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Randomly forced vibrations of a string

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SOMMAIRE. — Dans cet article on étudie les vibrations forcées d'une corde soumise à l'action de forces aléatoires extérieures. Dans le cas d'une corde dont la longueur est infinie on étudie les vibrations stimulées soit par un bruit blanc plane soit par d'autres forces stationnaires ou non. Au moyen d'un taux des croisements en montant, conditionné et non conditionné on étudie la forme de la corde à chaque instant dans le cas où les vibrations sont dues à la rumeur blanche. On analyse aussi les vibrations lorsque les forces appliquées à la corde ont la forme $F(x, t) = W(x)Z(t)$, $W(x)$ étant une rumeur blanche à une dimension et $Z(t)$ étant un processus stationnaire ou un processus de Wiener. En analysant les vibrations d'une corde infinie forcée par des forces appliquées à un point isolé on donne des bornes à la probabilité qu'il y ait un point de la corde au-dessus d'un niveau quelconque.

Enfin on analyse les vibrations d'une corde de longueur finie stimulées par une rumeur blanche plane. On voit que chaque point de la corde bouge comme la superposition de deux processus aléatoires indépendantes dont le premier est formé par des ondes indépendantes avec amplitude à distribution de Rayleigh et la phase distribuée uniformément dans $(0, 2\pi)$. On donne des bornes supérieures à la probabilité qu'un point dépasse un niveau quelconque dans un intervalle $(0, t)$ et que la corde ait des points au-dessus d'un niveau arbitraire au temps t.

1. INTRODUCTION

This paper deals with vibrations of a perfectly flexible string forced by random forces $F(x, t)$. These are supposed to be applied either at all points x of the string or concentrated at some fixed point a.

The case of an infinite string vibrating under the action of diffused random forces is investigated when $F(x, t)$ is a plane gaussian white noise and when $F(x, t) = Z(t)W(x)$ with $Z(t)$ being an univariate white noise and $W(x)$ a stationary gaussian process independent of $Z(t)$.

It is proved that when $F(x, t)$ is a white noise the string has at any time t the form of an almost surely continuous stationary process.

By means of a suitably defined upcrossing rate it is shown that increasingly high levels are upcrossed with ever decreasing frequency and that displacements of the string tend to amplify as time passes.

In the case that the acting force is $F(x, t) = Z(t)W(x)$ with $W(x)$ being a white noise and $Z(t)$ a Wiener process it is shown that the string has the structure of a stationary process which is less correlated than in the plane white noise case.

This means that impulses of increasing amplitude make the string swing more and more abruptly.

Section 3 deals with vibrations of an infinite string which are forced at some point a by a stochastic force. It is seen that two symmetrical random wave trains travel the string in opposite directions making progressively further points vibrate.

If the applied force is a white noise the distribution of the instantaneous maximal displacement of the string is computed. Analogously the distribution of the maximal displacement undergone by each point x in any time span $(0, t)$ is evaluated.

The case of a stationary, correlated force is also considered. In particular, if vibrations are excited by an Ornstein-Uhlenbeck force a lower bound for the maximal displacement is determined.

The last section is concerned with the case of a finite-length string forced by diffused white noise impulses.

The displacement of point x at time t, denoted throughout by $u(x, t)$, is given as a random series with independent gaussian coefficients.

It is seen that the string takes the form of almost surely continuous processes and that each point moves describing continuous sample paths.

Some upper bound for the probability of the string to exceed any given level is evaluated and is seen to depend essentially on the product Lt, L being the length and t the time elapsed since vibrations began.

A decomposition of the random process $u(x, t)$, x assumed a fixed point, into two independent gaussian processes is derived.

The first component of motion is made up by a series of cosine waves with random amplitude and phase.

The second one is a process which starts off at the last nodal time before t, that is at the last time of the form $T = kL/c$ ($k = 1, 2, \ldots$). By c we denote the velocity at which the wave trains run the string and depending on the string's tension and density.

This additional term can be interpreted as the disturbance produced on vibrations by overlapping waves running in opposite directions.

Thanks to the above representation the exact distribution of the maximal displacement at nodal times is determined.

If the finite length string is forced to vibrate by a concentrated white noise it is seen that $u(x, t)$ is given by a series of dependent sine waves.

The case of a finite string vibrating under white noise impulses is tackled by Cabaña [2] by means of completely different techniques than those of this paper.

Studying physical systems by means of stochastic arguments dates back to the time when brownian motion, Langevin equation and shot noise were first introduced.

The idea underlying this paper is that whenever random forces are assumed classical mathematical physics is still a valid basis for stochastic analysis of phenomena.

2. RANDOM VIBRATIONS OF AN INFINITE STRING

A perfectly flexible string of unit density directed along the x axis in equilibrium position is considered.

It is kept stretched by a constant tension T and is forced to vibrate perpendicularly to the x axis under the action of a randomly varying external force $F(x, t)$.

It is assumed that vibrations take place in a single plane.

If $u(x, t)$ denotes the displacement of point x at time t it is well-known that it satisfies the following nonhomogeneous wave equation:

$$(1) \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t) \qquad -\infty < x < +\infty, \quad t > 0$$

and where $c = \sqrt{T}$.

Assuming the initial conditions:

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 \tag{2}$$

the solution of (1) can be written in the following manner:

$$u(x, t) = \int_0^t \int_{-\infty}^{+\infty} G(x, t; y, s)F(y, s)dyds \tag{3}$$

$G(x, t; y, s)$ being the Green function of the initial-value problem made up by equation (1) and conditions (2).

The function $G(x, t; y, s)$ is the solution of the problem considered when at time s and at a point y an impulse of unit intensity is applied. If the external force is a plane, stationary, gaussian white noise we are lead to consider by analogy with (3) the following random field:

$$u(x, t) = \int_0^t \int_{-\infty}^{+\infty} G(x, t; y, s)d\beta(y, s) \tag{3'}$$

The process (3′) can be interpreted as the displacement of the string receiving diffused white noise impulses.

The function $G(x, t; y, s)$ being:

$$G(x, t; y, s) = \begin{cases} \dfrac{1}{2c} & x - c(t-s) < y < x + c(t-s), \quad 0 < s < t \\ 0 & \text{otherwise} \end{cases}$$

the process (3′) can receive the following explicit form:

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{cs+x-ct}^{-cs+x+ct} d\beta(y, s) \tag{4}$$

The random field (4) is a plane Wiener integral. A rigorous definition of such integrals along with their properties can be found in Cabana [*1*].

We begin by evaluating the covariance of (4).

Let (x_1, y_1) be a whatever point of the half-strip $\{x, t : t > 0, -\infty < x < +\infty\}$ and let the characteristic lines emanating from $(x_1 + ct_1, 0)$ and $(x_1 - ct_1, 0)$ be drawn.

The upper half-plane is thus subdivided into six regions which will be denoted as in figure 1.

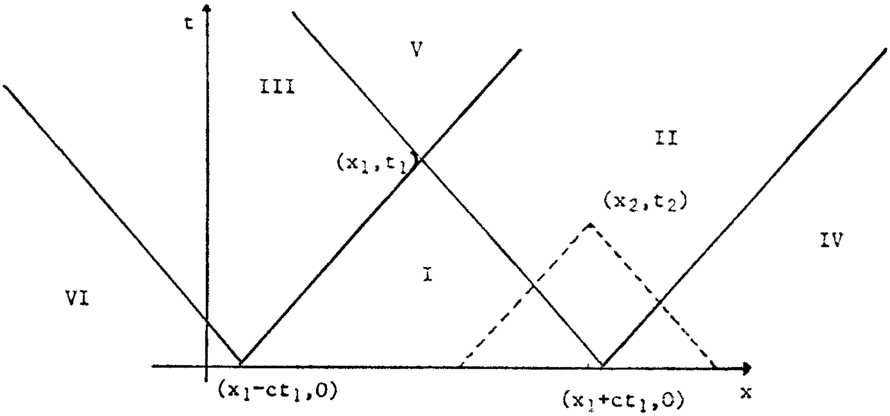

Fig. 1

Take now a point (x_2, t_2) and evaluate $\mathrm{E}u(x_1, t_1)u(x_2, t_2)$ when (x_2, t_2) is in any of the six regions indicated.

As an example of the calculations involved we suppose (x_2, t_2) to belong to region II.

Since $\mathrm{E}d\beta(x, t)d\beta(y, s) = 1$ iff $x = y$ and $t = s$ and is zero otherwise it results:

$$\text{(5)}\quad \mathrm{E}u(x_1,t_1)u(x_2,t_2) = \frac{1}{(2c)^2}\mathrm{E}\left\{\int_0^{t_1}\int_{x_1-ct_1+ct}^{x_1+ct_1-ct} d\beta(x,t)\int_0^{t_2}\int_{x_2-ct_2+cs}^{x_2+ct_2-cs} d\beta(y,s)\right\}$$

$$= \frac{1}{(2c)^2}\left\{\int_0^{t_1}\int_{x_1-ct_1+ct}^{x_1+ct_1-ct}\int_0^{t_2}\int_{x_2-ct_2+cs}^{x_2+ct_2-cs} \mathrm{E}(d\beta(x,t)d\beta(y,s))\right\}$$

$$= \frac{1}{(2c)^2}\int_0^{\frac{x_1-x_2}{2c}+\frac{t_1+t_2}{2}} dt\int_{ct+x_2-ct_2}^{-ct+x_1+ct_1} dx$$

$$= \frac{1}{16c^3}\{x_1 - x_2 + c(t_1 + t_2)\}^2$$

Analogously when (x_2, t_2) is in region III one obtains:

$$\text{(6)}\qquad \mathrm{E}u(x_1, t_1)u(x_2, t_2) = \frac{1}{16c^3}\{(x_2 - x_1) + c(t_1 + t_2)\}^2$$

With simpler calculations the following result can be obtained:

$$\text{(7)}\qquad \mathrm{E}u(x_1, t_1)u(x_2, t_2) = \begin{cases} \dfrac{t_2^2}{4c} & \text{when } (x_2, t_2) \text{ is in region I} \\[2ex] \dfrac{t_1^2}{4c} & \text{when } (x_2, t_2) \text{ is in region V} \end{cases}$$

Finally the covariance is zero when (x_2, t_2) belongs to one of the two remaining regions.

We start analysing the shape of the vibrating string at any time $t > 0$. By setting $t_1 = t_2 = t$ in the above formulae one obtains:

$$\mathrm{Cov}\,\{ u(x_1, t), u(x_2, t) \} = \frac{1}{16c^3}\{ 2ct - |x_1 - x_2| \}^2 \quad \text{if} \quad |x_1 - x_2| < 2ct$$

$$\mathrm{Var}\, u(x_1, t) = \mathrm{Var}\, u(x_2, t) = \frac{t^2}{4c}$$

$$r(|x_1 - x_2|) = \frac{\mathrm{Cov}\,\{ u(x_1, t) u(x_2, t) \}}{\mathrm{Var}\, u(x_1, t)} = \left(1 - \frac{|x_1 - x_2|}{2ct}\right)^2 \tag{8}$$

if $|x_1 - x_2| < 2ct$

At a fixed time $t = \bar{t}$, $u(x, \bar{t})$ is therefore a gaussian stationary process with autocorrelation function (8).

Gaussianity follows from the hypothesis that the external force is a gaussian white noise.

Formula (8) shows that two points of the string begin to be correlated when the initial wave train has run half their distance.

We show now that the process $u(x, \bar{t})$ has continuous sample paths with probability one.

We need the following classical theorem due to Kolmogorov (Cramer and Leadbetter, p. 63).

THEOREM 1. — Let $X(t)$ be a stochastic process defined for $t \in (a, b)$. Suppose that for all $(t, t + h)$ belonging to (a, b) it results that:

$$\mathrm{Prob}\,\{ |X(t + h) - X(t)| \geq g(h) \} \leq q(h) \tag{9}$$

where $g(h)$ and $q(h)$ are even, non decreasing functions of $h > 0$ such that:

$$\sum_{n=1}^{\infty} g(2^{-n}) < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} 2^n q(2^{-n}) < +\infty \tag{10}$$

In such conditions there always exist an equivalent process $Y(t)$ having continuous sample paths with probability one.

COROLLARY 1. — The gaussian stationary process $u(x, \bar{t})$, having zero mean, autocorrelation function (8), variance $Eu^2(x, \bar{t}) = \frac{\bar{t}^2}{4c}$ admits an equivalent process whose sample paths are almost surely continuous.

Proof. — Let

$$\sigma^2 = Eu^2(x, \bar{t}) = \frac{\bar{t}^2}{4c}$$

and observe that being:

$$r(h) = 1 - \frac{|h|}{ct} + \frac{h^2}{4c^2t^2} \qquad \text{whenever} \quad |h| < 2ct$$

it results that:

$$1 - r(h) \leq \frac{|h|}{ct}$$

On supposing for semplicity that $h > 0$ one obtains:

$$\begin{aligned}
&\text{Prob}\{ | u(x + h, \bar{t}) - u(x, \bar{t}) | \geq g(h) \} \\
&= \frac{2}{\sqrt{2\pi}} \int_{\frac{g(h)}{\sigma\sqrt{2(1-r(h))}}}^{\infty} \exp\left\{ -\frac{w^2}{2} \right\} dw \leq \frac{2}{\sqrt{2\pi}} \int_{cg(h)\sqrt{\frac{2}{ht}}}^{\infty} \exp\left\{ -\frac{w^2}{2} \right\} dw \\
&= \frac{2}{\sqrt{2\pi}} \int_{\frac{Ag(h)}{\sqrt{h}}}^{\infty} \exp\left\{ -\frac{w^2}{2} \right\} dw \qquad A = c\sqrt{\frac{2}{t}} \\
&\leq \frac{2\sqrt{h}}{Ag(h)} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{A^2g^2(h)}{2h} \right\} = q(h)
\end{aligned}$$

Let $g(h) = |h|^a$ with $0 < a < \frac{1}{2}$. The function $q(h)$

$$q(h) = \frac{1}{A}\sqrt{\frac{2}{\pi}}\, h^{\frac{1}{2}-a} \exp\left\{ -\frac{A^2}{2} h^{2a-1} \right\}$$

can straightforwardly be seen to be increasing for $h > 0$. By denoting by $B = \frac{1}{A}\sqrt{\frac{2}{\pi}}$ one has:

$$\sum_{n=1}^{\infty} 2^n q(2^{-n}) = B \sum_{n=1}^{\infty} 2^{n\left(\frac{1}{2}+a\right)} \exp\left\{ -\frac{A^2}{2} 2^{-n(2a-1)} \right\} \tag{11}$$

By the ratio convergence criterion one has:

$$\lim_{n\to\infty} \frac{2^{(n+1)\left(\frac{1}{2}+a\right)} \exp\left\{ -\frac{1}{2} A^2 2^{-(n+1)(2a-1)} \right\}}{2^{n\left(\frac{1}{2}+a\right)} \exp\left\{ -\frac{1}{2} A^2 2^{-n(2a-1)} \right\}}$$

$$= \lim_{n\to\infty} 2^{\frac{1}{2}+a} \exp\left\{ -\frac{1}{2} A^2 2^{-n(2a-1)} (2^{-(2a-1)} - 1) \right\} = 0 \quad \text{being } 2a-1<0$$

Therefore series (11) converges and since $\sum_{n=1}^{\infty} g(2^{-n}) = \sum_{n=1}^{\infty} 2^{-na} < +\infty$ the corollary follows.

Remark. — With few changes in the above proof it can be seen that any stationary gaussian process whose correlation function has the form $r(h) = 1 - \alpha \mid h \mid^{\beta} + o(h)$ as $h \to 0$ and $0 < \beta \leq 2$ has continuous sample paths with probability one (cf. [*6*]).

To gain further insight into the shape of the vibrating string we give the following definition.

DEFINITION 1. — We define as upcrossing rate of level v by the stationary process $u(x, \bar{t})$ the function $\mu(v)$ given below:

$$\mu(v) = \lim_{h \to 0+} \frac{\text{Prob}\{ u(x, \bar{t}) < v, u(x + h, \bar{t}) > v \}}{\sqrt{h}} \tag{12}$$

Since definition (12) is an unusual one for the upcrossing rate it needs some remarks. The process $u(x, \bar{t})$ having correlation function (8) is not differentiable (see Cramer and Leadbetter [*3*] page 67 for a discussion on differentiability) and its sample paths appear as extremely jagged curves.

Evaluating the upcrossing rate by the usual Rice's formula is therefore impossible, but measuring somehow the upcrossing frequency is anyway of interest.

Since the probability in formula (12) goes to zero as $\sqrt{h}$ definition (12) becomes the natural one to obtain a sensible result.

The usual definition would give an infinite upcrossing frequency of any level, which corresponds to the fact that the string has locally a very kinky form, because of white noise impulses.

We evaluate now the following probability:

$$\text{Prob}\{ u(x, \bar{t}) < v, u(x + h, \bar{t}) > v \} = \int_{-\infty}^{v} \int_{v}^{\infty} f(x, y, \sigma_{xy}) dx dy$$

where:

$$f(x, y, \sigma_{xy}) = \frac{1}{2\pi\sqrt{\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2}} \exp\left\{ - \frac{x^2\sigma_y^2 + y^2\sigma_x^2 - 2xy\sigma_{xy}}{2(\sigma_x^2\sigma_y^2 - \sigma_{xy}^2)} \right\}$$

$$\sigma_{xy} = \mathrm{E}\{ u(x, \bar{t}), u(x + h, \bar{t}) \} \qquad \sigma_x^2 = \sigma_y^2 = \mathrm{E}u^2(u, \bar{t})$$

Following Cramer and Leadbetter [3] page 26 one obtains:

$$(13)\quad \text{Prob}\{u(x,\bar t) < v, u(x+h,\bar t) > v\}$$
$$= \text{Prob}\{u(x,\bar t) < v\}\,\text{Prob}\{u(x+h,\bar t) > v\} - \int_0^{\sigma_{xy}} \frac{dx}{2\pi\sqrt{(\sigma_x^2)^2 - x^2}} \exp\left\{-\frac{v^2}{\sigma_x^2 + x}\right\}$$
$$= \text{Prob}\{u(x,\bar t) < v\}\,\text{Prob}\{u(x+h,\bar t) > v\} - \int_0^{r(h)} \frac{dx}{2\pi\sqrt{1 - x^2}} \exp\left\{-\frac{v^2}{\sigma_x^2(1 + x)}\right\}$$

where:

$$r(h) = \frac{\sigma_{xy}}{\sigma_x^2} = \frac{\mathrm{E}u(x,\bar t)u(x+h,\bar t)}{\mathrm{E}u^2(x,\bar t)}$$

The upcrossing rate (12) is therefore:

$$(14)\ \mu(v) = \lim_{h\to 0+} \frac{\text{Prob}\{u(x,\bar t) < v\}\,\text{Prob}\{u(x+h,\bar t) > v\} - \int_0^{r(h)} \frac{dx}{2\pi\sqrt{1-x^2}} \exp\left\{-\frac{v^2}{\sigma_x^2(1+x)}\right\}}{\sqrt{h}}$$

$$= \lim_{h\to 0+} \frac{-\frac{r'(h)}{2\pi\sqrt{1-r^2(h)}} \exp\left\{-\frac{v^2}{\sigma_x^2(1+r(h)}\right\}}{\frac{1}{2\sqrt{h}}}$$

$$= \lim_{h\to 0+} \frac{1}{\pi ct}\left(1 - \frac{h}{2ct}\right)\frac{\sqrt{h}}{\sqrt{1-r^2(h)}} \exp\left\{-\frac{v^2}{\sigma_x^2(1+r(h))}\right\}$$

$$= \frac{1}{\pi\sqrt{2ct}} \exp\left\{-\frac{v^2}{2\sigma_x^2}\right\} = \frac{1}{\pi\sqrt{2ct}} \exp\left\{-\frac{2v^2c}{t}\right\}$$

Result (14) is somehow similar to Rice's expression for the upcrossing rate of gaussian stationary processes having autocorrelation function $r(h) = 1 - \frac{1}{2}\alpha h^2 + o(h)$ as $h \to 0$ ($\alpha > 0$).

Formula (14) states that the upcrossing frequency decreases as level v increases.

We give now a conditional zero-crossing rate which we denote by $\mu(0 \mid 0)$ and is defined as follows:

$$(15) \quad \mu(0 \mid 0) = \lim_{h \to 0+} \frac{\text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \mid u(0, \bar{t}) = 0 \}}{\sqrt{h}}$$

The choice of the conditioning point is immaterial (not of the conditioning level) and the origin is taken for convenience.

Evaluation of the probability in (15) requires some adjustments to the framework of calculations hinted above in the unconditional case.

We give therefore the result omitting the proof.

$$(16) \quad \text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \mid u(0, \bar{t}) = 0 \} = \frac{1}{4} - \frac{1}{2\pi} \arcsin \frac{r(h) - r(x)r(x+h)}{\sqrt{(1 - r^2(x))(1 - r^2(x+h))}}$$

A quick check of result (16) may be obtained by setting $\breve{u} = 0$ in (13) and replacing the correlation coefficient $r(h)$ by the correlation coefficient of the r. v. $(u(x, \bar{t}), u(x+h, \bar{t}) \mid u(0, \bar{t}) = 0)$.

In the particular case where $x = ct$ and $x + h = 2ct$ formula (16) gives:

$$\text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \mid u(0, \bar{t}) = 0 \} = \frac{1}{4} - \frac{1}{2\pi} \arcsin \frac{1}{\sqrt{15}}$$

while in the unconditional case it results:

$$\text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \} = \frac{1}{4} - \frac{1}{2\pi} \arcsin \frac{1}{4}$$

After some manipulations in (16) one gets:

$$(17) \quad \sin\{ 2\pi \,\text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \mid u(0, \bar{t}) = 0 \} \} \frac{1}{\sqrt{h}} = \frac{\sqrt{1 - r^2(x) - r^2(x+h) - r^2(h) + 2r(x)r(h)r(x+h)}}{\sqrt{(1 - r^2(x))(1 - r^2(x+h))}} \frac{1}{\sqrt{h}}$$

and since $r^2(h) = 1 - \dfrac{2h}{ct} + o(h)$ by passing to the limit as $h \to 0+$ it turns out that:

$$\lim_{h \to 0+} \frac{\sin\{ 2\pi \,\text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \mid u(0, \bar{t}) = 0 \} \}}{\sqrt{h}} = \sqrt{\frac{2}{ct}} \frac{1}{1 - r^2(x)}$$

For small x $\sin x \sim x$ and therefore the above result may be given in the following form:

$$(18)\quad \lim_{h \to 0+} \frac{\text{Prob}\{ u(x, \bar{t}) < 0, u(x+h, \bar{t}) > 0 \mid u(0, \bar{t}) = 0 \}}{\sqrt{h}} = \frac{1}{\pi\sqrt{2ct(1 - r^2(x))}}$$

When $x = ct$ (18) gives $\mu(0 \mid 0) = \frac{16}{15}\mu(0)$ that is the information that $u(0, \bar{t}) = 0$ makes the zero upcrossing rate a bit higher than the unconditional one at $x = ct$.

Both (14) and (18) show that the zero upcrossing frequency decreases as c increases, that is an extremely stretched string tends to have a smooth shape.

Let us now draw our attention on a fixed point of the string, say $\bar{x}$. The process $u(\bar{x}, t)$ denotes the displacement of point $\bar{x}$ as time passes. Its covariance and correlation functions are:

$$\begin{aligned} \text{Cov}\{ u(\bar{x}, t_1), u(\bar{x}, t_2) \} &= \frac{1}{4c} \min\{ t_1^2, t_2^2 \} \\ r(t_1, t_2) = \text{Corr}\{ u(\bar{x}, t_1), u(\bar{x}, t_2) \} &= \frac{\min\{ t_1^2, t_2^2 \}}{t_1 t_2} \end{aligned} \qquad (19)$$

The process $u(\bar{x}, t)$ is therefore a non-stationary one which is mean-square continuous but not mean-square differentiable. Almost sure continuity may easily be proved as in corollary one. It is remarkable that the correlation of displacements at a whatever point $\bar{x}$ depends only on time elapsed.

The assumption that vibrations of the string are forced by plane white noise may appear to be somehow unrealistic. The analysis of vibrations excited by other types of stochastic forces becomes however extremely complex.

Take, for example, the case when $F(x, t) = Z(t)W(x)$, where $W(x)$ is an univariate white noise and $Z(t)$ is a stationary zero-mean, unit variance gaussian process independent of $W(x)$.

Since $W(x)$ is the formal derivative of a brownian motion $\beta(x)$, the random field (4) can be written:

$$(20)\qquad u(x, t) = \frac{1}{2c} \int_0^t Z(s)ds \int_{cs+x-ct}^{-cs+x+ct} d\beta(y)$$

Let us now compute the autocovariance function of (20) when (x_1, t_1) is chosen as in figure 1 and (x_2, t_2) is a point of region II.

$$(21)\quad Eu(x_1, t_1)u(x_2, t_2) = \frac{1}{(2c)^2}\int_0^{t_1}\int_0^{t_2} EZ(t)Z(s)dtds\left\{\int_{ct+x_1-ct_1}^{-ct+x_1+ct_1}\int_{cs+x_2-ct_2}^{-cs+x_2+ct_2} E(d\beta(x)d\beta(y)\right\}$$

For given t and s the inner integral in (21) is non-zero if and only if the intervals $(ct+x_1-ct_1, -ct+x_1+ct_1)$ and $(cs+x_2-ct_2, -cs+x_2+ct_2)$ overlap.

This requires that $\max(t, s) \le \left\{\frac{x_1 - x_2}{2c} + \frac{t_1 + t_2}{2}\right\}$ as a quick inspection on figure 1 shows.

Denoting by $r(h) = EZ(t)Z(t + h)$ one gets from (21):

$$Eu(x_1, t_1)u(x_2, t_2) = \frac{1}{(2c)^2}\int_0^{\frac{x_1-x_2}{2c}+\frac{t_1+t_2}{2}}\int_0^{\frac{x_1-x_2}{2c}+\frac{t_1+t_2}{2}} r(t-s)\min\{x_1 - x_2 + c(t_1 + t_2) - 2ct,\ x_1 - x_2 + c(t_1 + t_2) - 2cs\}\,dtds$$

$$= \frac{2}{(2c)^2}\int_0^{\frac{x_1-x_2}{2c}+\frac{t_1+t_2}{2}} dt\int_t^{\frac{x_1-x_2}{2c}+\frac{t_1+t_2}{2}} (x_1 - x_2 + c(t_1 + t_2) - 2cs)r(t-s)ds$$

To make notations less cumbersome put now: $A = \frac{x_1 - x_2}{2c} + \frac{t_1 + t_2}{2}$ so that $Eu(x_1, t_1)u(x_2, t_2)$ becomes:

$$(22)\quad Eu(x_1, t_1)u(x_2, t_2) = \frac{1}{c}\int_0^{A}\int_t^{A} r(t - s)(A - s)dsdt = \frac{1}{2c}\int_{-A}^{0} r(s)(A + s)^2 ds = \frac{1}{2c}\int_0^{A} r(s)(A - s)^2 ds$$

No substantial change in (22) is required when (x_2, t_2) belongs to region III so that we can write down explicitely the covariance function of (20) when (x_2, t_2) is either a point of region III or II without additional calculations.

$$(23)\quad Eu(x_1, t_1)u(x_2, t_2) = \frac{1}{2c}\int_0^{\frac{t_1+t_2}{2}+\frac{|x_1-x_2|}{2c}} r(s)\left\{\frac{t_1+t_2}{2} - \frac{|x_1-x_2|}{2c} - s\right\}^2 ds$$

Take now $t_1 = t_2 = t$ and $|x_1 - x_2| < 2ct$. Formula (23) gives in this case:

$$(24)\qquad Eu(x_1, t)u(x_2, t) = \frac{1}{2c}\int_0^{t-\frac{|x_1-x_2|}{2c}} r(s)\left(t - \frac{|x_1-x_2|}{2c} - s\right)^2 ds$$

and therefore the process $u(x, t)$, when t is a fixed time, is stationary with covariance function (24).

If (x_2, t_2) is a point of region I it readily turns out that:

$$(25)\quad \mathrm{E}u(x_1, t_1)u(x_2, t_2) = \frac{1}{(2c)^2}\int_0^{t_2}\int_0^{t_2}(2c)\min\{t_2-t, t_2-s\}\,\mathrm{EZ}(t)\mathrm{Z}(s)dtds$$
$$= \frac{1}{c}\int_0^{t_2}\int_t^{t_2} r(t-s)(t_2-s)dsdt = \frac{1}{2c}\int_0^{t_2} r(s)(t_2-s)^2 ds$$

The shape of the string is stationary even if the process $\mathrm{Z}(t)$ is not.

Assume, for example, $\mathrm{Z}(t)$ to be a Wiener process with unit variance parameter.

The covariance function $\mathrm{E}u(x_1, t)u(x_2, t)$ when $|x_1 - x_2| \leq 2ct$ is:

$$(26)\quad \mathrm{E}u(x_1, t)u(x_2, t)$$
$$= \frac{1}{(2c)^2}\int_0^{t-\frac{|x_1-x_2|}{2c}}\int_0^{t-\frac{|x_1-x_2|}{2c}} \mathrm{EZ}(w)\mathrm{Z}(s)\min\{2ct - 2cs - |x_1 - x_2|,\ 2ct - 2cw - |x_1 - x_2|\}\,dwds$$
$$= \frac{2}{(2c)^2}\int_0^{t-\frac{|x_1-x_2|}{2c}} \min(s, w)\int_s^{t-\frac{|x_1-x_2|}{2c}} \{2ct - 2cw - |x_1 - x_2|\}\,dwds$$
$$= \frac{1}{2c}\int_0^{t-\frac{|x_1-x_2|}{2c}} s\left(t - \frac{|x_1 - x_2|}{2c} - s\right)^2 ds = \frac{1}{24c}\left(t - \frac{|x_1 - x_2|}{2c}\right)^4$$

The variance is:

$$(27)\quad \mathrm{E}u^2(x, t) = \frac{1}{(2c)^2}\int_0^t ds\int_0^t \min\{s, w\}\min\{2c(t-s), 2c(t-w)\}\,dw$$
$$= \frac{2}{2c}\int_0^t ds\int_s^t s(t-w)^2dw = \frac{1}{2c}\int_0^t s(t-s)^2 ds = \frac{t^4}{24c}$$

If we denote by $r_w(|x_1 - x_2|, t)$ the autocorrelation function of $u(x, t)$ at time t when $\mathrm{F}(x, t) = \beta(t)\dfrac{\partial\beta(x)}{\partial x}$, β being a Wiener process we have:

$$(28)\quad r_w(|x_1 - x_2|, t) = \frac{\mathrm{E}u(x_1, t)u(x_2, t)}{\mathrm{E}u^2(x, t)} = \left(1 - \frac{|x_1 - x_2|}{2ct}\right)^4 \quad \text{if } |x_1 - x_2| \leq 2ct$$

Comparing (28) with (8) it can be seen that two points of the string $|x_1 - x_2|$ units apart are less correlated in the case when $F(x, t) = \beta(t)\frac{\partial \beta(x)}{\partial x}$ than when $F(x, t)$ is a plane white noise.

An intuitive explanation of the above result is the following. On each point of the string acts a force whose intensity has an increasing variance with time. Therefore the impulses make the string swing more and more abruptly.

3. RANDOM VIBRATIONS EXCITED BY A CONCENTRATED FORCE

Suppose now that the random force $F(x, t)$ is applied at an isolated point a of the infinite string.

Writing $F(y, s) = P(a, s)\delta(y - a)$, $\delta(y - a)$ being the Dirac delta function and substituting in (3) one obtains:

$$(29) \quad u(x,t) = \int_0^t \int_{-\infty}^{+\infty} G(x,t;y,s)P(a,s)\delta(y-a)dyds = \int_0^t G(x,t;a,s)P(a,s)ds$$

The explicit form of $G(x, t; a, s)$ is:

$$G(x, t, a, s) = \begin{cases} \dfrac{1}{2c} & \text{for } 0 < s < t - \dfrac{|x-a|}{c} \\ 0 & \text{otherwise} \end{cases}$$

and therefore formula (29) becomes:

$$(30) \qquad u(x, t) = \frac{1}{2c}\int_0^{t-\frac{|x-a|}{c}} P(a, s)ds \qquad t > \frac{|x-a|}{c}$$

We firstly suppose $P(a, s)$ to be a stationary gaussian process with zero mean and unit variance.

Formula (30) shows that the displacement at a point x begins to be different from zero as soon as an impulse started at $t = 0$ from $x = a$ has travelled the distance $|x - a|$.

To determine the covariance function of field (30) let us draw the characteristic lines emanating from point $(a, 0)$ of the (x, t) plane (see figure 2 below).

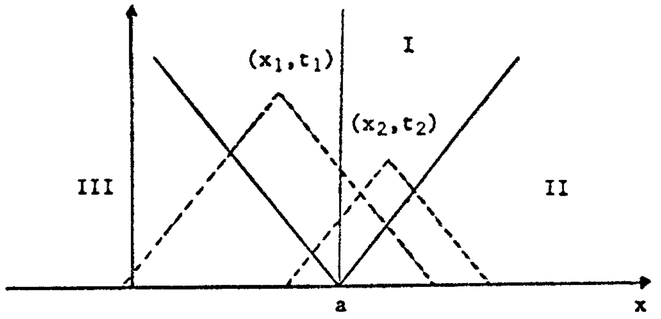

Fig. 2

If (x_1, t_1) is chosen in region I we evaluate $\mathrm{E}u(x_1, t_1)u(x_2, t_2)$ when (x_2, t_2) is any point of the (x, t) plane. If (x_2, t_2) is also a point belonging to region I it results:

$$\text{(31)} \quad \mathrm{E}u(x_2, t_2)u(x_1, t_1) = \frac{1}{(2c)^2} \int_0^{t_1 - \frac{|x_1 - a|}{c}} \int_0^{t_2 - \frac{|x_2 - a|}{c}} \mathrm{EP}(a, s)\mathrm{P}(a, z)dsdz$$

$$\text{if} \quad t_1 - \frac{|x_1 - a|}{c} \geq t_2 - \frac{|x_2 - a|}{c}$$

$$= \frac{1}{(2c)^2} \left\{ \int_0^{t_2 - \frac{|x_2 - a|}{c}} \int_0^{t_2 - \frac{|x_2 - a|}{c}} \mathrm{EP}(a, s)\mathrm{P}(a, z)dsdz + \int_0^{t_2 - \frac{|x_2 - a|}{c}} ds \int_{t_2 - \frac{|x_2 - a|}{c}}^{t_1 - \frac{|x_1 - a|}{c}} \mathrm{EP}(a, s)\mathrm{P}(a, z)dz \right\}$$

$$= \frac{1}{(2c)^2} \left\{ \int_0^{t_2 - \frac{|x_2 - a|}{c}} r(u)\left(t_2 - \frac{|x - a|}{c} - u\right)du + \int_0^{t_2 - \frac{|x_2 - a|}{c}} \int_{t_2 - \frac{|x_2 - a|}{c}}^{t_1 - \frac{|x_1 - a|}{c}} r(s - z)dsdz \right\}$$

In the case where $t_1 - \frac{|x_1 - a|}{c} < t_2 - \frac{|x_1 - a|}{c}$ a straightforward change in the integrals gives the covariance $\mathrm{E}u(x_2, t_2)u(x_1, t_1)$.

As a particular example we assume $\mathrm{P}(a, s)$ to be an Ornstein-Uhlenbeck process with covariance function $r(h) = \exp\{-|h|\}$.

In this particular case the random field (30) can be written as follows:

$$(32)\qquad u(x,t)=\begin{cases}\dfrac{1}{2c}\displaystyle\int_0^{t-\frac{|x-a|}{c}} e^{-s}\beta(e^{2s})ds & \text{if } t>\dfrac{|x-a|}{c}\\ 0 & \text{otherwise}\end{cases}$$

$\beta(s)$ denotes a Wiener process with zero mean and unit variance parameter.

A lower bound for the probability of $\{\max_{\frac{|x-a|}{c}\le z\le t} u(x,z)<\alpha\}$ is easily determined by observing that this event is implied by:

$$\max_{0\le s\le t-\frac{|x-a|}{a}} \beta(e^{2s}) < \frac{2c\alpha}{1-\exp\left\{-\left(t-\frac{|x-a|}{c}\right)\right\}}$$

Therefore by known results on the Wiener process it follows that:

$$(33)\quad \text{Prob}\left\{\max_{0\le s\le t-\frac{|x-a|}{c}} \beta(e^{2s}) < \frac{2c\alpha}{1-\exp\left\{-\left(t-\frac{|x-a|}{c}\right)\right\}}\right\}$$

$$=2\,\text{Prob}\left\{0\le\beta\left(\exp\left\{2\left(t-\frac{|x-a|}{c}\right)\right\}\right)\le\frac{2c\alpha}{1-\exp\left\{-\left(t-\frac{|x-a|}{c}\right)\right\}}\right\}$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^{\frac{2c\alpha\exp\left\{-\left(t-\frac{|x-a|}{c}\right)\right\}}{1-\exp\left\{-\left(t-\frac{|x-a|}{c}\right)\right\}}}\exp\left\{-\frac{z^2}{2}\right\}dz\le\text{Prob}\left\{\max_{\frac{|x-a|}{c}\le z\le t} u(x,z)<\alpha\right\}$$

By similar arguments one obtains a lower bound for the

$$\text{Prob}\left\{\max_{x:\frac{|x-a|}{c}\le t} u(x,t)<\alpha\right\}$$

Since:

$$(34)\qquad \max_{a-ct\le x\le a+ct} u(x,t)=\max_{0\le z\le t}\frac{1}{2c}\int_0^z e^{-s}\beta(e^{2s})ds$$

it results:

$$(35)\quad \text{Prob}\left\{\max_{a-ct\le x\le a+ct} u(x,t)<\alpha\right\}=\text{Prob}\left\{\max_{0\le z\le t} u(a,z)<\alpha\right\}$$

$$\ge\text{Prob}\left\{\max_{0\le z\le t}\beta(e^{2z})<\frac{2c\alpha}{1-e^{-t}}\right\}=\frac{2}{\sqrt{2\pi}}\int_0^{\frac{2c\alpha e^{-t}}{1-e^{-t}}}\exp\left\{-\frac{z^2}{2}\right\}dz$$

Equality (34) can be justified by observing that at time t all points of the interval $(a - ct, a + ct)$ have a non-zero displacement and their domain of influence intersects a segment of the line $x = a$ (see figure 2). Each segment corresponds to the displacement that point a has undergone in some instant of $(0, t)$.

Let us now consider the case where $P(a, s)$ is a gaussian white noise, that is $P(a, s) = \frac{\partial \beta(s)}{\partial s}$ where $\beta(s)$ is as usual a Wiener process.

The field (30) can be written as a Wiener process.

$$(36)\quad u(x, t) = \begin{cases} \dfrac{1}{2c}\displaystyle\int_0^{t - \frac{|x-a|}{c}} d\beta(s) = \dfrac{1}{2c}\beta\left(t - \dfrac{|x - a|}{c}\right) & \text{for } t > \dfrac{|x - a|}{c} \\ 0 & \text{otherwise} \end{cases}$$

Formula (36) shows that a point x vibrates as a delayed Wiener process. From (36) it results:

$$\begin{aligned} \mathrm{E}u(x_1, t_1)u(x_2, t_2) &= \frac{1}{(2c)^2}\mathrm{E}\beta\left(t_1 - \frac{|x_1 - a|}{c}\right)\beta\left(t_2 - \frac{|x_2 - a|}{c}\right) \\ &= \frac{1}{(2c)^2}\min\left\{t_1 - \frac{|x_1 - a|}{c}, t_2 - \frac{|x_2 - a|}{c}\right\} \end{aligned}$$

and:

$$\mathrm{E}u^2(x_1, t_1) = \frac{1}{(2c)^2}\left(t_1 - \frac{|x_1 - a|}{c}\right)$$

If $\min\left\{t - \frac{|x_1 - a|}{c}, t - \frac{|x_2 - a|}{c}\right\} > 0$ we can therefore write:

$$(37)\quad \mathrm{Corr}\{u(x_1, t), u(x_2, t)\} = r(x_1, x_2) = \frac{\min\left\{t - \frac{|x_1 - a|}{c}, t - \frac{|x_2 - a|}{c}\right\}}{\sqrt{\left(t - \frac{|x_1 - a|}{c}\right)\left(t - \frac{|x_2 - a|}{c}\right)}}$$

Let now a point x_1 be fixed. If $\frac{|x_1 - a|}{c} < \frac{|x_2 - a|}{c}$ formula (37) gives:

$$\mathrm{Corr}\{u(x_1, t), u(x_2, t)\} = \left\{\frac{t - \frac{|x_2 - a|}{c}}{t - \frac{|x_1 - a|}{c}}\right\}^{\frac{1}{2}}$$

and if $\frac{|x_1 - a|}{c} > \frac{|x_2 - a|}{c}$

$$\operatorname{Corr}\{u(x_1, t), u(x_2, t)\} = \left\{ \frac{t - \frac{|x_1 - a|}{c}}{t - \frac{|x_2 - a|}{c}} \right\}^{\frac{1}{2}}$$

We evidentiate the above results in figure 3 where the correlation function (37) is depicted when it is assumed that x_1 is a fixed point.

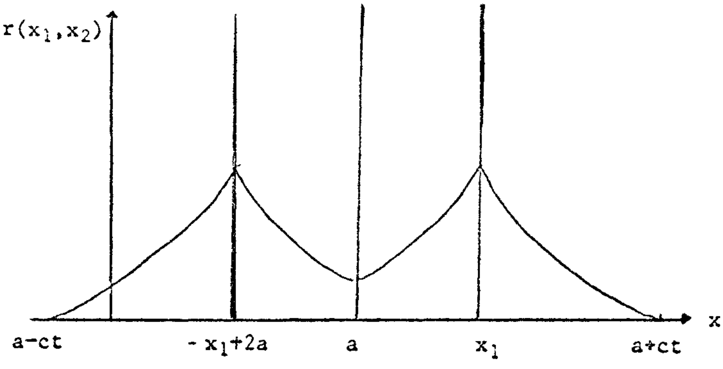

Fig. 3

Figure 3 shows that two symmetrical wave trains are stimulated by the force acting at point a. The stochastic process $u(x, t)$ is therefore symmetric with respect to point a at any time t. Two points, $2|x_1 - a|$ units afar, are therefore perfectly correlated.

If a white noise acts at point a, the representation (36) makes possible further considerations.

Since:

$$\{ \max_{a-ct \leq x \leq a+ct} u(x, t) \} = \left\{ \max_{0 \leq z \leq t} \frac{\beta(z)}{2c} \right\}$$

it can be written that:

$$(38) \quad \operatorname{Prob}\{ \max_{a-ct \leq x \leq a+ct} u(x, t) > \alpha \} = \operatorname{Prob}\left\{ \max_{0 \leq z \leq t} \frac{\beta(z)}{2c} > \alpha \right\}$$
$$= 2 \operatorname{Prob}\left\{ \frac{\beta(t)}{2c} > \alpha \right\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{2c\alpha}{\sqrt{t}}}^{\infty} \exp\left\{ -\frac{w^2}{2} \right\} dw$$

If x is a given point of the string it can be written also:

$$(39)\quad \text{Prob}\left\{ \max_{\frac{|x-a|}{c}\le s\le t} u(x, s) > \alpha \right\} = \text{Prob}\left\{ \max_{0\le s\le t-\frac{|x-a|}{c}} \frac{\beta(s)}{2c} > \alpha \right\}$$

$$= 2\,\text{Prob}\left\{ \beta\left(t - \frac{|x-a|}{c}\right) > 2c\alpha \right\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{2c\alpha}{\sqrt{t-\frac{|x-a|}{c}}}}^{\infty} \exp\left\{ -\frac{w^2}{2} \right\} dw$$

It is possible to analyse the case where a nonstationary force acts at point a as was done in the previous section with nonconcentrated forces. We drop this subject for the sake of brevity.

4. THE FINITE STRING

This section is concerned with vibrations of a finite string of length L stirred by plane white noise.

We are therefore considering the solution of the boundary-value problem:

$$(40)\quad \begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} + F(x, t) \\ u(x, 0) &= \frac{\partial u}{\partial t}(x, 0) = 0 \\ u(0, t) &= u(L, t) = 0 \end{aligned}$$

The string is supposed at rest at time $t = 0$ and fixed at both extremities.

As in section 2 the solution of the boundary-value problem (40) can be written:

$$(41)\quad u(x, t) = \int_0^t \int_0^L G(x, t; z, s) F(z, s)\, dz\, ds$$

By physical analogy with the deterministic case we are lead to consider the following random field when vibrations are forced by plane white noise:

$$(41')\quad u(x, t) = \int_0^t \int_0^L G(x, t; z, s)\, d\beta(z, s)$$

The explicit form of the Green function $G(x, t; z, s)$ is:

$$G(x, t; z, s) = \sum_{n=1}^{\infty} \frac{2}{L\omega_n} \sin \omega_n (t - s) \sin \frac{n\pi z}{L} \sin \frac{n\pi x}{L} \qquad \omega_n = \frac{n\pi c}{L}$$

as an application of the finite Fourier transform to (40) will show (for details see for example [8] page 149).

Since the Green function can be split up by elementary trigonometric formulae into four series of the form $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi w}{L}$ which are the Fourier expansions of line functions of w, the integral and the series in (41′) can be exchanged by theorem 13.2 page 107 of [8]. The process $u(x, t)$ can thus be reset as follows:

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L} \tag{42}$$

where:

$$T_n(t) = \frac{2}{L\omega_n} \int_0^t \int_0^L \sin \omega_n(t-s) \sin \frac{n\pi z}{L} \, d\beta(s, z) \tag{43}$$

We begin our analysis by evaluating the covariances of process (43).

$$\begin{aligned}
(44)\quad & ET_n(t+k)T_m(t) \\
&= \frac{4}{L^2\omega_n\omega_m} E\left\{ \int_0^{t+k} \int_0^L \sin \omega_n(t+k-s) \sin \frac{n\pi z}{L} \, d\beta(s, z) \int_0^t \int_0^L \sin \omega_m(t-y) \right. \\
&\quad \left. \sin \frac{m\pi w}{L} \, d\beta(y, w) \right\} \\
&= \frac{4}{L^2\omega_n\omega_m} \int_0^{t+k} \int_0^t \int_0^L \int_0^L \sin \omega_n(t+k-s) \sin \omega_m(t-y) \sin \frac{n\pi z}{L} \\
&\quad \sin \frac{m\pi w}{L} E\{ d\beta(s, z) d\beta(y, w) \} \\
&= \frac{4}{L^2\omega_n\omega_m} \int_0^t \int_0^L \sin \omega_n(t+k-s) \sin \omega_m(t-s) \sin \frac{n\pi z}{L} \sin \frac{m\pi z}{L} \, dz ds
\end{aligned}$$

The last step in (44) follows from the fact that $E\{ d\beta(s, z) d\beta(y, w) \} = 1$ if $z = w$ and $s = y$ and is zero otherwise.

Since:

$$\int_0^L \sin \frac{n\pi z}{L} \sin \frac{m\pi z}{L} \, dz = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{otherwise} \end{cases}$$

it results:

$$(45)\quad ET_n(t+k)T_m(t) = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{1}{L\omega_n^2}\left\{ t\cos\omega_n k + \dfrac{\sin\omega_n k - \sin\omega_n(2t+k)}{2\omega_n} \right\} & \text{if } m = n \end{cases}$$

From (45) it can be seen that the solution of problem (40) appears as a random series of independent sine waves. The amplitude of each term, that is $T_n(t)$ is a gaussian process with zero mean and covariance function (45)

The variance of $u(x, t)$ is easily seen to be:

$$(46)\quad \operatorname{Var} u(x,t) = Eu^2(x,t) = \sum_{n=1}^{\infty} \frac{2}{L\omega_n^2}\left\{ \int_0^t \sin^2\omega_n(t-s)ds \right\} \sin^2\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \frac{1}{L\omega_n^2}\left\{ t - \frac{\sin 2\omega_n t}{2\omega_n} \right\} \sin^2\frac{n\pi x}{L}$$

Since $\omega_n = \dfrac{cn\pi}{L}$ it results:

$$(47)\quad Eu^2(x,t) = \sum_{n=1}^{\infty} ET_n^2(t)\sin^2\frac{n\pi x}{L} \leq \frac{2t}{L}\sum_{n=1}^{\infty}\left(\frac{L}{cn\pi}\right)^2$$

$$= \frac{2tL}{c^2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{tL}{3c^2} < +\infty$$

A classical result states that a random series of independent r. v.'s with finite variance converges almost surely (see, for example Métivier [*5*], page 193).

When t is fixed the process $u(x, t)$ has therefore continuous sample paths by the so called Billard's theorem (see Kahane [*4*], page 49).

The shape of the vibrating string can therefore be thought of as a continuous curve at any time t. An analogous result is needed for the process $u(x, t)$ when x is fixed.

COROLLARY 2. — When x is fixed the process $u(x, t)$ defined by (42) and (43) has continuous sample paths with probability one.

Proof. — We apply theorem I following the framework of proof of corollary I. We evaluate the probability:

$$(48)\quad \operatorname{Prob}\{\, |u(x,t+h) - u(x,t)| > g(h) \,\} = 2\int_{\frac{g(h)}{\sqrt{\sigma(h)}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{w^2}{2} \right\} dw$$

where:

$$\sigma(h) = E(u(x, t+h) - u(x, t))^2 = E\left\{\sum_{n=1}^{\infty} T_n(t+h)\sin\frac{n\pi x}{L} - \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{L}\right\}^2$$

$$= \sum_{n=1}^{\infty} \sin^2\frac{n\pi x}{L} E(T_n(t+h) - T_n(t))^2$$

$$= \sum_{n=1}^{\infty} \sin^2\frac{n\pi x}{L}\left(\frac{2}{L\omega_n}\right)^2 E\left\{\int_0^L\int_t^{t+h} \sin\omega_n(t-s)\sin\frac{n\pi z}{L}\,d\beta(s, z)\right\}^2$$

It can readily be seen that:

$$E\left\{\int_t^{t+h}\int_0^L \sin\omega_n(t-s)\sin\frac{n\pi z}{L}\,d\beta(s, z)\right\}^2$$

$$= \int_t^{t+h}\int_0^L \sin^2\frac{n\pi z}{L}\sin^2\omega_n(t-s)dsdz = \frac{L}{2}\int_t^{t+h}\sin^2\omega_n(t-s)ds \leq \frac{L}{2}h$$

Therefore:

$$(49)\quad \sigma(h) \leq \frac{2h}{L}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}\sin^2\frac{n\pi x}{L} = \hat{\sigma}_x^2 h \text{ where } \hat{\sigma}_x^2 = \frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}\sin^2\frac{n\pi x}{L} < +\infty$$

Bearing in mind inequality (49) the probability (48) may be bounded as follows:

$$\text{Prob}\{\,|u(x, t+h) - u(x, t)| > g(h)\} \leq \sqrt{\frac{2}{\pi}}\int_{\frac{g(h)}{\hat{\sigma}_x\sqrt{h}}}^{\infty}\exp\left\{-\frac{w^2}{2}\right\}dw$$

$$\leq \sqrt{\frac{2}{\pi}}\frac{\hat{\sigma}_x\sqrt{h}}{g(h)}\exp\left\{-\frac{1}{2}\frac{g^2(h)}{\hat{\sigma}_x^2 h}\right\} = q(h)$$

Choosing $g(h) = |h|^a$ with $0 < a < \frac{1}{2}$ the proof of corollary II follows the same lines of corollary I.

As a consequence of this result we can imagine each point of the string to perform continuous sample paths when its vibrations are stirred by white noise.

We are now in a position to tackle the problem of finding bounds for the following probability

$$\text{Prob}\{\max_{0\leq s\leq t} u(x, s) > \alpha\}$$

We set our first result in theorem II.

THEOREM II. — The probability that a point x of a finite string upcrosses level α at least once during $(0, t)$ has the following upper bound:

(50) $$\text{Prob}\left\{ \max_{0 \leq s \leq t} u(x, s) > \alpha \right\} \leq \frac{2}{\sqrt{2\pi}} \int_{\frac{\alpha}{\bar{\sigma}_x}}^{\infty} \exp\left\{ -\frac{w^2}{2} \right\} dw$$

where:

(51) $$\bar{\sigma}_x^2 = \frac{1}{L} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2} \left\{ t - \frac{\sin 2\omega_n t}{2\omega_n} \right\} \sin^2 \frac{n\pi x}{L}$$

Proof. — We first split up the interval $(0, t)$ into 2^K disjoint subintervals in such a way that the r. v. $T_n(t)$ defined by (43) can be written:

(52) $$T_n(t) = \sum_{r=0}^{2^K-1} \frac{2}{L\omega_n} \int_{rt2^{-K}}^{(r+1)t2^{-K}} \int_0^L \sin \omega_n(t-s) \sin \frac{n\pi z}{L} d\beta(z, s) = \sum_{r=0}^{2^K-1} Y_{r,n}$$

The r. v.'s $Y_{r,n}$ $(r = 0, 1, \ldots, 2^K - 1)$ are clearly independent, zero mean normal variates. If $j = 1, 2, \ldots, 2^K$ it results:

(53) $$T_n(jt2^{-K}) = \sum_{r=0}^{j-1} Y_{r,n} = \sum_{r=0}^{j-1} \frac{2}{L\omega_n} \int_{rt2^{-K}}^{(r+1)t2^{-K}} \int_0^L \sin \omega_n(t-s) \sin \frac{n\pi z}{L} d\beta(z, s)$$
$$= \frac{2}{L\omega_n} \int_0^{jt2^{-K}} \int_0^L \sin \omega_n(t - s) \sin \frac{n\pi z}{L} d\beta(z, s)$$

By means of (52) the process $u(x, t)$, x being a fixed value, can be set in the following form:

(54) $$u(x, t) = \sum_{n=1}^{\infty} \sum_{r=0}^{2^K-1} Y_{r,n} \sin \frac{n\pi x}{L} = \sum_{r=0}^{2^K-1} \left\{ \sum_{n=1}^{\infty} Y_{r,n} \sin \frac{n\pi x}{L} \right\} = \sum_{r=0}^{2^K-1} Z_r$$

It can easily be checked that the r. v.'s Z_r are independent gaussian variates centered at zero, each defined by almost surely convergent series.

Analogously one can write:

$$u(x, jt2^{-K}) = \sum_{n=1}^{\infty} T_n(jt2^{-K}) \sin \frac{n\pi x}{L} = \sum_{r=0}^{j-1} \left\{ \sum_{n=1}^{\infty} Y_{r,n} \sin \frac{n\pi x}{L} \right\}$$
$$= \sum_{r=0}^{j-1} Z_r \qquad j = 1, \ldots, 2^K$$

By Lévy's inequality it can therefore be written:

(55) $$\text{Prob}\left\{ \max_{1 \leq j \leq 2^K} u(x, jt2^{-K}) > \alpha \right\} \leq 2 \,\text{Prob}\left\{ u(x, t) > \alpha \right\}$$

Since inequality (55) holds for whatever large K, we can let $K \to \infty$ thus getting the upper bound (50).

It is easy to find an upper bound to the probability that at time t the displacements of the string exceed a whatever level α.

Since:

$$\operatorname{Var}\{u(x,t)\} \leq \operatorname{Var}\left\{\sum_{n=1}^{\infty} T_n(t)\right\}$$

it results:

$$(56)\quad \operatorname{Prob}\{\max_{0\leq x\leq L} |u(x,t)| > \alpha\} \leq \operatorname{Prob}\left\{\left|\sum_{n=1}^{\infty} T_n(t)\right| > \alpha\right\} = \frac{2}{\sqrt{2\pi}}\int_{\frac{\alpha}{\sigma_x}}^{\infty} \exp\left\{-\frac{1}{2}w^2\right\}dw$$

and where:

$$\sigma_x^2 = \frac{1}{L}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}\left\{t - \frac{\sin 2\omega_n t}{2\omega_n}\right\}$$

Combining the above results one can obtain the following upper bound:

$$(57)\quad \operatorname{Prob}\{\max_{\substack{0\leq x\leq L\\ 0\leq s\leq t}} |u(x,s)| > \alpha\} \leq \operatorname{Prob}\left\{\max_{0\leq s\leq t}\left|\sum_{n=1}^{\infty} T_n(s)\right| > \alpha\right\}$$

$$\leq 2\operatorname{Prob}\left\{\left|\sum_{n=1}^{\infty} T_n(t)\right| > \alpha\right\} = \frac{4}{\sqrt{2\pi}}\int_{\frac{\alpha}{\sigma_x}}^{\infty} \exp\left\{-\frac{1}{2}w^2\right\}dw$$

where, again, $\sigma_x^2 = \frac{1}{L}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2}\left\{t - \frac{\sin 2\omega_n t}{2\omega_n}\right\} < +\infty$

For a deeper insight into the behaviour of the randomly vibrating string we give another representation of process $u(x, t)$, where x, is supposed a fixed value.

Let $T_k = \frac{k\pi}{\omega_1}$ $(k = 1, 2, \ldots)$ be a sequence of times which we call nodal times of vibrations.

We have now our third theorem.

THEOREM 3. — Let T be any nodal time, $\{\Theta_n\}$ a sequence of independent r. v.'s with uniform distribution in $(0, 2\pi)$, $\{A_n(T)\}$ a sequence of i. i. d. r. v.'s having the following Rayleigh density $\phi(s, T)$:

$$(58)\quad \phi(s,T) = \begin{cases} \frac{s}{T}\exp\left\{-\frac{s^2}{2T}\right\} & s > 0 \\ 0 & \text{otherwise} \end{cases}$$

The process:

$$(59)\quad \tilde{u}(x, t + \mathrm{T}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\mathrm{L}\omega_n}} \mathrm{A}_n(\mathrm{T}) \cos(\omega_n t + \Theta_n) \sin \frac{n\pi x}{\mathrm{L}} + \sum_{n=1}^{\infty} \frac{2}{\mathrm{L}\omega_n} \sin \frac{n\pi x}{\mathrm{L}} \int_{\mathrm{T}}^{\mathrm{T}+t} \int_0^{\mathrm{L}} \sin \omega_n(\mathrm{T} + t - s) \sin \frac{n\pi y}{\mathrm{L}} \, d\beta(s, y)$$

is equivalent to $u(x, t + \mathrm{T})$ in the sense that both have the same distributions.

The sequences $\{ \mathrm{A}_n(\mathrm{T}) \}$ and $\{ \Theta_n \}$ are intended to be mutually independent and further, independent from the random series appearing in the second term of (59).

Proof. — Since $\mathrm{A}_n(\mathrm{T}) \cos(\omega_n t + \Theta_n)$ is a gaussian r. v. both series of (59) are gaussian zero-mean processes. To prove equivalency it suffices to show that variances and covariances of $u(x, t + \mathrm{T})$ and $\tilde{u}(x, t + \mathrm{T})$ coincide.

From independency it results:

$$(60)\quad \mathrm{E}\tilde{u}(x, \mathrm{T} + t)\tilde{u}(x, \mathrm{T} + r)$$

$$= \sum_{n=1}^{\infty} \frac{1}{\mathrm{L}\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}} \mathrm{E}(\mathrm{A}_n^2(\mathrm{T}))\mathrm{E}\{ \cos(\omega_n t + \Theta_n) \cos(\omega_n r + \Theta_n) \}$$

$$+ \sum_{n=1}^{\infty} \frac{4}{\mathrm{L}^2\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}} \mathrm{E}\left\{ \int_{\mathrm{T}}^{\mathrm{T}+t} \int_0^{\mathrm{L}} \sin \omega_n(\mathrm{T} + t - s) \sin \frac{n\pi w}{\mathrm{L}} \, d\beta(s, w) \int_{\mathrm{T}}^{\mathrm{T}+r} \int_0^{\mathrm{L}} \sin \omega_n(\mathrm{T} + r - z) \sin \frac{n\pi y}{\mathrm{L}} \, d\beta(z, y) \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\mathrm{L}\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}} (2\mathrm{T}) \frac{1}{2} \cos \omega_n(t - r)$$

$$+ \sum_{n=1}^{\infty} \frac{4}{\mathrm{L}^2\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}} \int_{\mathrm{T}}^{\mathrm{T}+\min\{t,r\}} \int_0^{\mathrm{L}} \sin \omega_n(\mathrm{T} + t - s) \sin \omega_n(\mathrm{T} + r - s) \sin^2 \frac{n\pi y}{\mathrm{L}} \, dyds$$

$$= \sum_{n=1}^{\infty} \frac{\mathrm{T}}{\mathrm{L}\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}} \cos \omega_n(t - r) + \sum_{n=1}^{\infty} \frac{2}{\mathrm{L}\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}} \int_{\mathrm{T}}^{\mathrm{T}+\min\{t,r\}} \sin \omega_n(\mathrm{T} + t - s) \sin \omega_n(\mathrm{T} + r - s) ds$$

Suppose now to speed up calculations that min $\{ t, r \} = t$. By evaluating the last integral it results:

$$(61)\quad E\tilde{u}(x, T + t)\tilde{u}(x, T + r) = \sum_{n=1}^{\infty} \frac{T}{L\omega_n^2} \sin^2 \frac{n\pi x}{L} \cos \omega_n(t - r)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{L\omega_n^2} \sin^2 \frac{n\pi x}{L} \left\{ t \cos \omega_n(t - r) + \frac{\sin \omega_n(r - t) - \sin \omega_n(r + t)}{2\omega_n} \right\}.$$

By simply replacing in (45) $t + k$ with $T + r$ and t by $T + t$ the coincidence of covariances is verified, when the fact that T is a nodal time is considered.

The check that covariances coincide is carried out by simply setting $r = t$ in (61).

The representation (59) shows that at any time $T + t$ the process $u(x, t + T)$ can be thought of as the superposition of two independent processes.

The first one is a series of cosine waves while the second one starts off at the nodal time T. At any given time t (59) shows that one can take the displacement at point x as the result of a series of cosine waves plus an additional random term depending on the time span between the last nodal time before t and t.

It should be pointed out that the motion of point x up to nodal time T is summed up by the amplitude of waves appearing in the first random series of (59).

It is intuitive that since the second term in (59) starts from scratch at each nodal time its contribution to the overall variance of the process thins out as time passes.

Obviously if $t = 0$ formula (59) becomes:

$$(62)\qquad \tilde{u}(x, T) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{L}\omega_n} A_n(T) \cos \Theta_n \sin \frac{n\pi x}{L}$$

The random series (62) is equivalent to the r. v.:

$$(63)\qquad \hat{u}(x, T) = Z(T)\sigma_x$$

with $\sigma_x^2 = \sum_{n=1}^{\infty} \frac{1}{L\omega_n^2} \sin^2 \frac{n\pi x}{L}$ and Z(T) being a gaussian r. v. centered at zero with $EZ^2(T) = T$.

Equivalency must be understood in the sense that $\hat{u}(x, \mathrm{T})$ and $\tilde{u}(x, \mathrm{T})$ have the same distribution as can easily be seen.

From (63) it results that:

(64) $$\operatorname{Prob}\{\max_{0\leq x\leq \mathrm{L}} |u(x, \mathrm{T})| > \alpha\} = \operatorname{Prob}\{\max_{0\leq x\leq \mathrm{L}} |\hat{u}(x, \mathrm{T})| > \alpha\}$$

The function $\sigma_x^2 = \sum_{n=1}^{\infty} \frac{1}{\mathrm{L}\omega_n^2} \sin^2 \frac{n\pi x}{\mathrm{L}}$ is maximum at $x = \frac{\mathrm{L}}{2}$ where it takes the value $\frac{\mathrm{L}}{24c^2}$.

From the above remarks we obtain the exact distribution of $\max_{0\leq x\leq \mathrm{L}} |u(x, \mathrm{T})|$ that is:

(65) $$\operatorname{Prob}\{\max_{0\leq x\leq \mathrm{L}} |u(x, \mathrm{T})| > \alpha\} = \frac{2}{\sqrt{2\pi}} \int_{\alpha c\sqrt{\frac{24}{\mathrm{LT}}}}^{\infty} \exp\left\{-\frac{1}{2}w^2\right\} dw$$

For growing values of the product LT exceedances over level α become more and more likely while ever increasing values of c make excursions more and more rare. This accords both with intuition and results obtained in the previous section in analysing infinitely long strings.

We add now some remarks on vibrations of a finite string stirred by a concentrated white noise.

In this case by setting $\mathrm{F}(z, s) = \delta(z - a)\mathrm{P}(a, s)$ in (41) one obtains:

(66) $$u(x, t) = \int_0^t \mathrm{G}(x, t; a, s)\mathrm{P}(a, s)ds$$

In the case $\mathrm{P}(a, s)$ is a white noise applied at point a:

(67) $$u(x, t) = \int_0^t \mathrm{G}(x, t; a, s)d\beta(s)$$

and the processes $\mathrm{T}_n(t)$ defined by (43) take on the following form:

(68) $$\mathrm{T}_n(t) = \frac{2}{\mathrm{L}\omega_n} \sin \frac{n\pi a}{\mathrm{L}} \int_0^t \sin \omega_n(t - s)d\beta(s)$$

It can easily be verified that the processes $\mathrm{T}_n(t)$ are dependent since their cross-covariances are usually different from zero.

For this reason the investigation of the problem was dropped since the Fourier approach to the vibrations problem involves random series of dependent terms.

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