The Annals of Mathematical Statistics
1972, Vol. 43, No. 6, 1749-1766

RANK SPECTRAL PROCESSES AND TESTS FOR SERIAL DEPENDENCE[1]

BY R. J. BERAN

University of California, Berkeley

Rank analogues of the integrated periodogram spectral process are introduced and used to generate distribution-free tests for independence of a set of random variables. Under simple autoregressive alternatives, the rank spectral process with normal scores yields a test of Kolmogorov-Smirnov type whose local asymptotic efficiency relative to the analogous test based on the integrated periodogram is at least one. Moreover, the same rank test has good local asymptotic efficiency relative to tests based on optimally lagged rank serial correlation coefficients.

1. Introduction. The basic statistical problem studied in this paper is that of testing a set of observations for randomness against alternatives under which the observations are serially dependent. Simple tests that have been proposed for the problem include the classical serial correlation test and its rank analogues (Wald and Wolfowitz [15], Aiyar [1]). While such tests can be effective against alternatives of practical importance, their successful use requires knowledge of the proper lag to be introduced into the serial correlation coefficient. In the absence of such precise information, it is prudent to use tests which have reasonable power over a larger set of alternatives. The development of spectral methods in time series analysis has provided a means of constructing tests that satisfy this requirement.

Let $X_1, X_2, \cdots, X_N$ be the observed random variables and let H_N be the hypothesis of randomness, under which the $\{X_i\}$ are independent identically distributed with common continuous distribution function F. Suppose in addition that $E_{H_N}(X_i) = \mu$, $\mathrm{Var}_{H_N}(X_i) = \sigma^2$, and μ, σ^2 are known. Then, under H_N, the random variables $\{Z_i = \sigma^{-1}(X_i - \mu), 1 \leqq i \leqq N\}$ form a sample from a stationary random sequence with mean 0 and spectral distribution function $G(\lambda) = \lambda/\pi$ on $[0, \pi]$. The integrated periodogram estimate of $G(\lambda)$ is (see Grenander and Rosenblatt [7])

$$G_N(\lambda) = (\pi N)^{-1} \int_0^\lambda |\sum_{j=1}^N Z_j e^{-i\mu j}|^2 \, d\mu = N^{-1} \sum_{j=1}^N \sum_{k=1}^N a_{|j-k|}(\lambda) Z_j Z_k , \tag{1.1}$$

where

$$a_m(\lambda) = \frac{\sin(\lambda m)}{\pi m} \quad m \geqq 1; \qquad = \frac{\lambda}{\pi} \quad m = 0 . \tag{1.2}$$

Received September 13, 1971; revised April 24, 1972.
[1] This research was partially supported by National Science Foundation Grant GP-31091.

Tests for H_N versus serial dependence can therefore be based upon appropriate functionals of the spectral process $N^{\frac{1}{2}}[G_N(\lambda) - \lambda/\pi]$. Under mild conditions, such tests are consistent against stationary alternatives with spectral distribution function differing from λ/π.

The weak convergence in $C[0, \pi]$ of $N^{\frac{1}{2}}[G_N(\lambda) - G(\lambda)]$ to a Gaussian process has been studied under assumptions on the $\{Z_i\}$ more general than independence. Grenander and Rosenblatt [7] treated the case of a linear process for which the eighth moment exists. Ibragimov [10] considered stationary Gaussian processes with square integrable spectral densities. Brillinger [3] proved weak convergence in a topology finer than the uniform topology, assuming stationarity, a mixing condition, and existence of all moments. These results are applicable to the testing problem described above with μ, σ^2 known.

In practice, however, the moments μ, σ^2 are not known. One way of overcoming this difficulty in testing for H_N is to substitute the estimates $\bar{X} = N^{-1} \sum_{i=1}^{N} X_i$, $s^2 = N^{-1} \sum_{i=1}^{N} (X_i - \bar{X})^2$ for μ, σ^2 in $N^{\frac{1}{2}}[G_N(\lambda) - \lambda/\pi]$. Together with a rescaling, this step yields the spectral process

$$(1.3) \qquad V_N(\lambda) = (N^{\frac{1}{2}} s^2)^{-1} \sum\sum_{j \neq k}^{N} c_{|j-k|}(\lambda)(X_j - \bar{X})(X_k - \bar{X}) ,$$

where

$$(1.4) \qquad c_m(\lambda) = \frac{\sin(m\lambda)}{2^{\frac{1}{2}}\pi m} , \qquad m \geqq 1 .$$

Another approach, the one studied in this paper, is to replace the observations by their ranks $(R_{N1}, R_{N2}, \cdots, R_{NN})$ and form a rank analogue of $V_N(\lambda)$:

$$(1.5) \qquad S_N(\lambda) = (N^{\frac{1}{2}} ||\varphi||^2)^{-1} \sum\sum_{j \neq k}^{N} c_{|j-k|}(\lambda) a_N(R_{Nj}, R_{Nk}) .$$

The scores $\{a_N(\cdot, \cdot)\}$ satisfy regularity conditions and $||\varphi||^2$ is a normalizing constant (see Section 2).

Tests for H_N against serial dependence can be based upon functionals of the spectral processes $S_N(\lambda)$ or $V_N(\lambda)$. The tests based upon $S_N(\lambda)$ have the advantage of being distribution-free, making it practical to tabulate their exact critical values for small sample sizes. In addition, our results in this paper suggest that for testing H_N versus autoregressive alternatives, the spectral process $S_N(\lambda)$ with normal scores is at least as efficient asymptotically as is $V_N(\lambda)$. This dominance is established formally for tests based on $\max_\lambda S_N(\lambda)$ over their counterparts based on $\max_\lambda V_N(\lambda)$. Furthermore, it is shown that these particular rank tests fare reasonably well in comparison with asymptotically most powerful rank tests, at least for the low-order autoregressive alternatives of greatest practical interest.

The remainder of this paper is organized as follows. Section 2 describes the principal formal results: the weak convergence, asymptotic power, and asymptotic efficiency theorems which justify the remarks above. Proofs of these results are given in Sections 3 and 4. Section 3 deals with proofs under H_N while

Section 4 treats proofs under simple autoregressive alternatives. Similar methods can be used to study signed-rank spectral processes and rank cross-spectral processes.

2. Principal results. Before stating the main results of the paper, we introduce some notation and assumptions. Let $(\cdot, \cdot)$ and $||\cdot||$ denote, respectively, the inner product and norm in $L_2[0, 1]$. Let $X_1, X_2, \cdots$ be a sequence of real-valued random variables defined on the space and σ-algebra $(\Omega, \mathscr{A})$ and let R_{Ni} denote the rank of X_i among $(X_1, X_2, \cdots, X_N)$. Under H_N, the hypothesis of randomness for N observations, $(X_1, X_2, \cdots, X_N)$ are assumed to be independent identically distributed with common continuous distribution function F.

If $U_i = F(X_i)$ for $1 \leqq i \leqq N$, the random variables $(U_1, U_2, \cdots, U_N)$ are independent and uniformly distributed on $(0, 1)$ under H_N. Corresponding to each $\varphi \in L_2[0, 1]$, a sequence of rank scores $\{a_N^\varphi(\cdot, \cdot)\}$ may be defined by

$$a_N^\varphi(j, k) = E_{H_N}[\varphi(U_1)\varphi(U_2) \mid R_{N1} = j,\ R_{N2} = k]\,, \qquad 1 \leqq j \neq k \leqq N\,. \tag{2.1}$$

The rank spectral process $S_N^\varphi(\lambda)$ associated with these scores is

$$S_N^\varphi(\lambda) = (N^{\frac{1}{2}}||\varphi||^2)^{-1} \sum\sum_{j \neq k}^{N} c_{|j-k|}(\lambda) a_N^\varphi(R_{Nj}, R_{Nk})\,, \tag{2.2}$$

where the $\{c_m(\lambda)\}$ are defined as in (1.4). The sample paths of $S_N^\varphi(\lambda)$ belong to $C[0, \pi]$, the space of all continuous functions whose domain is the interval $[0, \pi]$. The Gaussian process $Z(\lambda)$ defined on $[0, \pi]$, satisfying

$$E[Z(\lambda)] = 0 \tag{2.3}$$

$$E[Z(\lambda)Z(\mu)] = \min\left(\frac{\lambda}{\pi}, \frac{\mu}{\pi}\right) - \frac{\lambda\mu}{\pi^2}\,,$$

and having all its sample paths in $C[0, \pi]$ is called the Brownian bridge on $[0, \pi]$.

THEOREM 2.1. *Suppose that $\varphi \in L_{2+\delta}[0, 1]$ for some $\delta > 0$ and that $(\varphi, 1) = 0$. Then, under H_N as $N \to \infty$, $S_N^\varphi(\lambda)$ converges in distribution in $C[0, \pi]$ to the Brownian bridge.*

Thus, if h is a continuous functional defined on $C[0, \pi]$ and if the conditions of Theorem 2.1 are satisfied, then the asymptotic distribution of $h(S_N^\varphi(\lambda))$ coincides with the distribution of $h(Z(\lambda))$. In particular, the statistics $\max_\lambda S_N^\varphi(\lambda)$, $\max_\lambda |S_N^\varphi(\lambda)|$ and $\pi^{-1} \int_0^\pi [S_N^\varphi(\lambda)]^2\, d\lambda$ have the same limiting distributions under H_N as, respectively, the one-sided and two-sided Kolmogorov-Smirnov statistics and the Cramér-von Mises statistic. Each of these three functionals of $S_N^\varphi(\lambda)$ will provide a test for H_N versus serial dependence. Another expression for the statistic of Cramér-von Mises type is

$$\pi^{-1} \int_0^\pi [S_N^\varphi(\lambda)]^2\, d\lambda = (4\pi^2)^{-1} \sum_{t=1}^{N-1} \gamma_N^2(t)/t^2\,, \tag{2.4}$$

where

$$\gamma_N(t) = (N^{\frac{1}{2}}||\varphi||^2)^{-1} \sum\sum_{|j-k|=t}^{N} a_N^\varphi(j, k)\,. \tag{2.5}$$

In the class of score functions $\{a_N^\varphi(\cdot, \cdot): \varphi \in L_2[0, 1]\}$, two choices stand out. Simplicity suggests taking $\varphi(u) = u - \frac{1}{2}$, which yields

$$a_N^\varphi(j, k) = \frac{jk + \min(j, k)}{(N+1)(N+2)} - \frac{j+k}{2(N+1)} + \tfrac{1}{4}. \tag{2.6}$$

Another choice, $\varphi(u) = \Phi^{-1}(u)$, where Φ is the standardized normal distribution function, is supported by the power considerations to which we turn now.

Let K_N denote the following alternative to the hypothesis of randomness: under K_N, the joint density of $(X_1, X_2, \cdots, X_N)$ is $\prod_{i=1}^N f(x_i - \theta_N x_{i-r})$, where $r \geqq 1$, $\theta_N = N^{-\frac{1}{2}}\theta$, $\theta \neq 0$, $x_0 = x_{-1} = \cdots = x_{1-r} = 0$, and f is the density of F, which is now assumed absolutely continuous. In other words, the observations are generated by a simple rth order autoregressive scheme. If f is absolutely continuous, we may define on (0, 1) the function

$$\varphi_F(u) = -\frac{f' \cdot F^{-1}(u)}{f \cdot F^{-1}(u)}. \tag{2.7}$$

THEOREM 2.2. *Suppose that f is absolutely continuous, that for some $\delta > 0$, $\varphi, \varphi_F, F^{-1} \in L_{2+\delta}[0, 1]$, and that $(\varphi, 1) = 0$. Let*

$$\mu_\theta(\lambda) = \left[\frac{2^{\frac{1}{2}}\theta \sin(r\lambda)}{\pi r||\varphi||^2}\right] (\varphi, F^{-1})(\phi, \phi_F). \tag{2.8}$$

Then under K_N as $N \to \infty$, $S_N^\varphi(\lambda) - \mu_\theta(\lambda)$ converges in distribution in $C[0, \pi]$ to the Brownian bridge.

Let h be a continuous functional on $C[0, \pi]$ and let $D(\alpha)$ be a Borel set for which $P[h(Z(\lambda)) \in D(\alpha)] = \alpha$. The test which rejects H_N if $h(S_N^\varphi(\lambda)) \in D(\alpha)$ has asymptotic size α when Theorem 2.1 applies. The asymptotic power of this test under K_N is defined to be

$$\beta(\alpha, \theta, f) = \lim_{N\to\infty} P_{K_N}[h(S_N^\varphi(\lambda)) \in D(\alpha)]. \tag{2.9}$$

If the assumptions of Theorem 2.2 are satisfied,

$$\beta(\alpha, \theta, f) = P[h(Z(\lambda) + \mu_\theta(\lambda)) \in D(\alpha)]. \tag{2.10}$$

The evaluation of (2.10) is very complex in practice. A method described by Hájek and Šidák [8] yields $\beta_s'(\alpha, 0, f)$, the derivative at $\theta = 0$ of the asymptotic power of the test which rejects H_N whenever $\max_\lambda S_N^\varphi(\lambda) \geqq (-2^{-1}\log\alpha)^{\frac{1}{2}}$. Since $P[\max_\lambda Z(\lambda) \geqq x] = \exp(-2x^2)$ for all $x \geqq 0$, this test has asymptotic size α. Higher derivatives and a Taylor expansion for the asymptotic power can be obtained using results by Anděl [2].

THEOREM 2.3. *Suppose that f is absolutely continuous, that for some $\delta > 0$, $\varphi, \varphi_F, F^{-1} \in L_{2+\delta}[0, 1]$, and that $(\varphi, 1) = 0$. Let*

$$\tau = (\varphi, F^{-1})(\varphi, \varphi_F)||\varphi||^{-2} \tag{2.11}$$

and

$$\psi(\alpha, t) = 2\Phi[(-2^{-1}\log\alpha)^{\frac{1}{2}}(1 - 2t)[t(1-t)]^{-\frac{1}{2}}] - 1. \tag{2.12}$$

Then

$$\beta_s'(\alpha, 0, f) = 2\tau\alpha(-\log \alpha)^{\frac{1}{2}} \textstyle\int_0^1 \cos(\pi r t)\psi(\alpha, t)\, dt \qquad \textit{if } r \textit{ is odd} \tag{2.13}$$
$$= 0 \qquad \textit{if } r \textit{ is even.}$$

The vanishing of $\beta_s'(\alpha, 0, f)$ when r is even reflects the fact that for such r, the test with critical region $\{\max_\lambda S_N^\varphi(\lambda) \geqq (-2^{-1}\log \alpha)^{\frac{1}{2}}\}$ has reasonable power against alternatives $\theta < 0$ as well as $\theta > 0$. (This may be checked heuristically by considering the spectral distribution function of a stationary rth order autoregressive process.) Thus, to characterize the local behavior of $\beta_s(\alpha, \theta, f)$ for even r, it would be necessary to evaluate the second derivative at $\theta = 0$.

Suppose $T_N^{(1)}$, $T_N^{(2)}$ are two test statistics which generate one-sided tests for H_N versus K_N with asymptotic powers $\beta_1(\alpha, \theta, f)$, $\beta_2(\alpha, \theta, f)$ respectively. If both $\beta_1'(\alpha, 0, f)$ and $\beta_2'(\alpha, 0, f)$ are not zero, the local asymptotic efficiency of $T_N^{(1)}$ relative to $T_N^{(2)}$ at size α is defined to be

$$e(\alpha; T_N^{(1)}, T_N^{(2)}) = \left[\frac{\beta_1'(\alpha, 0, f)}{\beta_2'(\alpha, 0, f)}\right]^2. \tag{2.14}$$

This concept, introduced in [8], is compatible with the notion of Pitman efficiency for one-sided tests in the following sense: if the limiting distributions of $T_N^{(1)}$, $T_N^{(2)}$ are normal under H_N and under K_N, then $e(\alpha; T_N^{(1)}, T_N^{(2)})$ equals the Pitman efficiency for all $\alpha \in (0, 1)$.

Theorem 2.3 gives a way of calculating the local asymptotic efficiency of the test with critical region $\{\max_\lambda S_N^\varphi(\lambda) \geqq (-2^{-1}\log \alpha)^{\frac{1}{2}}\}$ relative to other tests for H_N versus K_N, provided r is odd. Two competitors of interest are the test with critical region $\{\max_\lambda V_N(\lambda) \geqq (-2^{-1}\log \alpha)^{\frac{1}{2}}\}$ and the test with critical region $\{W_N \geqq \Phi^{-1}(1-\alpha)\}$, where

$$W_N = (N^{\frac{1}{2}}\|\varphi_F\|\,\|F^{-1}\|)^{-1} \sum_{k-j=r}^{N}\sum b_N^F(R_{Nj}, R_{Nk}) \tag{2.15}$$

and

$$b_N^F(j, k) = E_{H_N}[F^{-1}(U_1)\varphi_F(U_2) \mid R_{N1} = j, R_{N2} = k], \quad 1 \leqq j \neq k \leqq N. \tag{2.16}$$

The latter rank test is asymptotically most powerful at level α for H_N versus K_N when $\theta > 0$.

Analogues for $V_N(\lambda)$ of Theorems 2.1 through 2.3 may be obtained by replacing φ with F^{-1} in all assumptions and conclusions except $(\varphi, 1) = 0$, and by noting that $(\varphi_F, F^{-1}) = 1$. Thus, the asymptotic power $\beta_V(\alpha, 0, f)$ of the test with critical region $\{\max_\lambda V_N(\lambda) \geqq (-2^{-1}\log \alpha)^{\frac{1}{2}}\}$ satisfies

$$\beta_V'(\alpha, 0, f) = 2\alpha(-\log \alpha)^{\frac{1}{2}} \textstyle\int_0^1 \cos(\pi r t)\psi(\alpha, t)\, dt \qquad \text{if } r \text{ is odd} \tag{2.17}$$
$$= 0 \qquad \text{if } r \text{ is even.}$$

On the other hand, the asymptotic power of the test with critical region $\{W_N \geqq \Phi^{-1}(1-\alpha)\}$ is

$$\beta_W(\alpha, \theta, f) = 1 - \Phi[\Phi^{-1}(1-\alpha) - \theta\|\varphi_F\|\,\|F^{-1}\|]. \tag{2.18}$$

THEOREM 2.4. *Suppose that f is absolutely continuous, that for some $\delta > 0$, $\varphi, \varphi_F, F^{-1} \in L_{2+\delta}[0, 1]$, that $(\varphi, 1) = 0$, and that r is odd. Then*

$$e(\alpha; \max_\lambda S_N^\varphi(\lambda), \max_\lambda V_N(\lambda)) = [(\varphi, \varphi_F)(\varphi, F^{-1})||\varphi||^{-2}]^2 \tag{2.19}$$

and

$$\begin{aligned}\min_F e(\alpha; \max_\lambda S_N^\varphi(\lambda), \max_\lambda V_N(\lambda)) &= \frac{9\pi^4}{1024} \quad \text{if} \quad \varphi(u) = u - \tfrac{1}{2} \\ &= 1 \quad \text{if} \quad \varphi(u) = \Phi^{-1}(u)\,.\end{aligned} \tag{2.20}$$

The second value in (2.20) is attained only if F is normal. Moreover

$$\begin{aligned}&e(\alpha; \max_\lambda S_N^\varphi(\lambda), W_N) \\ &\quad = 8\pi\rho_1^2\rho_2^2\alpha^2(-\log\alpha)\exp\{[\Phi^{-1}(1-\alpha)]^2\} \textstyle\int_1^0 \cos(\pi rt)\psi(\alpha, t)\, dt\,,\end{aligned} \tag{2.21}$$

where

$$\rho_1 = (\varphi, F^{-1})||\varphi||^{-1}||F^{-1}||^{-1}\,, \qquad \rho_2 = (\varphi, \varphi_F)||\varphi||^{-1}||\varphi_F||^{-1}\,.$$

Also

$$\lim_{\alpha\to 0} e(\alpha; \max_\lambda S_N^\varphi(\lambda), W_N) = (8/\pi^2 r^2)\rho_1^2\rho_2^2\,. \tag{2.22}$$

Let

$$J_N = (N^{\frac{1}{2}}||\varphi||^2)^{-1} \sum_{k-j=r}^{N}\sum a_N^\varphi(R_{Nj}, R_{Nk})\,,$$

with $a_N^\varphi(\cdot, \cdot)$ defined by (2.1). The asymptotic efficiency of the test with critical region $\{J_N \geqq \Phi^{-1}(1-\alpha)\}$ relative to the analogous test based on W_N is $\rho_1^2\rho_2^2$. Thus, the factor $\rho_1^2\rho_2^2$ in (2.22) represents the efficiency loss due to choice of score function $a_N^\varphi(\cdot, \cdot)$. On the other hand, the factor $8/\pi^2 r^2$ in (2.22) is attributable to using a test based on $\max_\lambda S_N^\varphi(\lambda)$ in place of one based on J_N. In effect, this factor is the cost of not knowing the parameter r.

It appears likely that a rate of efficiency loss proportional to r^{-2} will occur also for tests based on other functionals of $S_N^\varphi(\lambda)$ or $V_N(\lambda)$. (Equation (2.4) is particularly suggestive in this respect.) The effect of this loss in practice may not be severe, since serial dependence in empirical time series often can be fitted with an autoregressive model that has some nonvanishing coefficients for low order lags.

It is interesting to contrast asymptotic power under K_N with that achieved under contiguous regression alternatives. Let K_N' be an alternative under which $X_1, X_2, \cdots, X_N$ are independent random variables with joint density $\prod_{i=1}^N f(x_i - c_{Ni})$, where $(c_{N1}, c_{N2}, \cdots, c_{NN})$ is a vector of constants. Let $\bar{c}_N = N^{-1}\sum_{i=1}^N c_{Ni}$. If $\lim_{N\to\infty}\max_{1\leq i\leq N}(c_{Ni} - \bar{c}_N)^2 = 0$, $\lim_{N\to\infty}\sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 = b^2 < \infty$, f is absolutely continuous, $\varphi, \varphi_F \in L_{2+\delta}[0, 1]$ for some $\delta > 0$, and $(\varphi, 1) = 0$, then under K_N' as $N \to \infty$, $S_N^\varphi(\lambda)$ converges in distribution to the Brownian bridge. Consequently, any test for H_N based on $S_N^\varphi(\lambda)$ has asymptotic power against K_N' that equals the asymptotic size. If φ is replaced by F^{-1} in the assumptions above, excluding $(\varphi, 1) = 0$, the same conclusion extends to $V_N(\lambda)$. This phenomenon is just the reverse of the insensitivity that linear rank tests for H_N versus K_N' display toward the contiguous autoregressive alternatives K_N when $(F^{-1}, 1) = 0$ (see Aiyar [1]).

THEOREM 2.5. *Suppose that* $\varphi \in L_2[0, 1]$, $(\varphi, 1) = 0$, $a_N^\varphi(\cdot, \cdot)$ *is defined as in* (2.1), *and* $a_N(\cdot, \cdot)$ *is another score function which satisfies*

$$\lim_{N\to\infty} N^{-\frac{3}{2}} \log(N) \sum\sum_{j\neq k}^{N} a_N(j, k) = 0 \tag{2.23}$$

$$\lim_{N\to\infty} N^{-2} (\log(N))^2 \sum\sum_{j\neq k}^{N} [a_N(j, k) - \bar{a}_N - a_N^\varphi(j, k)]^2 = 0\,, \tag{2.24}$$

where

$$\bar{a}_N = [N(N-1)]^{-1} \sum\sum_{j\neq k}^{N} a_N(j, k)\,. \tag{2.25}$$

Then Theorems 2.1 *through* 2.4 *apply also to the spectral process*

$$S_N(\lambda) = (N^{\frac{1}{2}}||\varphi||^2)^{-1} \sum\sum_{j\neq k}^{N} c_{|j-k|}(\lambda) a_N(R_{Nj}, R_{Nk})\,. \tag{2.26}$$

It may be noted that both (2.23) and (2.24) are implied by

$$\lim N^{-1}(\log(N))^2 \sum\sum_{j\neq k}^{N} [a_N(j, k) - a_N^\varphi(j, k)]^2 = 0\,. \tag{2.27}$$

This theorem can sometimes be used to replace the scores $a_N^\varphi(\cdot, \cdot)$ by more convenient ones. For example, if $\varphi(u) = u - \frac{1}{2}$, the scores

$$a_N(j, k) = \left(\frac{j}{N+1} - \frac{1}{2}\right)\left(\frac{k}{N+1} - \frac{1}{2}\right) \tag{2.28}$$

can be used in place of the corresponding $\{a_N^\varphi(\cdot, \cdot)\}$, given in (2.6).

3. Asymptotics under H_N. This section contains a number of ancillary results as well as a proof of Theorem 2.1.

LEMMA 3.1. *The following inequalities hold for all* $\lambda, \mu \in [0, \pi]$.

$$\sum_{t=1}^{\infty} \frac{\sin(\lambda t)\sin(\mu t)}{t^2} = \tfrac{1}{2}[\pi \min(\lambda, \mu) - \lambda\mu]\,, \tag{3.1}$$

$$N^{-1} \sum\sum_{j\neq k}^{N} \frac{\sin^2[\lambda(k-j)]}{(k-j)^2} \leqq \frac{\pi^2}{4}\,, \tag{3.2}$$

$$\left|\sum_{k\neq j}^{N} \frac{\sin[\lambda(k-j)]}{(k-j)}\right| \leqq 2(\pi + 1)\,, \qquad 1 \leqq j \leqq N\,. \tag{3.3}$$

PROOF. Equation (3.1) is well known (Bromwich [5]) and implies (3.2). Inequality (3.3) is a consequence of the bound

$$\left|\sum_{t=1}^{n} \frac{\sin(\lambda t)}{t}\right| \leqq \pi + 1\,, \tag{3.4}$$

which holds for all integer $n \geqq 1$, all $\lambda \in [0, \pi]$. This bound can be established by using the Fejér formula for Cesàro sums.

The next lemma is a variant of Lemma A1 in Jogdeo [11].

LEMMA 3.2. (Jogdeo) *Let the* $\{b_N(\cdot, \cdot)\}$ *be any scores and let the* $\{d_{jk}\}$ *be constants satisfying*

$$\sum_{j \neq k}^{N}\sum d_{jk}^2 < C_1 \tag{3.5}$$

$$\textstyle\sum_{j=1}^N (\sum_{k\neq j}^N d_{jk})^2 < C_2\,, \qquad \sum_{k=1}^N (\sum_{j\neq k}^N d_{jk})^2 < C_3\,, \tag{3.6}$$

where C_1, C_2, C_3 *are constants not depending on* N. *Let*

$$\bar{d} = [N(N-1)]^{-1} \sum_{j\neq k}^{N}\sum d_{jk}\,.$$

Then there exists a constant C_4, *depending only on* C_1, C_2, C_3, *such that*

$$E_{H_N}\Big[\sum_{j=k}^{N}\sum (d_{jk} - \bar{d}) b_N(R_{Nj}, R_{Nk})\Big]^2 \leqq C_4 E_{H_N}[b_N^2(R_{N1}, R_{N2})]\,. \tag{3.7}$$

PROOF. Because of (3.5), $|\bar{d}| \leqq C_1^{\frac{1}{2}}$. It is easily verified that (3.5) and (3.6) imply similar bounds in terms of the constants $\{d_{jk} - \bar{d}\}$, whereupon the result follows from the first part of the proof of Lemma A1 in [11].

LEMMA 3.3. *Suppose* $\varphi \in L_2[0, 1]$, $(\varphi, 1) = 0$, *and* $a_N^\varphi(\cdot, \cdot)$ *is defined as in* (2.1). *Then*

$$E_{H_N}[a_N^\varphi(R_{N1}, R_{N2})] = 0\,, \tag{3.8}$$

$$\lim_{N\to\infty} E_{H_N}[a_N^\varphi(R_{N1}, R_{N2}) - \varphi(U_1)\varphi(U_2)]^2 = 0\,. \tag{3.9}$$

PROOF. Let $\mathscr{F}_N \subset \mathscr{A}$ be the σ-algebra generated by $(R_{N1}, R_{N2}, \cdots, R_{NN})$ and let $\mathscr{F}_\infty \subset \mathscr{A}$ be the smallest σ-algebra containing $\bigcup_{N=1}^\infty \mathscr{A}_N$. Evidently

$$a_N^\varphi(R_{N1}, R_{N2}) = E_{H_N}[\varphi(U_1)\varphi(U_2) \,|\, \mathscr{F}_N]\,, \tag{3.10}$$

so that (3.8) is immediate.

By V, T18 of Meyer [12], $\{a_N^\varphi(R_{N1}, R_{N2}), \mathscr{F}_N;\, N \geqq 1\}$ is a uniformly integrable martingale. Moreover there exists an essentially unique $\mathscr{F}_\infty$ measurable random variable Y such that $a_N^\varphi(R_{N1}, R_{N2}) = E_{H_N}[Y \,|\, \mathscr{F}_N]$ and $\lim_{N\to\infty} a_N(R_{N1}, R_{N2}) = Y$ a.s. Since U_1, U_2 may be represented as a.s. limits of $\mathscr{F}_\infty$ measurable random variables (see [8] page 157), it follows that

$$\lim_{N\to\infty} a_N^\varphi(R_{N1}, R_{N2}) = \varphi(U_1)\varphi(U_2) \quad \text{a.s.} \tag{3.11}$$

By Jensen's inequality

$$[a_N^\varphi(R_{N1}, R_{N2})]^2 \leqq E_{H_N}[\varphi^2(U_1)\varphi^2(U_2) \,|\, \mathscr{F}_N] \quad \text{a.s.,} \tag{3.12}$$

which implies uniform integrability of the random variables $\{[a_N^\varphi(R_{N1}, R_{N2})]^2\}$. Hence (3.11) implies (3.9).

The next lemma is due to Whittle [16].

LEMMA 3.4. (Whittle) *Let $Z_1, Z_2, \cdots, Z_N$ be independent identically distributed random variables, each assuming the values ± 1 with respective probabilities $\frac{1}{2}$. Then for any set of constants $\{b_j\}$ and any $s \geqq 2$,*

$$E|\sum_{j=1}^N b_j Z_j|^s \leqq C(s)[\sum_{j=1}^N b_j^2]^{s/2}, \tag{3.13}$$

where

$$C(s) = 2^{s/2}\pi^{-\frac{1}{2}}\Gamma\left(\frac{s+1}{2}\right). \tag{3.14}$$

A useful bound on the moments of a quadratic form in independent identically distributed random variables that has no diagonal terms is given by the following lemma. It can be proved by simple modification of Whittle's [16] argument for his Theorem 2.

LEMMA 3.5. *Let $Z_1, Z_2, \cdots, Z_N$ be independent random variables, let the $\{d_{jk}\}$ be constants, and let*

$$A = \sum_{j \neq k}^{N}\sum d_{jk} Z_j Z_k .$$

Suppose that $s \geqq 2$ and $\mu_j(s) = E|Z_j|^s < \infty$ for $1 \leqq j \leqq N$. Then

$$E|A - E(A)|^s \leqq 2^{3s}C^2(s)\left[\sum_{j \neq k}^{N}\sum d_{jk}^2 \mu_j^{2/s}(s)\mu_k^{2/s}(s)\right]^{s/2}. \tag{3.15}$$

We establish next a central limit theorem for quadratic forms of this type. A related theorem for quadratic forms with nonvanishing diagonal terms was proved by Whittle [17], using a different method. The absence of diagonal terms reduces the moment conditions needed in the theorem.

THEOREM 3.1. *Let $Z_1, Z_2, \cdots$ be a sequence of independent identically distributed random variables defined on $(\Omega, \mathscr{A})$. Suppose that $E(Z_j) = 0$, $E(Z_j^2) = 1$, and that for some $\delta \in (0, 2]$, $E|Z_j|^{2+\delta} < \infty$. Let the $\{d_m\}$ be constants satisfying $\sum_{m=1}^\infty d_m^2 < \infty$. Then the distribution of*

$$A_N = \sum_{j \neq k}^{N}\sum d_{|j-k|} Z_j Z_k \tag{3.16}$$

is asymptotically normal $(0, N\gamma^2)$ with

$$\gamma^2 = 4 \sum_{m=1}^\infty d_m^2 . \tag{3.17}$$

PROOF. Let $\mathscr{G}_i \subset \mathscr{A}$ be the σ-algebra generated by $(A_1, A_2, \cdots, A_i)$. It may be verified, as in Varberg [14], that $\{A_i, \mathscr{G}_i; i \geqq 1\}$ is a martingale. Let

$$\begin{aligned} Y_1 &= A_1 = 0, \\ Y_i &= A_i - A_{i-1} = 2Z_i \sum_{j<i} d_{|i-j|} Z_j, \qquad i \geqq 2, \end{aligned} \tag{3.18}$$

and let $\gamma_N^2 = \sum_{i=1}^N E(Y_i^2)$. To establish the asymptotic normality of the martingale $\{A_i, \mathscr{G}_i\}$, it is sufficient to show (see Heyde and Brown [9]) that for some $\delta \in (0, 2]$,

$$\lim_{N\to\infty} \gamma_N^{-2-\delta} \sum_{i=1}^N E|Y_i|^{2+\delta} = 0 \tag{3.19}$$

and

$$\lim_{N\to\infty} E|\gamma_N^{-2} \sum_{i=1}^N Y_i^2 - 1|^{1+\delta/2} = 0 . \tag{3.20}$$

For notational convenience, set $s = 1 + \delta/2$ in the remainder of the proof. Since

$$\gamma_N^2 = 4N \sum_{m=1}^{N-1} d_m^2(1 - m/N) , \tag{3.21}$$

it follows from (3.17) that

$$\lim_{N\to\infty} N\gamma^2/\gamma_N^2 = 1 . \tag{3.22}$$

Applying Lemma 3.5, we see that for each $i \geqq 1$,

$$E|Y_i|^{2s} \leqq 2^{8s}C^2(2s)\mu^2(2s)[\sum_{m=1}^{\infty} d_m^2]^s = K_1 . \tag{3.23}$$

Consequently,

$$\gamma_N^{-2s} \sum_{i=1}^N E|Y_i|^{2s} \leqq K_1 N\gamma_N^{-2s} , \tag{3.24}$$

which yields (3.19) because of (3.22).

To prove (3.20), let $(V_1, V_2, \cdots, V_N)$ be an independent copy of $(Y_1, Y_2, \cdots, Y_N)$. By a well-known symmetrization inequality (see Rosén [13]),

$$\begin{aligned} E|\textstyle\sum_{i=1}^N Y_i^2 - E(\sum_{i=1}^N Y_i^2)|^s &\leqq E|\textstyle\sum_{i=1}^N (Y_i^2 - V_i^2)|^s \\ &= E|\textstyle\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^s . \end{aligned} \tag{3.25}$$

Let $\mathscr{B}_N \subset \mathscr{A}$ denote the σ-algebra generated by $(|Y_1 - V_1|, \cdots, |Y_N - V_N|, Y_1 + V_1, \cdots, Y_N + V_N)$. Since the conditional distribution of $(Y_1 - V_1, \cdots, Y_N - V_N)$ given $(Y_1 + V_1, \cdots, Y_N + V_N)$ is the same as that of $(V_1 - Y_1, \cdots, V_N - Y_N)$ given $(Y_1 + V_1, \cdots, Y_N + V_N)$, we may use Lemma 3.4 to justify the second inequality below

$$\begin{aligned} &E[|\textstyle\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^s \,|\, \mathscr{B}_N] \\ &\quad \leqq \{E[|\textstyle\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^{2s} \,|\, \mathscr{B}_N\}^{\frac{1}{2}} \\ &\quad \leqq C^{\frac{1}{2}}(2s)[\textstyle\sum_{i=1}^N |Y_i - V_i|^2|Y_i + V_i|^2]^{s/2} . \end{aligned} \tag{3.26}$$

From (3.26) and the c_r inequality, there follows

$$\begin{aligned} E|\textstyle\sum_{i=1}^N (Y_i - V_i)(Y_i + V_i)|^s &\leqq C^{\frac{1}{2}}(2s)E[\textstyle\sum_{i=1}^N |Y_i - V_i|^2|Y_i + V_i|^2]^{s/2} \\ &\leqq C^{\frac{1}{2}}(2s) \textstyle\sum_{i=1}^N E|Y_i^2 - V_i^2|^s . \end{aligned} \tag{3.27}$$

On the other hand, from (3.23)

$$E|Y_i^2 - V_i^2|^s \leqq 2^s K_1 = K_2 . \tag{3.28}$$

Combining (3.25), (3.27) and (3.28) shows that

$$E|\textstyle\sum_{i=1}^N Y_i^2 - E(\sum_{i=1}^N Y_i^2)|^s \leqq NK_3 . \tag{3.29}$$

From this and (3.22) follows (3.20) and hence the asymptotic normality. This completes the proof.

PROOF OF THEOREM 2.1. We prove first the weak convergence of all finite-dimensional distributions, then tightness. An arbitrary linear combination $\sum_{i=1}^{P} \alpha_i S_N^{\varphi}(\lambda_i)$ may also be written as

$$\sum_{i=1}^{P} \alpha_i S_N^{\varphi}(\lambda_i) = N^{-\frac{1}{2}} \sum_{j \neq k}^{N}\sum d_{|j-k|} a_N^{\varphi}(R_{Nj}, R_{Nk}) , \tag{3.30}$$

with

$$d_m = \sum_{i=1}^{P} \frac{\alpha_i \sin(m\lambda_i)}{2^{\frac{1}{2}}\pi m||\varphi||^2} . \tag{3.31}$$

The Cauchy-Schwarz inequality and (3.2) yield

$$N^{-1} \sum_{j \neq k}^{N}\sum d_{|j-k|}^2 \leqq K_4 , \tag{3.32}$$

while another simple argument using (3.3) gives

$$N^{-1} \sum_{j=1}^{N} (\sum_{k \neq j}^{N} d_{|j-k|})^2 = N^{-1} \sum_{k=1}^{N} (\sum_{j \neq k}^{N} d_{|j-k|})^2 \leqq K_5 . \tag{3.33}$$

Let $T_N(\lambda)$ be defined as

$$T_N(\lambda) = (N^{\frac{1}{2}}||\varphi||^2)^{-1} \sum_{j \neq k}^{N}\sum c_{|j-k|}(\lambda)\varphi(U_j)\varphi(U_k) , \tag{3.34}$$

so that

$$\sum_{i=1}^{P} \alpha_i T_N(\lambda_i) = N^{-\frac{1}{2}} \sum_{j \neq k}^{N}\sum d_{|j-k|}\varphi(U_j)\varphi(U_k) . \tag{3.35}$$

Let T_{N1} be defined as

$$T_{N1} = N^{-\frac{1}{2}} \sum_{j \neq k}^{N}\sum (d_{|j-k|} - \bar{d})\varphi(U_j)\varphi(U_k) , \tag{3.36}$$

where

$$\bar{d} = [N(N-1)]^{-1} \sum_{j \neq k}^{N}\sum d_{|j-k|} . \tag{3.37}$$

If $\mathbf{U}^{(\cdot)}$ denotes the order statistics of $(U_1, U_2, \cdots, U_N)$, then (3.32), (3.33), Lemma 3.2, and (3.8) imply

$$\begin{aligned} &E_{H_N}[(\sum_{i=1}^{N} \alpha_i S_N^{\varphi}(\lambda_i) - T_{N1})^2 \,|\, \mathbf{U}^{(\cdot)} = u^{(\cdot)}] \\ &= E_{H_N}\Big[N^{-\frac{1}{2}} \sum_{j \neq k}^{N}\sum (d_{jk} - \bar{d})[a_N^{\varphi}(R_{Nj}, R_{Nk}) - \varphi(u^{(R_{Nj})})\varphi(u^{(R_{Nk})})]\Big]^2 \\ &\leqq K_6 E_{H_N}[(a_N^{\varphi}(R_{N1}, R_{N2}) - \varphi(U_1)\varphi(U_2))^2 \,|\, \mathbf{U}^{(\cdot)} = \mathbf{u}^{(\cdot)}] . \end{aligned} \tag{3.38}$$

Hence (3.9) and (3.38) yield

$$\lim_{N\to\infty} E_{H_N}[\sum_{i=1}^{P} \alpha_i S_N^{\varphi}(\lambda_i) - T_{N1}]^2 = 0 . \tag{3.39}$$

On the other hand, it follows from (3.3) that $|\bar{d}| \leqq K_7(N-1)^{-1}$. Therefore

$$\begin{aligned} E_{H_N}[T_{N1} - \sum_{i=1}^{P} \alpha_i T_N(\lambda_i)]^2 &= (N-1)||\varphi||^4\bar{d}^2 \\ &\leqq K_8(N-1)^{-1} . \end{aligned} \tag{3.40}$$

From (3.39) and (3.40) we conclude that

$$\lim_{N\to\infty} E_{H_N}[\sum_{i=1}^{P} \alpha_i S_N{}^{\varphi}(\lambda_i) - \sum_{i=1}^{P} \alpha_i T_N(\lambda_i)]^2 = 0 . \tag{3.41}$$

Now (3.1) implies that

$$\sum_{m=1}^{\infty} d_m{}^2 = (4||\varphi||^4)^{-1} \sum_{i=1}^{P} \sum_{j=1}^{P} \alpha_i \alpha_j [\min(\lambda_i/\pi, \lambda_j/\pi) - \lambda_i \lambda_j/\pi^2] . \tag{3.42}$$

Therefore, by Theorem 3.1, $\sum_{i=1}^{P} \alpha_i T_N(\lambda_i)$ is asymptotically normal $(0, 4||\varphi||^4 \sum_{m=1}^{\infty} d_m{}^2)$. From (3.42) and (3.41), we find that the asymptotic distribution of $\sum_{i=1}^{N} \alpha_i S_N{}^{\varphi}(\lambda_i)$ coincides with the distribution of $\sum_{i=1}^{N} \alpha_i Z(\lambda_i)$.

To prove tightness of the distributions of the $\{S_N{}^{\varphi}(\lambda)\}$, let $\mathscr{F}_N \subset \mathscr{A}$ denote the σ-algebra generated by $(R_{N1}, R_{N2}, \cdots, R_{NN})$. Since $S_N{}^{\varphi}(\lambda) = E_{H_N}[T_N \mid \mathscr{F}_N]$ by (2.1), Jensen's inequality on conditional expectations implies that for all $\lambda, \mu \in [0, \pi]$

$$E_{H_N}|S_N{}^{\varphi}(\lambda) - S_N{}^{\varphi}(\mu)|^{2s} \leqq E_{H_N}|T_N(\lambda) - T_N(\mu)|^{2s} , \tag{3.43}$$

where $s = 1 + \delta/2$. By Lemma 3.5 and (3.1),

$$\begin{aligned} E_{H_N}|T_N(\lambda) - T_N(\mu)|^{2s} &\leqq K_9 N^{-s}\Big[\sum_{j\neq k}^{N}\sum (c_{|j-k|}(\lambda) - c_{|j-k|}(\mu))^2\Big]^s \\ &\leqq 2^s K_9[\sum_{m=1}^{\infty} (c_m(\lambda) - c_m(\mu))^2]^s \\ &\leqq K_{10}|\lambda - \mu|^s . \end{aligned} \tag{3.44}$$

Tightness follows from (3.43) and (3.44). This completes the proof (see Billingsley [3]).

REMARK 3.1. Under H_N and the assumption that for some $\delta > 0$, $F^{-1} \in L_{2+\delta}[0, 1]$, the spectral process $V_N(\lambda)$ defined in (1.3) also converges in $C[0, \pi]$ to the Brownian bridge. To verify this, assume also without loss in generality, that $E(X_i) = 0$, $\mathrm{Var}(X_i) = 1$. Let

$$V_{N1}(\lambda) = (N^{\frac{1}{2}} s_N{}^2)^{-1} \sum_{j\neq k}^{N}\sum c_{|j-k|}(\lambda) X_j X_k \tag{3.45}$$

where

$$s_N{}^2 = N^{-1} \sum_{i=1}^{N} (X_i - \bar{X})^2 . \tag{3.46}$$

Since $s_N{}^2 \to 1$ a.s. under H_N, an argument like that for Theorem 2.1 establishes the weak convergence of $V_{N1}(\lambda)$ to the Brownian bridge.

On the other hand, if

$$\begin{aligned} \gamma_{N1}(t) &= N^{-\frac{1}{2}} \sum_{|j-k|=t}^{N}\sum X_j X_k \\ \gamma_{N2}(t) &= N^{-\frac{1}{2}} \sum_{|j-k|=t}^{N}\sum (X_j - \bar{X})(X_k - \bar{X}) , \end{aligned} \tag{3.47}$$

then

$$\begin{aligned} &E_{H_N}[\sup_\lambda |s_N{}^2(V_{N1}(\lambda) - V_N(\lambda))|] \\ &\qquad \leqq (2^{\frac{1}{2}}\pi)^{-1} \sum_{t=1}^{N-1} t^{-1} E_{H_N}|\gamma_{N1}(t) - \gamma_{N2}(t)| . \end{aligned} \tag{3.48}$$

Since

$$\sum_{t=1}^{N-1} t^{-1} \leqq 1 + \log(N-2) \tag{3.49}$$

and, by direct calculation,

$$E_{H_N}|\gamma_{N1}(t) - \gamma_{N2}(t)| \leqq 6N^{-\frac{1}{2}}, \tag{3.50}$$

the bound in (3.48) tends to zero as $N \to \infty$. Therefore $V_N(\lambda)$ and $V_{N1}(\lambda)$ have the same limiting distribution in $C[0, \pi]$.

4. Asymptotics under K_N. Under K_N, the joint density of $(X_1, X_2, \cdots, X_N)$ is assumed to be $\prod_{i=1}^N f(x_i - \theta_N x_{i-r})$, where $r \geqq 1$, $\theta_N = N^{-\frac{1}{2}}\theta$, $\theta \neq 0$, $x_0 = x_{-1} = \cdots = x_{1-r} = 0$, and f is the density of F. To prove Theorem 2.2 by contiguity methods, it is necessary to study the asymptotic behavior under H_N of the log-likelihood ratio

$$L_N = \sum_{i=1}^N \log[f(X_i - \theta_N X_{i-r})/f(X_i)]. \tag{4.1}$$

This in turn requires study of the statistics

$$B_N = 2\sum_{i=1}^N \{[f(X_i - \theta_N X_{i-r})/f(X_i)]^{\frac{1}{2}} - 1\}, \tag{4.2}$$

and

$$C_N = -\sum_{i=1}^N \theta_N X_{i-r} f'(X_i)/f(X_i). \tag{4.3}$$

Since L_N, B_N, C_N are each sums of r-dependent random variables, it is advantageous to introduce also

$$L_{Nj} = \sum_{i \equiv j (\mathrm{mod}\, r+1)} \log[f(X_i - \theta_N X_{i-r})/f(X_i)], \qquad 1 \leqq j \leqq r+1 \tag{4.4}$$

and analogously defined sums B_{Nj}, C_{Nj}. The summands in L_{Nj} are mutually independent under H_N and

$$L_N = \sum_{j=1}^{r+1} L_{Nj}. \tag{4.5}$$

Similar expressions hold for B_N, C_N in terms of the $\{B_{Nj}\}$ and $\{C_{Nj}\}$ respectively.

Let φ_F be defined as in (2.7). The next three lemmas obtain for the alternatives K_N the type of result already known for translation alternatives (see Hájek and Šidák [8]). Aiyar [1] treated the case $r = 1$ under stronger assumptions.

LEMMA 4.1. *Suppose that f is absolutely continuous and φ_F, $F^{-1} \in L_2[0, 1]$. Then for $1 \leqq j \leqq r + 1$,*

$$\lim_{N\to\infty} E_{H_N}[B_{Nj} - C_{Nj} + (4(r+1))^{-1}\theta^2\|\varphi_F\|^2\|F^{-1}\|^2]^2 = 0. \tag{4.6}$$

PROOF. Let $w(x) = [f(x)]^{\frac{1}{2}}$. Under the assumptions of the lemma, w is absolutely continuous,

$$w'(x) = [4f(x)]^{-\frac{1}{2}}f'(x), \qquad \int_{-\infty}^{\infty}[w'(x)]^2\,dx = 4^{-1}\|\varphi_F\|^2,$$

$$\int_{-\infty}^{\infty}\left[\frac{w(x-h) - w(x)}{h}\right]^2 dx \leqq 4^{-1}\|\varphi_F\|^2 < \infty \tag{4.7}$$

for all $h \neq 0$, and

(4.8) $$\lim_{h\to 0} \int_{-\infty}^{\infty} \left[\frac{w(x-h)-w(x)}{h} - w'(x)\right]^2 dx = 0 .$$

Because the summands are independent,

(4.9) $$\begin{aligned} \operatorname{Var}_{H_N}&(B_{Nj} - C_{Nj}) \\ &\leqq 4 \sum_{i\equiv j(\mathrm{mod}\, r+1)} E_{H_N}\left[\frac{w(X_i - \theta_N X_{i-r})}{w(X_i)} - 1 + 2^{-1}\theta_N X_{i-r}\frac{f'(X_i)}{f(X_i)}\right]^2 \\ &= [4 \textstyle\sum_{i\equiv j(\mathrm{mod}\, r+1)} \theta_N^2] \int_{-\infty}^{\infty} y^2 f(y) \\ &\quad \times \int_{-\infty}^{\infty}\left[\frac{w(x-\theta_N y) - w(x)}{\theta_N y} - w'(x)\right]^2 dx\, dy .\end{aligned}$$

By integrating separately over $\{|y| \leqq c\}$ and $\{|y| > c\}$ and making use of (4.7), (4.8) and the assumptions, we arrive at

(4.10) $$\lim_{N\to\infty} \operatorname{Var}_{H_N}(B_{Nj} - C_{Nj}) = 0 .$$

On the other hand, $E(C_{Nj}) = 0$ because $(\varphi_F, 1) = 0$, and

(4.11) $$\begin{aligned} E_{H_N}(B_{Nj}) &= 2 \sum_{i\equiv j(\mathrm{mod}\, r+1)} E_{H_N}\left[\frac{w(X_i - \theta_N X_{i-r})}{w(X_i)} - 1\right] \\ &= [-\textstyle\sum_{i\equiv j(\mathrm{mod}\, r+1)}^{N} \theta_N^2] \int_{-\infty}^{\infty} y^2 f(y) \\ &\quad \times \int_{-\infty}^{\infty}\left[\frac{w(x-\theta_N y) - w(x)}{\theta_N y}\right]^2 dx\, dy .\end{aligned}$$

From this a simple argument shows that

(4.12) $$\lim_{N\to\infty} E_{H_N}(B_{Nj}) = -(4(r+1))^{-1}\theta^2 \|F^{-1}\|^2 \|\varphi_F\|^2 .$$

The lemma is a consequence of (4.10) and (4.12).

LEMMA 4.2. *Suppose that f is absolutely continuous and $\varphi_F, F^{-1} \in L_2[0, 1]$. Then for any $\varepsilon > 0$,*

(4.13) $$\lim_{N\to\infty} \max_{1\leqq i\leqq N} P_{H_N}\left[\left|\frac{f(X_i - \theta_N X_{i-r})}{f(X_i)} - 1\right| > \varepsilon\right] = 0 .$$

PROOF. Since f is absolutely continuous,

(4.14) $$\begin{aligned} E_{H_N}&\left|\frac{f(X_i - \theta_N X_{i-r})}{f(X_i)} - 1\right| \\ &\leqq \int_{-\infty}^{\infty} f(y) \operatorname{sgn}(\theta_N y) \int_{-\infty}^{\infty}\int_0^{\theta_N y} |f'(x-t)|\, dt\, dx\, dy .\end{aligned}$$

Applying first Fubini's theorem to the inner pair of integrals and then the Cauchy-Schwarz inequality yields

(4.15) $$E_{H_N}\left|\frac{f(X_i - \theta_N X_{i-r})}{f(X_i)} - 1\right| \leqq N^{-\frac{1}{2}}|\theta|\, \|\varphi_F\|\, \|F^{-1}\| .$$

The lemma follows.

LEMMA 4.3. *Suppose that f is absolutely continuous and φ_F, $F^{-1} \in L_2[0, 1]$. Then for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} P_{H_N}[|L_N - C_N + 2^{-1}\theta^2||\varphi_F||^2||F^{-1}||^2| > \varepsilon] = 0 . \tag{4.16}$$

Moreover L_N is asymptotically normal $(-2^{-1}\theta^2||\varphi_F||^2||F^{-1}||^2, \theta^2||\varphi_F||^2||F^{-1}||^2)$ under H_N and the alternatives K_N are contiguous to H_N.

PROOF. Let $\beta^2 = \theta^2||\varphi_F||^2||F^{-1}||^2$. Since C_{Nj} is a sum of independent identically distributed random variables and $E_{H_N}(C_{Nj}) = 0$, $\mathrm{Var}_{H_N}(C_{Nj}) = (r+1)^{-1}\beta^2$, the distribution of C_{Nj} is asymptotically normal $(0, (r+1)^{-1}\beta^2)$. Then, because of Lemma 4.1, B_{Nj} is asymptotically normal $(-(4(r+1))^{-1}\beta^2, (r+1)^{-1}\beta^2)$. From this fact, Lemma 4.2, and a slight extension of LeCam's second lemma as given in [8], we conclude that for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P_{H_N}[|L_{Nj} - B_{Nj} + (4(r+1))^{-1}\beta^2| > \varepsilon] = 0 , \quad 1 \leqq j \leqq r+1 . \tag{4.17}$$

The assertions of the lemma follow from (4.17), Lemma 4.1, LeCam's first lemma, and a central limit theorem for r-dependent random variables (see [6]).

PROOF OF THEOREM 2.2. We prove first the weak convergence of all finite-dimensional distributions, then tightness. Let D_N be an arbitrary linear combination

$$D_N = \sum_{i=1}^{P} \alpha_i S_N^{\varphi}(\lambda_i) + \alpha_{P+1} L_N . \tag{4.18}$$

By (3.41) and (4.16), $D_N + 2^{-1}\alpha_{P+1}\theta^2||\varphi_F||^2||F^{-1}||^2$ has the same asymptotic distribution under H_N as

$$\begin{aligned} E_N &= \sum_{i=1}^{P} \alpha_i T_N(\lambda_i) + \alpha_{P+1} C_N \\ &= N^{-\frac{1}{2}}\Big[\sum\sum_{j\neq k} d_{|j-k|}\varphi(U_j)\varphi(U_k) + \alpha_{P+1}\theta \sum_{j=1}^{N} F^{-1}(U_{j-r})\varphi_F(U_j)\Big] \end{aligned} \tag{4.19}$$

where d_m is defined as in (3.31) and $F^{-1}(U_0) = F^{-1}(U_{-1}) = \cdots = F^{-1}(U_{1-r}) = 0$. If

$$\begin{aligned} c^2 = &\sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_i\alpha_j[\min(\lambda_i/\pi, \lambda_j/\pi) - \lambda_i\lambda_j/\pi^2] \\ &+ \alpha_{p+1}^2\theta^2||F^{-1}||^2||\varphi_F||^2 + 4\alpha_{p+1}\theta d_r(\varphi, \varphi_F)(\varphi, F^{-1}) , \end{aligned} \tag{4.20}$$

an argument analogous to that used in proving Theorem 3.1 shows that E_N is asymptotically normal $(0, c^2)$ under H_N and consequently, D_N is asymptotically normal $(-2^{-1}\alpha_{p+1}\theta^2||\varphi_F||^2||F^{-1}||^2, c^2)$ under H_N. In particular, therefore, the joint asymptotic distribution of $(\sum_{i=1}^{p} \alpha_i S_N^{\varphi}(\lambda_i), L_N)$ under H_N is bivariate normal (μ, Σ) with

$$\begin{aligned} \mu &= (0, -2^{-1}\theta^2||\varphi_F||^2||F^{-1}||^2)' \\ \Sigma &= (\sigma_{ij}) \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \sigma_{11} &= \sum_{i=1}^{p}\sum_{j=1}^{p} \alpha_i\alpha_j[\min(\lambda_i/\pi, \lambda_j/\pi) - \lambda_i\lambda_j/\pi^2] \\ \sigma_{22} &= \theta^2||\varphi_F||^2||F^{-1}||^2 \\ \sigma_{12} &= 2\theta d_r(\varphi, \varphi_F)(\varphi, F^{-1}) = \sum_{i=1}^{p} \alpha_i \mu_\theta(\lambda_i) . \end{aligned} \tag{4.22}$$

It follows from LeCam's third lemma (see [8]) that the asymptotic distribution of $\sum_{i=1}^{p} \alpha_i [S_N^{\varphi}(\lambda_i) - \mu_\theta(\lambda_i)]$ under K_N coincides with the distribution of $\sum_{i=1}^{p} \alpha_i Z(\lambda_i)$. Thus, finite-dimensional distributions of $S_N^{\varphi}(\lambda)$ converge as asserted under K_N.

To establish tightness of the distributions under K_N of the $\{S_N^{\varphi}(\lambda)\}$, it is enough to show (see [3]) that for every $\varepsilon > 0$, $\eta > 0$, there exist $\delta \in (0, \pi)$ and integer N_0 such that

$$P_{K_N}[\sup_{|\lambda-\mu|<\delta} |S_N(\lambda) - S_N(\mu)| > \varepsilon] \leqq \eta \qquad \text{for } N \geqq N_0 . \tag{4.23}$$

The analogous statement under H_N was implied by (3.43) and (3.44). Since K_N is contiguous to H_N according to Lemma 4.3, (4.23) follows. This completes the proof of Theorem 2.2.

REMARK 4.1. An analogue of Theorem 2.2 for the process $V_N(\lambda)$ defined in (1.3) is obtained by replacing φ with F^{-1} in both assumptions and conclusions. The result may be justified as follows. Suppose without loss of generality that $E_{H_N}(X_i) = 0$ and $\mathrm{Var}_{H_N}(X_i) = 1$. Then $s_N^2 = N^{-1} \sum_{i=1}^{N} (X_i - \bar{X})^2$ converges in probability to 1 under K_N as well as under H_N because of contiguity. Let $V_{N1}(\lambda)$ be the process defined in (3.45) and let

$$\nu_\theta(\lambda) = \frac{2^{\frac{1}{2}}\theta \sin(r\lambda)}{\pi r} . \tag{4.24}$$

The proof of Theorem 2.2, the observation above concerning s_N^2, and the fact $(\varphi_F, F^{-1}) = 1$ imply that under K_N, $V_{N1}(\lambda) - \nu_\theta(\lambda)$ converges in distribution in $C[0, \pi]$ to the Brownian bridge.

Moreover, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P_{K_N}[\sup_\lambda |V_N(\lambda) - V_{N1}(\lambda)| > \varepsilon] = 0 , \tag{4.25}$$

because the corresponding result under H_N was established in Remark 3.1 and K_N is contiguous to H_N by Lemma 4.3. Hence $V_N(\lambda) - \nu_\theta(\lambda)$ and $V_{N1}(\lambda) - \nu_\theta(\lambda)$ have the same asymptotic distribution on $C[0, \pi]$ under K_N.

PROOF OF THEOREM 2.3. This follows from Theorems 2.1, 3.1, and 4.1 of Anděl [2], applied to the result in Theorem 2.2 of this paper.

REMARK 4.2. Consider the statistic W_N defined in (2.15). Under H_N and the assumption $\varphi \in L_2[0, 1]$, it follows from Lemma 3.2, Lemma 3.3, and a central limit theorem for r-dependent random variables (Diananda [6]) that the asymptotic distribution of W_N is $N(0, 1)$. Under K_N and the additional assumptions that f is absolutely continuous, $\varphi_F, F^{-1} \in L_2[0, 1]$, the asymptotic distribution of W_N is normal $(\theta||\varphi_F|| \, ||F^{-1}||, 1)$. This is proved by contiguity arguments akin to those used for Theorem 2.2. Consequently the asymptotic power of the test which rejects H_N whenever $W_N \geqq \Phi^{-1}(1 - \alpha)$ is

$$\beta_w(\alpha, \theta, f) = 1 - \Phi[\Phi^{-1}(1 - \alpha) - \theta||\varphi_F|| \, ||F^{-1}||] , \tag{4.26}$$

as asserted in (2.18).

On the other hand, Lemma 4.3 and LeCam's third lemma imply that (4.26) is also the asymptotic power of the size α likelihood ratio test, provided $\theta > 0$. Hence, W_N generates an asymptotically most powerful test at level α for H_N versus K_N when $\theta > 0$.

PROOF OF THEOREM 2.4. The validity of (2.17) is established by the argument of Theorem 2.3 applied to the results concerning $V_N(\lambda)$ noted in Remark 4.1. Then the efficiency (2.19) is immediate from (2.13) and (2.17). The minimization in (2.20) has been carried out in a different context by Aiyar [1]. For details of a limit computation similar to (2.22), see Hájek and Šidák [8] page 272.

REMARK 4.3. Consider now the statistic

$$J_N = (N^{\frac{1}{2}}||\varphi||^2)^{-1} \sum_{k-j=r}^{N}\sum a_N^{\varphi}(R_{Nj}, R_{Nk})\,. \tag{4.27}$$

An argument like that sketched in Remark 4.2 for W_N shows that the asymptotic power of the test which rejects H_N whenever $J_N \geqq \Phi^{-1}(1-\alpha)$ is

$$\beta_J(\alpha, \theta, f) = 1 - \Phi[\Phi^{-1}(1-\alpha) - (\varphi, \varphi_F)(\varphi, F^{-1})||\varphi||^{-2}]\,. \tag{4.28}$$

Comparison of (4.26) and (4.28) gives

$$e(\alpha; J_N, W_N) = \rho_1^2\rho_2^2\,, \tag{4.29}$$

as stated in the remarks following Theorem 2.4.

PROOF OF THEOREM 2.5. It is sufficient to show that

$$\lim_{N\to\infty} E_{H_N}[\sup_\lambda |S_N^{\varphi}(\lambda) - S_N(\lambda)|] = 0\,, \tag{4.30}$$

because contiguity of K_N to H_N then implies that for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P_{K_N}[\sup_\lambda |S_N^{\varphi}(\lambda) - S_N(\lambda)| > \varepsilon] = 0\,. \tag{4.31}$$

To prove (4.30), let

$$\gamma_{N3}(t) = N^{-\frac{1}{2}} \sum_{k-j=t}^{N}\sum a_N^{\varphi}(R_{Nj}, R_{Nk}) \tag{4.32}$$

$$\gamma_{N4}(t) = N^{-\frac{1}{2}} \sum_{k-j=t}^{N}\sum a_N(R_{Nj}, R_{Nk})\,.$$

Since

$$S_N(\lambda) = (2^{\frac{1}{2}}\pi||\varphi||^2)^{-1} \sum_{|t|=1}^{N-1} t^{-1}\sin(\lambda t)\gamma_{N4}(t) \tag{4.33}$$

and a similar expression is possible for $S_N^{\varphi}(\lambda)$, we find

$$\begin{aligned} E_{H_N}&[\sup_\lambda |S_N^{\varphi}(\lambda) - S_N(\lambda)|] \\ &\leqq (2^{\frac{1}{2}}\pi||\varphi||^2)^{-1} \sum_{|t|=1}^{N-1} t^{-1}E_{H_N}|\gamma_{N3}(t) - \gamma_{N4}(t)|\,. \end{aligned} \tag{4.34}$$

Let

$$\bar{\gamma}_N(t) = E_{H_N}[\gamma_{N4}(t)]\,. \tag{4.35}$$

Applying Lemma 3.2 with

$$d_{jk} = N^{-\frac{1}{2}} \quad \text{if} \quad k - j = t \qquad (4.36)$$
$$= 0 \quad \text{otherwise}$$

leads to the bound

$$E_{H_N}[\gamma_{N3}(t) - \gamma_{N4}(t) + \tilde{\gamma}_N(t)]^2 \qquad (4.37)$$
$$\leqq K_{13}[N(N-1)]^{-1} \sum_{j \neq k}^{N}\sum [a_N{}^{\varphi}(j, k) - a_N(j, k) + \bar{a}_N]^2 .$$

By direct calculation,

$$|\tilde{\gamma}_N(t)| \leqq N^{-\frac{3}{2}} \left| \sum_{j \neq k}^{N}\sum a_N(j, k) \right| . \qquad (4.38)$$

The limit in (4.30) follows from (4.34), (4.37), (4.38), (3.49), the assumptions of the theorem.

REFERENCES

[1] Aiyar, R. J. (1969). On some tests for trend and autocorrelation. Ph. D. dissertation, University of California, Berkeley.

[2] Anděl, J. (1967). Local asymptotic power and efficiency of tests of Kolmogorov-Smirnov type. *Ann. Math. Statist.* **38** 1705–1725.

[3] Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[4] Brillinger, D. (1969). Asymptotic properties of spectral estimates of second order. *Biometrika* **56** 375–390.

[5] Bromwich, T. J. (1926). *Introduction to the Theory of Infinite Series*. Macmillan, London.

[6] Diananda, P. H. (1953). Some probability limit theorems with statistical applications. *Proc. Cambridge Phil. Soc.* **49** 239–246.

[7] Grenander, U. and Rosenblatt, M. (1957). *Statistical Analysis of Stationary Time Series*. Wiley, New York.

[8] Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests*. Academic Press, New York.

[9] Heyde, C. C. and Brown, B. M. (1970). On the departure from normality of certain martingales. *Ann. Math. Statist.* **41** 2161–2165.

[10] Ibragimov, I. A. (1963). On estimation of the spectral function of a stationary Gaussian process. *Theor. Probability Appl.* **8** 366–401.

[11] Jogdeo, K. (1968). Asymptotic normality in nonparametric methods. *Ann. Math. Statist.* **39** 905–922.

[12] Meyer, P. A. (1966). *Probability and Potentials*. Blaisdell, Waltham.

[13] Rosén, B. (1970). On bounds on the central moments of even order of a sum of independent random variables. *Ann. Math. Statist.* **41** 1074–1077.

[14] Varberg, D. (1966). Convergence of quadratic forms in independent random variables. *Ann. Math. Statist.* **37** 567–576.

[15] Wald, A. and Wolfowitz, J. (1943). An exact test of randomness in the nonparametric case based on serial correlation. *Ann. Math. Statist.* **14** 378–388.

[16] Whittle, P. (1960). Bounds for the moments of linear and quadratic forms in independent variables. *Teor. Verojatnost. i Primenen.* **5** 331–334.

[17] Whittle, P. (1964). On the convergence to normality of quadratic forms in independent variables. *Teor. Verojatnost. i Primenen.* **9** 113–118.

Department of Statistics
University of California
Berkeley, California 94720