Randomly coloring sparse random graphs with fewer colors than the maximum degree

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October 11, 2005

Abstract

We analyze Markov chains for generating a random k-coloring of a random graph $G_{n,d/n}$. When the average degree d is constant, a random graph has maximum degree $\Theta(\log n/\log\log n)$, with high probability. We show that, with high probability, an efficient procedure can generate an almost uniformly random k-coloring when $k = \Theta(\log\log n/\log\log\log n)$, i.e., with many fewer colors than the maximum degree. Previous results hold for a more general class of graphs, but always require more colors than the maximum degree.

1 Introduction

We study Markov Chain Monte Carlo algorithms for generating a random (vertex) k-coloring of an input graph $G = (V, E)$. We will work with G and k where it is possible to generate *some* proper coloring in polynomial time. Our challenge will be to generate a random coloring that is selected almost uniformly from the set of proper colorings.

1.1 Prior work

In previous work, simple Markov chains, such as the *Glauber dynamics*, have been proven effective. The Glauber dynamics produce a Markov chain on proper colorings where at each step we randomly recolor a random vertex. More precisely, from a k-coloring X_t at time t, the transition $X_t \to X_{t+1}$ is defined as follows. First, a random vertex v_t is chosen. We then set $X_{t+1}(v_t)$ to a color chosen uniformly at random from those colors not appearing in the neighborhood of v_t in X_t. For all $w \neq v_t$, we set $X_{t+1}(w) = X_t(w)$. The stationary distribution of the Glauber dynamics is uniformly distributed over k-colorings. We are interested in the *mixing time* of such Markov chains, meaning the number of steps until the chain is within variation distance 1/4 of the stationary distribution, for any initial k-coloring X_0 (see Jerrum [14] for background on finite Markov chains).

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Jerrum [14] proved that whenever $k > 2\Delta$ the mixing time of the Glauber dynamics is $O(n \log n)$. Vigoda [20] improved Jerrum's result, by analyzing a more complicated chain, reducing the lower bound on k to $11\Delta/6$. This is still the best lower bound on k for general graphs.

Subsequent work, beginning with Dyer and Frieze [7], developed the notion of "burn-in", and used it to analyze the Glauber dynamics on restricted classes of graphs. Building upon [7, 17, 11], Hayes and Vigoda [12] proved the Glauber dynamics has $O(n \log n)$ mixing time when $k > (1+\epsilon)\Delta$ for any constant $\epsilon > 0$, assuming G has girth > 9 and $\Delta = \Omega(\log n)$. Dyer, Frieze, Hayes and Vigoda [8] reduced the lower bound on Δ to a sufficiently large constant, assuming $k > 1.489\Delta$ and girth $g > 5$. Further improvements were recently obtained for amenable graphs (without any lower bound on Δ) by Goldberg, Martin and Paterson [10], and for "locally sparse" graphs (assuming $\Delta = \Omega(\log n)$) by Frieze and Vera [9] (which extends work of Hayes and Vigoda [13]).

1.2 Our work

In many classes of graphs, such as random graphs and planar graphs, the chromatic number is intimately related to the average degree, as opposed to the maximum degree. This paper focuses on randomly coloring sparse random graphs. These graphs have constant average degree d and much larger maximum degree Δ. We randomly color such graphs with many fewer than Δ colors.

$G_{n,p}$ is the random graph with vertex set $[n] = \{1, 2, \ldots, n\}$ and where each possible edge is independently included with probability p. We work with $G = G_{n,p}$ where $p = d/n$ and d is a constant with $d > 1$. Such graphs have vertices of degree $\Theta(\log n/ \log\log n)$, but have relatively few such vertices (see, for example, [4, Theorem 3.7, p. 66]). Thus, it would seem that we might be able to randomly color such a graph with many fewer than Δ colors. We will prove this in our main theorem below.

The main difficulty caused by large degree vertices in the analysis of algorithms for randomly coloring graphs is that in many colorings, these vertices have few color choices, i.e., almost all of the colors might appear in their neighborhood. Thus, the color choice of the neighbor of a high degree vertex v can have a large influence on the color choice of v when v's color is updated. To avoid this, we cluster the high degree vertices into sets of nearby vertices. We then pad these sets with a radius r of low degree vertices. The radius r is chosen sufficiently large so that these padded sets are not overly influenced by the color choices of their neighbors. We analyze a Markov chain tailored to our clustering of high degree vertices.

We need some notation before formally defining the Markov chain we analyze. For $b \geq 1$, let $L_b = \{v : \deg(v) \geq b\}$ denote those vertices with degree at least b. For $r \geq 1$, let N_b denote those vertices at distance at most r from some vertex in L_b. Finally, let H_b be the subgraph of G induced by $V_b = L_b \cup N_b$.

In addition to the Glauber dynamics defined earlier, we consider the following Markov chain, which we refer to as the *modified Glauber dynamics*. Let $\lambda(v) = 1$ if $v \notin V_b$ and $\lambda(v) = 1/|C|$ if $v \in V_b$ and C is the component of H_b containing v. Let $\Lambda = \sum_{v \in V} \lambda(v)$. The transitions of the modified Glauber dynamics are defined as follows. From a coloring X_t, we choose $v_t \in V$ with probability $\lambda(v_t)/\Lambda$. If $v_t \in V_b$, then we randomly re-color the component C which contains v_t by choosing uniformly among the colorings of C that are consistent with the way X_t has colored the vertices in $V \setminus C$. Otherwise we randomly re-color v_t by choosing uniformly among the colors that do not appear in the neighborhood of v_t under coloring X_t.

We can now state our main theorem.

Theorem 1. *For all $d \geq 1$, with probability $1 - o(1)$, the random graph $G = G_{n,d/n}$ is such that*

(a) *if*

$$r = \ln\ln n, \quad b = \frac{(3 + \ln d)r}{\ln r} \text{ and } k \geq 12b \tag{1}$$

then

- *Modified Glauber dynamics has mixing time $O(n \log n)$.*
- *A step of the modified Glauber dynamics can be implemented in time polynomial in $\log n$.*

(b) *if $0 < \alpha < 1$ and*

$$r = 2(1 + 1/\alpha), \quad b = (\ln n)^\alpha \text{ and } k \geq 12b \tag{2}$$

then

- *Modified Glauber dynamics has mixing time $O(n \log n)$.*
- *A step of the modified Glauber dynamics can be implemented in time polynomial in $\log n$.*
- *Glauber dynamics has mixing time polynomial in n.*

Note that if $d < 1$ then **whp**[1] G consists of trees and unicyclic components of size $O(\log n)$ and then it is trivial to randomly color G. Also, we can allow d to grow with n, but once $d = \Omega(\log n)$ the result is subsumed by Jerrum's result.

It is well known that the maximum degree of $G = \Theta\left(\frac{\ln n}{\ln\ln n}\right)$ **whp** (see, for example, [4, Theorem 3.7, p. 66]). Thus the number of colors required for rapid mixing is $o(\log \Delta)$ in our first case, and $o(\Delta)$ in the second.

1.3 Outline of what follows

Section 2 shows that **whp** the Glauber dynamics yield an ergodic Markov chain for $G_{n,d/n}$ when $k > d + 2$. In Section 3 we state a collection of properties which hold **whp** that will be useful in the proof of Theorem 1 (the proof that these properties hold **whp** is deferred to Section 6). In Sections 4 and 5.1, we use the properties from Section 3 to prove Theorem 1(a) and Theorem 1(b) respectively. In Section 6, we prove that the properties stated in Section 3 hold **whp**. Section 7 concludes with some related open problems.

1.4 Notational reference

For convenient reference, we collect the definitions and parameters above here.

[1]Throughout this paper, we use the term with high probability, denoted **whp**, to refer to events which occur with probability $1 - o(1)$ as $n \to \infty$.

Notation

G**,** the graph, $G = (V, E)$ where $V = [n]$ is the set of vertices and $E \subseteq V \times V$ is the set of edges

d**,** the expected degree of G, a constant > 1

Δ**,** the maximum degree of G, which grows at rate $\Theta(\log n / \log \log n)$

k**,** the number of colors in the coloring, which is at least $12b$ in both case (a) and (b) of Theorem 1

b**,** the degree above which vertices are "bad", equal to $\frac{(3+\ln d)r}{\ln r}$ in case (a) and $(\ln n)^\alpha$ in case (b) of Theorem 1

r**,** the radius around bad vertices for which the Glauber dynamics will be modified, equal to $\ln \ln n$ in case (a) and $2(1 + 1/\alpha)$ in case (b) of Theorem 1

α**,** a constant less than 1 which controls value of b and hence the number of colors k in case (b) of Theorem 1

L_b**,** the set of vertices of degree at least b

N_b**,** the set of vertices at distance at most r from some vertex in L_b

V_b**,** the union of L_b and N_b

H_b**,** the subgraph of G induced by the vertices in V_b

$\lambda(v)$**,** the weight of vertex v in Modified Glauber Dynamics; $\lambda(v) = 1$ if $v \notin V_b$ and $\lambda(v) = 1/|C|$ if $v \in V_b$ and C is the component of H_b containing v

Λ**,** the normalization constant for $\lambda(v)$'s, $\Lambda = \sum_{v \in V} \lambda(v)$

Glauber Dynamics Transition:

Input: X_t, a k-coloring of G

Output: X_{t+1}, formed by selecting a vertex $v_t \in V$ uniformly at random, setting $X_{t+1}(v_t)$ to a color chosen uniformly at random among the colors that do not appear in the neighborhood of v_t under coloring X_t, and setting $X_{t+1}(w) = X_t(w)$ for all $w \neq v_t$.

Modified Glauber Dynamics Transition:

Input: X_t, a k-coloring of G

Output: X_{t+1}, formed as follows. Select a vertex $v_t \in V$ randomly according to the distribution $\mathbf{Pr}[v_t = v] = \lambda(v)/\Lambda$. If $v_t \in V_b$, then let C be the component of H_b containing v_t, and generate $X_{t+1}(C)$ by choosing uniformly at random among the colorings of C that are consistent with the way X_t has colored the vertices in $V \setminus C$. Otherwise $v_t \notin V_b$, and we set $X_{t+1}(v_t)$ by choosing uniformly at random among the colors that do not appear in the neighborhood of v_t under coloring X_t. In either case, we complete the coloring by setting $X_{t+1}(w) = X_t(w)$ for all w not already set.

2 Ergodicity

We first show that Glauber dynamics (and hence modified Glauber dynamics) is ergodic **whp** for a random graph $G_{n,d/n}$ when $k \geq d+2$.

For a graph $G = (V, E)$, the α-core is the unique maximal set $S \subseteq V$ such that the induced subgraph on S has minimum degree at least α. It follows from work of Pittel, Spencer and Wormald [18] that **whp** G has no α-core for $\alpha \geq d$. A graph without a α-core is α-*degenerate* i.e. its vertices can be ordered as $v_1, v_2, \ldots, v_n$ so that v_i has fewer than α neighbors in $\{v_1, v_2, \ldots, v_{i-1}\}$. To see this, let v_n be a vertex of minimum degree and then apply induction.

Lemma 2. *If $G = (V, E)$ has no α-core, then, for all $k \geq \alpha+2$, the Glauber dynamics for k-coloring yields an ergodic Markov chain.*

Proof. Let $v_1, \ldots, v_n$ denote an ordering of V such that the degree of v_i is less than α in G_i, defined as the induced subgraph on $\{v_1, v_2, \ldots, v_i\}$. For $1 \leq i \leq n$, let Ω_i denote the k-colorings of G_i.

We need to show that the set Ω_n is connected with respect to transitions of the Glauber dynamics. We will prove the claim by induction. The claim is trivial for $n = 1$. Assume the set Ω_j, for all $j < i$, is connected. Consider a pair of colorings $X, Y \in \Omega_i$. Let X', and Y' respectively, denote the projection of these colorings on G_{i-1}.

By induction, we know there exists a path of Glauber dynamics transitions (for G_{i-1}) connecting X' to Y'. Consider any such path, say it has length ℓ. Let (w_j, c_j) denote the $(vertex, color)$ update at step j of this path. We construct a path (of length $\leq 2\ell$) from X to Y along Glauber transitions for G_i.

For $j = 1, 2, \ldots, \ell$, we will re-color w_j to color c_j, if such a transition is valid (i.e., no neighbor of w_j has color c_j). If it is not valid, then v_i must be the only neighbor of w_j that is colored c_j. Since v_i has degree less than α in G_i, there exists a new color for v_i which does not appear in its neighborhood. Thus, we first re-color v_i to any new (valid) color, and then we re-color w_j to c_j. Hence, the length of the path at most doubles. □

3 Structure results

In this section we will define some useful graph properties and claim that G has these properties **whp**. It will be useful to define the notation G^r to denote the graph with an edge $\{u, v\}$ iff G contains a (u, v) path of length at most r.

3.1 Case (a)

The graph properties of interest are the following:

P1a The maximum component size in H_b is at most $C_{\max} = (\ln n)^2(2d)^r = (\ln n)^{O(1)}$.

P2 G contains no d-core.

P3 If $v \notin H_b$ and C is a component of H_b then v has at most 2 neighbors in C.

P4 Each component C of H_b has at most $|V(C)|$ edges.

P5 $|H_b| = o(n)$, and so $\Lambda = n(1 - o(1))$.

P6 There does not exist $S \subseteq L_b$ such that $|S| \geq s = 2r^{-1} \ln n$ and S induces a connected subgraph in G^r.

Theorem 3. *Under the hypotheses of Theorem 1(a), with probability* $1 - o(1)$ *properties* **P1a-P6** *hold.*

3.2 Case (b)

We modify our claims about the structure of G under the hypotheses of Theorem 1(b):

P1b The maximum component size in H_b is at most $C_{\max} = (10d)^r \ln n$.

Theorem 4. *Under the hypotheses of Theorem 1(b), with probability* $1 - o(1)$ *properties* **P1b-P5** *hold.*

We prove Theorem 3 in Section 6.1 and Theorem 4 in Section 6.2.

3.3 Implementing modified Glauber dynamics

Implementing a transition of the modified Glauber dynamics is equivalent to generating a random list coloring of the updated component C. In the list coloring problem every vertex $v \in C$ has a set $L(v)$ of valid colors, where $|L(v)| \subseteq \{1, 2, \ldots, k\}$, and v can only receive a color in $L(v)$. In our case, $L(v)$ are those colors not appearing in $N(v) \setminus C$.

For a tree on ℓ vertices, using dynamic programming we can exactly compute the number of list colorings in time ℓk. Therefore, we can also generate a random list coloring of a tree. By property **P4**, our components are trees or unicyclic. For a unicyclic component, we can simply consider all $\leq k^2$ colorings for the endpoints of the extra edge, and then recurse on the remaining tree. By property **P1** (**a** or **b**), this implies that the modified Glauber dynamics can be efficiently implemented.

4 Coupling Analysis: Proof of Theorem 1(a)

In this section, we prove Theorem 1(a), using the structure results from Theorem 3.

We use path coupling [5]. For a pair of colorings X, Y, our metric $d(X,Y)$ is Hamming distance:

$$d(X,Y) = \sum_{v \in V} 1_{X(v) \neq Y(v)},$$

We are therefore obliged to extend the state space to include improper colorings as transient states. For all (X_t, Y_t) where $d(X_t, Y_t) = 1$, we define a coupling $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$ such that

$$\mathbf{E}[d(X_{t+1}, Y_{t+1}) \mid X_t, Y_t] < (1 - 1/2n) d(X_t, Y_t).$$

This will imply mixing time $O(n \log n)$ by a standard application of path coupling [5, Theorem 1].

Each chain chooses the same random vertex w, and both chains re-color w, if $w \notin V_b$, or re-color the component $C_w \ni w$, if $w \in V_b$. The choices will be coupled as described below. We divide the coupling analysis into two cases, depending on whether X_t and Y_t differ at a (unique) vertex $v \in V_b$, or at a vertex $v \notin V_b$. Recall that $\Lambda = n - o(n)$, from **P5**.

When re-coloring a single vertex w at step t, we will frequently couple the chains X and Y maximally, meaning in a manner which maximizes the probability that $X_{t+1}(w) = Y_{t+1}(w)$. To define this formally, let A_X denote the set of colors not appearing in the neighborhood of w under coloring X_t, and let A_Y denote the set of colors not appearing in the neighborhood of w under coloring Y_t. The *maximal coupling* transition is defined as follows [17]. We take two mappings $f_X : [0,1] \to A_X$, $f_Y : [0,1] \to A_Y$ such that

- for each $c \in A_X$, $|f^{-1}(c)| = 1/|A_X|$ and similarly for Y, and
- $\{x : f_X(x) \neq f_Y(x)\}$ is as small as possible in measure.

Then we take a uniform random real $x \in [0,1]$ and choose color $f_X(x)$ for $X_{t+1}(w)$ and $f_Y(x)$ for $Y_{t+1}(w)$.

Case 1: For v such that $X_t(v) \neq Y_t(v)$ we have $v \in V_b$.

Let C_v be the connected component containing v. If we re-color component C_v, then both chains can choose the same coloring and $X_{t+1} = Y_{t+1}$. Consider $w \in N(v)$. If $w \in V_b$, then $w \in C_v$ and $X_{t+1}(w) = Y_{t+1}(w)$. If $w \notin V_b$ then $\deg(w) < b$, and there are at least $k - b$ colors not appearing in $X_t(N(w))$, and similarly for $Y_t(N(w))$. Using the maximal coupling, there is at most one choice for $X_{t+1}(w)$ which results in $X_{t+1}(w) \neq Y_{t+1}(w)$, i.e $X_{t+1}(w) = Y_t(v)$. It follows that

$$\mathbf{Pr}\left[X_{t+1}(w) \neq Y_{t+1}(w) \mid \xi_t = w\right] \leq \frac{1}{k-b}$$

where ξ_t is the random vertex chosen at step t.

We can now bound the expected change in distance after a coupled transition,

$$\begin{aligned} \mathbf{E}[d(X_{t+1}, Y_{t+1}) - d(X_t, Y_t)] &= -\mathbf{Pr}\left[\xi_t \in C_v\right] + \sum_{w \in N(v) \setminus C_v} \mathbf{Pr}\left[\xi_t = w \wedge X_{t+1}(w) \neq Y_{t+1}(w)\right] \\ &\leq -\frac{1}{\Lambda} + \frac{b}{(k-b)\Lambda} \\ &\leq -\frac{1}{2n} \quad \text{for } k > 4b \text{ and } n \text{ sufficiently large.} \end{aligned}$$

Case 2: For v such that $X_t(v) \neq Y_t(v)$ we have $v \notin V_b$.

For $w \in N(v) \setminus V_b$, using the maximal coupling, the probability w receives a different color in the two chains is bounded by $((k-b)\Lambda)^{-1}$, as above.

For $w \in V_b$, we will couple the colorings of C_w in X and Y, as described below, so as to have few disagreements. Let $\Phi(w)$ be the expected number of disagreements between X_{t+1} and Y_{t+1} in C_w, i.e.

$$\mathbf{E}\big[d(X_{t+1}(C_w), Y_{t+1}(C_w)) \mid \xi_t = w\big] = \Phi(w),$$

and $\Phi = \max_w \Phi(w)$. Then, we bound the expected change in distance by

$$\begin{aligned}
\mathbf{E}[d(X_{t+1}, Y_{t+1}) - d(X_t, Y_t)] &\leq -\mathbf{Pr}\left[\xi_t = v\right] + \frac{|N(v) \setminus V_b|}{(k-b)\Lambda} + \sum_{w \in N(v) \cap V_b} \mathbf{Pr}\left[\xi_t \in C_w\right] \Phi(w) \\
&\leq -\frac{1}{\Lambda} + \frac{b}{(k-b)\Lambda} + \frac{1}{\Lambda} \sum_{w \in N(v) \cap V_b} \Phi(w) \\
&\leq -\frac{1}{\Lambda} + \frac{b}{(k-b)\Lambda} + \frac{b\Phi}{\Lambda} \\
&\leq -\frac{1}{2n} \qquad \text{for } k \geq 6b, \quad b\Phi \leq \frac{1}{4}, \text{ and } n \text{ large.}
\end{aligned}$$

It remains to show that $b\Phi \leq \frac{1}{4}$. We use the "disagreement percolation" coupling construction of van den Berg and Maes [1, Theorem 1]. We wish to couple $X_{t+1}(C_w)$ and $Y_{t+1}(C_w)$ as closely as possible, but the identity coupling is precluded by the disagreement at v. The technique of [1] assembles the coupling in a stepwise fashion working away from w. In our case, it may be viewed as follows. From **P3** we know that C_w is a tree with at most one additional edge. Also, from the definition of H_b, it has degree at most b except for a central "kernel" of higher-degree vertices at distance r from its boundary. The disagreement at v propagates into C_w along paths from w. A disagreement at vertex $x \in C_w$ at (edge) distance ℓ from w propagates to a neighbor z at distance $\ell + 1$ if $X_{t+1}(z) \neq Y_{t+1}(z)$. The distributions of $X_{t+1}(z)$, $Y_{t+1}(z)$ are invariant under a Glauber dynamics transition. Thus, if z is not in the kernel, we may couple $X_{t+1}(z), Y_{t+1}(z)$, using the maximal coupling, to have $\mathbf{Pr}\left[X_{t+1}(z) \neq Y_{t+1}(z)\right] \leq 2/(k-b) =: \zeta$, since

(i) z can have at most two neighbors which disagree, since C_w is a tree plus 1 edge,

(ii) each such neighbor of z will have at most one disagreement,

(iii) there are at least $k - b$ colors available at z.

The disagreement percolation is dominated by an independent process. Thus a disagreement propagates to a vertex at distance $\ell < r$ from w with probability at most $\zeta^{\ell+1}$. Moreover there are at most b^ℓ such vertices. It propagates to a vertex in the kernel with probability at most $\zeta^{r+1} \ln n$ for large n, using **P6**. If this happens, we couple arbitrarily with the remaining probability, and concede $|C_w|$ disagreements. Since $k \geq 12b$, it follows that

$$\begin{aligned}
\Phi(w) &\leq \sum_{\ell=0}^{r-1} b^\ell \zeta^{\ell+1} + |C_w| \zeta^{r+1} \ln n \\
&\leq \zeta \sum_{\ell=0}^{\infty} \left(\frac{2b}{k-b} \right)^\ell + o(\zeta), \\
&\leq \frac{2}{9b} \big(1 + o(1)\big), \\
&\leq \frac{1}{4b} \quad \text{for } n \text{ large},
\end{aligned}$$

using $|C_w| = (\log n)^{O(1)}$ (from **P1a**), and $r = \Omega(\log \log n)$ so $\zeta^{-r} = (\log n)^{\Omega(\log \log \log n)}$. □

5 Proof of Theorem 1: Part (b)

We first analyze the mixing time of the modified Glauber dynamics in Section 5.1. Then, in Section 5.2, we use the comparison method of Diaconis and Saloff-Coste [6] to bound the mixing time of the Glauber dynamics.

5.1 Coupling Analysis

In this section we bound the mixing time of the modified Glauber dynamics as claimed in Theorem 1(b).

We follow the argument of Section 4. The only place we might run into trouble is showing that $|C_w| \zeta^{r+1} \ln n = o(\zeta)$, noting that the kernel now has size $O(\log n)$. The remaining parts of the argument are unchanged. But for large n we have

$$|C_w| \zeta^{r+1} \ln n \leq (\ln n)^2 \left(\frac{20d}{11 (\ln n)^\alpha} \right)^{2(1+1/\alpha)} = o(\zeta).$$

5.2 Comparison: Part(b)

We now bound the mixing time of the Glauber dynamics. Let τ_G denote the mixing time of the Glauber dynamics, and τ_M denote the mixing time of the modified Glauber dynamics. Let P_G and P_M denote their corresponding transition matrices. Let Ω denote the k-colorings of the graph of interest. Let $\pi(Z) = 1/|\Omega|$ denote the probability of coloring Z under the stationary distribution.

Lemma 5. *Under the hypotheses of Theorem 1(b),*

$$\tau_G \leq d^{O(\log n)} \tau_M \log |\Omega|,$$

for the dynamics on $G_{n,d/n}$ **whp**.

Proof. We will use the comparison technique of Diaconis and Saloff-Coste [6] (see also [19]). For all $I, F \in \Omega$ where $P_M(I,F) > 0$, we will define a path $\gamma_{IF} = (Z_0 = I, Z_1, \ldots, Z_\ell = F)$ such that $P_G(Z_i, Z_{i+1}) > 0$, for all $1 \le i < \ell$. For $t = (Z, Z') \in \Omega^2$ where $P_G(Z, Z') > 0$, let

$$cp(t) = \{(I,F) \in \Omega^2 : t \in \gamma_{IF}\},$$

denote the set of canonical paths which contain t. We are interested in its congestion:

$$\begin{aligned}
\rho(t) &= \frac{1}{\pi(Z)P_G(Z,Z')} \sum_{(I,F)\in cp(t)} |\gamma_{IF}|\pi(I)P_M(I,F) \\
&\le nk \sum_{(I,F)\in cp(t)} |\gamma_{IF}|P_M(I,F) && (3)\\
&\le nk|cp(t)|\gamma_{\max}, && (4)
\end{aligned}$$

where $\gamma_{\max} = \max_{(I,F)\in\Omega^2} |\gamma_{IF}|$. Let

$$\rho = \max_t \rho(t).$$

Then, by [19, Proposition 1],

$$\tau_G \le 4(2 + \log|\Omega|)\rho\tau_M.$$

Consider a Glauber transition t which re-colors a vertex v. We only need to consider the case $v \in H_b$. Say v is in a component S of H_b. Fix an arbitrary coloring σ of $\bar{S} = V \setminus S$. Let $\Omega(S)$ denote the set of colorings of S consistent with σ.

We'll begin with an easy bound on $\rho(t)$, which suffices when $k = O(1)$. Clearly,

$$cp(t) \subseteq \Omega(S)^2.$$

Since $|S| = O(\log n)$, we trivially have

$$|cp(t)| \le |\Omega(S)|^2 \le k^{2|S|} = k^{O(\log n)}.$$

Using the canonical paths implied by the ergodicity proof implies $\gamma_{\max} \le 2^{|S|}$. Hence, from (4), we have

$$\rho \le \exp(O(\log n \log k)).$$

And, for a constant number of colors k, we have a polynomial bound on the mixing time of the Glauber dynamics.

We'd like to get a polynomial bound when np is constant, and $k = \Omega(1)$. So we'll fine-tune the above argument, and use (3).

By property **P2**, our input graph has no d-core. Fix an ordering $(v_1, \ldots, v_\ell), \ell = |S|$ such that v_i has degree less than d in the induced subgraph on $S_i = \{1, \ldots, i\}$. Let G_i denote the induced subgraph on $S_i \cup \bar{S}$. Note that vertex v_i has degree less than $\delta := d + b$ in G_i. Hence, in any coloring of G_i, vertex v_i has at least two valid color choices. Let Ω_j denote the colorings of S_j in G_j ($\bar{S}$ has the fixed coloring σ).

Consider a pair of colorings $I, F \in \Omega_i$. We'll inductively define the canonical path $\gamma_i(I,F)$ along Glauber transitions for G_i. Let I', F' denote the projections of I, F onto G_{i-1}. We inductively have a path $\gamma_{i-1}(I', F')$ connecting I', F'. Let (v_j, c_j) denote the j-th transition on $\gamma_{i-1}(I', F')$. We will attempt the same transitions, in order, with a possible recoloring of v_i before and after each transition of $\gamma_{i-1}(I', F')$, in order to: (i) free up v_i's color for a neighbor of v_i (as in the ergodicity proof), and (ii) keep v_i colored with $I(v_i)$ unless a neighbor of v_i has color $I(v_i)$.

More precisely, consider the j-th transition, updating v_j to color c_j, and let Z denote the current coloring. Before the update (v_j, c_j), if $v_i \in N(v_j)$ and $c_j = Z(v_i)$, then choose an arbitrary new valid color for v_i. This ensures that the recoloring of v_j to c_j is valid. After the update (v_j, c_j), if $v_j \in N(v_i)$ and $I(v_i) \notin Z(N(v_i))$, we recolor v_i to $I(v_i)$. This, of course, may be redundant if v_i already has color $I(v_i)$. We are trying to "remember" the initial coloring. Finally, after all of the transitions of the path $\gamma_{i-1}(I', F')$, we recolor v_i to $F(v_i)$.

Note, the length of these paths are at most $3^{|S|}$. We bound the congestion with a similar inductive construction. For a Glauber transition t_i in G_i, let $cp_i(t_i)$ denote the set of canonical paths crossing t_i. We inductively assume, for all $j < i$, all t_j,

$$|cp_j(t_j)| \leq |\Omega_j|(1+\delta)^{2j}. \tag{5}$$

Moreover, consider an injective map, or "encoding",

$$\eta_{t_j} : cp_j(t_j) \rightarrow \Omega_j \times \{0, \ldots, \delta\}^j \times \{0, \ldots, \delta\}^j.$$

Consider a transition $t_i = Z \rightarrow Z'$ in G_i. Suppose t_i re-colors a vertex $v_j \neq v_i$. Then, let t_{i-1} denote the corresponding transition in G_{i-1}.

For $(I, F) \in cp_i(t_i)$, we define $\eta_{t_i}(I, F)$ by a simple modification of $\eta_{t_{i-1}}(I', F')$. Let

$$\eta_{t_{i-1}}(I', F') = (C', \{\alpha_1, \ldots, \alpha_{i-1}\}, \{\beta_1, \ldots, \beta_{i-1}\}),$$

where $C' \in \Omega_{i-1}$, and for all $1 \leq j < i$, $\alpha_j, \beta_j \in \{0, \ldots, \delta\}$. Now we'll define a coloring $C \in \Omega_i$ and α_i, β_i, which will define η_{t_i}. Let $w_1, \ldots, w_{d'}, d' \leq \delta$, denote the neighbors of v_i.

The coloring C is the same as C' for all $v_j \neq v_i$. If no neighbor of v_i has color $F(v_i)$, we set $C(v_i) = F(v_i)$ and set $\alpha_i = 0$. Otherwise, we color v_i to an arbitrary valid color, and set

$$\alpha_i = \min\{1 \leq j \leq d' : C'(w_j) = F(v_i)\},$$

to "remember" the color $F(v_i)$.

Similarly, we set $\beta_i = 0$ if $Z(v_i) = I(v_i)$. (Recall, the transition is $t_i = Z \rightarrow Z'$.) Otherwise, set

$$\beta_i = \min\{1 \leq j \leq d' : Z(w_j) = I(v_i)\},$$

to "remember" the color $I(v_i)$. Note, we defined our canonical paths so that, for all colorings W on the path, $W(v_i) = I(v_i)$ or a neighbor of v_i has color $I(v_i)$ in W.

From the encoding and the transition t_i we can uniquely recover $(I, F) \in cp_i(t_i)$. Hence, our new mapping is again injective. For a transition t_i which recolors v_i, define the encoding identically to the adjacent transition which recolors some $v_j \in N(v_i)$.

We can now bound the congestion via (3). Note, for all $Z, Z' \in \Omega$, $P_M(Z, Z') = 1/|\Omega|$. Hence, applying (5) with (3), we have

$$\rho \leq nk3^{\ell}(1+\delta)^{2\ell}$$

This completes the proof of the lemma. $\square$

6 Proof of Structure Results

To show that these properties hold **whp**, it is convenient to define some additional properties and prove that they also hold **whp**.

Q1 For all $s \leq n/(2e^3d^2)$, there is no subgraph of G with s vertices which contains more than $2s$ edges.

Q2 For $v \in V$ let $B(v,r)$ denote the set of vertices at distance at most r from vertex v. Then $|B(v,r)| \leq 6(2d)^r \ln n$ for all $v \in V$.

Lemma 6. *Under the hypotheses of Theorem 1(a) and (b), with probability* $1-o(1)$ *properties* **Q1** *and* **Q2** *hold.*

First we will show how Lemma 6 implies Theorem 3 and Theorem 4. Then we perform the calculations necessary to verify Lemma 6. It is convenient to prove that the properties hold in the order that follows.

6.1 Proof of Theorem 3

In this section we assume r, b, k are defined as in Theorem 1(a), so $r = \ln\ln n$, $b = \frac{(3+\ln d)r}{\ln r}$, and $k \geq 12b$.

P5: $|H_b| = o(n)$, *and so* $\Lambda = n(1-o(1))$.

Note that

$$\mathbf{E}(|H_b|) \leq n\sum_{i=0}^{r} n^i (d/n)^i \mathbf{Pr}(\mathrm{Bin}(n, d/n) \geq b-1). \tag{6}$$

We verify **P5** by showing that the RHS of (6) is $o(n)$ and using the Markov inequality. (The RHS of (6) bounds the expected number of vertices within distance r of a vertex in L_b). But, $\mathbf{Pr}(\mathrm{Bin}(n, d/n) \geq b-1) \leq (de/(b-1))^{b-1} = o(\ln n^{-1/2})$, since the numerator of this quantity is a constant, and $b - 1 = \Omega(\ln\ln n/\ln\ln\ln n)$.

P6: *There does not exist* $S \subseteq L_b$ *such that* $|S| \geq s = 2r^{-1}\ln n$ *and* S *induces a connected subgraph in* G^r.

If S exists then we can assume that $|S| = s$ and that there exists a tree T in G such that (i) $T \cap L_b = S$, (ii) $t = |T| \leq sr$ and (iii) the leaves of T are in S. We can also assume that S contains at most $2s$ edges, from Property **Q1**.

Suppose that T has leaves L and $|L| = \ell$. We use the identity

$$\ell = 2 + \sum_{v \in T\setminus L} (\deg_T(v) - 2). \tag{7}$$

Let $T_b = (T \setminus L) \cap L_b$ and $D = \sum_{v \in L_b} \deg_T(v)$. Then (7) implies $\ell \geq D - 2(s-\ell)$ and from this we deduce that $D \leq 2s$.

Then let M be the number of edges joining $L \cup T_b$ to $V \setminus T$. We need a bound on M.

$$M \geq \ell(b-1) + (s-\ell)b - D - 2s \geq (b-5)s$$

(where the term $2s$ is subtracted to account for edges in S).

So,

$$\begin{aligned}
\mathbf{Pr}(\exists S) &\leq \sum_{t=s}^{sr} \binom{n}{t}\binom{t}{s} t^{t-2}p^{t-1}\binom{s(n-s-t)}{(b-5)s} p^{(b-5)s} \\
&\leq \sum_{t=s}^{sr} \left(\frac{ne}{t}\right)^t 2^t t^{t-2} p^{t-1} \left(\frac{s(n-s-t)ep}{(b-5)s}\right)^{(b-5)s} \\
&\leq \sum_{t=s}^{sr} n(2ed)^t (3db^{-1})^{(b-5)s} \\
&\leq 2n((2ed)^r(3db^{-1})^{b-5})^s \qquad (8) \\
&= 2n\left(e^{(\ln 2e+\ln d)\ln\ln n} e^{-(3+\ln d-o(1))\ln\ln n}\right)^s \\
&\leq 2ne^{-sr(1-o(1))} \\
&= o(1).
\end{aligned}$$

P1a: *The maximum component size in H_b is at most $(\ln n)^2 b^r$* **whp**.

Let C be a component of H_b. Let $K = C \cap L_b$. Then from **Q2** we have $|C| \leq 6|K|(2d)^r \ln n$. But **P6** implies that $|K| \leq 2r^{-1}\ln n$ and so **P1a** also holds **whp**. □

P2: *G contains no d-core.*

As mentioned in Section 2, this follows from the work of Pittel, Spencer, and Wormald [18].

P3: *If $v \notin H_b$ then* **whp** *v has at most 2 neighbors in the same component of H_b.*
P4: *Each component C of H_b has at most $|C|$ edges.*

Let $N_{i,b}$ denote the set of vertices within distance i of L_b. Thus, $N_b = \bigcup_{i=1}^r N_{i,b}$. To prove these properties, we fix a "typical" degree sequence $\mathbf{d} = d_1, d_2, \ldots, d_n$ for $G_{n,p}$ and generate a random graph with this degree sequence using the configuration model as described in Bollobás [2]. Let $m = (d_1 + \cdots + d_n)/2$. We construct a random pairing F of the points in $W = \bigcup_{i=1}^n W_i$, $|W_i| = d_i$ and interpret them as edges of a (multi-)graph on $[n]$. A typical degree sequence is such that the probability it is simple is bounded away from zero by a function of d only. We first expose all the pairs $\{x_1, x_2\}$ in F such that $\{x_1, x_2\} \cap \left(\bigcup_{i\in L_b} W_i\right) \neq \emptyset$. This will reveal $N_{1,b}$. Then we expose all pairs $\{x_1, x_2\} \in F$ such that $\{x_1, x_2\} \cap N_{1,b} \neq \emptyset$. This will reveal $N_{2,b}$. Continuing in this way we reveal $N_{i,b}, i = 1, 2, \ldots, r$ and then the components of H_b are determined. In order for a component C to have $|C| + 1$ edges, there must be some vertex in some $N_{i,b}$ which connects to a component induced by $N_{i-1,b}$ with at least 2 edges. So if **P1a** and **P5** hold, then the probability that any component C gets $|C| + 1$ edges is at most

$$\sum_{v\in N_b} \left(\binom{\Delta}{2}\left(\frac{C_{\max}}{2m - o(n)}\right)^2\right) \leq n\left(\frac{(\Delta C_{\max})^2}{(2m - o(n))^2}\right) = o(1).$$

We continue generating G by exposing all remaining pairs $\{x_1, x_2\}$ for which both points x_j lie in $\bigcup_{i\in N_b} W_i$. The rest of F will be a random pairing of the points of W which are (i) not incident with $\bigcup_{i\in L_b} W_i$ and (ii) meet $X = \bigcup_{i\in N_b} W_i$ in at most one point. We may generate this by randomly pairing the unpaired points in X and then randomly pairing up the remaining points. We consider one component C of H_b and estimate the probability that 3 vertices have a common neighbor outside H_b. Now, since all the vertices with edges still unassigned are not in L_b, each has at most b edges left to assign. So, if **P1a** and **P5** hold and $m = |W| \geq dn/3$, then the probability that there exists a vertex $v \notin H_b$ with 3 neighbors in a component C of H_b is at most the sum over v and C of the expected number of triples of vertices in $C \cap N_b$ which are adjacent to v, which is at most

$$n \sum_C \binom{|C \cap N_b|}{3} \left(\frac{b^2}{2m - o(n)} \right)^3 = O(n^{-1+o(1)}).$$

Let $\mathbf{R1} = (\neg\mathbf{P3} \cup \neg\mathbf{P4}) \cap \mathbf{P1a} \cap \mathbf{P5}$. Then, we have shown that $\mathbf{Pr}[\mathbf{R1} \mid \mathbf{d}] = o(1)$.

We obtain the result unconditionally by this summing over values of $\mathbf{d}$ for which $\frac{1}{2}\sum_{v=1}^n \mathbf{d}_v \geq dn/3$ and then

$$\begin{aligned} \mathbf{Pr}[\neg\mathbf{P3} \cup \neg\mathbf{P4}] &\leq \mathbf{Pr}[\neg\mathbf{P1a}] + \mathbf{Pr}[\neg\mathbf{P5}] + \mathbf{Pr}[m \leq dn/3] + \sum_{\mathbf{d}:\, m \geq dn/3} \mathbf{Pr}[\mathbf{R1} \mid \mathbf{d}]\mathbf{Pr}[\mathbf{d}] \\ &= o(1). \end{aligned}$$

□

6.2 Proof of Theorem 4

In this section we assume r, b, k are defined as in Theorem 1(b), so $r = 2(1 + 1/a)$, $b = (\ln n)^\alpha$, and $k \geq 12b$.

Proof. **P5:** *$|H_b| = o(n)$, and so $\Lambda = n(1 - o(1))$.*

Since $\alpha \in (0, 1)$, the value of r is smaller now than in Section 6.1 and the value of b is larger. These changes can only decrease the size of H_b, so **P5** holds by the same argument as in Section 6.1.

(Theorem 4 does not require **P6**.)

P1b: *The maximum component size in H_b is at most $C_{\max} = (10d)^r \ln n$.*

If **P1b** fails to hold then there exists sets of vertices $S, T_0, T_1, \ldots T_r$ such that (i) $S \subseteq L_b$ and S connected in G^{2r} and T_0 is a minimal set such that $T = S \cup T_0$ contains a tree in G with leaves $L \subseteq S$ (so T_0 are vertices witnessing that S is connected in G^r, as in the proof of **P6** above) and (ii) T_1 is the neighbor set of T and T_{i+1} is the neighbor set of T_i for $0 \leq i < r$ and (iii) $|S| + |T_0| + \cdots + |T_r| \geq (10d)^r \ln n$.

We will now argue that **whp** $|T| \leq \ln n$ for any $S \subseteq V_b$ which is connected in G^r. Suppose $|S| = s$. Then, since S is connected in G^{2r}, $|T_0| \leq 2r(s-1)$. Let S_1 be the vertices of S which have degree less than $b/2$ in T. Then $|S_1|$ exceeds the number of leaves in T, so by (7), we have

$$s_1 := |S_1| \geq 2 + \sum_{v \in T \setminus leaves} (\deg_T(v) - 2) \geq \sum_{v \in S \setminus S_1} (b/2 - 2) = (s - s_1)(b/2 - 2),$$

and so $s_1 = s(1 - o(1))$.

The probability that S, T_0 exist so that $|T| \geq \ln n$ is bounded above by the expected number of trees T on sets S and T_0 with $0 \leq |T_0| \leq 2rs$ and $(\ln n)/4r \leq |S| \leq \ln n$ such that the vertices $v \in S$

with $\deg_T(v) < b/2$ have at least $b/2$ edges to vertices in $V \setminus T$. By using the Chernoff bound for $R \geq 6d$ we have $\mathbf{Pr}[\mathrm{Bin}(n(1-o(1)), d/n) \geq R] \leq 2^{-R}$, we obtain the following as an upper bound of the expectation

$$\sum_{s=(\ln n)/4r}^{\ln n} \binom{n}{s} \sum_{t=0}^{2rs} \binom{n}{t} (s+t)^{s+t-2} \left(\frac{d}{n}\right)^{s+t-1} \left(2^{-(1-o(1))b/2}\right)^{(1-o(1))s} = o(1).$$

Now we prove that **whp** there does not exist a set Y such that (i) Y induces a connected subset of G and (ii) $|Y| \leq (\ln n)^2$ and (iii) $|N(Y)| \geq 9d|Y| + \ln n$. This will complete the verification of **P1**, since we have already shown that **whp** $|T| \leq \ln n$.

$$\begin{aligned} \mathbf{Pr}(\exists Y) &\leq \sum_{t=1}^{(\ln n)^2} \binom{n}{t} t^{t-2} p^{t-1} \binom{t(n-t)}{9dt + \ln n} p^{9dt+\ln n} \\ &\leq n \sum_{t=1}^{(\ln n)^2} (de)^t (e/9)^{9dt+\ln n} \\ &= o(1). \end{aligned}$$

□

P2: *G contains no d-core.*

Property **P2** does not depend on the parameters that have changed, so it holds by the arguments in Section 6.1.

P3: *If $v \notin H_b$ then* **whp** *v has at most 2 neighbors in the same component of H_b.*
P4: *Each component C of H_b has at most $|C|$ edges.*

Repeating exactly the proof of Properties **P3,P4** in Theorem 3 with the new values of r and b, and the value of $C_{\max}$ given by property **P1b** completes the theorem.

6.3 Proof of Lemma 6

Recall that d is a constant, and we wish to show that these properties for r, b, and k are as defined in Theorem 1(a) and (b), so either $r = \ln\ln n$, $b = \frac{(3+\ln d)r}{\ln r}$, and $k \geq 12b$ or $r = 2(1+1/a)$, $b = (\ln n)^\alpha$, and $k \geq 12b$.

Q1: *For all $s \leq n/(2e^3d^2)$, there is no subgraph of G with s vertices which contains more than $2s$ edges.*

Let $\mu_d = n/(2e^3d^2)$ and define $\mathcal{B}$ to the event that there exists a set S with $|S| = s \leq \mu_d$ such that

S contains at least $2s$ edges. Then

$$\begin{aligned}
\mathbf{Pr}(\mathcal{B}) &\le \sum_{s=1}^{\mu_d} \binom{n}{s} \binom{\binom{s}{2}}{2s} p^{2s} \\
&\le \sum_{s=1}^{\mu_d} \left(\frac{ne}{s}\right)^s \left(\frac{s^2 e}{4s}\right)^{2s} \left(\frac{d}{n}\right)^{2s} \\
&= \sum_{s=1}^{\mu_d} \left(\frac{e^3 d^2 s}{16n}\right)^s \\
&= \sum_{s=1}^{\log_2 n} \left(\frac{e^3 d^2 s}{16n}\right)^s + \sum_{s=\log_2 n}^{\mu_d} \left(\frac{e^3 d^2 s}{16n}\right)^s \\
&\le (\log_2 n) \left(\frac{e^3 d^2 \log_2 n}{16n}\right) + \mu_d \left(\frac{1}{32}\right)^{\log_2 n} \\
&= o(1).
\end{aligned}$$

Q2: $|B(v,r)| \le 6(2d)^r \ln n$ *for all* $v \in V$.

Fix $v \in V$ and let $B_i = B(v,i)$. We first observe that since $|B_{i+1}|$ is stochastically dominated by $\mathrm{Bin}(n|B_i|,p)$ we have

$$\begin{aligned}
\mathbf{Pr}(|B_{i+1}| \ge 2d|B_i| \mid |B_i| \ge 6\ln n) &\le e^{-2d\ln n} \le n^{-2}. \\
\mathbf{Pr}(|B_{i+1}| \ge 12\ln n \mid |B_i| \le 6d^{-1}\ln n) &\le e^{-2d\ln n} \le n^{-2}.
\end{aligned}$$

Now **whp** $|B_1| = o(\ln n)$ and then **whp** either $|B(v,r)| \le 12\ln n$ or there exists i_0 such that $|B_{i_0}| \in [6d^{-1}\ln n, 12\ln n]$. In the both cases we see that **whp** $|B(v,r)| \le 6(2d)^r \ln n$ as required.

7 Open questions

There are two natural questions that we would like to resolve:

1. Can we prove that the modified Glauber mixes rapidly if $k = O(d)$?
2. What can we say about the mixing time of the Glauber dynamics under the hypotheses of Theorem 1(a)?

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