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ON THE FUNDAMENTAL GROUPS OF POSITIVELY CURVED MANIFOLDS

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0. Introduction

In 1965, S. S. Chern posed the following question [7, p.167] (sometimes called Chern's conjecture [12, p.671]; see also [6]): Let M be a compact Riemannian manifold of positive sectional curvature. Is it true that every abelian subgroup of $\pi_1(M)$ is cyclic? Since $\pi_1(M)$ is finite, this is equivalent to saying that the cohomology ring $H^*(\pi_1, \mathbb{Z})$ is periodic (cf.[2]). In this note we will point out that there exist infinitely many counterexamples by observing that the normal homogeneous Aloff-Wallach space $N_{1,1}$ (cf.[10]) and the Eschenburg space $M_{1,1}$ (cf.[3])[1] both admit free, isometric $\mathrm{SO}(3)$ actions. Curiously enough $N_{1,1}$ was precisely the one missed in the classification of positively curved normal homogeneous spaces (cf.[1]). So, the motivation for posing the question possibly came from looking at metric space forms (cf.[11]) or more generally (?) $\frac{1}{4}$-pinched manifolds (cf.[4]) where the fundamental groups all have periodic cohomology, and from manifolds of negative curvature where the statement is true (Preissman's Theorem).

1. Free, isometric $\mathrm{SO}(3)$ actions

Following Wilking [10] we represent the normal homogeneous Aloff-Wallach space $N_{1,1}$ as the quotient $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\,\mathrm{U}^*(2)$. Here $\mathrm{U}^*(2)$ is the image under the embedding $(i, \pi) : \mathrm{U}(2) \hookrightarrow \mathrm{SU}(3) \times \mathrm{SO}(3)$ given

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[1]In [3], the space $M_{1,1}$ is denoted as $M'_{1,1}$.

by the natural inclusion

$$i(A) = \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix} \quad \text{for } A \in \mathrm{U}(2)$$

and the projection $\pi : \mathrm{U}(2) \to \mathrm{U}(2)/\mathrm{S}^1 \cong \mathrm{SO}(3)$, where $\mathrm{S}^1 \subset \mathrm{U}(2)$ is the center of $\mathrm{U}(2)$. The metric being normal homogeneous, the entire group $\mathrm{SU}(3) \times \mathrm{SO}(3)$ acts isometrically on $N_{1,1}$ on the left. In particular, the subgroup $\{\mathrm{id}\} \times \mathrm{SO}(3)$ acts isometrically on the left.

Proposition 1.1. *The group* $\{\mathrm{id}\} \times \mathrm{SO}(3)$ *acts freely on* $N_{1,1}$.

Proof. The action is free if and only if $(\{\mathrm{id}\} \times \mathrm{SO}(3)) \cap \mathrm{Ad}(g)(\mathrm{U}^*(2))$ is trivial for all g in $\mathrm{SU}(3) \times \mathrm{SO}(3)$. This is equivalent to saying that $\mathrm{Ad}(g)(\{\mathrm{id}\} \times \mathrm{SO}(3)) \cap \mathrm{U}^*(2)$ is trivial for all g. But

$$\mathrm{Ad}(g)(\{\mathrm{id}\} \times \mathrm{SO}(3)) = \{\mathrm{id}\} \times \mathrm{SO}(3)$$

and $(i, \pi)(\mathrm{U}(2)) \cap \{\mathrm{id}\} \times \mathrm{SO}(3)$ is clearly trivial. q.e.d.

The Eschenburg space $M_{1,1}$ is constructed as follows: Start with the group $\mathrm{U}(3)$ and perturb the bi-invariant metric to a normal homogeneous metric that is left invariant and $\mathrm{Ad}(\mathrm{U}(2) \times \mathrm{U}(1))$-invariant. Consider the subgroups

$$Z' = \{\mathrm{diag}(z, z, \bar{z}) | z \in \mathrm{S}^1\}$$

and

$$U_{p,q} = \{\mathrm{diag}(z^p, z^q, 1) | z \in \mathrm{S}^1\},$$

where $\gcd(p, q) = 1$, and $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ denotes the matrix with diagonal entries $a_1, a_2, \ldots, a_n$. It is shown in [3] that if $p \cdot q > 0$, then the double coset manifold $M_{p,q} = U_{p,q} \backslash \mathrm{U}(3)/Z'$ (also called a biquotient) has positive curvature for the submersed metric. Since the group $\mathrm{U}(2) \times Z'$ acts freely and isometrically on $\mathrm{U}(3)$, it follows that there is a Riemannian fibration (see [3] for details)

$$(\mathrm{U}(2) \times Z')/(U_{p,q} \times Z') \to M_{p,q} \to \mathbf{C}P^2.$$

When $p = q = 1$, $\mathrm{U}(2) \times Z'$ induces an isometric but non-effective action on $M_{1,1}$ (since $U_{1,1}$ is the center of $\mathrm{U}(2)$) with kernel $U_{1,1} \times Z'$. The resulting isometric action by $\mathrm{SO}(3) = (\mathrm{U}(2) \times Z')/(U_{1,1} \times Z')$ is clearly free, and we get

$$\mathrm{SO}(3) \to M_{1,1} \to \mathbf{C}P^2.$$

Proposition 1.2. *The Eschenburg space $M_{1,1}$ admits a free, isometric* SO(3) *action.*

In fact, it was consideration of this fibration that led to the original observation by the author.

2. Remarks

1. Up to conjugacy the finite subgroups of SO(3) are

$$\mathbb{Z}_n,\ \ n \geq 1 \qquad D_m,\ \ m \geq 2 \qquad A_4 \qquad S_4 \qquad A_5$$

where D_m is the dihedral group of order $2m$, S_n denotes the permutation group on n letters, and $A_n \subset S_n$ is the subgroup of even permutations. Since $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \approx D_2 \hookrightarrow D_{2m}$ for all $m \geq 1$, we get two infinite families of counterexamples: one covered by $N_{1,1}$ and one covered by $M_{1,1}$. This answers Chern's question in the negative.

2. A finite group is said to satisfy the m-condition if any subgroup of order m is cyclic. It is shown in [5] that a finite group acts freely on a topological sphere if and only if it satisfies all $2p$- and p^2-conditions where p is any prime that divides the order of the group. Note that satisfying all p^2-conditions is equivalent to the condition in Chern's problem. The above examples show that neither the $2p$- nor the 2^2-conditions need hold for fundamental groups of positively curved manifolds. However, it is not known whether the p^2-condition remains true for odd primes p. We may formulate the following:

Question. *Let M be a compact Riemannian manifold of positive sectional curvature. Is it true that every abelian subgroup of $\pi_1(M)$ of odd order is cyclic?*

3. In the context of the question, some partial answers are known when the dimension of the manifold is fixed; see for instance [8].

4. The remaining proper, closed subgroups of SO(3) are SO(2) $\approx S^1$ and O(2). The quotients by SO(2) are $N_{1,1}/S^1 = F$ and $M_{1,1}/S^1 = F'$ where F is the space of flags over $\mathbf{C}P^2$, and F' is the "twisted" Eschenburg flag (cf.[3]). The quotients by O(2) then give positively curved manifolds with fundamental group $\mathbb{Z}_2$. They are isometric $\mathbb{Z}_2$ quotients of F and F' respectively. It follows from Synge's theorem that they are nonorientable. It can be shown without too much difficulty that the quaternionic flag Sp(3)/(Sp(1) $\times$ Sp(1) $\times$ Sp(1)) and the Cayley flag

$F_4/\mathrm{Spin}(8)$ also admit isometric $\mathbb{Z}_2$ quotients. In summary all known simply connected, even dimensional manifolds with positive curvature admit isometric $\mathbb{Z}_2$ quotients if they do so topologically, since the remaining known examples are the compact, rank one, symmetric spaces (cf. [9]).

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References

[1] M. Berger, *Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive*, Ann. Scuola. Norm. Sup. Pisa **15** (1961) 179-246.

[2] K. S. Brown, *Cohomology of groups*, Springer, New York, 1982.

[3] J.-H. Eschenburg, *Inhomogeneous spaces of positive curvature*, Differential Geom. Appl. **2** (1992) 123-132.

[4] W. Klingenberg, *Über Mannigfaltigkeiten mit positiver Krümmung*, Comm. Math. Helv. **35** (1961) 47-54.

[5] I. Madsen, C. B. Thomas & C. T. C. Wall, *The topological spherical space form problem–II*, Topology **15** (1976) 375-382.

[6] P. Petersen, *Comparison geometry problem list*, Fields Inst. Monograph 4, Amer. Math. Soc., Providence, RI, 1996.

[7] *Proc. US-Japan Seminar in Differential Geometry, Kyoto, Japan*, 1965.

[8] X. Rong, *The almost cyclicity of the fundamental groups of positively curved manifolds*, Invent. Math. **126** (1996) 47-64.

[9] N. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. of Math. **96** (1972) 277-295.

[10] B. Wilking, *The normal homogeneous space* $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\mathrm{U}^*(2)$ *has positive sectional curvature*, Proc. Amer. Math. Soc., to appear.

[11] J. A. Wolf, *Spaces of constant curvature*, 5th ed., Publish or Perish, Houston, 1984.

[12] S.-T. Yau, *Seminar on differential geometry*, Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1982.

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