Rank-width and Vertex-minors

Sang-il Oum

Program in Applied and Computational Mathematics, Princeton University
Fine Hall, Washington Road, Princeton, NJ 08544 USA

Abstract

The *rank-width* is a graph parameter related in terms of fixed functions to *clique-width* but more tractable. Clique-width has nice algorithmic properties, but no good "minor" relation is known analogous to graph minor embedding for *tree-width*. In this paper, we discuss the vertex-minor relation of graphs and its connection with rank-width. We prove a relationship between vertex-minors of bipartite graphs and minors of binary matroids, and as an application, we prove that bipartite graphs of sufficiently large rank-width contain certain bipartite graphs as vertex-minors. The main theorem of this paper is that for fixed k, there is a finite list of graphs such that a graph G has rank-width at most k if and only if no graph in the list is isomorphic to a vertex-minor of G. Furthermore, we prove that a graph has rank-width at most 1 if and only if it is *distance-hereditary*.

Key words: clique-width; rank-width; vertex-minor; local complementation; pivoting; branch-width; binary matroid

1 Introduction

This paper is motivated by the following open problem:

> For fixed $k > 3$, find a polynomial-time algorithm to decide whether an input graph has *clique-width* at most k.

The notion of *clique-width* was defined by Courcelle and Olariu [1]. It has good algorithmic properties; many NP-hard graph problems can be solved in polynomial time, if the input graphs have clique-width at most some fixed k.

Email address: `sangil@princeton.edu` (Sang-il Oum).

It is interesting to compare clique-width to *tree-width*. Tree-width was developed in the series of papers by Robertson and Seymour, some of which are [2,3,4]. Like clique-width, if the input graphs have tree-width at most some fixed k, then many NP-hard problems can be solved in polynomial time.

An analogy between tree-width and clique-width shows hope of results for clique-width similar to those for tree-width. To do so, it would be desirable to have something similar to *graph minors*. A graph H is a *minor* of a graph G if H can be obtained by a sequence of contractions of edges, deletions of edges, and deletions of vertices. It was shown that the tree-width of a minor of G is at most the tree-width of G and moreover for each k, there is a finite list of graphs such that a graph G has tree-width at most k if and only if no graph in the list is isomorphic to a minor of G. For each k, the finiteness of this list and a polynomial-time algorithm to check the minor containment can be used to construct a polynomial-time algorithm to decide whether the tree-width of a graph is at most k.

To do similar things for clique-width, we need an appropriate containment relation on graphs, having the similar properties for clique-width. Certainly, minor containment is not appropriate for clique-width because every graph G is a minor of the complete graph K_n of $n = |V(G)|$ vertices, and K_n has clique-width 2 if $n > 1$.

Courcelle and Olariu [1] showed that if H is an induced subgraph of a graph G, then the clique-width of H is at most that of G. But, induced subgraph containment is not rich enough; Corneil et al. wrote the following comment in their paper [5].

> Unfortunately, there does not seem to be a succinct forbidden subgraph characterization of graphs with clique-width at most 3, similar to the P_4-free characterization of graphs with clique-width at most 2. In fact every cycle C_n with $n \geq 7$ has clique-width 4, thereby showing an infinite set of minimal forbidden induced subgraphs for Clique-width≤ 3.

We did not yet find an appropriate containment relation for clique-width, but we found that a certain graph containment relation which we call the *vertex-minor* relation is interesting in connection with *rank-width*. The notion of *rank-width*, denoted by $\mathrm{rwd}(G)$, is defined by Oum and Seymour [6] so as to yield an approximation algorithm for clique-width. They also show that rank-width and clique-width are in a sense approximately equal; more precisely, the following inequality [6] links rank-width to clique-width: If the clique-width of G is k, then

$$\log_2(k+1) - 1 \leq \mathrm{rwd}(G) \leq k.$$

Thus, a set of graphs of bounded rank-width is also of bounded clique-width and vice versa. For a graph G and $v \in V(G)$, performing the *local comple-*

mentation at v consists in replacing a subgraph induced on the neighbors of v by its edge-complement graph. The graph obtained by applying local complementation at v to G is denoted by $G * v$. A graph H is a *vertex-minor* of G if H can be obtained by applying a sequence of local complementations and deletions of vertices to G. We will show that if H is a vertex-minor of G, then rank-width of H is at most that of G.

The notion of *branch-width* for both graphs and matroids was defined by Robertson and Seymour [4]. A *fundamental graph* of a binary matroid $\mathcal{M}$ is a bipartite graph with a bipartition $(B, E(\mathcal{M}) \setminus B)$ such that B is a basis of $\mathcal{M}$, and $e \in B$ and $f \in E(\mathcal{M}) \setminus B$ are adjacent if and only if e is in the fundamental circuit of f with respect to B. We will show that the branch-width of a binary matroid is one more than the rank-width of its fundamental graph. It turns out that a fundamental graph of a minor of a binary matroid $\mathcal{M}$ is a vertex-minor of a fundamental graph of $\mathcal{M}$. This allows us to think of generalizing theorems about branch-width of binary matroids to rank-width of graphs.

The main theorem of this paper is the following.

Theorem. Let $k \geq 1$. The set of graphs having rank-width at most k is characterized by excluded vertex-minors with at most $(6^{k+1} - 1)/5$ vertices.

This implies that for each k, there is a finite list of graphs, such that a graph G has rank-width at most k if and only if no graph in the list is isomorphic to a vertex-minor of G. This will be used by Courcelle and Oum [7] to find a polynomial-time algorithm to decide whether rank-width is at most k for fixed k. This is an exact analog to the corresponding theorem for tree-width.

The paper is organized as follows. In the next section, we review the notion of rank-width and we define the vertex-minor relation. In Section 3, we discuss the vertex-minor relation and rank-width of bipartite graphs in connection with the minor relation and branch-width of binary matroids. This enables us to translate a theorem for binary matroids into a theorem for bipartite graphs. In Section 4, we prove useful inequalities which will be used in both Section 5 and 6. In Section 5, we prove that the set of graphs having rank-width at most k is characterized by excluded vertex-minors of bounded size for fixed $k \geq 1$. In Section 6, we show one example of generalizing a theorem for binary matroids to general graphs; we generalize Tutte's linking theorem about minors in matroids to a theorem about vertex-minors in graphs. In Section 7, we characterize graphs of rank-width at most one and obtain another proof that distance-hereditary graphs have clique-width at most three.

2 Definitions

In this section, we review the notion of rank-width and we introduce the vertex-minor relation. In this paper, we assume that graphs are simple undirected and finite.

Let us first review the definition of rank-width, introduced by Oum and Seymour [6]. For a matrix $M = (m_{ij} : i \in R, j \in C$ over a field F, let $\mathrm{rk}(M)$ denote its linear rank. If $X \subseteq R$ and $Y \subseteq C$, then let $M[X, Y]$ be the submatrix $(m_{ij} : i \in X, j \in Y)$ of M. We assume that adjacency matrices of graphs are matrices over GF(2).

Definition. Let G be a graph and A, B be disjoint subsets of $V(G)$. Let M be the adjacency matrix of G over GF(2). We define the *rank of* (A, B), $\mathrm{rk}_G(A, B)$, as $\mathrm{rk}(M[X, Y])$. The *cut-rank*, $\mathrm{cutrk}_G(A)$ of $A \subseteq V(G)$, is defined by

$$\mathrm{cutrk}_G(A) = \mathrm{rk}_G(A, V(G) \setminus A).$$

A *subcubic tree* is a tree such that every vertex has exactly one or three incident edges. We call (T, L) a *rank-decomposition* of G if T is a subcubic tree and and L is a bijection from $V(G)$ to the set of leaves of T.

For an edge e of T, the two connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a rank-decomposition (T, L) is $\mathrm{cutrk}_G(L^{-1}(X))$. The *width* of (T, L) is the maximum width of all edges of T. The *rank-width* of G, denoted by $\mathrm{rwd}(G)$, is the minimum width of all rank-decompositions of G. (If $|V(G)| \leq 1$, we define $\mathrm{rwd}(G) = 0$.)

Now, we define *local complementation*, *pivoting*, *vertex-minors*, and *pivot-minors*. In fact, vertex-minor containment was called *l-reduction* by Bouchet [8], but the author thinks "vertex-minor" is a better name, because of the many analogies with matroid minors discussed in Section 3.

For two sets A and B, let $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Definition. Let $G = (V, E)$ be a graph and $v \in V$. The graph obtained by applying *local complementation* at v to G is

$$G * v = (V, E \Delta \{xy : xv, yv \in E, x \neq y\}).$$

For an edge $uv \in E$, the graph obtained by *pivoting* uv is defined by $G \wedge uv = G * u * v * u$. We call H is *locally equivalent* to G if G can be obtained by applying a sequence of local complementations to G. We call H is a *vertex-minor* of G if H can be obtained by applying a sequence of vertex deletions and local complementations to G. We call H is a *pivot-minor* of G if H can be obtained by applying a sequence of vertex deletions and pivotings. A vertex-minor H of G is called a *proper* vertex-minor if H has fewer vertices than G. and similarly a pivot-minor H of G is called a *proper* pivot-minor if H has fewer vertices than G.

A pivoting is well-defined because $G * u * v * u = G * v * u * v$ if u and v are adjacent. To prove this, we prove the following proposition that describes pivoting directly.

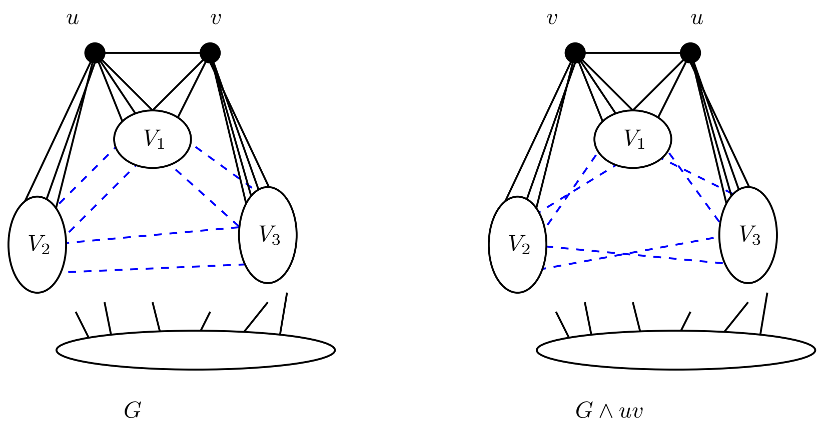

Fig. 1. Pivoting

Proposition 2.1. For a graph H and $u, v \in V(H)$, let H_{uv} be a graph obtained by exchanging u and v in H. For $X, Y \subseteq V(H)$, let $H * (X, Y)$ be the graph $(V(H), E')$ where $E' = E(H)\Delta\{xy : x \in X, y \in Y, x \neq y\}$. Let $G = (V, E)$ be a graph. For $x \in V$, let $N(x)$ be the set of neighbors of x in G. For $uv \in E$, let $V_1 = N(u) \cap N(v)$, $V_2 = N(u) \setminus N(v) \setminus \{v\}$, and $V_3 = N(v) \setminus N(u) \setminus \{v\}$. Then

$$G \wedge uv = (G * (V_1, V_2) * (V_2, V_3) * (V_3, V_1))_{uv}.$$

Proof. Note that V_1, V_2, V_3 are disjoint subsets of $V(G)$. For a graph H and $X \subseteq V(H)$, let $H * (X)^2 = H * (X, X)$.

Let us first consider the neighbors of u and v in $G * u * v * u$. The set of neighbors of u in G is $N(u) = V_1 \cup V_2 \cup \{v\}$. The set of neighbors of v in $G * u$ is $N(v)\Delta(N(u) \setminus \{v\}) = V_2 \cup V_3 \cup \{u\}$. The set of neighbors of u in $G * u * v$ is $N(u)\Delta(V_2 \cup V_3) = V_1 \cup V_3 \cup \{v\}$. Therefore, $G * u * v * u = G * (V_1 \cup V_2 \cup \{v\})^2 * (V_2 \cup V_3 \cup \{u\})^2 * (V_1 \cup V_3 \cup \{v\})^2$.

Now, we use the simple facts that $G * (X \cup Y)^2 = G * (X)^2 * (Y)^2 * (X, Y)$ for $X \cap Y = \emptyset$, $G * (X, Y) * (Z, W) = G * (Z, W) * (X, Y)$, $G * (X, Y) * (X, Y) = G$, and $G * (\{x\})^2 = G$. So, $G * (V_1 \cup V_2 \cup \{v\})^2 = G * (V_1)^2 * (V_2)^2 * (V_1, V_2) * (V_1, \{v\}) * (V_2, \{v\})$.

By applying these, we obtain the following.

$$
\begin{aligned}
&G * u * v * u \\
&= G * (V_1, V_2) * (V_2, V_3) * (V_3, V_1) \\
&\quad * (V_1, \{v\}) * (V_2, \{v\}) * (V_2, \{u\}) * (V_3, \{u\}) * (V_1, \{v\}) * (V_3, \{v\}) \\
&= G * (V_1, V_2) * (V_2, V_3) * (V_3, V_1) * (V_2, \{v\}) * (V_2, \{u\}) * (V_3, \{v\}) * (V_3, \{u\}) \\
&= (G * (V_1, V_2) * (V_2, V_3) * (V_3, V_1))_{uv}.
\end{aligned}
$$

□

In other words, pivoting uv is an operation that,

(1) for each $(x, y) \in (V_1 \times V_2) \cup (V_2 \times V_3) \cup (V_3 \times V_1)$, adds a new edge xy if $xy \notin E(G)$ or deletes it otherwise,
(2) and then, exchanges u and v (so that u is adjacent to vertices in $V_1 \cup V_3$, and v is adjacent to vertices in $V_1 \cup V_2$).

Corollary 2.2. If G is a graph and $uv \in E(G)$, then $G*u*v*u = G*v*u*v$.

Proof. This is immediate from Proposition 2.1. □

Corollary 2.3. If a graph G is bipartite and $uv \in E(G)$, $G \wedge uv$ is also bipartite.

Proof. Let V_1, V_2, and V_3 be sets defined in Proposition 2.1. Since G is bipartite, $V_1 = \emptyset$. It does not break bipartiteness to add edges between V_2 and V_3. □

For a graph H, let $x \simeq_H y$ denote that either $x = y$ or they are adjacent in G. Let $a \oplus b$ denote $(a \wedge \neg b) \vee (\neg a \wedge b)$. This operation is usually called the logical "exclusive or" operation. (Note that we use the $\wedge$ symbol with two meanings: one for pivoting and another for the logical "and" operation.)

The next corollary is reformulation of the above proposition.

Corollary 2.4. Let G be a graph and let $uv \in E(G)$. For all $x, y \in V(G)$, $x \simeq_{G \wedge uv} y$ if and only if

$$(x \simeq_G y) \oplus (x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u).$$

Proof. If $x = y$, then it is clear because $(x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u)$ is always false.

Suppose $\{x, y\} \cap \{u, v\} = \emptyset$ and $x \neq y$. Let V_1, V_2, and V_3 be sets defined in Proposition 2.1. We add or remove an edge xy if and only if there exist

$i, j \in \{1, 2, 3\}$ such that $x \in V_i$, $y \in V_j$, and $i \neq j$. It is equivalent to say that $(x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u)$ is true.

Now, consider when one of x or y is u or v. We may assume that $x = u$ without loss of generality. Then

$$\begin{aligned}
&(x \simeq_G y) \oplus (x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u) \\
&= (u \simeq_G y) \oplus (y \simeq_G v) \oplus (y \simeq_G u) \qquad \text{because } u \text{ is adjacent to } v. \\
&= y \simeq_G v \\
&= y \simeq_{G \wedge uv} u \qquad \text{because we exchanged } u \text{ and } v. \\
&= x \simeq_{G \wedge uv} y
\end{aligned}$$

□

The following proposition is essentially equivalent to [9, Lemma 10] and [10, Proposition 5], but our proof is a routine application of the previous corollary, whereas [9] and [10] show it by reducing the problem into a certain graph of 11 vertices.

Proposition 2.5. If $vv_1, vv_2 \in E(G)$ are two distinct edges incident to v, then,

$$G \wedge vv_1 \wedge v_1v_2 = G \wedge vv_2,$$

and therefore $G \wedge vv_1 \setminus v$ is locally equivalent to $G \wedge vv_2 \setminus v$.

Proof. First of all, $G \wedge vv_1 \wedge v_1v_2$ is well-defined because v_1 and v_2 are adjacent in $G \wedge vv_1$. Let $G' = G \wedge vv_1$. Corollary 2.4 implies that $x \simeq_{G \wedge uv} y$ if and only if

$$(x \simeq_G y) \oplus (x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u).$$

For simplicity, we write $\simeq$ instead of $\simeq_G$.

$$x \simeq_{G' \wedge v_1v_2} y = (x \simeq_{G'} y) \oplus (x \simeq_{G'} v_1 \wedge y \simeq_{G'} v_2) \oplus (x \simeq_{G'} v_2 \wedge y \simeq_{G'} v_1) \tag{1}$$

$$x \simeq_{G'} y = (x \simeq y) \oplus (x \simeq v \wedge y \simeq v_1) \oplus (x \simeq v_1 \wedge y \simeq v) \tag{2}$$

$$x \simeq_{G'} v_1 = x \simeq v \tag{3}$$

$$y \simeq_{G'} v_2 = (y \simeq v_2) \oplus (y \simeq v_1) \oplus (y \simeq v \wedge v_2 \simeq v_1) \tag{4}$$

$$x \simeq_{G'} v_2 = (x \simeq v_2) \oplus (x \simeq v_1) \oplus (x \simeq v \wedge v_2 \simeq v_1) \tag{5}$$

$$y \simeq_{G'} v_1 = y \simeq v \tag{6}$$

Now, let us apply (2) — (6) to (1). We use the fact that $a \wedge (b \oplus c) =$

$(a \wedge b) \oplus (a \wedge c)$.

$$\begin{aligned}
&x \simeq_{G' \wedge v_1 v_2} y \\
&= (x \simeq_{G'} y) \oplus (x \simeq_{G'} v_1 \wedge y \simeq_{G'} v_2) \oplus (x \simeq_{G'} v_2 \wedge y \simeq_{G'} v_1) \\
&= (x \simeq y) \oplus (x \simeq v \wedge y \simeq v_1) \oplus (x \simeq v_1 \wedge y \simeq v) \\
&\quad \oplus (x \simeq v \wedge y \simeq v_2) \oplus (x \simeq v \wedge y \simeq v_1) \oplus (x \simeq v \wedge y \simeq v \wedge v_2 \simeq v_1) \\
&\quad \oplus (x \simeq v_2 \wedge y \simeq v) \oplus (x \simeq v_1 \wedge y \simeq v) \oplus (x \simeq v \wedge y \simeq v \wedge v_2 \simeq v_1) \\
&= (x \simeq y) \oplus (x \simeq v \wedge y \simeq v_2) \oplus (x \simeq v_2 \wedge y \simeq v) \\
&= x \simeq_{G \wedge v v_2} y
\end{aligned}$$

Therefore, $x \simeq_{G \wedge v v_1 \wedge v_1 v_2} y$ if and only if $x \simeq_{G \wedge v v_2} y$. □

The following observation is fundamental.

Proposition 2.6. Let $G' = G * v$. Then for every $X \subseteq V(G)$,

$$\mathrm{cutrk}_G(X) = \mathrm{cutrk}_{G'}(X).$$

Proof. We may assume that $v \in X$ by the symmetry of cut-rank. Let M, M' be the adjacency matrix of G, G' respectively. Let $N = M[X, V(G) \setminus X]$ and $N' = M[X, V(G) \setminus X]$. It is easy to see that N' is obtained from N by adding the row of v to the rows of its neighbors in X. Therefore, $\mathrm{cutrk}_G(X) = \mathrm{rk}(N) = \mathrm{rk}(N') = \mathrm{cutrk}_{G'}(X)$. □

Corollary 2.7. If H is locally equivalent to G, then the rank-width of H is equal to the rank-width of G. If H is a vertex-minor of G, then the rank-width of H is at most the rank-width of G.

Proof. The first statement is obvious. Since vertex deletion does not increase cut-rank, it does not increase rank-width, and therefore the second statement is true. □

3 Bipartite graphs and Binary matroids

In this section, we discuss the relation between branch-width of binary matroids and rank-width of bipartite graphs. We will also discuss further properties relating binary matroids and bipartite graphs. As an example, we will show the implication of the grid theorem for binary matroids by Geelen, Gerards, and Whittle [11].

Let us review matroid theory first. For general matroid theory, we refer to Oxley's book [12]. We call $\mathcal{M} = (E, \mathcal{I})$ a *matroid* if E is a finite set and $\mathcal{I}$ is a collection of subsets of E, satisfying

(1) $\emptyset \in \mathcal{I}$
(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(3) For every $Z \subseteq E$, maximal subsets of Z in $\mathcal{I}$ all have the same size $r(Z)$. We call $r(Z)$ the *rank* of Z.

An element of $\mathcal{I}$ is called *independent* in $\mathcal{M}$. We let $E(\mathcal{M}) = E$. We call $B \subseteq E$ a *base* if it is maximally independent. A matroid may also be defined by axioms on the set of bases. We call $B' \subseteq E$ a *cobase* if $E \setminus B'$ is a base. The *dual matroid* $\mathcal{M}^*$ of $\mathcal{M}$ is the matroid on $E(\mathcal{M})$ such that the set of cobases of $\mathcal{M}$ is equal to the set of bases of $\mathcal{M}^*$.

A matroid $\mathcal{M} = (E, \mathcal{I})$ is *binary* if there exists a matrix N over GF(2) such that E is a set of column vectors of N and

$$\mathcal{I} = \{X \subseteq E : X \text{ is linearly independent}\}.$$

For $e \in E(\mathcal{M})$, $\mathcal{M} \setminus e$ is the matroid $(E \setminus \{e\}, \mathcal{I}')$ such that

$$\mathcal{I}' = \{X \subseteq E(\mathcal{M}) \setminus \{e\} : X \in \mathcal{I}\}.$$

This operation is called *deletion* of e. For $e \in E(\mathcal{M})$, $\mathcal{M}/e = (\mathcal{M}^* \setminus e)^*$ and this operation is called *contraction* of e. A matroid $\mathcal{N}$ is called a *minor* of $\mathcal{M}$ if $\mathcal{N}$ can be obtained from $\mathcal{M}$ by applying a sequence of deletions and contractions.

The *connectivity* function $\lambda_{\mathcal{M}}$ of $\mathcal{M}$ is

$$\lambda_{\mathcal{M}}(X) = r(X) + r(E \setminus X) - r(E) + 1.$$

We call (T, L) a *branch-decomposition* of $\mathcal{M}$ if T is a subcubic tree and and L is a bijection from $E(\mathcal{M})$ to the set of leaves of T. For an edge e of T, the two connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a branch-decomposition (T, L) is $\lambda_{\mathcal{M}}(L^{-1}(X))$. The *width* of (T, L) is the maximum width of all edges of T. The *branch-width* $\mathrm{bw}(\mathcal{M})$ of $\mathcal{M}$ is the minimum width of all branch-decompositions of $\mathcal{M}$. (If $|E(\mathcal{M})| \leq 1$, we define $\mathrm{bw}(\mathcal{M}) = 1$.)

Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let M be the adjacency matrix of G. Let $Bin(G, A, B)$ be the binary matroid on V, represented by the $A \times V$ matrix $\begin{pmatrix} I_A & M[A,B] \end{pmatrix}$, where I_A is the $A \times A$ identity matrix. If $\mathcal{M} = Bin(G, A, B)$, then G is called a *fundamental graph* of $\mathcal{M}$.

Here is a major observation, which gives a relation between connectivity of binary matroids and cut-rank of bipartite graphs.

Proposition 3.1. Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then for every $X \subseteq V$, $\lambda_{\mathcal{M}}(X) = \mathrm{cutrk}_G(X) + 1$.

Proof. Let M be the adjacency matrix of G. First note that

$$M[X, V \setminus X] = \begin{pmatrix} 0 & M[X \cap A, (V \setminus X) \cap B] \\ M[X \cap B, (V \setminus X) \cap A] & 0 \end{pmatrix}.$$

Therefore, $\mathrm{cutrk}_G(X) = \mathrm{rk}(M[X \cap B, (V \setminus X) \cap A]) + \mathrm{rk}(M[X \cap A, (V \setminus X) \cap B])$. Consequently,

$$\begin{aligned}
\lambda_{\mathcal{M}}(X) &= r(X) + r(V \setminus X) - r(V) + 1 \\
&= \mathrm{rk} \begin{pmatrix} 0 & M[(V \setminus X) \cap A, X \cap B] \\ I_{X \cap A} & M[X \cap A, X \cap B] \end{pmatrix} \\
&\quad + \mathrm{rk} \begin{pmatrix} 0 & M[X \cap A, (V \setminus X) \cap B] \\ I_{(V \setminus X) \cap A} & M[(V \setminus X) \cap A, (V \setminus X) \cap B] \end{pmatrix} - |A| + 1 \\
&= \mathrm{rk}(M[(V \setminus X) \cap A, X \cap B] + \mathrm{rk}(M[X \cap A, (V \setminus X) \cap B] + 1 \\
&= \mathrm{cutrk}_G(X) + 1.
\end{aligned}$$

□

An easy corollary of Proposition 3.1 is the following.

Corollary 3.2. Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then the branch-width of $\mathcal{M}$ is one more than the rank-width of G.

Proof. This is trivial because (T, L) is a branch-decomposition of $\mathcal{M}$ of width $k + 1$ if and only if it is a rank-decomposition of G of width k. □

Now, let us discuss the relation between minors of matroids and vertex-minors of graphs.

Proposition 3.3. Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then

(1) $Bin(G, B, A) = \mathcal{M}^*$,
(2) For $uv \in E(G)$, $Bin(G \wedge uv, A \Delta \{u, v\}, B \Delta \{u, v\}) = \mathcal{M}$.
(3) $Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) = \begin{cases} \mathcal{M}/v & \text{if } v \in A, \\ \mathcal{M} \setminus v & \text{if } v \in B. \end{cases}$

Proof. Let M be the adjacency matrix of G. Then, $\mathcal{M}$ is represented by a matrix $\begin{pmatrix} I & M[A,B] \end{pmatrix}$.

(1): It is known that $\mathcal{M}^*$ is represented by a matrix $\begin{pmatrix} M[B,A] & I \end{pmatrix}$. Therefore, $\mathcal{M}^* = Bin(G, B, A)$

(2): We may assume that $u \in A$, $v \in B$. Let $R = (r_{ij} : i \in A, j \in V) = \begin{pmatrix} I & M[A,B] \end{pmatrix}$ be a matrix over GF(2). (So, $r_{ij} = 1$ if $j \in B$ and $ij \in E(G)$ or $i = j$, and $r_{ij} = 0$ otherwise.) We know that elementary row operations on R do not change the associated matroid $\mathcal{M}$.

By adding the row vector of u, that is $(r_{uj} : j \in V)$, to the rows of neighbors of u in A, we obtain another matrix $R' = (r'_{ij} : i \in A, j \in V)$ representing the same matroid. We observe that the column vector of u, v in R' is equal to the column vector of v, u in R respectively, and therefore $R'[A, (A \setminus \{u\}) \cup \{v\}]$ is an identity matrix. Moreover for $i \neq u$ and $j \in B \setminus \{v\}$, $r'_{ij} \neq r_{ij}$ if and only if $r_{uj} = 1$ and $r_{iv} = 1$, or equivalently $iv, ju \in E(G)$. By Proposition 2.1, we know that for $i \in A \setminus \{u\}$ and $j \in B \setminus \{v\}$, ij belongs to exactly one of $E(G)$ and $E(G \wedge uv)$ if and only if $iv, ju \in E(G)$. (Because G is bipartite, $iu, jv \notin E(G)$.) Moreover the set of neighbors of u, v in $G \wedge uv$ is equal to the set of neighbors of v, u in G respectively. Therefore, we conclude that $\mathcal{M} = Bin(G \wedge uv, A\Delta\{v,w\}, B\Delta\{v,w\})$.

(3): If $v \in B$, by deleting the column of v in $\begin{pmatrix} I & M[A,B] \end{pmatrix}$, we obtain a matrix representation of $\mathcal{M} \setminus v$ and therefore $\mathcal{M} \setminus v = Bin(G \setminus v, A, B \setminus \{v\})$.

If $v \in A$, then $\mathcal{M}^* = Bin(G, B, A)$, and therefore $\mathcal{M}^* \setminus v = Bin(G, B, A \setminus \{v\})$ and $\mathcal{M}/v = Bin(G, A \setminus \{v\}, B)$. □

Corollary 3.4. Let $\mathcal{M}$ be a binary matroid and G be the fundamental graph of $\mathcal{M}$ with a bipartition $V(G) = A \cup B$ such that $\mathcal{M} = Bin(G, A, B)$. If v has no neighbor in G, then

$$\mathcal{M} \setminus v = M/v = Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}).$$

Otherwise let w be a neighbor of v.

(1) $$\mathcal{M} \setminus v = \begin{cases} Bin(G \wedge vw \setminus v, A\Delta\{v,w\}, B\Delta\{v,w\} \setminus \{v\}) & \text{if } v \in A, \\ Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) & \text{otherwise.} \end{cases}$$

(2) $$\mathcal{M}/v = \begin{cases} Bin(G \wedge vw \setminus v, A\Delta\{v,w\} \setminus \{v\}, B\Delta\{v,w\}) & \text{if } v \in B, \\ Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) & \text{otherwise.} \end{cases}$$

Note that the matroid $Bin(G \wedge vw \setminus v, A\Delta\{v,w\} \setminus \{v\}, B\Delta\{v,w\} \setminus \{v\})$ is independent of the choice of w by Proposition 2.5 and (2) of Proposition 3.3.

Proof. If v has no neighbor in G, then v is a loop or a coloop of $\mathcal{M}$, and therefore $M \setminus v = M/v$. By (3) of Proposition 3.3, we deduce that $Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) = M \setminus v = M/v$.

Now we assume that w is a neighbor of v. By (1) of Proposition 3.3, it is enough to show (1). If $v \in B$, then by (3) of Proposition 3.3, we obtain that $\mathcal{M} \setminus v = Bin(G \setminus v, A, B \setminus \{v\})$. If $v \in A$, then $\mathcal{M} = Bin(G \wedge vw, A\Delta\{v, w\}, B\Delta\{v, w\})$, and therefore $\mathcal{M} \setminus v = Bin(G \wedge vw, A\Delta\{v, w\}, B\Delta\{v, w\} \setminus \{v\})$. □

Corollary 3.5. If G, H are bipartite graphs with bipartitions $A \cup B = V(G)$ and $A' \cup B' = V(H)$ and $Bin(H, A', B') = Bin(G, A, B)$, then H can be obtained by applying a sequence of pivotings to G, and therefore H is locally equivalent to G.

Proof. We proceed by induction on $|A'\Delta A|$.

Let $\mathcal{M} = Bin(G, A, B) = Bin(H, A', B')$. If $A' = A$, then $G = H$ because $\mathcal{M}$ determines every fundamental circuit with respect to A.

Now, we may assume that $A' \neq A$. Since A and A' are bases of $\mathcal{M}$, we may pick $w \in A' \setminus A$ and $v \in A \setminus A'$ such that w is in the fundamental circuit of v with respect to A', and therefore $vw \in E(H)$. Let $H' = H \wedge vw$. By (2) of Proposition 3.3, $\mathcal{M} = Bin(H', A'\Delta\{v, w\}, B'\Delta\{v, w\})$. By induction, H' can be obtained by applying a sequence of pivotings to G. Since $H = H' \wedge vw$, H can be obtained by applying a sequence of pivotings to G. □

Corollary 3.6.

(1) Let $\mathcal{N}$, $\mathcal{M}$ be binary matroids, and H, G be fundamental graphs of $\mathcal{N}$, $\mathcal{M}$ respectively. If $\mathcal{N}$ is a minor of $\mathcal{M}$, then H is a pivot-minor of G, and therefore H is a vertex-minor of G.
(2) Let G be a bipartite graph with a bipartition $A \cup B = V(G)$. If H is a pivot-minor of G, then there is a bipartition $A' \cup B' = V(H)$ of H such that $Bin(H, A', B')$ is a minor of $Bin(G, A, B)$.

Proof. (1) We proceed by induction on $|E(\mathcal{M}) \setminus E(\mathcal{N})|$. By Corollary 3.5, we may assume that $\mathcal{M} \neq \mathcal{N}$. By induction, it is enough to show it when $\mathcal{N} = \mathcal{M} \setminus v$ or $\mathcal{N} = \mathcal{M}/v$ for $v \in V(G)$. By Corollary 3.4, either $G \wedge vw \setminus v$ for some $w \in V(G)$ or $G \setminus v$ is a fundamental graph of $\mathcal{N}$. By Corollary 3.5, H can be obtained from either $G \wedge vw \setminus v$ or $G \setminus v$ by applying a sequence of pivotings.

(2): By (2) and (3) of Proposition 3.3, we obtain a bipartition (A', B') of H such that $Bin(H, A', B')$ is a minor of $Bin(G, A, B)$. $\square$

By Proposition 3.3, theorems about branch-width of binary matroids give corollaries about rank-width of bipartite graphs. One of the recent theorems about branch-width of binary matroids is proved by Geelen, Gerards, and Whittle. Let us recall their theorem in the context of binary matroids. The $n \times n$ grid is a graph on the vertex set $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$.

Theorem 3.7 (Grid theorem for binary matroids [11]). For every positive integer k, there is an integer l such that if $\mathcal{M}$ is a binary matroid with branch-width at least l, then $\mathcal{M}$ contains a minor isomorphic to the cycle matroid of the $k \times k$ grid.

To make corollaries about rank-width from this theorem, it is helpful to replace the $k \times k$ grid by a planar graph whose cycle matroid has a simpler fundamental graph. We define a planar graph $R_k = (V, E)$ (Fig. 2) as following:

$$
\begin{aligned}
V &= \{v_1, v_2, \cdots, v_{k^2}\}, \\
E &= \{v_i v_{i+1} : 1 \le i \le k^2 - 1\} \cup \{v_i v_{i+k} : 1 \le i \le k^2 - k\}.
\end{aligned}
$$

We can obtain a minor of R_k isomorphic to the $k \times k$ grid by deleting edges $v_{ik} v_{ik+1}$ for all $1 \le i \le k - 1$. To show that R_k is isomorphic to a minor of $l \times l$ grid for a big l, let us cite a useful lemma by Robertson, Seymour, and Thomas.

Lemma 3.8 ([13, (1.5)]). If H is a planar graph with $|V(H)| + 2|E(H)| \le n$, then H is isomorphic to a minor of the $2n \times 2n$ grid.

By this lemma, R_k is isomorphic to a minor of the $6k^2 \times 6k^2$ grid. Therefore, Theorem 3.7 is still true if R_k is used instead of the $k \times k$ grid.

Now, let us construct a fundamental graph S_k of the cycle matroid of R_k. Since edges of R_k represent elements of the cycle matroid of R_k, they are vertices of S_k. Let $a_i = v_i v_{i+1}$ and $b_i = v_i v_{i+k}$. Let $A = \{a_i : 1 \le i \le k^2 - 1\}$ and $B = \{b_i : 1 \le i \le k^2 - k\}$ so that A is the set of edges of a spanning tree of R_k. For each $b_j \in B$, $a_i b_j \in E(S_k)$ if and only if a_i is in the fundamental cycle of b_j with respect to the spanning tree of R_k with the edge set A. In summary, S_k is a bipartite graph with $V(S_k) = A \cup B$ such that $a_i b_j \in E(S_k)$ if and only if $i \le j < i + k$ (Fig. 2). By Corollary 3.6, we obtain the following.

Corollary 3.9. For every positive integer k, there is an integer l such that if a bipartite graph G has rank-width at least l, then it contains a vertex-minor isomorphic to S_k.

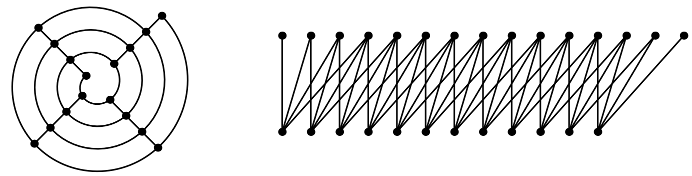

Fig. 2. R_4 and S_4

This corollary will be used by Courcelle and Oum [7] to prove a slight weakening of Seese's conjecture.

4 Inequalities on cut-rank and vertex-minors

Submodularity plays an important role in many places of combinatorics. In this section, we prove several inequalities concerning the cut-rank function. The following proposition is called the submodular inequality of the matrix rank function, and implies that the cut-rank function is submodular [6].

Proposition 4.1. Let M be a matrix over a field F. Let C be the set of column indexes of M, and R the set of row indexes of M. Then for all $X_1, X_2 \subseteq R$ and $Y_1, Y_2 \subseteq C$, we have

$$\mathrm{rk}(M[X_1, X_2]) + \mathrm{rk}(M[Y_1, Y_2]) \geq \mathrm{rk}(M[X_1 \cup Y_1, X_2 \cap Y_2]) + \mathrm{rk}(M[X_1 \cap Y_1, X_2 \cup Y_2]).$$

Proof. See [14, Proposition 2.1.9], [15, Lemma 2.3.11] or [16]. □

Corollary 4.2 ([6]). If G is a graph and $X, Y \subseteq V(G)$, then

$$\mathrm{cutrk}_G(X) + \mathrm{cutrk}_G(Y) \geq \mathrm{cutrk}_G(X \cap Y) + \mathrm{cutrk}_G(X \cup Y).$$

Proof. Let M be the adjacency matrix of G over GF(2). Then

$$\mathrm{cutrk}_G(X) = \mathrm{rk}(M[X, V(G) \setminus X]).$$

Apply Proposition 4.1. □

Proposition 4.3. Let $G = (V, E)$ be a graph and let $v \in V$ and $Y_1 \subseteq V$. Let $M = A(G)$ be the adjacency matrix of G over GF(2). Then

$$\mathrm{cutrk}_{G*v\setminus v}(Y_1) = \mathrm{rk}\begin{pmatrix} 1 & M[\{v\}, V \setminus Y_1 \setminus \{v\}] \\ M[Y_1, \{v\}] & M[Y_1, V \setminus Y_1 \setminus \{v\}] \end{pmatrix} - 1.$$

Moreover, if w is a neighbor of v, then

$$\mathrm{cutrk}_{G\wedge vw\setminus v}(Y_1) = \mathrm{rk}\begin{pmatrix} 0 & M[\{v\}, V\setminus Y_1\setminus\{v\}] \\ M[Y_1,\{v\}] & M[Y_1, V\setminus Y_1\setminus\{v\}] \end{pmatrix} - 1.$$

Proof. We will use elementary row operations on matrices to prove the claim. For a graph H, let $A(H)$ denote the adjacency matrix of H. Let N be the set of neighbors of v in G. Let J_A^B be a matrix $(1)_{i\in A, j\in B}$. We will write J instead of J_A^B if it is not confusing. Let $V = V(G)$. Let $Y_2 = V\setminus Y_1\setminus\{v\}$. Let $L_{11} = M[Y_1\cap N, Y_2\cap N]$, $L_{12} = M[Y_1\cap N, Y_2\setminus N]$, $L_{21} = M[Y_1\setminus N, Y_2\cap N]$, and $L_{22} = M[Y_1\setminus N, Y_2\setminus N]$. Then

$$\begin{aligned}
\mathrm{cutrk}_{G*v\setminus v}(Y_1) &= \mathrm{rk}(A(G*v)[Y_1,Y_2]) \\
&= \mathrm{rk}\begin{pmatrix} L_{11}+J & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} 1\ 1\ 1\ 1\ \cdots 1 & 0\ 0\ 0 \cdots 0 \\ \mathbf{0} \quad L_{11}+J & L_{12} \\ \mathbf{0} \quad L_{21} & L_{22} \end{pmatrix} - 1 \\
&= \mathrm{rk}\begin{pmatrix} 1\ 1\ 1\ 1\ \cdots 1 & 0\ 0\ 0 \cdots 0 \\ J \quad L_{11} & L_{12} \\ \mathbf{0} \quad L_{21} & L_{22} \end{pmatrix} - 1 \\
&= \mathrm{rk}\begin{pmatrix} 1 & M[\{v\}, Y_2] \\ M[Y_1,\{v\}] & M[Y_1,Y_2] \end{pmatrix} - 1.
\end{aligned}$$

Let W be the set of neighbors of w. We may assume that $w\in Y_1$ by symmetry. Consequently $w\in Y_1\cap(N\setminus W)$. Let $N_1 = N\setminus W\setminus\{w\}$, $N_2 = N\cap W$, $N_3 = W\setminus N$, $N_4 = V\setminus N\setminus W\setminus\{w\}$. Let $M_{ij} = M[Y_1\cap N_i, Y_2\cap N_j]$ for all $i,j\in\{1,2,3,4\}$. Then

$$\begin{aligned}
\mathrm{cutrk}_{G\wedge vw\setminus v}(Y_1) &= \mathrm{rk}(A(G\wedge vw)[Y_1,Y_2]) \\
&= \mathrm{rk}\begin{pmatrix} 1\ 1\ 1\cdots 1 & 1\ 1\ 1\cdots 1 & 0\ 0\ 0\cdots 0 & 0\ 0\ 0\cdots 0 \\ M_{11} & M_{12}+J & M_{13}+J & M_{14} \\ M_{21}+J & M_{22} & M_{23}+J & M_{24} \\ M_{31}+J & M_{32}+J & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}
\end{aligned}$$

$$= \operatorname{rk}\begin{pmatrix} 1\ 0\ 0\ 0\cdots 0\ 1\ 1\ 1\cdots 1\ 1\ 1\ 1\cdots 1\ 0\ 0\ 0\cdots 0 \\ 0\ 1\ 1\ 1\cdots 1\ 1\ 1\ 1\cdots 1\ 0\ 0\ 0\cdots 0\ 0\ 0\ 0\cdots 0 \\ \begin{matrix} \mathbf{0} & M_{11} & M_{12}+J & M_{13}+J & M_{14} \\ \mathbf{0} & M_{21}+J & M_{22} & M_{23}+J & M_{24} \\ \mathbf{0} & M_{31}+J & M_{32}+J & M_{33} & M_{34} \\ \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{matrix} \end{pmatrix} - 1$$

$$= \operatorname{rk}\begin{pmatrix} 1\ 0\ 0\ 0\cdots 0\ 1\ 1\ 1\cdots 1\ 1\ 1\ 1\cdots 1\ 0\ 0\ 0\cdots 0 \\ 0\ 1\ 1\ 1\cdots 1\ 1\ 1\ 1\cdots 1\ 0\ 0\ 0\cdots 0\ 0\ 0\ 0\cdots 0 \\ \begin{matrix} J & M_{11} & M_{12} & M_{13} & M_{14} \\ J & M_{21}+J & M_{22}+J & M_{23} & M_{24} \\ \mathbf{0} & M_{31}+J & M_{32}+J & M_{33} & M_{34} \\ \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{matrix} \end{pmatrix} - 1$$

$$= \operatorname{rk}\begin{pmatrix} 0\ 1\ 1\ 1\cdots 1\ 1\ 1\ 1\cdots 1\ 0\ 0\ 0\cdots 0\ 0\ 0\ 0\cdots 0 \\ 1\ 0\ 0\ 0\cdots 0\ 1\ 1\ 1\cdots 1\ 1\ 1\ 1\cdots 1\ 0\ 0\ 0\cdots 0 \\ \begin{matrix} J & M_{11} & M_{12} & M_{13} & M_{14} \\ J & M_{21} & M_{22} & M_{23} & M_{24} \\ \mathbf{0} & M_{31} & M_{32} & M_{33} & M_{34} \\ \mathbf{0} & M_{41} & M_{42} & M_{43} & M_{44} \end{matrix} \end{pmatrix} - 1$$

$$= \operatorname{rk}\begin{pmatrix} 0 & M[\{v\}, Y_2] \\ M[Y_1, \{v\}] & M[Y_1, Y_2] \end{pmatrix} - 1. \qquad \square$$

The following lemma is analogous to [17, (5.2)].

Lemma 4.4. Let G be a graph and $v \in V(G)$. Suppose that (X_1, X_2) and (Y_1, Y_2) are partitions of $V(G) \setminus \{v\}$. Then

$$\operatorname{cutrk}_{G\setminus v}(X_1) + \operatorname{cutrk}_{G*v\setminus v}(Y_1) \geq \operatorname{cutrk}_G(X_1 \cap Y_1) + \operatorname{cutrk}_G(X_2 \cap Y_2) - 1.$$

If w is a neighbor of v, then

$$\operatorname{cutrk}_{G\setminus v}(X_1) + \operatorname{cutrk}_{G\wedge vw\setminus v}(Y_1) \geq \operatorname{cutrk}_G(X_1 \cap Y_1) + \operatorname{cutrk}_G(X_2 \cap Y_2) - 1.$$

Proof. We use Proposition 4.3 and apply Proposition 4.1. Let M be the adjacency matrix of G over GF(2). Then

$$\begin{aligned}
&\mathrm{cutrk}_{G\setminus v}(X_1) + \mathrm{cutrk}_{G\wedge vw\setminus v}(Y_1)\\
&= \mathrm{rk}(M[X_1, X_2] + \mathrm{rk}(M[Y_1 \cup \{v\}, Y_2 \cup \{v\}]) - 1\\
&\geq \mathrm{rk}(M[X_1 \cap Y_1, X_2 \cup \{v\} \cup Y_2] + \mathrm{rk}(M[X_1 \cup \{v\} \cup Y_1, Y_2 \cap X_2]) - 1\\
&= \mathrm{cutrk}_G(X_1 \cap Y_1) + \mathrm{cutrk}_G(X_2 \cap Y_2) - 1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathrm{cutrk}_{G\setminus v}(X_1) + \mathrm{cutrk}_{G*v\setminus v}(Y_1)\\
&= \mathrm{rk}(M[X_1, X_2] + \mathrm{rk}\begin{pmatrix} 1 & M[\{v\}, Y_2] \\ M[Y_1, \{v\}] & M[Y_1, Y_2] \end{pmatrix} - 1\\
&\geq \mathrm{rk}(M[X_1 \cap Y_1, X_2 \cup \{v\} \cup Y_2]) + \mathrm{rk}(M[X_1 \cup \{v\} \cup Y_1, Y_2 \cap X_2]) - 1\\
&= \mathrm{cutrk}_G(X_1 \cap Y_1) + \mathrm{cutrk}_G(X_2 \cap Y_2) - 1. \qquad \square
\end{aligned}$$

5 Excluded vertex-minors

In this section, we show that for any fixed k, there is a finite set $\mathcal{C}_k$ of graphs such that for every graph G, $\mathrm{rwd}(G) \leq k$ if and only if no graph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G. Since the number of graphs with bounded number of vertices is finite up to isomorphism, it is enough to show that if a graph G has rank-width larger than k but every proper vertex-minor of G has rank-width at most k, then $|V(G)|$ is bounded by a function of k. We prove a stronger statement that if $\mathrm{rwd}(G) > k$ and every proper pivot-minor has rank-width at most k, then $|V(G)|$ is bounded by a function of k. The analogous result for matroids is proved by Geelen, Gerards, and Whittle [18] and we extend their method to graphs.

Let us begin with some additional definitions from [18]. Let G be a graph and (A, B) a partition of $V(G)$. A *branching* of B is a triple (T, r, L) where T is a subcubic tree with a fixed leaf node r and L is a bijection from B to the set of leaf nodes of T different from r. For an edge e of T of the branching (T, r, L), let T_e be the set of vertices in B mapped by L to nodes in the component of $T \setminus e$ not containing r. We say B is *k-branched* if there is a branching (T, r, L) of B such that for each edge e of T, $\mathrm{cutrk}_G(T_e) \leq k$. Note that if both A and B are k-branched, then the rank-width of G is at most k.

The following lemma is proved by Geelen et al. [18, Lemma 2.1] in terms of matroids. But their proof relies on the fact that $\lambda_{\mathcal{M}}$ is integer-valued submodular, and since cut-rank also has these properties, we can use basically the same argument.

Lemma 5.1. Let G be a graph of rank-width k. Let (A, B) be a partition of $V(G)$ such that $\text{cutrk}_G(A) \leq k$. If there is no partition (A_1, A_2, A_3) of A such that $\text{cutrk}(A_i) < \text{cutrk}(A)$ for all $i \in \{1, 2, 3\}$, then B is k-branched.

Proof. (Obvious modification of the proof of [18, Lemma 2.1])

Claim 5.1.1. If (X_1, X_2) is a partition of $V(G)$ with $\text{cutrk}_G(X_1) \leq k$, then either $\text{cutrk}_G(B \cap X_1) \leq k$ or $\text{cutrk}_G(B \cap X_2) \leq k$.

Proof of Claim 5.1.1. From the partition $(A \cap X_1, A \cap X_2, \emptyset)$ of A, either $\text{cutrk}_G(A \cap X_1) \geq \text{cutrk}_G(A)$ or $\text{cutrk}_G(A \cap X_2) \geq \text{cutrk}_G(A)$. We may assume that $\text{cutrk}_G(A \cap X_1) \geq \text{cutrk}_G(A)$. By submodularity, $\text{cutrk}_G(A \cup X_1) \leq \text{cutrk}_G(A) + \text{cutrk}_G(X_1) - \text{cutrk}_G(A \cap X_1) \leq k$. So, $\text{cutrk}_G(B \cap X_2) = \text{cutrk}_G(A \cup X_1) \leq k$. □

Let (T, L) be a rank-decomposition of G of width k. We may assume that T has degree-3 nodes, as otherwise it is trivial. We may also assume that $k > 0$. If v is a vertex of T and e is an edge of T, we let $X_{ev} = L^{-1}(\mathcal{X}_{ev})$ where $\mathcal{X}_{ev}$ is the set of leaves of T in the component of $T \setminus e$ not containing v.

Claim 5.1.2. There exists a degree-3 vertex s of T such that, for each edge e of T, $\text{cutrk}_G(X_{es} \cap B) \leq k$.

Proof of Claim 5.1.2. We construct an orientation of T. Let e be an edge of T, and let u and v be the ends of e. If $\text{cutrk}_G(X_{ev} \cap B) \leq k$, then we orient e from u to v. By Claim 5.1.1, each edge receives at least one orientation.

First, assume that there exists a node v of T such that every other node can be connected to v by a directed path on T. Since $k \geq 1$, each edge incident with a leaf has been oriented away from that leaf. Hence we may assume that v has degree 3. Then the claim follows with $s = v$.

Next, we assume that there is no vertex reachable from every other vertex. Then there exists a pair of edges e and f and a vertex w on the path connecting e and f such that neither e nor f is oriented toward w. Let $Y_1 = X_{ew}$, $Y_3 = X_{fw}$, and $Y_2 = V(G) \setminus (Y_1 \cup Y_2)$. Since e and f are oriented away from w, $\text{cutrk}_G((Y_2 \cup Y_3) \cap B) \leq k$ and $\text{cutrk}_G((Y_1 \cup Y_2) \cap B) \leq k$. By submodularity,

$$\begin{aligned} &\text{cutrk}_G(Y_1 \cap B) + \text{cutrk}_G(Y_3 \cap B) \\ &\qquad\qquad \leq \text{cutrk}_G((Y_2 \cup Y_3) \cap B) + \text{cutrk}_G((Y_1 \cup Y_2) \cap B) \leq 2k. \end{aligned}$$

This contradicts the fact that neither e nor f is oriented toward w. □

Let s be a vertex satisfying Claim 5.1.2, let e_1, e_2, and e_3 be the edges of T incident with s, and let X_i denote $X_{e_i s}$ for each $i \in \{1,2,3\}$. Note that $\mathrm{cutrk}_G(X_i \cap A) \geq \mathrm{cutrk}_G(A)$ for some $i \in \{1,2,3\}$; suppose that $\mathrm{cutrk}_G(X_1 \cap A) \geq \mathrm{cutrk}_G(A)$. Then by submodularity,

$$\begin{aligned}\mathrm{cutrk}_G((X_2 \cup X_3) \cap B) &= \mathrm{cutrk}_G(X_1 \cup A)\\ &\leq \mathrm{cutrk}_G(X_1) + \mathrm{cutrk}_G(A) - \mathrm{cutrk}_G(X_1 \cap A)\\ &\leq \mathrm{cutrk}_G(X_1) \leq k.\end{aligned}$$

Now we construct a branching (T', r, L') of B; let T be a tree obtained from the minimum subtree of T containing both e_1 and nodes in $L(B)$ by subdividing e_1 with a vertex b, adding a new leaf r adjacent to b, and contracting one of incident edges of each degree-2 vertex until no degree-2 vertices are left. For each $x \in B$, we define $L'(x)$ to be a leaf of T' induced by $L(x)$. Then (T', r, L') is a branching.

It is easy to see that $\mathrm{cutrk}_G(T'_e) \leq k$ for all e in T' by Claim 5.1.2. So, B is k-branched. □

We continue to follow [18]. Let $\mathbb{Z}^+$ be the set of nonnegative integers. Let $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function. A graph G is called (m,g)-*connected* if for every partition (A, B) of $V(G)$, $\mathrm{cutrk}_G(A) = l < m$ implies either $|A| \leq g(l)$ or $|B| \leq g(l)$.

Lemma 5.2. Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a nondecreasing function. Let G be a (m, f)-connected graph and let $v \in V(G)$ and $vw \in E(G)$. Then either $G \setminus v$ or $G \wedge vw \setminus v$ is $(m, 2f)$-connected.

Proof. The proof for matroids in [18, Lemma 3.1] works for general graphs. For the completeness of this paper, the proof is included here.

Suppose not. There are partitions (X_1, X_2), (Y_1, Y_2) of $V(G) \setminus \{v\}$ such that

$$\begin{aligned} a = \mathrm{cutrk}_{G\setminus v}(X_1) &< m, & |X_1| &> 2f(a), & |X_2| &> 2f(a),\\ b = \mathrm{cutrk}_{G\wedge vw\setminus v}(Y_1) &< m, & |Y_1| &> 2f(b), & |Y_2| &> 2f(b).\end{aligned}$$

We may assume that $a \geq b$ by replacing G by $G \wedge vw$. We may assume that $|X_1 \cap Y_1| > f(a)$ by swapping Y_1 and Y_2.

By Lemma 4.4, we obtain

$$\mathrm{cutrk}_G(X_1 \cap Y_1) + \mathrm{cutrk}_G(X_2 \cap Y_2) \leq a + b + 1.$$

Thus, either $\mathrm{cutrk}_G(X_1 \cap Y_1) \leq a$ or $\mathrm{cutrk}_G(X_2 \cap Y_2) \leq b$. So, either $|X_1 \cap Y_1| \leq f(a)$ or $|X_2 \cap Y_2| \leq f(b)$. By assumption, $|X_2 \cap Y_2| \leq f(b)$.

Similarly we apply the same inequality after swapping X_1 and X_2. Either $|X_2 \cap Y_1| \leq f(a)$ or $|X_1 \cap Y_2| \leq f(b)$. Since $|X_1 \cap Y_2| = |Y_2| - |Y_2 \cap X_2| > f(b)$, $|X_2 \cap Y_1| \leq f(a)$.

Then $|X_2| = |X_2 \cap Y_1| + |X_2 \cap Y_2| \leq f(a) + f(b) \leq 2f(a)$. This is a contradiction. $\square$

Let $g(n) = (6^n - 1)/5$. Note that $g(0) = 0$, $g(1) = 1$, and $g(n) = 6g(n-1) + 1$ for all $n \geq 1$.

Lemma 5.3. Let $k \geq 1$. If G has rank-width larger than k but every proper pivot-minor of G has rank-width at most k, then G is $(k+1, g)$-connected.

Proof. We continue to follow the proof of [18, Lemma 4.1] with a slight modification.

It is easy to see that G is $(1, g)$-connected, because if G is disconnected, then the rank-width of G is the maximum of the rank-width of each component.

Suppose that $m \leq k$ and G is (m, g)-connected and G is not $(m+1, g)$-connected. Then there exists a partition (A, B) with $\text{cutrk}_G(A) = m$ such that $|A|, |B| > g(m) = 6g(m-1) + 1$. Since G has rank-width greater than k, either A or B is not k-branched. We may assume that B is not k-branched. Let $v \in A$. Since G is connected, there is a neighbor w of v in G.

By Lemma 5.2, either $G \setminus v$ or $G \wedge vw \setminus v$ is $(m, 2g)$-connected. Since both $G \setminus v$ and $G \wedge vw \setminus v$ are proper pivot-minors of G, they have rank-width at most k.

We may assume that $G \setminus v$ is $(m, 2g)$-connected by swapping G and $G \wedge vw$. Let (A_1, A_2, A_3) be a partition of $A \setminus \{v\}$. Since $|A| > 6g(m-1)+1$, $|A_i| > 2g(m-1)$ for some $i \in \{1, 2, 3\}$. Since $G \setminus v$ is $(m, 2g)$-connected and $|B| > 2g(m-1)$,

$$\text{cutrk}_{G \setminus v}(A_i) \geq m \geq \text{cutrk}_{G \setminus v}(A \setminus \{v\}).$$

Therefore by Lemma 5.1, B is k-branched in $G \setminus v$. Since B is not k-branched in G, there exists $X \subseteq B$ such that

$$\text{cutrk}_G(X) = \text{cutrk}_{G \setminus v}(X) + 1.$$

Let M be the adjacency matrix of G. By submodular inequality (Proposition

4.1), we obtain

$$\begin{aligned}
\mathrm{cutrk}_{G\setminus v}(B) + \mathrm{cutrk}_G(X) &= \mathrm{rk}(M[B, V(G)\setminus B\setminus\{v\}]) + \mathrm{rk}(M[X, V(G)\setminus X])\\
&\geq \mathrm{rk}(M[B, V(G)\setminus B]) + \mathrm{rk}(M[X, V(G)\setminus X\setminus\{v\}])\\
&= \mathrm{cutrk}_G(B) + \mathrm{cutrk}_{G\setminus v}(X)\\
&= \mathrm{cutrk}_G(B) + \mathrm{cutrk}_G(X) - 1,
\end{aligned}$$

and therefore $\mathrm{cutrk}_{G\setminus v}(B) = \mathrm{cutrk}_G(B) - 1 = m-1$. But this is a contradiction because $G\setminus v$ is $(m, 2g)$-connected. □

Theorem 5.4. Let $k \geq 1$. If G has rank-width larger than k but every proper pivot-minor of G has rank-width at most k, then $|V(G)| \leq (6^{k+1}-1)/5$.

Proof. Let $v \in V(G)$. Since G is connected, pick w such that $vw \in E(G)$. We may replace G by $G \wedge vw$, and hence we may assume that $G\setminus v$ is $(k+1, 2g)$-connected. Since $G \setminus v$ has rank-width k, there exists a partition (X_1, X_2) of $V(G)\setminus\{v\}$ such that $|X_1|, |X_2| \geq \frac{1}{3}(|V(G)|-1)$ and $\mathrm{cutrk}_{G\setminus v}(X_1) \leq k$. By $(k+1, 2g)$-connectivity, either $|X_1| \leq 2g(k)$ or $|X_2| \leq 2g(k)$. Therefore, $|V(G)| - 1 \leq 6g(k)$ and consequently $|V(G)| \leq 6g(k) + 1 = g(k+1)$. □

One of the main corollary of the above theorem is the following corollary. This corollary is used by Courcelle and Oum [7] to show a polynomial-time algorithm to recognize graphs of rank-width at most k.

Corollary 5.5. For each $k \geq 0$, there is a finite list $\mathcal{C}_k$ of graphs having at most c_k vertices where

$$c_k = \begin{cases} (6^{k+1}-1)/5 & \text{if } k > 0,\\ 2 & \text{if } k = 0, \end{cases}$$

such that a graph has rank-width at most k if and only if no graph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G.

Proof. If $k = 0$, then we let K_2 be a graph with two vertices and one edge joining them and let $\mathcal{C}_0 = \{K_2\}$. Since a graph G has rank-width 0 if and only if G has no edge, the rank-width of G is 0 if and only if K_2 is not isomorphic to a vertex-minor of G. Now we may assume that $k \geq 1$.

Let $\mathcal{C}_k$ be the set of graphs H with $V(H) = \{1, 2, \ldots, n\}$ for some integer n such that $\mathrm{rwd}(H) > k$ and every proper vertex-minor has rank-width at most k. By Theorem 5.4, $\mathcal{C}_k$ is finite and each graph in $\mathcal{C}_k$ has at most $(6^{k+1}-1)/5$ vertices.

Suppose the rank-width of a graph G is at most k. Since every graph in $\mathcal{C}_k$ has rank-width larger than k, no graph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G.

Conversely, suppose that the rank-width of a graph G is larger than k. Let H be a proper vertex-minor of G with the minimum number of vertices such that $\mathrm{rwd}(H) > k$. Then there exists a graph $H' \in \mathcal{C}_k$ isomorphic to H. $\square$

By Corollary 3.6, Theorem 5.4 implies the following corollary, which is a special case of [18, Theorem 1.1].

Corollary 5.6. Let $k \geq 2$. If a binary matroid $\mathcal{M}$ has branch-width larger than k but every proper minor of $\mathcal{M}$ has branch-width at most k, then $|E(\mathcal{M})| \leq (6^k - 1)/5$.

6 Tutte's linking theorem

In this section, we show a theorem analogous to Tutte's linking theorem [19]. In the following theorem, we show that the minimum cut-rank of cuts separating two disjoint sets X, Y of vertices of a graph G is equal to the maximum cut-rank of X in all vertex-minors of G having $X \cup Y$ as the set of vertices. In particular, this theorem implies that it is in NP $\cap$ coNP to answer whether $\min_{X\subseteq Z\subseteq V(G)\setminus Y} \mathrm{cutrk}_G(Z) \geq k$ when a graph G and subsets X, Y of $V(G)$ and k are given as the input. In [20], the author obtains a direct combinatorial algorithm to solve this problem in polynomial time, which is essentially based on this theorem. We note that there are algorithms that can minimize any submodular function in polynomial time [21].

Theorem 6.1. Let G be a graph and X, Y be disjoint subsets of $V(G)$. The following are equivalent.

(1) $\min\limits_{X\subseteq Z\subseteq V(G)\setminus Y} \mathrm{cutrk}_G(Z) \geq k$.

(2) There exists a vertex-minor G' of G such that

$$V(G') = X \cup Y \text{ and } \mathrm{cutrk}_{G'}(X) \geq k.$$

(3) There exists a pivot-minor G' of G such that

$$V(G') = X \cup Y \text{ and } \mathrm{cutrk}_{G'}(X) \geq k.$$

Proof. (2)$\Rightarrow$(1): We may assume that G' is an induced subgraph of G by applying local complementations to G. For all Z satisfying $X \subseteq Z \subseteq V(G)\setminus Y$, we have $k \leq \mathrm{cutrk}_{G'}(X) = \mathrm{cutrk}^*_G(X, Y) \leq \mathrm{cutrk}^*_G(Z, V(G)\setminus Z) = \mathrm{cutrk}_G(Z)$.

(3)$\Rightarrow$(2): Trivial.

(1)⇒(3): We proceed by induction on $|V(G) \setminus (X \cup Y)|$. Suppose there is no such graph G'. If $X \cup Y = V(G)$, then it is trivial. Let $x \in V(G) \setminus (X \cup Y)$. If x has no neighbor, then for all $Z \subseteq V(G) \setminus \{x\}$,

$$\mathrm{cutrk}_{G\setminus x}(Z) = \mathrm{cutrk}_G(Z).$$

Therefore, $\min_{X\subseteq Z\subseteq V(G)\setminus Y} \mathrm{cutrk}_G(Z) = \min_{X\subseteq Z\subseteq V(G)\setminus\{x\}\setminus Y} \mathrm{cutrk}_{G\setminus x}(Z)$.

So, we may assume that x has a neighbor y. By induction, there exists $A \subseteq V(G) \setminus \{x\}$ such that

$$\mathrm{cutrk}_{G\setminus x}(A) \leq k - 1.$$

Also, there exists $B \subseteq V(G) \setminus \{x\}$ such that

$$\mathrm{cutrk}_{G\wedge xy\setminus x}(B) \leq k - 1.$$

By Lemma 4.4, either $\mathrm{cutrk}_G(A \cap B) \leq k - 1$ or $\mathrm{cutrk}_G(A \cup B) \leq k - 1$. Consequently, $\min_{X\subseteq Z\subseteq V(G)\setminus Y} \mathrm{cutrk}_G(Z) \leq k - 1$. □

We can deduce Tutte's linking theorem for binary matroids from the above theorem. Here is the statement of Tutte's linking theorem for binary matroids.

Corollary 6.2. Let $\mathcal{M} = (E, \mathcal{I})$ be a binary matroid and let X, Y be disjoint subsets of E. Then

$$\min_{X\subseteq Z\subseteq E\setminus Y} \lambda_{\mathcal{M}}(Z) \geq k$$

if and only if there is a minor $\mathcal{M}'$ of $\mathcal{M}$ such that $E(\mathcal{M}') = X \cup Y$ and $\lambda_{\mathcal{M}'}(X) \geq k$.

Proof. Let G be a bipartite graph with a bipartition $A \cup B = V(G)$ such that $Bin(G, A, B) = \mathcal{M}$. There exists a minor $\mathcal{M}'$ of $\mathcal{M}$ such that $E(\mathcal{M}') = X \cup Y$ and $\lambda_{\mathcal{M}'}(X) \geq k$ if and only if there exists a pivot-minor H of G such that $V(H) = X \cup Y$ and $\mathrm{cutrk}_H(X) \geq k - 1$ by Corollary 3.6. The remaining proof is routine by Proposition 3.1 and Proposition 6.1. □

7 Distance-hereditary graphs

We call a graph G *distance-hereditary* if and only if for every connected induced subgraph H of G, the distance between every pair of vertices in H is the same as in G. In this section, we show that a graph is distance-hereditary if and only if it has rank-width at most 1.

Two distinct vertices v, w are called *twins* of G if for every $x \in V(G) \setminus \{v, w\}$, v is adjacent to x if and only if w is adjacent to x. We call v a *pendant vertex* of G if it has only one incident edge in G.

Proposition 7.1. Let G be a graph. If $v, w \in V(G)$ are twins of G and $G \setminus v$ has at least one edge different from vw, then

$$\mathrm{rwd}(G \setminus v) = \mathrm{rwd}(G).$$

Note that we do not require that $vw \in E(G)$.

Proof. It is enough to show that $\mathrm{rwd}(G \setminus v) \geq \mathrm{rwd}(G)$. Let (T, L) be a rank-decomposition of $G \setminus v$ of width $\mathrm{rwd}(G \setminus v)$. Let $x = L(w)$ and let $y \in V(T)$ be such that $xy \in E(T)$.

Let T' be a tree obtained from T by deleting xy, adding two new vertices x', z, and adding three new edges yz, zx', zx. Let $L'(x') = v$ and $L'(u) = L(u)$ for all $u \neq x'$.

So, (T', L') is a rank-decomposition of G. For every edge e except zx' and zx in T', the width of e in (T', L') is equal to the width of e in (T, L), because v and w are twins. Both the width of zx and the width of zx' are at most 1. Since G has at least one edge $e \neq vw$ and v, w are twins, $G \setminus v$ has at least one edge and $\mathrm{rwd}(G \setminus v) \geq 1$, and therefore the width of (T', L') is $rwd(G \setminus v)$. Therefore, $\mathrm{rwd}(G \setminus v) \geq \mathrm{rwd}(G)$. □

Proposition 7.2. If G has rank-width at most 1 and $|V(G)| \geq 2$, then G has a pair of vertices v and w such that either they are twins or w has no neighbor different from v.

Proof. If $|V(G)| = 2$, then the claim is trivial, and so we may assume that $|V(G)| \geq 3$.

Let (T, L) be a rank-decomposition of G of width at most 1. Since the number of leaf nodes is at least 3, there exists a node x of T that is adjacent to two leaf nodes $L(v)$, $L(w)$ of T. Let y be the node of T adjacent to x different from $L(v)$ and $L(w)$. The partition of $V(G)$ induced by xy is $(\{v, w\}, V(G) \setminus \{v, w\})$. So, the width of xy is $\mathrm{cutrk}_G(\{v, w\}) \leq 1$. That means either v, w are twins or v has no neighbor different from w or w has no neighbor different from v. □

Proposition 7.3. G is distance-hereditary if and only if the rank-width of G is at most 1.

Proof. Bandelt and Mulder [22] showed that every distance-hereditary graph can be obtained by creating twins, adding an isolated vertex, or adding a pendant vertex to a distance-hereditary graph or is a graph with one vertex. So, the rank-width of every distance-hereditary graphs is at most 1 by Proposition 7.1. Conversely, if a graph has rank-width at most 1, then by Proposition 7.2, it is a distance-hereditary graph. □

Golumbic and Rotics [23] proved that distance-hereditary graphs have clique-width at most 3, and this can be proved as a corollary of Proposition 7.3.

Corollary 7.4. Distance-hereditary graphs have clique-width at most 3.

Proof. Use the inequality that the clique-width of a graph G is at most $2^{\mathrm{rwd}(G)+1} - 1$ [6]. $\square$

By Corollary 5.5, there is a finite list $\mathcal{C}_1$ of graphs having at most seven vertices such that a graph G is distance-hereditary graphs if and only if no graph in $\mathcal{C}_1$ is isomorphic to a vertex-minor of G. We may ask what $\mathcal{C}_1$ is. In fact, it is proved by Bouchet that a graph G is distance-hereditary if and only if it has no vertex-minor isomorphic to the 5-cycle [24,25].

8 Conclusion

We introduce vertex-minors of graphs by generalizing minors of binary matroids. Surprisingly, the branch-width of binary matroids is one more than the rank-width of their fundamental graph. Thus, all theorems on branch-width of binary matroids implies theorems on rank-width of bipartite graphs, and in many cases we are able to prove that the same theorems hold for general graphs. Section 5 and 6 are such examples. In [26], the author shows that graphs of bounded rank-width are *well-quasi-ordered* by the vertex-minor relation; this generalizes the theorem by Geelen et al. [17] stating that binary matroids of bounded branch-width are well-quasi-ordered by the matroid minor relation. However, it is still open whether Corollary 3.9 is true for general graphs.

In [7], Courcelle and Oum show that vertex-minor relation can be written in a certain kind of logic formulas, called *modulo-2 counting monadic second-order logic formulas*, and therefore for fixed graph H, it is possible to decide whether an input graph contains H in polynomial time if an input graph has rank-width at most k for fixed k. This theorem is combined with Corollary 3.9 to show the existence of polynomial-time algorithms to decide whether an input graph has rank-width at most k for fixed k. Moreover they use Corollary 3.9 to prove a slight weakening of Seese's conjecture.

Acknowledgment. The author would like to thank P. Seymour for our valuable discussions, A. Brandstädt for pointing out distance-hereditary graphs, and anonymous referee for numerous helpful comments and new references.

References

[1] B. Courcelle, S. Olariu, Upper bounds to the clique width of graphs, Discrete Appl. Math. 101 (1-3) (2000) 77–114.

[2] N. Robertson, P. Seymour, Graph minors. V. Excluding a planar graph, J. Combin. Theory Ser. B 41 (1) (1986) 92–114.

[3] N. Robertson, P. Seymour, Graph minors. IV. Tree-width and well-quasi-ordering, J. Combin. Theory Ser. B 48 (2) (1990) 227–254.

[4] N. Robertson, P. Seymour, Graph minors. X. Obstructions to tree-decomposition, J. Combin. Theory Ser. B 52 (2) (1991) 153–190.

[5] D. G. Corneil, M. Habib, J.-M. Lanlignel, B. Reed, U. Rotics, Polynomial time recognition of clique-width ≤ 3 graphs (extended abstract), in: Gonnet, Gastón H. (ed.) et al., LATIN 2000: Theoretical informatics. 4th Latin American symposium, Punta del Este, Uruguay, April 10-14, 2000., Vol. 1776 of Lecture Notes in Comput. Sci., Springer, Berlin, 2000, pp. 126–134.

[6] S. Oum, P. Seymour, Approximating clique-width and branch-width, submitted (2004).

[7] B. Courcelle, S. Oum, Vertex-minors, monadic second-order logic, and a conjecture by Sesse, submitted (2004).

[8] A. Bouchet, Circle graph obstructions, J. Combin. Theory Ser. B 60 (1) (1994) 107–144.

[9] R. Arratia, B. Bollabás, G. B. Sorkin, The interlace polynomial of a graph, J. Combin. Theory Ser. B 92 (2) (2004) 199–233.

[10] F. Genest, Word and set complementation of graphs, invertible graphs, submitted (2005).

[11] J. F. Geelen, A. M. H. Gerards, G. Whittle, Excluding a planar graph from GF(q)-representable matroids, manuscript (2003).
URL `http://homepages.cwi.nl/~bgerards/personal/papers/grid.ps`

[12] J. G. Oxley, Matroid theory, Oxford University Press, New York, 1992.

[13] N. Robertson, P. Seymour, R. Thomas, Quickly excluding a planar graph, J. Combin. Theory Ser. B 62 (2) (1994) 323–348.

[14] K. Murota, Matrices and matroids for systems analysis, Vol. 20 of Algorithms and Combinatorics, Springer-Verlag, Berlin, 2000.

[15] K. Truemper, Matroid decomposition, Academic Press Inc., Boston, MA, 1992.
URL `http://www.utdallas.edu/~klaus/mbook.html`

[16] K. Truemper, A decomposition theory for matroids. I. General results, J. Combin. Theory Ser. B 39 (1) (1985) 43–76.

[17] J. F. Geelen, A. M. H. Gerards, G. Whittle, Branch-width and well-quasi-ordering in matroids and graphs, J. Combin. Theory Ser. B 84 (2) (2002) 270–290.

[18] J. F. Geelen, A. M. H. Gerards, N. Robertson, G. Whittle, On the excluded minors for the matroids of branch-width k, J. Combin. Theory Ser. B 88 (2) (2003) 261–265.

[19] W. T. Tutte, Menger's theorem for matroids, J. Res. Nat. Bur. Standards Sect. B 69B (1965) 49–53.

[20] S. Oum, Approximating rank-width and clique-width quickly, submitted (2005).

[21] S. Iwata, L. Fleischer, S. Fujishige, A combinatorial strongly polynomial algorithm for minimizing submodular functions, Journal of the ACM (JACM) 48 (4) (2001) 761–777.

[22] H.-J. Bandelt, H. M. Mulder, Distance-hereditary graphs, J. Combin. Theory Ser. B 41 (2) (1986) 182–208.

[23] M. C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, Internat. J. Found. Comput. Sci. 11 (3) (2000) 423–443, selected papers from the Workshop on Theoretical Aspects of Computer Science (WG 99), Part 1 (Ascona).
URL `http://search.epnet.com/direct.asp?an=6618892&db=aph`

[24] A. Bouchet, Isotropic systems, European J. Combin. 8 (3) (1987) 231–244.

[25] A. Bouchet, Transforming trees by successive local complementations, J. Graph Theory 12 (2) (1988) 195–207.

[26] S. Oum, Rank-width and well-quasi-ordering, manuscript (2004).