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# RANDOMIZED URN MODELS REVISITED USING STOCHASTIC APPROXIMATION

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This paper presents the link between stochastic approximation and clinical trials based on randomized urn models investigated by Bai and Hu [*Stochastic Process. Appl.* **80** (1999) 87–101], Bai and Hu [*Ann. Appl. Probab.* **15** (2005) 914–940] and Bai, Hu and Shen [*J. Multivariate Anal.* **81** (2002) 1–18]. We reformulate the dynamics of both the urn composition and the assigned treatments as standard stochastic approximation (SA) algorithms with remainder. Then, we derive the a.s. convergence and the asymptotic normality [central limit theorem (CLT)] of the normalized procedure under less stringent assumptions by calling upon the ODE and SDE methods. As a second step, we investigate a more involved family of models, known as multi-arm clinical trials, where the urn updating depends on the past performances of the treatments. By increasing the dimension of the state vector, our SA approach provides this time a new asymptotic normality result.

**1. Introduction.** The aim of this paper is to illustrate the efficiency of stochastic approximation (SA) theory by revisiting several recent results on randomized urn models applied to clinical trials (especially [5–7]). We will first retrieve the a.s. convergence (strong consistency) and asymptotic normality results obtained in these papers under less stringent assumptions. Then we will take advantage of this more synthetic approach to establish a new central limit theorem (CLT) in the more sophisticated randomized urn model known as "multi-arm clinical test." In this model, the urn updating which produces the adaptive design is based on statistical estimators of the past efficiency of the assigned treatments.

In these adaptive models, the starting point is the equation which governs the urn composition updated after each newly treated patient. Basically, we will show that a normalized version of this urn composition can be formulated as a classical recursive stochastic algorithm with step $\gamma_n = \frac{1}{n}$ which classical stochastic approximation theory deals with. Doing so we will be in position to establish the a.s. convergence of the procedure by calling upon the so-called ordinary differential equation method (ODE method) and to derive the asymptotic normality (a CLT, to be precise) from the standard CLT for stochastic algorithms (sometimes called the

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stochastic differential equation method (SDE method) (see, e.g., [9, 14]). These two main theoretical results are recalled in a self-contained form in the Appendix. They can be found in all classical textbooks on SA [9, 13, 14, 22] and go back to [21] and [11]. SA theory is also used in clinical trials to solve dose-finding problems (see, e.g., [12] and citations therein).

Clinical trials essentially deal with the asymptotic behavior of the patient allocation to several treatments during the procedure. Adaptive designs in clinical trials aim at detecting "on line" which treatment should be assigned to more patients, while keeping randomness enough to preserve the basis of treatments. This adaptive approach relies on the cumulative information provided by the responses to treatments of previous patients in order to adjust treatment allocation to the new patients. To this end, many urn models have been suggested in the literature (see [15, 20, 25, 27] and [28]). The most widespread random adaptive model is the generalized Friedman urn (GFU) (see [2] and more recently [19, 24]), also called generalized Pólya urn (GPU). The idea of this modeling is that the urn contains balls of $d$ different types representative of the treatments. All random variables involved in the model are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote $Y_0 = (Y_0^i)_{i=1,\dots,d} \in \mathbb{R}_+^d \setminus \{0\}$ the initial composition of the urn, where $Y_0^i$ denotes the number of balls of type $i$, $i = 1, \dots, d$ (of course, a more realistic though not mandatory assumption would be $Y_0 \in \mathbb{N}^d \setminus \{0\}$). The allocation of the treatments is sequential and the urn composition at draw $n$ is denoted by $Y_n = (Y_n^i)_{i=1,\dots,d}$. When the $n$th patient presents, one draws randomly (i.e., uniformly) a ball from the urn with instant replacement. If the ball is of type $j$, then the treatment $j$ is assigned to the $n$th patient, $j = 1, \dots, d$, $n \geq 1$. The urn composition is updated by taking into account the response of the $n$th patient to the treatment $j$, or the responses of all patients up to the $n$th one (i.e., the efficiency of the assigned treatment), namely, by adding $D_n^{ij}$ balls of type $i$, $i = 1, \dots, d$. The procedure is iterated as long as patients present. Consequently, the larger the number of balls of a given type, the more efficient the treatment is. The urn composition at stage $n$, modeled by an $\mathbb{R}^d$-valued vector $Y_n$, satisfies the following recursive procedure:

$$Y_n = Y_{n-1} + D_n X_n, \qquad n \geq 1, \qquad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \tag{1.1}$$

where $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ is the addition rule matrix and $X_n$ is the result of the $n$th draw and $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \dots, e^d\}$ models the selected treatment ($\{e^1, \dots, e^d\}$ denotes the canonical basis of $\mathbb{R}^d$ and $e^j$ stands for treatment $j$). We assume that there is no extinction, that is, $Y_n \in \mathbb{R}_+^d \setminus \{0\}$ a.s. for every $n \geq 1$: so is the case if all the entries $D_n^{ij}$ are a.s. nonnegative, but other settings can also be taken under consideration (see Section 2). We model the drawing in the urn by setting

$$X_n = \sum_{j=1}^{d} \mathbb{1}_{\{(\sum_{\ell=1}^{j-1} Y_{n-1}^\ell)/(\sum_{\ell=1}^{d} Y_{n-1}^\ell) < U_n \leq (\sum_{\ell=1}^{j} Y_{n-1}^\ell)/(\sum_{\ell=1}^{d} Y_{n-1}^\ell)\}} e^j, \qquad n \geq 1, \tag{1.2}$$

where $(U_n)_{n\geq 1}$ is i.i.d. with distribution $U_1 \stackrel{\mathcal{L}}{\sim} \mathcal{U}_{[0,1]}$.

Let $\mathcal{F}_n = \sigma(Y_0, U_k, D_k, 1 \leq k \leq n)$ be the filtration of the procedure. The *generating matrices* are defined as the $\mathcal{F}_n$-compensator of the additions rule sequence, that is,

$$H_n = \left(\mathbb{E}\left[D_n^{ij} | \mathcal{F}_{n-1}\right]\right)_{1\leq i,j\leq d}, \qquad n \geq 1.$$

Other fields of application can be considered for such procedures like the adaptive asset allocation by an asset manager or a trader. Indeed this has already been done in [23] and successfully implemented with multi-armed bandit procedure. Imagine an asset manager who can trade the same financial instrument (tradable asset) on different trading venues. To optimize the execution of an inventory of this asset, she can split her orders across these trading destinations. She starts with the initial allocation vector $Y_0$. At stage $n$, she chooses a trading destination according to the distribution (1.2) of $X_n$, then evaluates its performance during one time step and modifies the urn composition (most likely virtually) and proceeds. Thus the normalized urn composition represents the allocation vector among the venues and the addition rule matrices model the successive re-allocations depending on the past performances of the different trading destinations.

One may also consider this type of procedure as a strategy to update the composition of a portfolio or even a whole fund, based on the (recent) past performances of the assets.

The first designs under consideration were the homogeneous GFU models where the addition rules $D_n$ are i.i.d. and the so-called generating matrices $H_n = H = \mathbb{E}D_n$ are identical, nonrandom, with nonnegative entries and irreducible. Hence, by the Perron–Frobenius theorem, $H$ has a unique and positive maximal eigenvalue and an eigenvector with positive components (see [2, 3, 17, 18]). But the homogeneity of the generating matrix is often not satisfied in practice and inhomogeneous GFU models have been introduced (see [5]) in which $H_n$ are not random but converge to a deterministic limit $H$, under the assumption that the total number of balls added at each stage is constant. As a third step, the homogeneous extended Pólya urn (EPU) models have been introduced in [26] in which only the mean total number of balls added at each stage is constant. This number is called the *balance* of the urn and the urn is said to be *balanced*.

Finally, in [6] the authors proposed a nonhomogeneous EPU model because in applications, the addition rule $D_n$ depends on the past history of previous trials (see [1]), so that the general generating matrix $H_n$ is usually random. Thus the entries of $H$ may not be all nonnegative (e.g., when there is no replacement after the draw diagonal terms may become negative), and they assume that the matrix $H$ has a unique maximal eigenvalue $\lambda$ with associated (right) eigenvector $v^* = (v^{*,i})_{i=1,\ldots,d}$ with $\sum_{i=1}^d v^{*,i} = 1$. Furthermore, the conditional expectation of the total number of balls added at each stage was constant.

The first theoretical investigations on these models focused on the asymptotic properties of the urn composition (consistency and asymptotic normality). However, for practical matter, it is clear that the asymptotic behavior of the vector $N_n := \sum_{k=1}^{n} X_k$ which stores the treatment allocation among the first $n$ patients is of high interest, especially its variance structure in order to compare several adaptive designs. Thus, in [6] is proved the strong consistency of both (normalized) quantities $Y_n/n$ and $N_n/n$ (under a summability assumption on the generating matrices).

By considering an appropriate recursive procedure for the normalized urn composition derived from (1.1) we prove by the ODE method its a.s. convergence toward $v^*$ under a significantly less stringent assumption, namely, the minimal requirement that $H_n \xrightarrow[n\to\infty]{\text{a.s.}} H$. The a.s. convergence of the treatment allocation frequency $N_n/n$ toward the same $v^*$ follows from the previous one.

Concerning asymptotic normality, separate results on these two quantities are obtained in [6] under an additional assumption on the rate of convergence of the generating matrices $H_n$ toward $H$. On our side we propose to consider a stochastic approximation procedure with remainder satisfied by the higher-dimensional vector $(Y_n/n, N_n/n)$. Then, the standard CLT for SA procedures with remainder directly provides the expected asymptotic normality result for the whole vector under an assumption on the $L^2$-rate of convergence of the generating matrices toward their limit [i.e., $\|\|H_n - H\|\| = o(n^{-1/2})$] which is again slightly less stringent than the original one. As a result, we obtain the asymptotic joint distribution with an explicit global covariance structure matrix.

In the end of [6], an application to multi-arm clinical trials randomized urn models is proposed. This adaptive design has already been introduced in [7] with first consistency results. This kind of model is clearly the most interesting for practitioners since it takes into account the past results of the assigned treatments in the addition rule matrices, denoted $S_n$ at time $n$ ($S_n^i$ denotes the number of cured patients by treatment $i$ among the $N_n^i$ treated ones). The above strong consistency results apply but none of the asymptotic normality works as stated since the generating matrices $H_n$ do not (in fact *cannot* as we will emphasize) converge at the requested rate. The reason is that they themselves satisfy a CLT. However, we can overcome this obstacle by once again increasing the structural dimension of the problem; we show that the triplet $(Y_n/n, N_n/n, S_n/n)$ can be written as a recursive SA algorithm with remainder satisfying a.s. convergence and a CLT (provided the limiting generating matrix is still irreducible, etc.). Thus we illustrate on this example that SA theory is a powerful tool to investigate this kind of adaptive design problem. The main difficulty is to exhibit the appropriate form for the recursion by making a priori the balance between significant asymptotic terms and remainder terms.

The paper is organized as follows. We rewrite the dynamics (1.1) of the urn composition as a stochastic approximation procedure with state variable for $\widetilde{Y}_n :=$

$Y_n/n$ in Section 2.1. In Section 2.2 the a.s. convergence of $\frac{1}{n}\sum_{i=1}^{d} Y_n^i$ is established which implies that of $\widetilde{Y}_n$ and $\widetilde{N}_n := N_n/n$ by using the ODE method of SA under slightly lighter assumption than in [6]. The rate of convergence is investigated in Section 2.3: we obtain a CLT, once again under slightly less stringent assumptions on the limit generating matrix $H$ than in [6]. Section 3 is devoted to multi-arm clinical tests. In Section 3.1 we briefly recall the Wei GFU model introduced [7, 27] where the generating matrices $H_n$ are not random. In this case, the strong consistency and the asymptotic normality follow from the results of Section 2 (as in [6]). In Section 3.2 we study the adaptive design proposed in [7] where the addition rule matrices depend on the responses of all the past patients. We use the results from Section 2.2 to prove the strong consistency. We prove in Section 3.3 a new CLT for this model, when the generating matrix $H_n$ satisfies itself a CLT, which relies again on stochastic approximation techniques.

NOTATION. $\forall u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$, $\|u\|$ denotes the canonical Euclidean norm of the column vector $u$ on $\mathbb{R}^d$, $w(u) = \sum_{k=1}^{d} u^k$ denotes its "weight," $u^t$ denotes its transpose; $|||A|||$ denotes the operator norm of the matrix $A \in \mathcal{M}_{d,q}(\mathbb{R})$ with $d$ rows and $q$ columns with respect to canonical Euclidean norms. When $d = q$, $\mathrm{Sp}(A)$ denotes the set of eigenvalues of $A$. $\mathbf{1} = (1 \cdots 1)^t$ denotes the unit column vector in $\mathbb{R}^d$, $I_d$ denotes the $d \times d$ identity matrix and $\mathrm{diag}(u) = [\delta_{ij} u_i]_{1\le i,j\le d}$, where $\delta_{ij}$ is the Kronecker symbol.

**2. Convergence and first rate result.** With the notation and definitions described in the the Introduction, we then formulate the main assumptions to establish the a.s. convergence of the urn composition

$$
(\mathrm{A1}) \equiv \begin{cases}
\text{(i)} & \textit{Addition rule matrix}\text{: for every } n \ge 1, \text{ the matrix } D_n \text{ a.s.}\\
& \text{has nonnegative entries.}\\
\text{(ii)} & \textit{Generating matrix}\text{: for every } n \ge 1, \text{ the generating matrices}\\
& H_n = (H_n^{ij})_{1\le i,j\le d} \text{ a.s. satisfies}\\
& \qquad \forall j \in \{1,\dots,d\} \qquad \displaystyle\sum_{i=1}^{d} H_n^{ij} = c > 0.\\
\text{(iii)} & \textit{Starting value}\text{: the starting urn composition vector}\\
& Y_0 \in \mathbb{R}_+^d \setminus \{0\}.
\end{cases}
$$

The constant $c$ is known as the balance of the urn. In fact, we may assume without loss of generality, up to a renormalization of $Y_n$, that $c = 1$ since $\widehat{Y}_n = \frac{Y_n}{c}$ and $\widehat{D}_{n+1} = \frac{D_{n+1}}{c}$, $n \ge 0$, formally satisfy the dynamic (1.1), namely,

$$
\widehat{Y}_n = \widehat{Y}_{n-1} + \widehat{D}_n X_n, \qquad n \ge 1, \qquad \widehat{Y}_0 \in \mathbb{R}_+^d \setminus \{0\}.
$$

From now on, throughout the paper, we will consider this normalized balance version. Nevertheless, we will still denote by $Y_n$ and $D_n$ the normalized quantities and assume that $c = 1$.

(A2) The addition rule $D_n$ is conditionally independent of the drawing procedure $X_n$ given $\mathcal{F}_{n-1}$ and satisfies

$$\forall 1 \le j \le d \qquad \sup_{n\ge 1} \mathbb{E}\big[\|D_n^{\cdot j}\|^2 | \mathcal{F}_{n-1}\big] < +\infty \qquad \text{a.s.,} \tag{2.1}$$

where $D_n^{\cdot j} = (D_n^{ij})_{i=1,\dots,d}$.

(A3) Assume that there exists an irreducible $d \times d$ matrix $H$ (with nonnegative entries) such that

$$H_n \xrightarrow[n\to\infty]{\text{a.s.}} H. \tag{2.2}$$

$H$ is called the *limit generating matrix*.

The combination of assumptions (A1)–(A3) guarantees that $H$ satisfies the assumptions of the Perron–Frobenius theorem (see [10]) so that 1 is the eigenvalue of $H$ with the highest norm (maximal eigenvalue) and that the components of its right eigenvector $v$ can be chosen all positive. Therefore, we may normalize this vector $v^*$ such that $w(v^*) = 1$.

A VARIANT INCLUDING POSSIBLE DEFINITE REMOVAL. We may relax assumption (A1) by allowing the removal of the drawn ball from its urn (see, e.g., [19]). Other relaxation of these requirements may be considered; it could be possible to remove balls other than the drawn one. This leads to *tenable* urns (studied notably in [4]; see also [24]) where an arithmetical assumption to the row of any negative diagonal entry in $D_n$ is added in order to avoid the urn extinction [see assumption (A′1) below]. Thus we may replace assumption (A1) (after renormalization) by

$$(\text{A}'1) \equiv \begin{cases} \text{(i)} & \textit{Addition rule matrix}\text{: For every } i \in \{1,\dots,d\}, \\ & \text{there exists } c_i \in (0,+\infty) \text{ such that,} \\ & \text{for every } n \ge 1, \forall i, j \in \{1,\dots,d\}, \frac{\delta_{ij}}{c_i} + D_n^{ij} \in \frac{\mathbb{N}}{c_i} \text{ a.s.} \\ & \text{and } \forall j \in \{1,\dots,d\}, \sum_{i=1}^d D_n^{ij} \ge 0 \text{ a.s.} \\ \text{(ii)} & \textit{Generating matrix}\text{: For every } n \ge 1, H_n \text{ a.s. satisfies} \\ & \qquad \forall j \in \{1,\dots,d\} \qquad \displaystyle\sum_{i=1}^d H_n^{ij} = 1. \\ \text{(iii)} & \textit{Starting value}\text{: The starting urn composition vector} \\ & \qquad Y_0 \in \left(\displaystyle\prod_{i=1}^d \frac{\mathbb{N}}{c_i}\right) \Big\backslash \{0\}. \end{cases}$$

In this case $H$ may have negative (diagonal) entries and the Perron–Frobenius theorem cannot be used, so we change assumption (A3) into

(A′3) 1 is the maximal eigenvalue of $H$ and $\exists v \in \mathbb{R}_+^d \setminus \{0\}$ such that $Hv = v$.

Throughout the paper, we may substitute (A′1)–(A′3) for (A1)–(A3) as recalled in each result.

The following preliminary lemma ensures that if (A′1) holds, then the urn extinction never occurs and its weight $w(Y_n)$ is nondecreasing.

LEMMA 2.1 (Preliminary). *If* (A′1) *holds, then* $w(Y_n)$ *is nondecreasing and positive.*

PROOF. We proceed by induction on $n \geq 0$. Assume $Y_{n-1} \in (\prod_{i=1}^{d} \frac{\mathbb{N}}{c_i}) \setminus \{0\}$. For every $i \in \{1, \dots, d\}$,

$$Y_n^i = Y_{n-1}^i + \sum_{j=1}^{d} D_n^{ij} \mathbb{1}_{\{X_n = e^j\}} \quad \text{and} \quad \{X_n = e^j\} \subset \{Y_{n-1}^j > 0\} = \{Y_{n-1}^j \geq 1/c_j\}.$$

Consequently, $Y_n^i \geq Y_{n-1}^i$ and $Y_n^i \in \frac{\mathbb{N}}{c_i} \setminus \{0\}$ on the event $\bigcup_{j \neq i}\{X_n = e^j\}$. On $\{X_n = e^i\}$, $\{Y_{n-1}^i \geq \frac{1}{c_i}\}$ so that $Y_n^i = Y_{n-1}^i + D_n^{ii} \geq \frac{1}{c_i} - \frac{1}{c_i} \geq 0$. Finally,

$$w(Y_n) = w(Y_{n-1}) + \sum_{j=1}^{d} \left( \sum_{i=1}^{d} D_n^{ij} \right) \mathbb{1}_{\{X_n = e^j\}} \geq w(Y_{n-1}) > 0. \qquad \square$$

2.1. *The dynamics as a stochastic approximation procedure.* Our aim in this section is to reformulate the dynamics (1.1)–(1.2) into a recursive stochastic algorithm. Then we aim at applying the most powerful tools of SA, namely the "ODE" and the "SDE" methods to elucidate the asymptotic properties (a.s. convergence and weak rate) of both the urn composition and the treatment allocation. We start from (1.1) with $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$. For $n \geq 1$,

$$Y_{n+1} = Y_n + D_{n+1} X_{n+1} = Y_n + \mathbb{E}[D_{n+1} X_{n+1} | \mathcal{F}_n] + \Delta M_{n+1}, \tag{2.3}$$

where

$$\Delta M_{n+1} := D_{n+1} X_{n+1} - \mathbb{E}[D_{n+1} X_{n+1} | \mathcal{F}_n]$$

is an $\mathcal{F}_n$-martingale increment. By the definition of the generating matrix $H_n$, we have

$$\begin{aligned} \mathbb{E}[D_{n+1} X_{n+1} | \mathcal{F}_n] &= \sum_{i=1}^{d} \mathbb{E}\big[D_{n+1} \mathbb{1}_{\{X_{n+1} = e^i\}} e^i | \mathcal{F}_n\big] \\ &= \sum_{i=1}^{d} \mathbb{E}[D_{n+1} | \mathcal{F}_n] \mathbb{P}\big(X_{n+1} = e^i | \mathcal{F}_n\big) e^i \\ &= H_{n+1} \sum_{i=1}^{d} \frac{Y_n^i}{w(Y_n)} e^i = H_{n+1} \frac{Y_n}{w(Y_n)} \end{aligned}$$

so that $Y_{n+1} = Y_n + H_{n+1}\frac{Y_n}{w(Y_n)} + \Delta M_{n+1}$.

Now we can derive a stochastic approximation for the normalized urn composition $Y_n$. First we have for every $n \geq 1$,

$$\frac{Y_{n+1}}{n+1} = \frac{Y_n}{n} + \frac{1}{n+1}\left(H_{n+1}\frac{Y_n}{w(Y_n)} - \frac{Y_n}{n}\right) + \frac{\Delta M_{n+1}}{n+1}.$$

Consequently, $\widetilde{Y}_n = \frac{Y_n}{n}$, $n \geq 1$, satisfies a canonical recursive stochastic approximation procedure

$$\begin{aligned}\widetilde{Y}_{n+1} &= \widetilde{Y}_n + \frac{1}{n+1}(H_{n+1} - I_d)\widetilde{Y}_n \\ &\quad + \frac{1}{n+1}\left(\Delta M_{n+1} + \left(\frac{n}{w(Y_n)} - 1\right)H_{n+1}\widetilde{Y}_n\right) \\ &= \widetilde{Y}_n - \frac{1}{n+1}(I_d - H)\widetilde{Y}_n + \frac{1}{n+1}(\Delta M_{n+1} + r_{n+1})\end{aligned} \tag{2.4}$$

with step $\gamma_n = \frac{1}{n}$ and a remainder term given by

$$r_{n+1} := \left(\frac{n}{w(Y_n)} - 1\right)H_{n+1}\widetilde{Y}_n + (H_{n+1} - H)\widetilde{Y}_n. \tag{2.5}$$

Furthermore, in order to establish the a.s. boundedness of $(\widetilde{Y}_n)_{n\geq 1}$, we will rely on the following recursive equation satisfied by $w(Y_n)$:

$$w(Y_{n+1}) = w(Y_n) + \frac{w(H_{n+1}Y_n)}{w(Y_n)} + w(\Delta M_{n+1}).$$

By the properties of the generating matrix $H_{n+1}$, we obtain

$$w(H_{n+1}Y_n) = \sum_{i=1}^{d}(H_{n+1}Y_n)_i = \sum_{i=1}^{d}\sum_{j=1}^{d} H_{n+1}^{ij}Y_n^j = \sum_{j=1}^{d}\left(\sum_{i=1}^{d} H_{n+1}^{ij}\right)Y_n^j = w(Y_n).$$

Consequently,

$$w(Y_{n+1}) = w(Y_n) + 1 + w(\Delta M_{n+1}). \tag{2.6}$$

2.2. *Convergence results.*

THEOREM 2.1. *Let* $(Y_n)_{n\geq 0}$ *be the urn composition sequence defined by* (1.1)–(1.2). *Under the assumptions* (A1), (A2) *and* (A3) [*or* (A′1), (A2) *and* (A′3)],

(a) $\frac{w(Y_n)}{n} \xrightarrow[n\to\infty]{a.s.} 1$ *and* $\frac{Y_n}{w(Y_n)} \xrightarrow[n\to\infty]{a.s.} v^*$.

(b) $\widetilde{N}_n := \frac{N_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{a.s.} v^*$.

REMARKS. • We simply need that $H_n \xrightarrow[n\to\infty]{\text{a.s.}} H$ while the assumption in [6] is

$$\sum_{n\geq 1} \frac{\|H_n - H\|_\infty}{n} < +\infty,$$

where $\|\cdot\|_\infty$ is the norm on $L^\infty_{\mathbb{R}^{d\times d}}(\mathbb{P})$.

• Assumption (A3) is not necessary to prove that $\frac{w(Y_n)}{n} \xrightarrow[n\to\infty]{\text{a.s.}} 1$.

PROOF OF THEOREM 2.1. We will first prove that (a) $\Rightarrow$ (b), then we will prove (a).

(a) $\Rightarrow$ (b). We have

$$\mathbb{E}[X_n|\mathcal{F}_{n-1}] = \sum_{i=1}^{d} \frac{Y^i_{n-1}}{w(Y_{n-1})} e^i = \frac{Y_{n-1}}{w(Y_{n-1})}$$

and by construction $\|X_n\|^2 = 1$ so that $\mathbb{E}[\|X_n\|^2|\mathcal{F}_{n-1}] = 1$. Hence, the martingale

$$\widetilde{M}_n = \sum_{k=1}^{n} \frac{X_k - \mathbb{E}[X_k|\mathcal{F}_{k-1}]}{k} \xrightarrow[n\to\infty]{\text{a.s. and } L^2} \widetilde{M}_\infty \in L^2$$

and by the Kronecker lemma we obtain

$$\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} \frac{Y_{k-1}}{w(Y_{k-1})} \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

This yields the announced implication owing to the Cesaro lemma.

(a) FIRST STEP: We have

$$D_{n+1}X_{n+1} = \sum_{j=1}^{d} D^{\cdot j}_{n+1} \mathbb{1}_{\{X_{n+1}=e^j\}}.$$

Therefore,

$$\|D_{n+1}X_{n+1}\|^2 = \sum_{j=1}^{d} \|D^{\cdot j}_{n+1}\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}},$$

so that

$$\begin{aligned}\mathbb{E}[\|D_{n+1}X_{n+1}\|^2|\mathcal{F}_n] &= \sum_{j=1}^{d} \mathbb{E}[\|D^{\cdot j}_{n+1}\|^2|\mathcal{F}_n]\mathbb{P}(X_{n+1} = e^j|\mathcal{F}_n) \\ &\leq \sup_{n\geq 0} \sup_{1\leq j\leq d} \mathbb{E}[\|D^{\cdot j}_{n+1}\|^2|\mathcal{F}_n] < +\infty \qquad \text{a.s.}\end{aligned}$$

Consequently, $\sup_{n\geq 1}\mathbb{E}[\|\Delta M_{n+1}\|^2|\mathcal{F}_n] < +\infty$ a.s. Therefore, thanks to the strong law of large numbers for conditionally $L^2$-bounded martingale increments, we have $\frac{M_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} 0$. Consequently, it follows from (2.6) that

$$\frac{w(Y_n)}{n} = 1 + \frac{w(Y_0)-1}{n} + \frac{w(M_n)}{n} \xrightarrow[n\to\infty]{\text{a.s.}} 1. \tag{2.7}$$

SECOND STEP: Since the components of $\widetilde{Y}_n = \frac{Y_n}{n}$ are nonnegative and $w(\widetilde{Y}_n) = \frac{w(Y_n)}{n} \xrightarrow[n\to\infty]{\text{a.s.}} 1$, it is clear that $(\widetilde{Y}_n)_{n\geq 1}$ is a.s. bounded and that a.s. the set $\mathcal{Y}_\infty$ of all its limiting value is contained in

$$\mathcal{V} = w^{-1}\{1\} = \{u \in \mathbb{R}^d_+ | w(u) = 1\}.$$

So we may try applying the ODE method (see the Appendix, Theorem A.1). Since $\widetilde{Y}_n$ and $H_{n+1}\widetilde{Y}_n$ are a.s. bounded, (2.7) and (A3) imply that $r_n \xrightarrow[n\to\infty]{\text{a.s.}} 0$.

The ODE associated to the recursive procedure reads

$$\text{ODE}_{I_d-H} \equiv \dot{y} = -(I_d - H)y.$$

Owing to assumption (A3), $I_d - H$ admits $v^*$ as unique zero in $\mathcal{V}$. The restriction of $\text{ODE}_{I_d-H}$ to the affine hyperplane $\mathcal{V}$ is the linear system $\dot{z} = -(I_d - H)z$, where $z = y - v^*$ takes values in $\mathcal{V}_0 = \{u \in \mathbb{R}^d | w(u) = 0\}$, since $\text{Sp}((I_d - H)_{|\mathcal{V}_0}) \subset \{\lambda \in \mathbb{C}, \Re e(\lambda) > 0\}$, owing to assumption (A3). As a consequence $v^*$ is a uniformly stable equilibrium for the restriction of $\text{ODE}_{I_d-H}$ to $\mathcal{V}$, the whole hyperplane, as an attracting area. The fundamental result derived from the ODE method (see Theorem A.1 in the Appendix and the notation therein, in particular, the remainder $r_n$) yields the expected result

$$\widetilde{Y}_n \xrightarrow[n\to\infty]{\text{a.s.}} v^*. \qquad \square$$

REMARK. If we assume that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy besides (A1), then we can directly write a stochastic approximation for $\frac{Y_n}{w(Y_n)}$ with step $\frac{1}{w(Y_n)}$ in which the remainder simply reads $r_{n+1} = (H_{n+1} - H)\frac{Y_n}{w(Y_n)}$ and prove the a.s. convergence under the same assumptions.

COMMENTS. We could apply directly the ODE method because we first proved that $(\widetilde{Y}_n)_{n\geq 1}$ is a.s. bounded without using the standard Lyapunov machinery developed in SA theory. That is why the assumption on the remainder sequence $(r_n)_{n\geq 1}$ simply reads

$$r_n \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

Another approach is the martingale one. It relies on the existence of a Lyapunov function $V : \mathbb{R}^d \to \mathbb{R}_+$ associated to the algorithm satisfying

$$\begin{aligned}&\exists a > 0, \forall y \in \mathbb{R}^d,\ y \neq v^* \\ &\langle \nabla V | I_d - H\rangle(y) > 0 \quad \text{and} \quad \langle \nabla V | I_d - H\rangle > a|\nabla V|^2.\end{aligned} \tag{2.8}$$

In this framework the existence of a Lyapunov function can be established. Hence, the natural condition on the remainder sequence $(r_n)_{n\geq 1}$ reads (see [13])

$$\sum_{n\geq 1} \frac{\|r_n\|^2}{n} < +\infty \qquad \text{a.s.}$$

In that perspective, the assumption on the generating matrices would read $\sum_{n\geq 1} \frac{\|\|H_n - H\|\|^2}{n} < +\infty$ a.s. which is still slightly less stringent than assumption on the generating matrices made in [6].

2.3. *Rate of convergence.* In the previous section we proved the a.s. convergence of both quantities of interest, namely, $\widetilde{Y}_n$ and $\widetilde{N}_n$, toward $v^*$. In this section we establish a "joint CLT" for the couple $\theta_n := (\widetilde{Y}_n, \widetilde{N}_n)^t$ with an explicit asymptotic joint normal distribution (including covariances). To this end we will show that $\theta_n$ satisfies an SA recursive procedure which [a.s. converges toward $\theta^* = (v^*, v^*)^t$ and] fulfills the assumptions of the CLT Theorem A.2 for SA algorithms (see the Appendix), with a special attention paid to condition (A.3) about the remainder term.

Concerning $\widetilde{Y}_n$, we derive from (2.4) that

$$\forall n \geq 1 \qquad \widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\big(I_d - \big(2 - w(\widetilde{Y}_n)\big)H\big)\widetilde{Y}_n + \frac{1}{n+1}(\Delta M_{n+1} + \bar{r}_{n+1}),$$

where $\bar{r}_{n+1} := (\frac{H_{n+1} - H}{w(\widetilde{Y}_n)} + \frac{(w(\widetilde{Y}_n)-1)^2}{w(\widetilde{Y}_n)} H)\widetilde{Y}_n$.

For $\widetilde{N}_n$ we have, still for every $n \geq 1$,

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\big(\widetilde{N}_n - \big(2 - w(\widetilde{Y}_n)\big)\widetilde{Y}_n\big) + \frac{1}{n+1}(\Delta\widetilde{M}_{n+1} + \widetilde{r}_{n+1})$$

with $\Delta\widetilde{M}_{n+1} := X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_{n+1} - \frac{Y_n}{w(Y_n)}$ and $\widetilde{r}_{n+1} := \frac{(w(\widetilde{Y}_n)-1)^2}{w(\widetilde{Y}_n)}\widetilde{Y}_n$.

Thus, we obtain a new recursive SA procedure, still with step $\gamma_n = \frac{1}{n}$, namely,

$$\theta_{n+1} = \theta_n - \frac{1}{n+1}h(\theta_n) + \frac{1}{n+1}(\Delta\mathbf{M}_{n+1} + R_{n+1}), \qquad n \geq 1,$$

with $\Delta\mathbf{M}_{n+1} := \binom{\Delta M_{n+1}}{\Delta\widetilde{M}_{n+1}}$, $R_{n+1} := \binom{\bar{r}_{n+1}}{\widetilde{r}_{n+1}}$ and

$$\forall\theta = \begin{pmatrix} y \\ \nu \end{pmatrix}, y \in \mathbb{R}^d, \nu \in \mathbb{R}^d$$

$$h(\theta) := \begin{pmatrix} \big(I_d - \big(2 - w(y)\big)H\big)y \\ \nu - \big(2 - w(y)\big)y \end{pmatrix} \qquad \text{with } h(\theta^*) = 0.$$

The function $h$ is differentiable on $\mathbb{R}^{2d}$ and its differential at point $\theta^*$ is given by

$$Dh(\theta^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & 0_{\mathcal{M}_d(\mathbb{R})} \\ v^*\mathbf{1}^t - I_d & I_d \end{pmatrix}.$$

To establish a CLT for the sequence $(\theta_n)_{n\geq 1}$ we need to make the following additional assumptions:

(A4) The addition rules $D_n$ a.s. satisfy

$$\forall 1 \leq j \leq d \qquad \begin{cases} \sup_{n\geq 1} \mathbb{E}\big[\|D_n^{\cdot j}\|^{2+\delta} | \mathcal{F}_{n-1}\big] \leq C < \infty, & \text{for a } \delta > 0, \\ \mathbb{E}\big[D_n^{\cdot j}(D_n^{\cdot j})^t | \mathcal{F}_{n-1}\big] \underset{n\to\infty}{\longrightarrow} C^j, \end{cases}$$

where $C^j = (C^j_{il})_{1\leq i,l\leq d}$, $j = 1, \dots, d$, are $d \times d$ positive definite matrices.

Note that (A4) $\Rightarrow$ (A2) since $\mathbb{E}[\|D_n^{\cdot j}\|^2 | \mathcal{F}_{n-1}] \leq (\mathbb{E}[\|D_n^{\cdot j}\|^{2+\delta} | \mathcal{F}_{n-1}])^{2/(2+\delta)}$.

(A5) The matrix $H$ satisfies

$$n\mathbb{E}\big[\|\|H_n - H\|\|^2\big] \underset{n\to\infty}{\longrightarrow} 0. \tag{2.9}$$

THEOREM 2.2. *Assume* (A1), (A3) [*or* (A′1), (A′3)], (A4) *and* (A5).

(a) *Assume furthermore that*

$$\Re e\big(\mathrm{Sp}(H) \setminus \{1\}\big) < 1/2. \tag{2.10}$$

*Then*, $\theta_n \to \theta^*$ *a.s. as* $n \to +\infty$ *and*

$$\sqrt{n}\big(\theta_n - \theta^*\big) \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0, \Sigma) \qquad \text{with } \Sigma = \int_0^{+\infty} e^{u(Dh(\theta^*)-I/2)}\Gamma e^{u(Dh(\theta^*)-I/2)^t}\, du$$

*and*

$$\begin{aligned} \Gamma &= \begin{pmatrix} \displaystyle\sum_{k=1}^d v^{*k}C^k - v^*(v^*)^t & H\big(\mathrm{diag}(v^*) - v^*(v^*)^t\big) \\ \big(\mathrm{diag}(v^*) - v^*(v^*)^t\big)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t \end{pmatrix} \\ &= a.s.\text{-}\lim_{n\to\infty} \mathbb{E}\big[\Delta\mathbf{M}_n \Delta\mathbf{M}_n^t | \mathcal{F}_{n-1}\big]. \end{aligned} \tag{2.11}$$

(b) *Denote by* $\lambda_{\max}$ *the eigenvalue, different from* 1, *of* $H$ *with the highest real part. If* $\lambda_{\max} = 1/2$, *then* $\theta_n \to \theta^*$ *a.s. as* $n \to +\infty$ *and*

$$\sqrt{\frac{n}{\log n}}\big(\theta_n - \theta^*\big) \underset{n\to\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0, \Sigma).$$

(c) *If* $\lambda_{\max} > 1/2$, *then* $\theta_n \to \theta^*$ *a.s. as* $n \to +\infty$ *and* $n^\beta(\theta_n - \theta^*)$ *a.s. converges as* $n \to +\infty$ *toward a finite random variable, where* $\beta = 1 - \lambda_{\max}$.

PROOF. (a) We will check the three assumptions of the CLT for SA algorithms recalled in the Appendix, Theorem A.2. First, the condition (A.4) on the spectrum of $Dh(\theta^*)$ requested for algorithms with step $\frac{1}{n}$ in Theorem A.2 reads $\Re e(\mathrm{Sp}(Dh(\theta^*))) > \frac{1}{2}$. This follows from our assumption (2.10) since by decomposing $\mathbb{R}^d = \mathbb{R}v^* \oplus \mathrm{Ker}(w)$, one checks that

$$\mathrm{Sp}\big(Dh(\theta^*)\big) = \{1\} \cup \big\{1-\lambda, \lambda \in \mathrm{Sp}(H) \setminus \{1\}\big\}.$$

Second, assumption (A4) ensures that condition (A.2) is satisfied since

$$\sup_{n\geq 1} \mathbb{E}\big[\|\Delta\mathbf{M}_n\|^{2+\delta}|\mathcal{F}_{n-1}\big] < +\infty \qquad \text{a.s.}$$

and

$$\mathbb{E}\big[\Delta\mathbf{M}_n\Delta\mathbf{M}_n^t|\mathcal{F}_{n-1}\big] \xrightarrow[n\to\infty]{\text{a.s.}} \Gamma \qquad \text{as } n\to\infty,$$

where $\Gamma$ is the symmetric nonnegative matrix given by (2.11) as established below. To this end we have to determine three blocks since $\Gamma$ reads

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^t & \Gamma_2 \end{pmatrix} \qquad \text{where } \Gamma_1, \Gamma_2, \Gamma_{12} \in \mathcal{M}_d(\mathbb{R}).$$

*Computation of* $\Gamma_1$.

$$\begin{aligned}
&\mathbb{E}\big[\Delta M_{n+1}\Delta M_{n+1}^t|\mathcal{F}_n\big]\\
&\quad= \sum_{q=1}^d \mathbb{P}\big(X_{n+1}=e^q|\mathcal{F}_n\big)\big(\mathbb{E}[D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t|\mathcal{F}_n]\\
&\qquad\qquad - \mathbb{E}[D_{n+1}X_{n+1}|\mathcal{F}_n]\mathbb{E}[D_{n+1}X_{n+1}|\mathcal{F}_n]^t\big)\\
&\quad= \sum_{q=1}^d \frac{Y_n^q}{w(Y_n)}\mathbb{E}\big(D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t|\mathcal{F}_n\big) - \left(H_{n+1}\frac{Y_n}{w(Y_n)}\right)\left(H_{n+1}\frac{Y_n}{w(Y_n)}\right)^t\\
&\quad\xrightarrow[n\to\infty]{\text{a.s.}} \Gamma_1 = \sum_{q=1}^d v^{*q}C^q - v^*(v^*)^t.
\end{aligned}$$

*Computation of* $\Gamma_2$.

$$\begin{aligned}
\mathbb{E}\big[\Delta\widetilde{M}_{n+1}\Delta\widetilde{M}_{n+1}^t|\mathcal{F}_n\big] &= \mathbb{E}\big[X_{n+1}X_{n+1}^t|\mathcal{F}_n\big] - \frac{Y_n}{w(Y_n)}\left(\frac{Y_n}{w(Y_n)}\right)^t\\
&= \mathrm{diag}\left(\frac{Y_n}{w(Y_n)}\right) - \frac{Y_n}{w(Y_n)}\left(\frac{Y_n^q}{w(Y_n)}\right)^t \xrightarrow[n\to\infty]{\text{a.s.}} \Gamma_2\\
&= \mathrm{diag}(v^*) - v^*(v^*)^t.
\end{aligned}$$

*Computation of* $\Gamma_{12}$.

$$\begin{aligned}
&\mathbb{E}\big[\Delta M_{n+1}\Delta\widetilde{M}_{n+1}^t|\mathcal{F}_n\big]\\
&\quad=\mathbb{E}\big[D_{n+1}X_{n+1}X_{n+1}^t|\mathcal{F}_n\big]-\mathbb{E}[D_{n+1}X_{n+1}|\mathcal{F}_n]\mathbb{E}[X_{n+1}|\mathcal{F}_n]^t\\
&\quad=\mathbb{E}[D_{n+1}|\mathcal{F}_n]\mathbb{E}\big[X_{n+1}X_{n+1}^t|\mathcal{F}_n\big]\\
&\qquad-\mathbb{E}[D_{n+1}|\mathcal{F}_n]\mathbb{E}[X_{n+1}|\mathcal{F}_n]\mathbb{E}[X_{n+1}|\mathcal{F}_n]^t\\
&\quad=H_{n+1}\operatorname{diag}\biggl(\frac{Y_n}{w(Y_n)}\biggr)-H_{n+1}\frac{Y_n}{w(Y_n)}\biggl(\frac{Y_n}{w(Y_n)}\biggr)^t\\
&\quad\xrightarrow[n\to\infty]{\text{a.s.}}\Gamma_{12}=H\big(\operatorname{diag}(v^*)-v^*(v^*)^t\big).
\end{aligned}$$

Finally, it remains to check that the remainder sequence $(R_n)_{n\geq 1}$ satisfies (A.3) for an $\epsilon>0$:

$$\mathbb{E}\big[(n+1)\|R_{n+1}\|^2\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\epsilon\}}\big]\xrightarrow[n\to\infty]{}0. \tag{2.12}$$

We note that $\|R_{n+1}\|^2=\|\bar{r}_{n+1}\|^2+\|\widetilde{r}_{n+1}\|^2$. It follows from the definition of $\bar{r}_{n+1}$ and the elementary facts $\|\widetilde{Y}_n-v^*\|\leq\|\theta_n-\theta^*\|$ and $w(\widetilde{Y}_n)\geq\|\widetilde{Y}_n\|$ that

$$\begin{aligned}
&\|\bar{r}_{n+1}\|^2\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}\\
&\quad\leq 2\biggl(\frac{(w(\widetilde{Y}_n)-1)^4}{\|v^*\|/2}+\frac{|||H_{n+1}-H|||^2}{\|v^*\|/2}\biggr)\frac{3}{2}\|v^*\|\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}\\
&\quad\leq 6\big((w(\widetilde{Y}_n)-1)^4+|||H_{n+1}-H|||^2\big)\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}.
\end{aligned}$$

But $w(\widetilde{Y}_n)-1=\frac{w(\Delta M_n)}{n}$ where $\sup_{n\geq 0}\mathbb{E}[|w(\Delta M_{n+1})|^{2+\delta}|\mathcal{F}_n]\leq C'$, $\delta>0$, owing to (A4). Now using that $|w(y)|\leq C_d\|y\|$,

$$\begin{aligned}
\mathbb{E}\big[n\big|w(\widetilde{Y}_n)-1\big|^4\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}\big]&\leq C_\delta^* n\mathbb{E}\big[\big|w(\widetilde{Y}_n)-1\big|^{2+\delta}\big]\\
&=\frac{C_d}{n^{1+\delta}}\mathbb{E}\big[\big|w(\Delta M_n)\big|^{2+\delta}\big]\\
&\leq\frac{C_d'}{n^{1+\delta}},
\end{aligned}$$

where $C_\delta^*>0$ is a real constant. Consequently

$$\mathbb{E}\big[\big|w(\widetilde{Y}_n)-1\big|^4\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}\big]=o\biggl(\frac{1}{n}\biggr).$$

Thus, by (A5) we obtain

$$\mathbb{E}\big[\|\bar{r}_{n+1}\|^2\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}\big]=o\biggl(\frac{1}{n}\biggr).$$

The same argument yields $\mathbb{E}[\|\widetilde{r}_{n+1}\|^2\mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\|v^*\|/2\}}]=o(\frac{1}{n})$, therefore (2.12) is satisfied.

Theorem 2.2(b) and (c) follows from Theorem A.2(b) and (c) in the Appendix (see also [14]). □

**3. Application to urn models for multi-arm clinical trials.** In this section, we consider urn models for multi-arm clinical trials introduced by Wei and generalized by Bai, Hu and Shen. In this context, the initial framework where the addition rule matrices have nonnegative entries is the only one to make sense.

3.1. *The Wei GFU model.* We consider here the model presented in [27] and in [7], where balls are added depending on the success probabilities of each treatment. Define an *efficiency indicator* as follows: let $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, be $d$ independent sequences of $[0, 1]$-valued i.i.d. random variables, independent of the i.i.d. *sampling* sequence $(U_n)_{n\geq 1}$ so that

$$\mathbb{E}[T_n^i] = p^i, \qquad 0 < p^i < 1, \qquad 1 \leq i \leq d. \tag{3.1}$$

REMARK. If $(T_n^i)_{n\geq 1}$, $1 \leq i \leq d$, is simply a *success indicator*, namely, $d$ independent sequences of i.i.d. $\{0, 1\}$-valued Bernoulli trials with respective parameter $p^i$, then the convention is to set $T_n^i = 1$ to indicate that the response of the $i$th treatment in the $n$th trial is a success and $T_n^i = 0$ otherwise.

In this framework one considers the filtration $\mathcal{F}_n = \sigma(Y_0, U_k, T_k, 1 \leq k \leq n)$, $n \geq 0$. Consider the following addition rules: a success on the treatment $i$ adds a ball of type $i$ to the urn and a failure on the treatment $i$ adds $\frac{1}{d-1}$ balls for each of the other $d - 1$ types. Thus the addition rule proposed in [27] is as follows:

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{1 - T_{n+1}^2}{d-1} & \cdots & \frac{1 - T_{n+1}^d}{d-1} \\ \frac{1 - T_{n+1}^1}{d-1} & T_{n+1}^2 & \cdots & \frac{1 - T_{n+1}^d}{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1 - T_{n+1}^1}{d-1} & \frac{1 - T_{n+1}^2}{d-1} & \cdots & T_{n+1}^d \end{pmatrix}$$

so that

$$H_{n+1} = \mathbb{E}[D_{n+1} | \mathcal{F}_n] = \mathbb{E} D_{n+1} = H = \begin{pmatrix} p^1 & \frac{q^2}{d-1} & \cdots & \frac{q^d}{d-1} \\ \frac{q^1}{d-1} & p^2 & \cdots & \frac{q^d}{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{q^1}{d-1} & \frac{q^2}{d-1} & \cdots & p^d \end{pmatrix},$$

where $q^i = 1 - p^i$, $1 \le i \le d$. The strong consistency has been first established in [3], then redone in [6]. It follows from Theorem 2.1 as well. The asymptotic normality

$$\frac{Y_n - nv^*}{\sqrt{n}} = \sqrt{n}\left(\frac{Y_n}{n} - v^*\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

results from Theorem 3.2 in [6] and from Theorem 2.2 of this paper. However, using Theorem 2.2 we obtain a joint CLT for $(\widetilde{Y}_n, \widetilde{N}_n)$. Furthermore, we know that

$$v^{*i} = \frac{1/q^i}{\sum_{j=1}^d 1/q^j}, \qquad 1 \le i \le d.$$

Note that if $p^i > p^j$, then $v^{*i} > v^{*j}$. Hence, the components $v^{*i}$ are ordered according to the increasing efficiency $p^i$ of the treatments. Furthermore, it is clear that, if $p^i \uparrow 1$ and all other probabilities $p^j$ stand still, then

$$\lim_{p^i \to 1} v^{*j} = \delta_{ij}.$$

Consequently, since $v^{*i}$ is the asymptotic probability of assigning treatment $i$ to a patient, the procedure asymptotically allocates more patients to the most efficient treatment(s). Following the practitioners, the fact that a marginal allocation of less efficient treatments is preserved is justified by some comparison matter.

However, this model only takes into account, in the addition rule matrix $D_n$, the response of the $n$th patient without considering the ones of past patients. This led the author to introduce [7] a new model based on statistical observations of the efficiency of the assigned treatments to all past patients.

3.2. *The Bai–Hu–Shen GFU model.* We consider now the model introduced in [7] (and considered again in [6]) where $(T_n^i)_{n\ge1}$, $1 \le i \le d$, are $d$ independent sequences of i.i.d. $\{0, 1\}$-valued Bernoulli trials satisfying (3.1) and the filtration $(\mathcal{F}_n)_{n\ge0}$ is defined as in the previous section. Let $N_n = (N_n^1, \dots, N_n^d)^t$ and $S_n = (S_n^1, \dots, S_n^d)^t$, where $N_n^i = N_{n-1}^i + X_n^i$, $n \ge 1$, still denotes the number of times the $i$th treatment is selected among the first $n$ stages and

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \qquad n \ge 1,$$

denotes the *number of successes* of the $i$th treatment among these $N_n^i$ trials, $i = 1, \dots, d$. However, to avoid degeneracy of the procedure, we will make the following initialization assumption

$$N_0^i = 1, \qquad S_0^i = 1, \qquad i = 1, \dots, d$$

(which makes the above interpretation of these quantities correct "up to one unit").

REMARK. Like with the Wei model, we can simply assume that $T_n^i$ is a $\{0, 1\}$-valued efficiency indicator.

Define $\Pi_n = (\Pi_n^1, \dots, \Pi_n^d)^t$, where $\Pi_n^i = \frac{S_n^i}{N_n^i}$, $i = 1, \dots, d$. In [7] the authors consider the following addition rule matrices:

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \dfrac{\Pi_n^1(1-T_{n+1}^2)}{\sum_{j\neq 2}\Pi_n^j} & \cdots & \dfrac{\Pi_n^1(1-T_{n+1}^d)}{\sum_{j\neq d}\Pi_n^j} \\ \dfrac{\Pi_n^2(1-T_{n+1}^1)}{\sum_{j\neq 1}\Pi_n^j} & T_{n+1}^2 & \cdots & \dfrac{\Pi_n^2(1-T_{n+1}^d)}{\sum_{j\neq d}\Pi_n^j} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\Pi_n^d(1-T_{n+1}^1)}{\sum_{j\neq 1}^d\Pi_n^j} & \dfrac{\Pi_n^d(1-T_{n+1}^2)}{\sum_{j\neq 2}^d\Pi_n^j} & \cdots & T_{n+1}^d \end{pmatrix},$$

that is, at stage $n+1$, if the response of the $j$th treatment is a success, then one ball of type $j$ is added in the urn. Otherwise, $\frac{\Pi_n^i}{\sum_{k\neq j}\Pi_n^k}$ (virtual) balls of type $i$, $i \neq j$, are added. This addition rule matrix clearly satisfies (A1)(i) and (A2). Then, one easily checks that the generating matrices are given by

$$H_{n+1} = \mathbb{E}[D_{n+1}|\mathcal{F}_n] = \begin{pmatrix} p^1 & \dfrac{\Pi_n^1(1-p^2)}{\sum_{j\neq 2}\Pi_n^j} & \cdots & \dfrac{\Pi_n^1(1-p^d)}{\sum_{j\neq d}\Pi_n^j} \\ \dfrac{\Pi_n^2(1-p^1)}{\sum_{j\neq 1}\Pi_n^j} & p^2 & \cdots & \dfrac{\Pi_n^2(1-p^d)}{\sum_{j\neq d}\Pi_n^j} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\Pi_n^d(1-p^1)}{\sum_{j\neq 1}\Pi_n^j} & \dfrac{\Pi_n^d(1-p^2)}{\sum_{j\neq 2}\Pi_n^j} & \cdots & p^d \end{pmatrix}$$

and satisfy (A1)(ii). As soon as $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, $H_n \xrightarrow{\text{a.s.}} H$ (see Lemma 3.1 below or [7] when $Y_0 \in (0,\infty)^d$) where

$$H = \begin{pmatrix} p^1 & \dfrac{p^1(1-p^2)}{\sum_{j\neq 2}p^j} & \cdots & \dfrac{p^1(1-p^d)}{\sum_{j\neq d}p^j} \\ \dfrac{p^2(1-p^1)}{\sum_{j\neq 1}p^j} & p^2 & \cdots & \dfrac{p^2(1-p^d)}{\sum_{j\neq d}p^j} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{p^d(1-p^1)}{\sum_{j\neq 1}p^j} & \dfrac{p^d(1-p^2)}{\sum_{j\neq 2}p^j} & \cdots & p^d \end{pmatrix}.$$

The matrix $H$ is clearly irreducible since $0 < p^i < 1$, $1 \leq i \leq d$ so that assumption (A3) is satisfied. Then, calling upon Theorem 2.1 (or following the direct proof from [7]), we obtain

$$\widetilde{Y}_n = \frac{Y_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} v^* \quad \text{and} \quad \widetilde{N}_n = \frac{N_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} v^*. \tag{3.2}$$

Note that the normalized maximal eigenvector $v^*$ (associated to the eigenvalue 1) is given by

$$v^{*i} = \frac{(p^i/(1-p^i))\sum_{k\neq i} p^k}{\sum_{1\le j\le d}(p^j/(1-p^j))\sum_{k\neq j} p^k}, \qquad i = 1, \dots, d.$$

Note that if $p^i > p^j$, $\frac{p^i}{p^j}\frac{\sum_{k\neq i} p^k}{\sum_{k\neq j} p^k} > 1$ and $\frac{1-p^j}{1-p^i} > 1$ so that $v^{*i} > v^{*j}$. Hence, the entries $v^{*i}$ are ordered according to the increasing efficiency $p^i$ of the treatments. This model can be considered as more ethical than the Wei model since a better treatment will be administrated to more patients. Indeed, when $d > 2$, for any $i \neq j$, $1 \le i, j \le d$, if $p^i > p^j$,

$$\frac{v^{*i}_{\mathrm{BHS}}}{v^{*j}_{\mathrm{BHS}}} > \frac{v^{*i}_W}{v^{*j}_W} > 1$$

(when $d = 2$ both matrices $H$ coincide).

REMARK. Note that in the Bai–Hu–Shen GFU model the "balls" in the urn become virtual since there exists no $N \in \mathbb{N}$ such that, for every $n \ge 1$, $ND_n \in \mathcal{M}_d(\mathbb{N})$.

3.3. *Asymptotic normality for multi-arm clinical trials for the Bai–Hu–Shen* (*BHS*) *GFU model.* In order to derive a CLT in [7], not with the bias $\mathbb{E}Y_n$ but with $nv^*$, from their own general asymptotic normality result (which statement is similar to Theorem 2.2) they need to fulfill the following convergence rate assumption for $H_n$:

$$\sum_{n\ge 1} \frac{\|H_n - H\|_\infty}{\sqrt{n}} < +\infty, \tag{3.3}$$

where $\|\cdot\|_\infty$ is the norm on $L^\infty_{\mathbb{R}^{d\times d}}(\mathbb{P})$. In [7], the a.s. rate of decay $|||H_n - H|||_\infty = o(n^{-1/4})$ is shown which is clearly not fast enough to fulfill (3.3).

However, by enlarging the dimension of the structure process of the procedure by considering the $3d$-dimensional random sequence

$$\widetilde{\theta}_n = \begin{pmatrix} \widetilde{Y}_n \\ \widetilde{N}_n \\ \widetilde{S}_n \end{pmatrix} \qquad \text{where } \widetilde{S}_n = \frac{S_n}{n}, \qquad n \ge 1,$$

we will establish that a CLT does hold for the BHS GFU model.

The first step is to notice that the generating matrix $H_{n+1}$ can be written as a function depending on $\widetilde{S}_n$ and $\widetilde{N}_n$, that is, $H_{n+1} = \Phi(\widetilde{S}_n, \widetilde{N}_n)$, where $\Phi : \mathbb{R}_+^d \times (0, \infty)^d \to \mathcal{M}_d(\mathbb{R})$ is a differentiable function defined by

$$\Phi(s, \nu) = \big(\Phi^{ij}(s, \nu)\big)_{1 \le i, j \le d},$$

where

$$\begin{cases} \Phi^{ii}(s, \nu) = p^i, & 1 \le i \le d, \\ \Phi^{ij}(s, \nu) = \dfrac{s^i/\nu^i}{\sum_{k \ne j} s^k/\nu^k} q^j, & 1 \le i, j \le d, i \ne j. \end{cases}$$

Then the following strong consistency and CLT hold for $(\widetilde{\theta}_n)_{n \ge 1}$.

THEOREM 3.1. *Assume that* $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$.

(a) *If* $\Re e(\mathrm{Sp}(H) \setminus \{1\}) < \frac{1}{2}$, *then*

$$\widetilde{\theta}_n \xrightarrow[n \to \infty]{\text{a.s.}} \widetilde{\theta}^* \quad \text{and} \quad \sqrt{n}\big(\widetilde{\theta}_n - \widetilde{\theta}^*\big) \xrightarrow[n \to \infty]{\mathcal{L}} \mathcal{N}(0, \widetilde{\Sigma}),$$

*where*

$$\widetilde{\theta}^* := \big(v^*, v^*, \mathrm{diag}(p) v^*\big)^t,$$
$$\widetilde{\Sigma} = \int_0^{+\infty} e^{u(D\widetilde{h}(\widetilde{\theta}^*) - I/2)} \widetilde{\Gamma} e^{u(D\widetilde{h}(\widetilde{\theta}^*) - I/2)^t} \, du$$

*with*

$$\widetilde{\Gamma} = \begin{pmatrix} \sum_{k=1}^d v^{*k} C^k - v^*(v^*)^t & H\big(\mathrm{diag}(v^*) - v^*(v^*)^t\big) & \big(\mathrm{diag}(v^*) - v^*(v^*)^t\big)\,\mathrm{diag}(p) \\ \big(\mathrm{diag}(v^*) - v^*(v^*)^t\big)^t H^t & \mathrm{diag}(v^*) - v^*(v^*)^t & \big(\mathrm{diag}(v^*) - v^*(v^*)^t\big)\,\mathrm{diag}(p) \\ \mathrm{diag}(p)\big(\mathrm{diag}(v^*) - v^*(v^*)^t\big)^t & \mathrm{diag}(p)\big(\mathrm{diag}(v^*) - v^*(v^*)^t\big)^t & \mathrm{diag}(p)\big(v^* - v^* v^{*t}\,\mathrm{diag}(p)\big) \end{pmatrix},$$

*where* $C^k = (C^k_{ij})_{1 \le i, j \le d}$, $1 \le k \le d$, *are* $d \times d$ *positive definite matrices with*

$$C^k_{ij} = \frac{p^i p^j (1 - p^k)}{(\sum_{\ell \ne k} p^\ell)^2} \mathbb{1}_{\{i, j \ne k\}} + p^k \mathbb{1}_{\{i = j = k\}}$$

*and*

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\dfrac{\partial}{\partial \nu}\big(\Phi(s,\nu)y\big)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\dfrac{\partial}{\partial s}\big(\Phi(s,\nu)y\big)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ \operatorname{diag}(p)\big(v^*\mathbf{1}^t - I_d\big) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix},$$

*which is invertible.*

(b) *Denote by* $\lambda_{\max}$ *the eigenvalue, different from* 1, *of* $H$ *with the highest real part. If* $\lambda_{\max} = 1/2$, *then*, $\theta_n \to \theta^*$ *a.s. as* $n \to +\infty$ *and*

$$\sqrt{\frac{n}{\log n}}(\theta_n - \theta^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

(c) *If* $\lambda_{\max} > 1/2$, *then* $n^\beta(\theta_n - \theta^*)$ *a.s. converges as* $n \to +\infty$ *toward a finite random variable, where* $\beta = 1 - \lambda_{\max}$.

PROOF. *Step* 1 (*Strong consistency*). We will show with Lemma 3.1 that $\widetilde{S}_n \xrightarrow[n\to\infty]{\text{a.s.}} \operatorname{diag}(p)v^*$ and we will deduce that $H_n \xrightarrow[n\to\infty]{\text{a.s.}} H$, that is, assumption (A3) holds. As we have already checked that assumptions (A1)(i)–(ii) and (A2) are satisfied, then by only adding (A1)(iii) we use Theorem 2.1 to prove that $\widetilde{\theta}_n \xrightarrow[n\to\infty]{\text{a.s.}} \widetilde{\theta}^*$.

LEMMA 3.1. *If the assumption* (1.1) *holds and* $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, *then*,

$$\Pi_n \xrightarrow{a.s.} p = (p^1, \dots, p^d) \qquad \text{as } n \to \infty$$

*so that assumption* (2.2) *holds, that is*, $H_n \xrightarrow[n\to\infty]{a.s.} H$.

REMARK. If we assume that $Y_0^i > 0$, $1 \le i \le d$, then we can prove that $\lim_n N_n^i = +\infty$ a.s., $1 \le i \le d$, faster than below by using that $Y_n^i \ge Y_0^i$, $1 \le i \le d$, $n \ge 1$. The following proof considers the more general case where $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$.

PROOF OF LEMMA 3.1. *Step* 1. It follows from the dynamics (1.1) and the definitions of $D_{n+1}$ and $H_{n+1}$ that, for every $n \ge 0$, $w(Y_n) = w(Y_0) + n$ and that, for every $i \in \{1, \dots, d\}$,

$$Y_{n+1}^i = Y_n^i + \sum_{j=1}^d H_{n+1}^{ij} \frac{Y_n^i}{w(Y_n)} + \Delta M_{n+1}^i,$$

where $(\Delta M_n^i)_{n\ge1}$ is a sequence of martingale increments satisfying

$$\sup_n \mathbb{E}\big[|\Delta M_n^i|^2 | \mathcal{F}_{n-1}\big] < +\infty$$

since the addition rule matrices satisfy (2.1). Now, using that $S_0^i = N_0^i = 1$ by convention, one derives that

$$\forall i \neq j \qquad H_{n+1}^{ij} \geq \frac{\kappa_0}{n} \qquad \text{with } \kappa_0 = \frac{1}{2d} \min_{1 \leq i \leq d} (p^i, 1 - p^i) > 0$$

so that, using that $H_{n+1}^{ii} = p^i$, there exists a deterministic integer $n_0$ such that for every $n \geq n_0$,

$$\begin{aligned} Y_{n+1}^i &\geq \left(1 + \frac{p_i}{n} - \frac{\kappa_0}{w(Y_n)}\right) Y_n^i + \frac{\kappa_0}{n} + \Delta M_{n+1}^i \\ &\geq \left(1 + \frac{p_i}{2w(Y_n)}\right) Y_n^i + \frac{\kappa_0}{n} + \Delta M_{n+1}^i. \end{aligned}$$

Standard computations show that, setting $a_n^i = \prod_{k=n_0}^{n-1} (1 + \frac{p_i}{2w(Y_n)})$, $i = 1, \dots, d$,

$$\forall n \geq n_0 \qquad \frac{Y_n^i}{a_n^i} \geq \frac{Y_{n_0}^i}{a_{n_0}^i} + \sum_{k=n_0+1}^{n} \frac{\kappa_0}{a_k^i} + \sum_{k=n_0+1}^{n} \frac{\Delta M_k^i}{a_k^i}.$$

Since there exists $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 n^{p^i/2} \leq a_n^i \leq \kappa_2 n^{p^i/2}$, one has

$$\forall \eta > 0 \qquad \sum_{k=n_0+1}^{n} \frac{\Delta M_k^i}{a_k^i} = o\big(n^{(1-p^i+\eta)/2}\big).$$

Finally, there exists a positive real constant $c'$ such that, for every $i = 1, \dots, d$,

$$Y_n^i \geq c' n^{p^i/2} \sum_{k=n_0+1}^{n} k^{-p^i/2} + o\big(n^{(1+\eta)/2}\big)$$

so that

$$\forall i \in \{1, \dots, d\} \qquad \liminf_n \widetilde{Y}_n^i \geq c' \int_0^1 u^{-p^i/2}\, du > 0$$

and, as a consequence, $\sum_{n \geq 1} \widetilde{Y}_n^i = +\infty$ a.s. Now, using that for every $i = 1, \dots, d$,

$$N_n^i = \sum_{k=1}^{n} \mathbb{1}_{\{X_k = e^i\}} \quad \text{and} \quad \mathbb{P}(X_n = e^i | \mathcal{F}_{n-1}) = \widetilde{Y}_{n-1}^i \left(1 - \frac{w(Y_0)}{w(Y_{n-1})}\right), \qquad n \geq 1,$$

we get by the conditional Borel–Cantelli lemma that $N_\infty^i = \lim_n N_n^i = +\infty$ a.s.

*Step* 2. First, we note that

$$\Pi_n^i = \frac{\sum_{k=1}^{n} T_k^i \Delta N_k^i}{N_n^i}$$

and we introduce the sequence $(\widetilde{\Pi}_n)_{n\ge 1}$ defined by

$$\widetilde{\Pi}^i_n = \sum_{k=1}^{n}(T^i_k - p^i)\frac{\Delta N^i_k}{N^i_{k-1}+1}, \qquad n\ge 1.$$

It is an $\mathcal{F}_n$-martingale since, $T^i_k$ being independent of $\mathcal{F}_{k-1}$ and $X_k$,

$$\mathbb{E}((T^i_k - p^i)\Delta N^i_k|\mathcal{F}_{k-1}) = \mathbb{E}(T^i_k - p^i)\mathbb{P}(X_k = e^i|\mathcal{F}_{k-1}) = 0.$$

It has bounded increments since $|T^i_k - p^i| \le 1$ and

$$\langle \widetilde{\Pi}^i\rangle_n \le \sum_{k=1}^{n}\frac{\mathbb{E}((\Delta N^i_k)^2|\mathcal{F}_{k-1})}{(N^i_{k-1}+1)^2}.$$

It follows, using $(\Delta N^i_k)^2 = \Delta N^i_k$, that, for every $n\ge 1$,

$$\mathbb{E}\langle \widetilde{\Pi}^i\rangle_n \le \mathbb{E}\left(\sum_{k=1}^{n}\frac{\Delta N^i_k}{(N^i_{k-1}+1)^2}\right) \le \mathbb{E}\left(\sum_{k=1}^{n}\frac{\Delta N^i_k}{N^i_{k-1}N^i_k}\right) \le \frac{1}{N^i_0} = 1.$$

Consequently, $\widetilde{\Pi}^i_n \to \widetilde{\Pi}^i_\infty \in L^1(\mathbb{P})$ a.s. as $n\to\infty$. This in turn implies by Kronecker's lemma that

$$\Pi^i_n \xrightarrow{\text{a.s.}} p^i \qquad \text{as } n\to\infty$$

since $N^i_n \to \infty$ by the first step. □

It follows from the lemma and Theorem 2.1 that $(\widetilde{Y}_n, \widetilde{N}_n) \to (v^*, v^*)$. Furthermore, $\operatorname{diag}(\widetilde{S}_n) = \operatorname{diag}(Q_n)\widetilde{N}_n \to \operatorname{diag}(p)v^* = u^*$ so that $\widetilde{\theta}_n \to \widetilde{\theta}^*$ as $n\to+\infty$.

*Step* 2 (*Asymptotic normality*). We will show now that $(\widetilde{\theta}_n)_{n\ge 1}$ satisfies an appropriate recursion to apply Theorem A.2 (CLT). First, we write a recursive procedure for $\widetilde{S}_n$. Having in mind that $S_n = 1 + \sum_{1\le k\le n}\operatorname{diag}(T_k)X_k$, we get

$$\begin{aligned}
\widetilde{S}_{n+1} &= \widetilde{S}_n - \frac{1}{n+1}(\widetilde{S}_n - \operatorname{diag}(T_{n+1})X_{n+1})\\
&= \widetilde{S}_n - \frac{1}{n+1}\left(\widetilde{S}_n - \operatorname{diag}(p)\frac{\widetilde{Y}_n}{w(\widetilde{Y}_n)}\right) + \frac{1}{n+1}\Delta\widehat{M}_{n+1}\\
&= \widetilde{S}_n - \frac{1}{n+1}(\widetilde{S}_n - \operatorname{diag}(p)(2-w(\widetilde{Y}_n))\widetilde{Y}_n)\\
&\quad + \frac{1}{n+1}(\Delta\widehat{M}_{n+1} + \widehat{r}_{n+1}),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\Delta\widehat{M}_{n+1} &:= \operatorname{diag}(T_{n+1})X_{n+1} - \mathbb{E}[\operatorname{diag}(T_{n+1})X_{n+1}|\mathcal{F}_n]\\
&= \operatorname{diag}(T_{n+1})X_{n+1} - \operatorname{diag}(p)\frac{\widetilde{Y}_n}{w(Y_n)}
\end{aligned}$$

is an $\mathcal{F}_n$-martingale increment and $\widehat{r}_{n+1} = \operatorname{diag}(p)\frac{(w(\widetilde{Y}_n)-1)^2}{w(\widetilde{Y}_n)}\widetilde{Y}_n$. Then we rewrite the dynamics satisfied by $\widetilde{Y}_n$ as

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1}\big(I_d - \big(2 - w(\widetilde{Y}_n)\big)H_{n+1}\big)\widetilde{Y}_n + \frac{1}{n+1}(\Delta M_{n+1} + \check{r}_{n+1}), \tag{3.5}$$

where $\check{r}_{n+1} := \frac{(w(\widetilde{Y}_n)-1)^2}{w(\widetilde{Y}_n)} H_{n+1}\widetilde{Y}_n$. Finally, we get the following recursive procedure for $\widetilde{\theta}_n$:

$$\widetilde{\theta}_{n+1} = \widetilde{\theta}_n - \frac{1}{n+1}\widetilde{h}(\widetilde{\theta}_n) + \frac{1}{n+1}(\Delta\widetilde{\mathbf{M}}_{n+1} + \widetilde{R}_{n+1}), \qquad n \geq 1,$$

where, for every $\widetilde{\theta} = (y, \nu, s)^t \in \mathbb{R}_+^{3d}$,

$$\widetilde{h}(\widetilde{\theta}) := \begin{pmatrix} \big(I_d - \big(2 - w(y)\big)\Phi(s,\nu)\big)y \\ \nu - \big(2 - w(y)\big)y \\ s - \big(2 - w(y)\big)\operatorname{diag}(p)y \end{pmatrix}, \qquad \Delta\widetilde{\mathbf{M}}_{n+1} := \begin{pmatrix} \Delta M_{n+1} \\ \Delta\widetilde{M}_{n+1} \\ \Delta\widehat{M}_{n+1} \end{pmatrix}$$

and

$$\widetilde{R}_{n+1} := \begin{pmatrix} \check{r}_{n+1} \\ \widetilde{r}_{n+1} \\ \widehat{r}_{n+1} \end{pmatrix}.$$

Let us check that the addition rule matrices satisfy (A4). For every $j \in \{1, \dots, d\}$, let set $C_n^j = \mathbb{E}[D_{n+1}^{\cdot j}(D_{n+1}^{\cdot j})^t | \mathcal{F}_n]$. We have that

$$\begin{aligned}(C_n^j)_{ii'} &= \mathbb{E}\big[D_{n+1}^{ij}(D_{n+1}^{i'j})^t | \mathcal{F}_n\big] \\ &= \frac{Q_n^i Q_n^{i'}}{(\sum_{k\neq j} Q_n^k)^2}\mathbb{E}\big[(1 - T_{n+1}^j)^2 | \mathcal{F}_n\big]\mathbb{1}_{\{i,i'\neq j\}} + \mathbb{E}\big[(T_{n+1}^j)^2 | \mathcal{F}_n\big]\mathbb{1}_{\{i=i'=j\}},\end{aligned}$$

because $T_{n+1}^j(1 - T_{n+1}^j) = 0$. Then, owing to Lemma 3.1, $C_n^j \xrightarrow[n\to+\infty]{\text{a.s.}} C^j$ with

$$C_{ii'}^j = \frac{p^i p^{i'}(1 - p^j)}{(\sum_{k\neq j} p^k)^2}\mathbb{1}_{\{i,i'\neq j\}} + p^j\mathbb{1}_{\{i=i'=j\}}.$$

We can check that $C^j$ is a positive definite matrix. Consequently, (A4) holds.

The function $\Phi$ being differentiable at the equilibrium point $\widetilde{\theta}^*$, we have

$$D\widetilde{h}(\widetilde{\theta}^*) = \begin{pmatrix} I_d - H + v^*\mathbf{1}^t & -\frac{\partial}{\partial\nu}\big(\Phi(s,\nu)y\big)_{|\widetilde{\theta}=\widetilde{\theta}^*} & -\frac{\partial}{\partial s}\big(\Phi(s,\nu)y\big)_{|\widetilde{\theta}=\widetilde{\theta}^*} \\ v^*\mathbf{1}^t - I_d & I_d & 0_{\mathcal{M}_d(\mathbb{R})} \\ \operatorname{diag}(p)\big(v^*\mathbf{1}^t - I_d\big) & 0_{\mathcal{M}_d(\mathbb{R})} & I_d \end{pmatrix},$$

which is invertible since, by Schur complement, we have $\det(D\widetilde{h}(\widetilde{\theta}^*)) = \det(I_d - H + v^*\mathbf{1}^t)$ thanks to $\frac{\partial}{\partial v}(\Phi(s,\nu)y)_{|\widetilde{\theta}=\widetilde{\theta}^*} = -\operatorname{diag}(p)\frac{\partial}{\partial s}(\Phi(s,\nu)y)_{|\widetilde{\theta}=\widetilde{\theta}^*}$.

At this stage, the proof follows the lines of that of Theorem 2.2: the computation of the covariance matrix $\widetilde{\Gamma}$ and the treatment of the remainder term uses the same tools as before. The three results of convergence rate follows from Theorem A.2 in the Appendix. The details are left to the reader. □

REMARK. The asymptotic variances of $\widetilde{Y}_n$ and $\widetilde{N}_n$ in Theorem 3.1 are different from those in Theorem 2.2 because the differential matrices $Dh(\theta^*)$ and $D\widetilde{h}(\widetilde{\theta}^*)$ are not the same.

COROLLARY 3.1. *Under the assumptions of Theorem 3.1,*

$$\sqrt{n}(H_n - H) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0; \Gamma_H),$$

*where $\Gamma_H$ is a $d^2 \times d^2$ matrix given by $\Gamma_H = D\Phi(u^*, v^*)[\widetilde{\Sigma}_{i+d,j+d}]_{1\le i,j\le 2d} \times D\Phi(u^*, v^*)^t$.*

PROOF. This is an easy consequence of the so-called $\Delta$-method since

$$\begin{aligned} H_n &= \Phi(\widetilde{S}_n, \widetilde{N}_n) \\ &= \Phi(u^*, v^*) + D\Phi(u^*, v^*)(\widetilde{S}_n - u^*, \widetilde{N}_n - v^*) \\ &\quad + \|(\widetilde{S}_n - u^*, \widetilde{N}_n - v^*)\|\varepsilon(\widetilde{S}_n, \widetilde{N}_n) \end{aligned}$$

with $\lim_{y\to(u^*,v^*)}\varepsilon(y) = 0$. Consequently,

$$\sqrt{n}(H_n - H) = D\Phi(u^*, v^*)\bigl(\sqrt{n}(\widetilde{S}_n - u^*), \sqrt{n}(\widetilde{N}_n - v^*)\bigr) + \varepsilon_{\mathbb{P}}(n),$$

where $\varepsilon_{\mathbb{P}}(n)$ goes to 0 in probability (as the product of a tight sequence and an a.s. convergent sequence). The proof is complete. □

REMARK. This corollary shows a posteriori that it was hopeless to try applying Theorem 2.2 in its standard form to establish asymptotic normality for multi-arm clinical trials since the assumption (A5) cannot be satisfied. Our global SA approach breaks the vicious circle.

NUMERICAL EXAMPLE: BHS MODEL. We consider the case $d = 2$, so that $v^*$ has the same form as in the example in Section 2.3. The results of the simulation are reproduced in Figure 1 (the parameters are included in the caption).

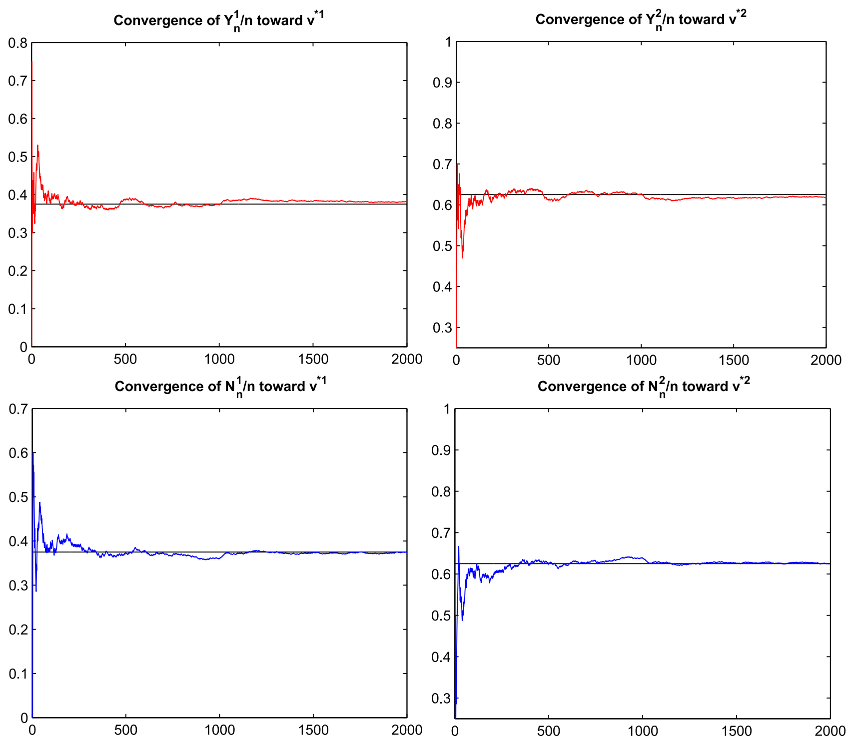


FIG. 1. *Convergence of* $\frac{Y_n}{n}$ *toward* $v^*$ (*up-windows*) *and of* $\frac{N_n}{n}$ *toward* $v^*$ (*down-windows*): $d = 2$, $n = 2.10^3$, $p^1 = 0.5$, $p^2 = 0.7$, $Y_0 = (0.5, 0.5)^t$ *and* $N_0 = (1, 1)^t$.

## APPENDIX: BASIC TOOLS OF STOCHASTIC APPROXIMATION

Consider the following recursive procedure defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

$$\text{(A.1)} \qquad \forall n \geq n_0 \qquad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1}(\Delta M_{n+1} + r_{n+1}),$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function, $\theta_{n_0}$ an $\mathcal{F}_{n_0}$-measurable finite random vector and, for every $n \geq n_0$, $\Delta M_{n+1}$ is an $\mathcal{F}_n$-martingale increment and $r_n$ is an $\mathcal{F}_n$-adapted remainder term.

THEOREM A.1 (A.s. convergence with ODE method; see, e.g., [8, 9, 14, 16, 22]). *Assume that $h$ is locally Lipschitz, that*

$$r_n \xrightarrow[n\to\infty]{a.s.} 0 \quad \textit{and} \quad \sup_{n\geq n_0} \mathbb{E}\big[\|\Delta M_{n+1}\|^2 | \mathcal{F}_n\big] < +\infty \qquad a.s.$$

*and that $(\gamma_n)_{n\geq 1}$ is a positive sequence satisfying*

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

*Then the set* $\Theta^\infty$ *of its limiting values as* $n \to +\infty$ *is a.s. a compact connected set, stable by the flow of*

$$\mathrm{ODE}_h \equiv \dot{\theta} = -h(\theta).$$

*Furthermore, if* $\theta^* \in \Theta^\infty$ *is a uniformly stable equilibrium on* $\Theta^\infty$ *of* $\mathrm{ODE}_h$, *then*

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^*.$$

**Comments.** By uniformly stable we mean that

$$\sup_{\theta\in\Theta^\infty} \left|\theta(\theta_0, t) - \theta^*\right| \longrightarrow 0 \qquad \text{as } t \to +\infty,$$

where $\theta(\theta_0, t)_{\theta_0\in\Theta^\infty, t\in\mathbb{R}_+}$ is the flow of $\mathrm{ODE}_h$ on $\Theta^\infty$.

THEOREM A.2 (Rate of convergence (see [14], Theorem 3.III.14, page 131; for CLT see also, e.g., [9, 22])). *Let* $\theta^*$ *be an equilibrium point of* $\{h = 0\}$. *Assume that the function* $h$ *is differentiable at* $\theta^*$ *and all the eigenvalues of* $Dh(\theta^*)$ *have positive real parts. Assume that for some* $\delta > 0$,

$$\text{(A.2)} \qquad \begin{aligned} \sup_{n\ge n_0} \mathbb{E}\big[\|\Delta M_{n+1}\|^{2+\delta} | \mathcal{F}_n\big] &< +\infty \qquad a.s., \\ \mathbb{E}\big[\Delta M_{n+1} \Delta M_{n+1}^t | \mathcal{F}_n\big] &\xrightarrow[n\to\infty]{a.s.} \Gamma, \end{aligned}$$

*where* $\Gamma$ *is a deterministic symmetric definite positive matrix and for an* $\epsilon > 0$,

$$\text{(A.3)} \qquad \mathbb{E}\big[(n+1)\|r_{n+1}\|^2 \mathbb{1}_{\{\|\theta_n-\theta^*\|\le\epsilon\}}\big] \xrightarrow[n\to\infty]{} 0.$$

*Specify the gain parameter sequence as*

$$\text{(A.4)} \qquad \forall n \ge 1 \qquad \gamma_n = \frac{1}{n}.$$

(a) *If* $\Lambda := \Re e(\lambda_{\min}) > \frac{1}{2}$, *where* $\lambda_{\min}$ *denotes the eigenvalue of* $Dh(\theta^*)$ *with the lowest real part, then, the above a.s. convergence is ruled on the convergence set* $\{\theta_n \longrightarrow \theta^*\}$ *by the following central limit theorem*:

$$\sqrt{n}(\theta_n - \theta^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{2\Lambda - 1}\Sigma\right)$$

*with*

$$\Sigma := \int_0^{+\infty} \big(e^{-(Dh(\theta^*)-I_d/2)u}\big)^t \Gamma e^{-(Dh(\theta^*)-I_d/2)u}\, du.$$

(b) *If* $\Lambda = \frac{1}{2}$, *then*

$$\sqrt{\frac{n}{\log n}}(\theta_n - \theta^*) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

(c) *If* $\Lambda < \frac{1}{2}$, *then* $n^\Lambda(\theta_n - \theta^*)$ *a.s. converges as* $n \to +\infty$ *toward a finite random variable.*

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