much of the structure introduced in preceding sections is preserved under the projection operators. Thus, for instance, if X has an infinitely divisible distribution in n dimensions, then the distribution of $\prod_m^n X$ is infinitely divisible in m dimensions. The projection operators are useful in that they connect random walks in different numbers of dimensions; an example of the way they can be used will be given in the next section. They can also be used to explain, for instance, the reason why the recurrence conditions (56) and (69) take the form

$$n \leqslant n_0.$$

We note in passing the formula

$$\prod_m^n \left(\prod_n^p X\right) \doteq \prod_m^p X, \tag{75}$$

valid in the range $1 \leqslant m < n < p < \infty$.

If we return to the geometrical picture of a symmetric random walk in an integral number of dimensions, we see that instead of projecting the walk onto a single linear subspace, we could project it simultaneously onto two (or more) orthogonal subspaces. The two (or more) projections will not, of course, be independent in general. Thus, if X is the length of a symmetric random n-vector $\mathbf{X}$, we can consider the projections ξ_1, ξ_2 of $\mathbf{X}$ onto two orthogonal subspaces of dimensions m_1, m_2. Clearly we have

$$\xi_1 = X \cos\alpha \cos\beta, \quad \xi_2 = X \cos\alpha \sin\beta,$$

where α is the angle which $\mathbf{X}$ makes with the join of the two subspaces, and β the angle which the projection of $\mathbf{X}$ onto this join makes with the first of these subspaces. Because of the spherical symmetry α and β are independent, α having the distribution which a uniformly distributed unit n-vector makes with an (m_1+m_2)-dimensional subspace, and β the distribution of the angle which a uniformly distributed (m_1+m_2)-vector makes with an m_1-dimensional subspace.

Now, if $\mathbf{Y}$ is a random unit vector in N dimensions having a uniform (i.e. spherically symmetric) distribution, then the angle which $\mathbf{Y}$ makes with an M-dimensional linear subspace ($M < N$) has probability density

$$C_{M,N} \cos^{M-1}\phi \sin^{N-M-1}\phi \quad (0 \leqslant \phi \leqslant \tfrac{1}{2}\pi), \tag{76}$$

where the constant $C_{M,N}$ is given by

$$C_{M,N} = \frac{2\,(\frac{1}{2}N-1)!}{(\frac{1}{2}M-1)!\,(\frac{1}{2}N-\frac{1}{2}M-1)!} \tag{77}$$

Now (76) defines a probability distribution whenever M and N are real and $1 \leqslant M < N < \infty$; this distribution will be denoted by $\mathcal{D}(M,N)$. We shall extend the definition by taking $\mathcal{D}(N,N)$ to be the distribution concentrated at $\phi = 0$

It follows that the projections ξ_1, ξ_2 of the random n-vector $\mathbf{X}$ satisfy

$$(\xi_1, \xi_2) \doteq X \cos\alpha\,(\cos\beta, \sin\beta), \tag{78}$$

where α, β are independent, $\alpha \in \mathcal{D}(m_1+m_2, n)$, $\beta \in \mathcal{D}(m_1, m_1+m_2)$. (The notation $\alpha \in \mathcal{D}$ means that α has the distribution $\mathcal{D}$). More generally, if $\xi_1, \ldots, \xi_k$ are the projections of the spherically symmetric random n-vector $\mathbf{X}$ onto a set of k mutually orthogonal subspaces of dimensions $m_1, \ldots, m_k$, then it is not difficult to see that

$$(\xi_1, \ldots, \xi_k) \doteq X(\cos\alpha \cos\beta,\ \cos\alpha \sin\beta \cos\gamma,\ \cos\alpha \sin\beta \sin\gamma \cos\delta,\ \ldots,\ \cos\alpha \sin\beta \sin\gamma \ldots \sin\theta), \tag{79}$$

where $X, \alpha, \beta, \ldots, \theta$ are all independent, with

$$\left.\begin{array}{l} \alpha \in \mathcal{D}(m_1 + \ldots + m_k, n), \\ \beta \in \mathcal{D}(m_1, m_1 + \ldots + m_k), \\ \gamma \in \mathcal{D}(m_2, m_2 + \ldots + m_k), \\ \quad\vdots \\ \theta \in \mathcal{D}(m_{k-1}, m_{k-1} + m_k). \end{array}\right\} \tag{80}$$

So far we have discussed the case in which n, m_j are integers. It is, however, natural to extend the definition to the case when n, m_j are any real numbers with

$$m_j \geqslant 1, \quad m_1 + m_2 + \ldots + m_k \leqslant n.$$

Then, if X is a non-negative random variable, we *define* the simultaneous projection of X from n to $(m_1, \ldots, m_k)$ dimensions to be the vector

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k) = X\ (\cos\alpha \cos\beta,\ \cos\alpha \sin\beta \cos\gamma,\ \ldots) \tag{81}$$

as in (79), where $X, \alpha, \ldots, \theta$ are independent, the distributions of $\alpha, \ldots, \theta$ being given by (80). Thus $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = X\boldsymbol{\zeta}, \tag{82}$$

where $\boldsymbol{\zeta}$ is a random k-vector distributed over $|\boldsymbol{\zeta}| \leqslant 1$. If $m_1 + m_2 + \ldots + m_k = n$, then $|\boldsymbol{\zeta}| = 1$ and $X = |\boldsymbol{\xi}|$. In this case the components of $\boldsymbol{\xi}$ can be looked upon as generalised coordinates of X.

It is easy to see that Sonine's second finite integral ([16], 376) implies that, if

$$\phi \in \mathcal{D}\{2(1+\varrho),\ 2(1+\varrho) + 2(1+\sigma)\},$$

then

$$E\{\Lambda_\varrho(u \cos\phi)\ \Lambda_\varrho(v \sin\phi)\} = \Lambda_{\varrho+\sigma+1}\{(u^2 + v^2)^{\frac{1}{2}}\}. \tag{83}$$

Now consider the simultaneous projection of X from n to $(m_1, \ldots, m_k)$ dimensions, and put

$$n = 2(1+s), \quad m_j = 2(1+r_j).$$

Then repeated application of (83) yields the formula

$${}^n\Psi_X(t) = E\{\Lambda_{r_1}(t_1\xi_1) \ldots \Lambda_{r_k}(t_k \xi_k)\}, \tag{84}$$

where $t^2 = t_1^2 + \ldots + t_k^2$.

This formula is analogous to the formula (for n an integer)

$${}^n\Psi_X(t) = E\{\exp(it_1 X_1) \ldots \exp(it_n X_n)\}$$

which lies at the basis of the use of the radial characteristic function. It shows that the concept of simultaneous projection has a symmetry which is not apparent from its definition. Thus, if π is a permutation of $(1,2,\ldots,k)$, and if $(\eta_1,\ldots,\eta_k)$ is the simultaneous projection of X from n to $(m_{\pi 1},\ldots,m_{\pi k})$ dimensions, then

$$(\eta_1,\ldots,\eta_k) \doteq (\xi_{\pi 1},\ldots,\xi_{\pi k}).$$

As an illustration of the use of simultaneous projections, we obtain a characterisation of the Rayleigh distribution, which is the analogue of the property of the symmetric normal distribution that its components are independent. Let $k \geqslant 2$ be an integer, m_j $(j=1,\ldots,k)$ any real numbers not less than 1, and $n \geqslant \sum m_j$. Let X be a non-negative random variable, and let $\boldsymbol{\xi}$ be its simultaneous projection from n to $(m_1,\ldots,m_k)$ dimensions. Now suppose that $\xi_1,\ldots,\xi_k$ are independent. We show that this implies that X has an n-dimensional Rayleigh distribution. In fact, it follows easily from (84) that the radial characteristic function $\psi(t)$ of X satisfies

$$\psi\{(t_1^2+\ldots+t_k^2)^{\frac{1}{2}}\}=\psi(t_1)\ldots\psi(t_k),$$

and this together with the continuity of $\psi(t)$ shows that $\psi(t)=e^{-at^2}$ for some a.

To conclude this section, we remark that, when X is concentrated at c, (84) takes the form

$$\Lambda_s(ct)=\int \Lambda_{r_1}(ct_1\xi_1)\ldots\Lambda_{r_k}(ct_k\zeta_k)\,g(n,m_j;\boldsymbol{\zeta})\,d\boldsymbol{\zeta} \tag{85}$$

where the function g can be written down. This gives a whole family of integrals, which are generalisations of the finite integrals of Sonine.

10. The class of radial characteristic functions

In this section we consider briefly the problem of deciding which functions can be radial characteristic functions. We recall the notations $\mathfrak{F}_n, \mathfrak{I}_n$ and $\mathcal{S}_n(=\mathcal{S})$ for the class of n-dimensional radial characteristic functions and its subclasses corresponding to infinitely divisible and stable distributions. The problem is to identify $\mathfrak{F}_n$, $\mathfrak{I}_n$ and $\mathcal{S}_n$. As far as $\mathcal{S}_n$ is concerned, the answer is given by Theorem 8; $\mathcal{S}_n$ consists exactly of the functions

$$\exp(-ct^{\alpha})$$

for $c \geqslant 0$, $0<\alpha\leqslant 2$.

The following inclusion relations hold trivially for any n;

$$\mathcal{S} \subset \mathfrak{I}_n \subset \mathfrak{F}_n. \tag{86}$$

(These inclusions are strict; weak inclusions are here denoted by $\subseteq$.) Further inclusion relations are given by the following theorem.

THEOREM 12. *The sequences* $\{\mathfrak{F}_n\}$, $\{\mathfrak{I}_n\}$ *are strictly decreasing in* $n \geqslant 1$, *and their limits*

$$\mathfrak{F}_\infty = \bigcap_{n\geqslant 1} \mathfrak{F}_n, \; \mathfrak{I}_\infty = \bigcap_{n\geqslant 1} \mathfrak{I}_n \tag{87}$$

are non-empty with

$$\mathcal{S} \subseteq \mathfrak{I}_\infty \subseteq \mathfrak{F}_\infty. \tag{88}$$

Proof. Let $m < n$. Then, if $\psi \in \mathfrak{F}_n$, there exists X with

$$\psi(t) = {}^n\Psi_t[X],$$

and so

$$\psi(t) = {}^m\Psi_t[\textstyle\prod_m^n X] \in \mathfrak{F}_m.$$

Hence $\mathfrak{F}_n \subseteq \mathfrak{F}_m$. If X is infinitely divisible in n dimensions, then so is $\prod_m^n X$ in m dimensions, and so $\mathfrak{I}_n \subseteq \mathfrak{I}_m$.

Now consider the generalised Poisson distribution for which

$$\psi(t) = \exp\{\Lambda_r(t) - 1\} \quad (m = 2(1+r)).$$

Then $\psi(t) \in \mathfrak{I}_m \subseteq \mathfrak{F}_m$. Now suppose that $\psi(t) \in \mathfrak{F}_n$. Then

$$\psi(t) = {}^m\Psi_t[\textstyle\prod_m^n X],$$

and since $\prod_m^n X = \mu X$ where μ has an absolutely continuous distribution, μX can have no atoms except possibly at zero. But $\psi(t) = {}^m\Psi_t[Y]$, where Y has atoms at 0 and 1. Thus we have a contradiction, showing that $\mathfrak{I}_m$ is not a subset of $\mathfrak{F}_n$. It follows that the two inclusions

$$\mathfrak{F}_n \subseteq \mathfrak{F}_m, \quad \mathfrak{I}_n \subseteq \mathfrak{I}_m$$

are strict.

The inclusions (88) follow at once from (86), and since $\mathcal{S}$ is non-empty, so are $\mathfrak{F}_\infty$ and $\mathfrak{I}_\infty$. Hence the theorem is proved.

The classical criterion for a function to be a characteristic function is given by Bochner's theorem, which can be made to characterise $\mathfrak{F}_n$ when n is an integer. Thus (cf.[7]), $\psi(t) \in \mathfrak{F}_n$ (n integral) if and only if it is continuous, $\psi(0) = 1$, and the matrix

$$(\psi(d_{ij}); \quad i, j = 1, 2, \ldots, N)$$

is positive-semi-definite whenever there exist points $A_1, \ldots, A_N$ in Euclidean n-space R^n such that d_{ij} is the distance from A_i to A_j.

Thus the main condition for $\psi(t)$ to belong to $\mathfrak{F}_n$, when n is an integer, is that the matrix $(\psi(d_{ij}))$ be positive-semi-definite whenever the d_{ij} satisfy a certain condition, and this condition is the only point at which the dimension n enters. As the dimension increases, this latter condition weakens, so that $\mathcal{J}_{n+1} \subseteq \mathcal{J}_n$ in accordance with Theorem 12.

Unfortunately, it does not seem to be possible to modify the condition on the d_{ij} so that it becomes meaningful when n is not an integer. In fact, for any given N, the condition allows exactly

$$\tfrac{1}{2}n(2N-n-1) \quad (n \leqslant N-1)$$

of the d_{ij} to be assigned freely, and this alone seems to make it unlikely that the condition can be so modified. Hence it may not be possible to obtain a simple characterisation of the Bochner type for $\mathcal{J}_n$ when n is not an integer. The best we can do is to say that it lies between $\mathcal{J}_m$ and $\mathcal{J}_{m+1}$, where m is the integral part of n.

The class $\mathcal{J}_\infty$ can be written in the alternative form

$$\mathcal{J}_\infty = \bigcap_{n=1}^{\infty} \mathcal{J}_n,$$

and we can appeal to a known theorem ([7], 24) to show that $\psi \in \mathcal{J}_\infty$ if and only if there exists a distribution function G on $[0, \infty)$ such that

$$\psi(t) = \int_0^\infty e^{-ct^2} dG(c). \tag{89}$$

Equivalently, $\psi(t) \in \mathcal{J}_\infty$ if and only if $\psi(t)$ is continuous in $t \geqslant 0$ and infinitely differentiable in $t > 0$, $\psi(0) = 1$, and $\psi(x^{\frac{1}{2}})$ is a completely monotonic function of x. In particular, since the function

$$\exp(-t^\alpha) \quad (0 < \alpha < 2)$$

belongs to $\mathcal{S}$ and hence to $\mathcal{J}_\infty$, it follows that the function

$$\exp(-x^\beta) \quad (0 < \beta < 1)$$

is completely monotonic, a result due to Bochner [1]. Notice that (89) shows that $\mathcal{J}_\infty$ is exactly the (pointwise) closed convex hull of $\mathcal{S}$, and that the extreme points both of $\mathcal{J}_\infty$ and of $\mathcal{S}$ are the functions e^{-ct^2} $(c \geqslant 0)$. In fact, a number of the results obtained in this paper can be placed within the context of the extreme-point theory of convex sets.

It is possible to sharpen the inclusions (88) to give

$$\mathcal{S} \subset \mathfrak{J}_\infty \subset \mathcal{J}_\infty. \tag{90}$$

The details are straightforward and will be omitted.

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