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Ranked increments of a stable subordinator and ranked excursion lengths of Brownian motion

Andreas Lindell [*]and Anders Martin-Löf [†]

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Consider a stable subordinator $\{S(t)\}$ with characteristic exponent $0 < \alpha < 1$ and jumps, with increments $X(t) = S(t) - S(t-)$, occurring at random times. The first passage time when $S(t)$ crosses the value 1 is denoted by T. We study Laplace transforms of the ranked increments until time T. Also the tied down situation, conditioning on the event $S(T) \in (1, 1 + \epsilon)$, is considered. Using these Laplace transforms for the special case $\alpha = 1/2$, we study densities for ranked excursion lengths of free Brownian motion and the Brownian bridge. The densities are given in terms of infinite sums involving a special Kummer function. Accurate approximations, using only a few terms of the sums, are available for $0 \leq x \leq 1/2$. Since explicit formulae are available for $1/2 \leq x \leq 1$ we have a good description of the densities for all values.

KEYWORDS: Stable distribution; Brownian motion; Ranked excursion lengths; Ranked increments; Stable subordinator

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1. Introduction

Consider a stable subordinator $\{S(t)\}$ with characteristic exponent $0 < \alpha < 1$ and jumps, with increments $X(t) = S(t) - S(t-)$, occurring at random times. The first passage time when $S(t)$ crosses the value 1 is denoted by T. We are interested in the distributions of the ranked increments until time T, including the truncated increment at time T. Denote these ranked increments by $L_1 > L_2 > \ldots \geq 0$. The corresponding variables for the tied down situation, conditioning on the event $S(T) \in (1, 1+\epsilon)$, are denoted $L_1^* > L_2^* > \ldots \geq 0$. To obtain such results, we study a related situation and show that the results concerning the related situation are equal in distribution to the original situation. Let $S = S(1)$, and let $\{X_i\}_{i=1}^{\infty}$ be nonnegative iid random variables belonging to the domain of attraction of the stable random variable S. Define the sum $S_n = \sum_{i=1}^{n} X_i$ where the random variable X_i is referred to as the ith contribution to the sum. Since S is stable we have $\frac{S_n}{a_n} \xrightarrow{d} S$, $n \to \infty$, where $a_n = n^{1/\alpha}$. No centering is required for $0 < \alpha < 1$. Denote the ranked contributions $X_{(1)} > X_{(2)} > \ldots > X_{(n)} \geq 0$. The corresponding asymptotic random variables are denoted $\{V_i : \frac{X_{(i)}}{a_n} \xrightarrow{d} V_i, \quad i = 1, 2, \ldots\}$. The Laplace transforms of S/V_i are equivalent to the Laplace transforms of $1/L_i$. Corresponding results for the tied down situation are also presented.

The main contribution of this paper is as follows: (1) previous results are collected for easy reference; (2) a straightforward derivation of the Laplace transforms using a Poisson characterization is provided; (3) formulae for the densities are given also in the special case of ranked excursion lengths of free Brownian motion; (4) the results are illustrated graphically using accurate approximations for $0 \leq x \leq 1/2$ and explicit formulae for $1/2 \leq x \leq 1$.

There are many papers considering the Laplace transforms mentioned above. The earliest results, due to Darling, go back as far as the 50's. Darling's results have been generalized and extended by Wendel, Rosén, Pitman, Yor, Csáki, Hu, Lindell, Holst and others. Our ambition is to give an easy reference to the most important results, and also to provide an easily accessible proof of the main results. For the special case of Brownian motion and the Brownian bridge, there are either accurate approximation formulae or explicit formulae available for the densities, and thus we obtain a good description for all values. This is illustrated graphically. It is pointed out that, in principle, it is not more difficult to obtain similar results regarding densities for arbitrary $0 < \alpha < 1$. Our technique to derive the Laplace transforms is a straightforward calculation based on a result for stable random variables stating that the number of contributions of a certain length falling into a specified interval $(x, x+dx)$ is Poisson distributed with intensity specified by the Lévy measure $l(x)dx = \frac{c\alpha dx}{x^{\alpha+1}}$. In principle, it is the same technique as used by Pitman and Yor [14], but our calculations are more explicit. The technique used to calculate the densities is to invert the Laplace transforms by calculus of residues. The densities are given in terms of infinite sums involving a certain Kummer function. Using mathematical software, such as Maple, it is possible to find the 0s and to plot the densities.

Section 2 recapitulates general theory about stable distributions, Lévy processes and Brownian motion that is required in the following sections. Section 3 provides a survey of previous results. In Section 4 we derive the Laplace transforms and in Section 5 we describe how the results can be applied to ranked excursion lengths of Brownian motion. These results are also illustrated graphically. Concluding remarks are given in Section 6. Appendix A provides a detailed description of the Kummer function.

2. Preliminaries

2.1. Stable distributions

The definitions below follow the definitions in Feller [8]. Let $\{X_i\}_{i=1}^n$ be iid variables with the common distribution G and let $S_n = \sum_{i=1}^n X_i$.

Definition 1. The distribution G is *stable in the broad sense* if there exist constants $c_n > 0$, γ_n such that

$$S_n \stackrel{d}{=} c_n X + \gamma_n \tag{1}$$

and G is not concentrated at one point. G is *stable in the strict sense* if (1) holds with $\gamma_n = 0$. In the sequel we always assume that a stable distribution is stable in the strict sense.

The norming constants are of the form $c_n = n^{1/\alpha}$ with $0 < \alpha \leq 2$, see e.g. Feller [8]. The constant α is referred to as the characteristic exponent of G.

Definition 2. The distribution F of the independent random variables X_i belongs to the *domain of attraction* of a distribution G if there exist norming constants $a_n > 0$, b_n such that the distribution of $a_n^{-1}(S_n - b_n)$ tends to G.

The statement in Definition 2 can be reformulated to the effect that *a distribution G possesses a domain of attraction iff it is stable.*

Laplace transform of a stable distribution: The Laplace transform is a convenient tool when analyzing stable distributions. See e.g. Bondesson and Holst [2] for results on the weak convergence of iid sums to stable variables, using Laplace transforms. For a nonnegative random variable with a stable distribution, the Laplace transform is given by

$$\gamma_\alpha(\lambda) = \exp\left(-c\lambda^\alpha\right), \tag{2}$$

see e.g. Bondesson and Holst [2]. The parameter α is the characteristic exponent. For a stable variable S, we also have the Laplace transform

$$\mathbf{E}\left(e^{-\lambda S}\right) = \exp\left(-\int_0^\infty \left(1 - e^{-\lambda x}\right) l(x)dx\right), \tag{3}$$

where $l(x) = \frac{c\alpha}{x^{\alpha+1}}$, $x > 0, 0 < \alpha < 1$. See e.g. Resnick [15].

2.2. Lévy processes

The definitions below are from e.g. Kyprianou [10].

Definition 3. A process $\{Y(t)\}$, defined for $t \geq 0$, is said to be a *Lévy process* if it possesses the following properties:

1. The paths of Y are $\mathbf{P}$-almost surely cádlág.
2. $\mathbf{P}\left(Y(0) = 0\right) = 1$.
3. For $0 \leq s \leq t$, $Y(t) - Y(s)$ is equal in distribution to $Y(t-s)$.
4. For $0 \leq s \leq t$, $Y(t) - Y(s)$ is independent of $\{Y(u) : u \leq s\}$.

Definition 4. A *stable process* $\{S(t)\}$ is a Lévy process where $S(1)$ is a stable distribution with characteristic exponent α, and for all λ, $\{S(\lambda t) : t \geq 0\}$ has the same distribution as $\{\lambda^{1/\alpha} S(t) : t \geq 0\}$.

Definition 5. A *stable subordinator* $\{S(t)\}$ is a stable process where all increments are nonnegative.

2.3. Brownian motion

Definition 6. Standard *Brownian motion*, $\{B(t)\}$, is a Lévy process with normally distributed increments and $B(1) \sim \mathrm{N}(0,1)$.

Definition 7. The process $\{B^*(t)\}$ defined by

$$B^*(t) := B(t) - tB(1),\ 0 \le t \le 1,$$

is referred to as the *Brownian bridge*, i.e. a Brownian motion visiting the origin both at time 0 and 1.

The distribution with density

$$f(x) = \frac{1}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2x}\right), \quad x > 0 \tag{4}$$

is stable with $\alpha = 1/2$, see e.g. Feller [8]. Let $\{X_i\}_{i=1}^{\infty}$ be the lengths of the excursions in a symmetric simple random walk. The X_i's are iid random variables where the distribution is given by

$$X_i = 2Y_i, \quad \mathbf{P}(Y = k) = \frac{1}{2k-1}\binom{2k}{k}\frac{1}{2^{2k}},\ k = 1, 2, \dots\ ,$$

see e.g. Feller [7] page 78. Let $S_n = \sum_{i=1}^{n} X_i$ and $a_n = n^2$. Bondesson and Holst [2] give a proof of the following asymptotic result

$$\mathbf{P}\left(\frac{S_n}{a_n} \le x\right) \to \int_0^x f(y)dy, \quad n \to \infty, \tag{5}$$

where $f(y)$ is described by (4). Hence, the distribution of the random variables $\{X_i\}$ belongs to the domain of attraction of the stable distribution with characteristic exponent $\alpha = \frac{1}{2}$.

Definition 8. Let $\{B(s)\}$ be standard Brownian motion. Then,

$$T_t = \sup\{s \le t : B(s) = 0\}$$

is the time of the *last visit to the origin*, before time t.

The distribution of T_1 is given by an arc sine law and hence, the density function is

$$f_{T_1}(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad 0 \le x \le 1, \tag{6}$$

see e.g. Feller [7].

3. Survey of previous results

First, we provide results for the Laplace transforms or generating functions of the random random variables of interest. Then we proceed with explicit formulae for the distributions.

3.1. Transforms

First, we present a general result obtained by Wendel [16]. Wendel's set-up and notation is reformulated slightly here. We interpret the results for the special case of stable subordinators. Let $\{S(t)\}$ be a stable subordinator with characteristic exponent $0 < \alpha < 1$. Let Z be the set of values which the subordinator $S(t)$ assumes. The open set $[0, 1) \setminus Z$ is the union of countably many increments of the subordinator, which we take to be arranged in decreasing order of size L_n. Denote the first passage time, when $S(t)$ crosses the value 1, by T. We note that also the truncated increment at time T is included in the sequence $\{L_n\}$. Let $F_n(x) = \mathbf{P}(L_n \leq x)$ be the distribution functions corresponding to L_n. Define $C(\epsilon)$ as the event that Z meets the interval $(1, 1+\epsilon)$, i.e. $S(T) \in (1, 1+\epsilon)$, and say that the subordinator is tied down if we condition on the event $C(\epsilon)$. In the sequel we let the superscript $*$ distinguish the tied down situation from the free one. For the case where the process is tied down, let L_n^* be the increments in decreasing order and let $F_n^*(x) = \mathbf{P}(L_n^* \leq x) = \lim_{\epsilon \to 0} \mathbf{P}(L_n \leq x | C(\epsilon))$, $0 \leq x \leq 1$. Through this relation, Wendel links the tied down situation with the free one. Let

$$\Phi(\lambda) := \int_1^\infty e^{-\lambda y} \frac{\alpha dy}{y^{\alpha+1}}, \tag{7}$$

$$\Psi(\lambda) := 1 + \int_0^1 \left(1 - e^{-\lambda y}\right) \frac{\alpha dy}{y^{\alpha+1}} = \Phi(\lambda) + \lambda^\alpha \Gamma(1-\alpha). \tag{8}$$

$\Psi(\lambda) = M(-\alpha, 1-\alpha, -\lambda) = e^{-\lambda} M(1, 1-\alpha, \lambda)$ where the function $M(a, b, s)$ is known as the Kummer function, see Appendix A. Wendel obtains the following results for $0 < \alpha < 1$:

Theorem 3.1.

$$\int_0^\infty e^{-\lambda x} d(1 - F_n(1/x)) = \int_0^\infty e^{-\lambda/x} dF_n(x) = e^{-\lambda} \frac{\Phi(\lambda)^{n-1}}{\Psi(\lambda)^n}. \tag{9}$$

$$\int_0^\infty e^{-\lambda x} x^{\alpha-1} (1 - F_n^*(1/x)) dx = \frac{\Gamma(\alpha)\Phi(\lambda)^n}{\lambda^\alpha \Psi(\lambda)^n}. \tag{10}$$

Equation (9) is also recovered in the paper by Pitman and Yor [14] for stable subordinators using Poisson random measures. For related results, see Lamperti [11]. In Section 4, we present a proof of (9) following the same lines as in the paper by Pitman and Yor [14]; however in greater detail. By exploiting the random walk and weak convergence, Csáki and Hu [3] obtain (9) for the special case $\alpha = 1/2$. Invariance principles concerning the weak convergence are obtained in Csáki and Hu [4]. Using characteristic functions, Darling [5] provides a proof of (9) for the more general case $0 < \alpha \leq 2$ when $n = 1$. We also note that (10) is related to the results obtained by Lindell and Holst [12]; results which are valid for the tied down situation in the special case $\alpha = 1/2$. By exploiting weak convergence of the random walk and inverting generating functions, Lindell and Holst obtain explicit formulae for the *joint* and *marginal* distributions, see below and also Section 3.2. Introduce the Kummer function $K(\lambda) := M(-\frac{1}{2}, \frac{1}{2}, -\lambda)$. The following theorem for the Brownian bridge, due to Lindell and Holst, is their key result which gives the generating function for the number of excursions $\{N_j^*\}_{j=1}^r$ with lengths in the respective intervals $(x_j, x_{j-1}]$, $0 < x_r \leq x_{r-1} \leq \cdots \leq x_1 < x_0 = \infty$.

Theorem 3.2.

$$\mathbf{E}\left(z_1^{N_1^*} \cdots z_r^{N_r^*}\right) = \frac{1}{2i\sqrt{x_r}} \int_{-i\infty}^{i\infty} \frac{e^{\lambda/x_r}}{K(\lambda) - z_1 A(\lambda) - \sum_{j=2}^r z_j B_j(\lambda)} \, d\lambda$$

where

$$A(\lambda) = K\left(\frac{\lambda x_1}{x_r}\right)\sqrt{\frac{x_r}{x_1}} - \sqrt{\pi\lambda},$$

$$B_j(\lambda) = K\left(\frac{\lambda x_j}{x_r}\right)\sqrt{\frac{x_r}{x_j}} - K\left(\frac{\lambda x_{j-1}}{x_r}\right)\sqrt{\frac{x_r}{x_{j-1}}}.$$

3.2. Inverting the transforms

Wendel [16] provides a formula for the distribution of the largest increment of a stable subordinator, $0 < \alpha < 1$. The formula is obtained by inverting (10). The special case $\alpha = 1/2$ is due to Rosén, i.e the distribution for the largest excursion length of the Brownian bridge. Inspired by Roséns result, Lindell and Holst [12] investigate the joint and marginal distributions for the n largest excursion lengths of the Brownian bridge. Let $V_1^* > V_2^* > \ldots > V_n^* > 0$ be the ranked excursion lengths of the Brownian bridge. The following results are due to Rosén for $n = 1$, and due to Lindell and Holst for

$n = 2, 3$. The results are obtained by inversion of the generating function in Theorem 3.2.

Corollary 3.1. *The distributions of the three longest excursions of the Brownian bridge are given by the following expressions.*

n=1: For $0 \le x \le 1$

$$\mathbf{P}(V_1^* \le x) = \frac{2\pi}{\sqrt{x}} \sum_{k=-\infty}^{\infty} -\lambda_k e^{\lambda_k(1+\frac{1}{x})},$$

where the λ_k*'s are the* $0s$ *of* $K(\lambda)$*, see Appendix A.*

n=2: For $0 \le x \le 1/2$

$$\begin{aligned} \mathbf{P}(V_2^* \le x) &= 2\mathbf{P}(V_1^* \le x) \\ &+ 2\pi \sum_{k=-\infty}^{\infty} -\lambda_k e^{2\lambda_k} \left(2e^{\frac{\lambda_k}{x}} \left(1 + \lambda_k + \frac{\lambda_k}{x}\right) K\left(\frac{\lambda_k}{x}\right) - 1\right). \end{aligned}$$

n=3: For $0 \le x \le 1/3$

$$\begin{aligned} \mathbf{P}(V_3^* \le x) &= 3(\mathbf{P}(V_2^* \le x) - \mathbf{P}(V_1^* \le x)) \\ &- \frac{2\pi^2}{\sqrt{x}} \sum_{k=-\infty}^{\infty} \lambda_k^2 e^{\lambda_k(\frac{1}{x}+3)} \left(\lambda_k^2 \left(\frac{2}{x^2} + \frac{6}{x} + 4\right) + \lambda_k \left(\frac{7}{x} + 9\right) + 3\right). \end{aligned}$$

n=1,2: For $0 \le x_2 \le x_1 \le 1,\ 0 \le x_2 \le 1/2$

$$\begin{aligned} &\mathbf{P}(V_1^* \le x_1, V_2^* \le x_2) = 2\mathbf{P}(V_1^* \le x_2) \\ &+ \frac{2\pi}{\sqrt{x_1}} \sum_{k=-\infty}^{\infty} -\lambda_k e^{\lambda_k\left(\frac{1-x_1}{x_2}+2\right)} \left(2e^{\lambda_k \frac{x_1}{x_2}} K\left(\frac{\lambda_k x_1}{x_2}\right) \left(1 + \lambda_k + \frac{\lambda_k}{x_2}\right) - 1\right). \end{aligned}$$

Using only a few terms of the sums, these expressions give accurate approximations for $0 \le x \le 1/2$, see Lindell [13]. Explicit formulae for the density of the longest excursion length of the Brownian bridge, V_1^*, are obtained by e.g Wendel [16], Pitman and Yor [14], and Gourdon [9]. These explicit formulae are valid for the interval $1/4 \le x \le 1$. In particular, the density for the longest excursion of the Brownian bridge, $h_1^*(x)$, is given by

$$h_1^*(x) = \frac{1}{2x^{3/2}}, \quad 1/2 \le x \le 1. \tag{11}$$

In Section 5, for the special case $\alpha = 1/2$, we calculate the densities also for the ranked excursion lengths of free Brownian motion, on the unit interval. The technique used to invert the Laplace transforms is the same technique as used by Lindell and Holst [12]. It is a straightforward calculation of integrals using calculus of residues and a close examination of the special Kummer function. As expected, the results for the Brownian motion and for the Brownian bridge do not coincide, see graphical illustrations in Section 5.2.

We note here that it should also be possible to invert the transforms for arbitrary $0 < \alpha < 1$. The same procedure may be used, and the 0s for the corresponding Kummer functions are required in the formulae.

4. Ranked contributions – Derivation of Laplace transforms using a Poisson characterization

The convergence of the normalized sum of iid variables to a stable distribution can be conveniently analyzed in a Poisson process framework, see e.g. Section 2.7 in Durrett [6]. For a description of the Poisson characterization of the increments of a Lévy process, see e.g. Section 2.2 in Kyprianou [10].

It turns out that $\{\frac{X_{(i)}}{a_n}\}_{i=1}^n$ converges to a Poisson process $\{V_i\}_{i=1}^\infty$ where the intensity is specified in terms of the Lévy measure

$$l(v)dv = \frac{c\alpha dv}{v^{\alpha+1}}, \quad v > 0. \tag{12}$$

Recall that $\{V_i\}_{i=1}^\infty$ are the ranked contributions to S, $V_1 > V_2 \cdots > V_n > \ldots > 0$. Using the convergence to a Poisson process, it is possible to prove

$$\frac{S_n}{a_n} \to S, \quad n \to \infty, \tag{13}$$

where $S = \sum_{i=1}^\infty V_i$.

Below, we derive the Laplace transforms of the random variables $\frac{S}{V_i}$. First we provide a proof for the special case $\frac{S}{V_1}$ and then we proceed with the general case. **Largest contribution:** We have

$$\frac{X_{(1)}}{a_n} \to V_1, \quad n \to \infty, \tag{14}$$

where V_1 is a non-degenerate distribution having

$$\mathbf{P}\left(V_1 \le v\right) = \mathbf{P}\left(\text{no points} > v\right) = \exp - \int_v^\infty l(v)dv = \exp\left(-cv^{-\alpha}\right).$$

Hence,

$$\mathbf{P}\left(V_1 \in dv\right) = l(v)dv \exp\left(-cv^{-\alpha}\right).$$

Recalling the Laplace transform for S, given by (3), conditioning on the event $V_1 \in dv$, we have

$$\begin{aligned}
\mathbf{E}\left(e^{-\lambda S}|V_1 \in dv\right) &= \exp\left(-\lambda v + \int_0^v \left(e^{-\lambda y} - 1\right) l(y)dy\right) \\
&= \exp\left(-\lambda v + \int_0^v \left(e^{-\lambda y} - 1\right) \frac{c\alpha dy}{y^{\alpha+1}}\right) \\
&= \{\text{Change of variables } y \to vy\} \\
&= \exp\left(-\lambda v + \int_0^1 \left(e^{-\lambda vy} - 1\right) \frac{c\alpha dy}{v^\alpha y^{\alpha+1}}\right).
\end{aligned}$$

Hence,

$$\mathbf{E}\left(e^{-\lambda \frac{S}{v}}|V_1 \in dv\right) = \exp\left(-\lambda + \int_0^1 \left(e^{-\lambda y} - 1\right) \frac{c\alpha dy}{y^{\alpha+1}v^\alpha}\right)$$

and

$$\begin{aligned}
&\mathbf{E}\left(e^{-\lambda \frac{S}{V_1}}\right) = \mathbf{E}\left(\mathbf{E}\left(e^{-\lambda \frac{S}{v}}|V_1 \in dv\right)\right) = \int_0^\infty \mathbf{E}\left(e^{-\lambda \frac{S}{v}}|V_1 \in dv\right)\mathbf{P}\left(V_1 \in dv\right) \\
&= \int_0^\infty \frac{c\alpha dv}{v^{\alpha+1}} \exp\left(-cv^{-\alpha} - \lambda + \int_0^1 \left(e^{-\lambda y} - 1\right) \frac{c\alpha dy}{y^{\alpha+1}v^\alpha}\right) \\
&= \{\text{Change of variables } t = cv^{-\alpha}\} \\
&= e^{-\lambda} \int_0^\infty dt \exp t \left[\int_0^1 \left(e^{-\lambda y} - 1\right) \frac{\alpha}{y^{\alpha+1}} dy - 1\right] \\
&= \frac{e^{-\lambda}}{1 + \int_0^1 \left(1 - e^{-\lambda y}\right) \frac{\alpha dy}{y^{\alpha+1}}} = \frac{e^{-\lambda}}{\Psi(\lambda)}.
\end{aligned}$$

Note that the expression above does not involve c.

n largest contributions: Let

$$S = \sum_{i=1}^\infty V_i = \sum_{i=1}^n V_i + R,$$

We have

$$\mathbf{P}(V_1 \in dv_1, \cdots, V_n \in dv_n) = l(v_n)dv_n \cdots l(v_1)dv_1 \exp\left(-cv_n^{-\alpha}\right), \quad 0 < v_n < \cdots < v_1.$$

Conditional on $V_n \in dv_n$, the points $V_1, \ldots, V_{n-1}$ are distributed over the interval

$[v_n, \infty)$ and can be arranged in $(n-1)!$ ways. Thus, for $V_n \in dv_n$, we have

$$\begin{aligned}
& \mathbf{E}\left(e^{-\lambda \sum_{i=1}^{n-1} V_i}, V_n \in dv_n\right) = \\
= & \int_{v_n < v_{n-1} < \cdots < v_1} e^{-\lambda \sum_{i=1}^{n-1} v_i} l(v_n) dv_n l(v_{n-1}) dv_{n-1} \cdots l(v_1) dv_1 \exp\left(-c v_n^{-\alpha}\right) \\
= & \exp\left(-c v_n^{-\alpha}\right) \frac{\left(\int_{v_n}^{\infty} e^{-\lambda v} l(v) dv\right)^{n-1}}{(n-1)!} l(v_n) dv_n .
\end{aligned}$$

Conditional on $V_n \in dv_n$, $R = \sum_{i=n+1}^{\infty} V_i$ is independent of the expression above. Recall the Laplace transform for $S = \sum_{i=1}^{\infty} V_i$, given by (3). We have

$$\begin{aligned}
& \mathbf{E}\left(e^{-\lambda S}, V_n \in dv_n\right) = \\
= & e^{-\lambda v_n} \exp\left(-\int_0^{v_n} \left(1 - e^{-\lambda v}\right) l(v) dv\right) \exp\left(-c v_n^{-\alpha}\right) \frac{\left(\int_{v_n}^{\infty} e^{-\lambda v} l(v) dv\right)^{n-1}}{(n-1)!} l(v_n) dv_n ,
\end{aligned}$$

and, dividing by v_n,

$$\begin{aligned}
& \mathbf{E}\left(e^{-\lambda \frac{S}{v_n}}, V_n \in dv_n\right) = \\
= & \exp\left(-\lambda - \int_0^{v_n} \left(1 - e^{-\lambda \frac{v}{v_n}}\right) l(v) dv\right) \exp\left(-c v_n^{-\alpha}\right) \frac{\left(\int_{v_n}^{\infty} e^{-\lambda \frac{v}{v_n}} l(v) dv\right)^{n-1}}{(n-1)!} l(v_n) dv_n \\
= & \{\text{Change of variables } v = v_n y\} \\
= & \exp\left(-\lambda - v_n^{-\alpha} \int_0^1 \left(1 - e^{-\lambda y}\right) l(y) dy\right) \exp\left(-c v_n^{-\alpha}\right) \frac{\left(\int_1^{\infty} e^{-\lambda y} l(y) dy\right)^{n-1}}{v_n^{\alpha(n-1)} (n-1)!} l(v_n) dv_n \\
= & \exp\left(-\lambda - c v_n^{-\alpha} \left[\alpha \int_0^1 \left(1 - e^{-\lambda y}\right) \frac{dy}{y^{\alpha+1}} + 1\right]\right) \frac{\left(c\alpha \int_1^{\infty} e^{-\lambda y} y^{-\alpha-1} dy\right)^{n-1}}{v_n^{\alpha(n-1)} (n-1)!} \frac{c\alpha dv_n}{v_n^{\alpha+1}} .
\end{aligned}$$

We need the following property of the Gamma function. For all integers n,

$$\int_0^{\infty} e^{-y\gamma} y^{n-1} dy = \frac{\Gamma(n)}{\gamma^n} = \frac{(n-1)!}{\gamma^n} .$$

Integrating over v_n, and substituting $t = cv_n^{-\alpha}$, we have

$$
\begin{aligned}
& \mathbf{E}\left(e^{-\lambda \frac{S}{V_n}}\right) = \\
= \quad & \int_0^\infty \exp\left(-\lambda - t\left[\alpha \int_0^1 \left(1-e^{-\lambda y}\right)\frac{dy}{y^{\alpha+1}} + 1\right]\right) t^{n-1} \frac{\left(\alpha \int_1^\infty e^{-\lambda y} y^{-\alpha-1} dy\right)^{n-1}}{(n-1)!} dt \\
= \quad & \{\text{Let } \gamma = 1 + \alpha \int_0^1 \left(1-e^{-\lambda y}\right)\frac{dy}{y^{\alpha+1}}\} \\
= \quad & e^{-\lambda} \int_0^\infty e^{-t\gamma} t^{n-1} dt \frac{\left(\int_1^\infty e^{-\lambda y} y^{-\alpha-1} \alpha dy\right)^{n-1}}{(n-1)!} \\
= \quad & \frac{e^{-\lambda}\left(\int_1^\infty e^{-\lambda y}\frac{\alpha dy}{y^{\alpha+1}}\right)^{n-1}}{\left(1+\int_0^1 \left(1-e^{-\lambda y}\right)\frac{\alpha dy}{y^{\alpha+1}}\right)^n} = e^{-\lambda}\frac{\Phi(\lambda)^{n-1}}{\Psi(\lambda)^n}.
\end{aligned}
$$

Note again that the expression above does not involve c. We have the same expression as in the first part of Theorem 3.1, due to Wendel. Note that Wendel studied the ranked increments of a stable subordinator until the first passage time T, including the truncated increment at time T. Here, we have derived the Laplace transform for the ranked increments contributing to the value of the stable subordinator $S = S(1)$, i.e. at time 1. Obviously, since the expressions for the Laplace transforms coincide, the distributions of both cases agree. For a complete alternative proof of Wendel's result we must provide a rigorous argument proving that it is sufficient to consider one of the situations in order to obtain a proof also for the other. Fortunately, such a result is provided by Proposition 6 in Pitman and Yor [14]. See also further references therein.

5. Ranked excursion lengths of Brownian motion

5.1. Densities for ranked excursion lengths of Brownian motion

The zero-free intervals of Brownian motion in the unit interval can be described as consisting of two parts; the first part is a Brownian bridge and the second part is a path starting at the origin ending up on either the positive side or the negative side without touching the time axis. The time of the last visit to the origin is described by the density in (6). This is the special case of Wendel's results for a stable subordinator, where also the truncated increment is included in the sequence $\{L_n\}$. Integrating over possible values for the last visit to the origin, using (6) and (11), we have the explicit density

$$h_1(x) = \frac{1}{\pi x^{3/2}\sqrt{1-x}}, \quad 1/2 \le x \le 1, \tag{15}$$

for the longest excursion of Brownian motion in the unit interval. Consider the special case of Theorem 3.1 where $\alpha = \frac{1}{2}$. Recall $K(\lambda) := M(-\frac{1}{2}, \frac{1}{2}, -\lambda)$. Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we have

$$\mathbf{E}\left(e^{-\lambda \frac{S}{V_n}}\right) = e^{-\lambda} \frac{\left(K(\lambda) - \sqrt{\lambda\pi}\right)^{n-1}}{K(\lambda)^n}. \tag{16}$$

It is possible to calculate the densities $g_n(x)$ of the random variables $\frac{S}{V_n}$ by the Laplace inversion formula, i.e.

$$g_n(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda(x-1)} \frac{\left(K(\lambda) - \sqrt{\lambda\pi}\right)^{n-1}}{K(\lambda)^n} d\lambda. \tag{17}$$

These integrals are evaluated in Lindell and Holst [12] for $n = 1, 2, 3$ by calculus of residues. Here we only perform the calculations for $n = 1$ to describe the technique. The densities $h_n(x)$ of the random variables $\frac{V_n}{S}$ are easily obtained by the transformation theorem. We obtain the following corollary.

Corollary 5.1. *The densities of* $\frac{1}{L_1} \stackrel{d}{=} \frac{S}{V_1}$ *and* $L_1 \stackrel{d}{=} \frac{V_1}{S}$ *respectively are*

$$g_1(x) = -2\sum_{k=-\infty}^{\infty} \lambda_k e^{\lambda_k x}, \quad 1 \le x \le \infty.$$

$$h_1(x) = -2\sum_{k=-\infty}^{\infty} \lambda_k \frac{e^{\lambda_k/x}}{x^2}, \quad 0 \le x \le 1.$$

Proof. By the Residue theorem

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{z_k \text{ inside } \Gamma} \operatorname{Res}(f; z_k),$$

where the sum is taken over all those singularities z_k of f that lie inside the closed contour Γ. Consider a contour Γ consisting of the parts $-iA < \lambda < iA$ and $\{\lambda = Ae^{i\Theta}, \frac{\pi}{2} \le \Theta \le \frac{3\pi}{2}\}$. Letting $A \to \infty$ and applying the Residue theorem, we can calculate the integral in (17) for $n = 1$.

$$g_1(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{\lambda(x-1)}}{K(\lambda)} d\lambda = \sum_{\lambda_k \text{ inside } \Gamma} \operatorname{Res}\left(\frac{e^{\lambda(x-1)}}{K(\lambda)}; \lambda_k\right).$$

The residues are given by $\{\frac{e^{\lambda_k}}{\frac{d}{d\lambda}K(\lambda)|_{\lambda=\lambda_k}}, \quad k \text{ integer valued}\}$, where $\frac{d}{d\lambda}K(\lambda_k) = \frac{-e^{-\lambda_k}}{2\lambda_k}$. See Appendix A. Since the results hold asymptotically for the normalized excursion lengths of the simple symmetric random walk, we have proved the assertion for Brownian motion. The transformation theorem finishes the proof of the assertion, for $h_1(x)$.

Corollary 5.2. *The densities of* $\frac{1}{L_2} \stackrel{d}{=} \frac{S}{V_2}$ *and* $L_2 \stackrel{d}{=} \frac{V_2}{S}$ *respectively are:*
For $2 \le x \le \infty$

$$\begin{aligned} g_2(x) &= g_1(x) - \frac{2}{\sqrt{x-1}} \\ &\times \sum_{k=-\infty}^{\infty} \lambda_k e^{2\lambda_k} \left(2e^{\lambda_k(x-1)}\left(1+\lambda_k x\right) K\left(\lambda_k(x-1)\right) - 1\right). \end{aligned}$$

For $0 \le x \le 1/2$

$$\begin{aligned} h_2(x) &= h_1(x) - \frac{2}{x^{3/2}\sqrt{1-x}} \\ &\times \sum_{k=-\infty}^{\infty} \lambda_k e^{2\lambda_k} \left(2e^{\lambda_k(1/x-1)}\left(1+\frac{\lambda_k}{x}\right) K\left(\lambda_k(1/x-1)\right) - 1\right). \end{aligned}$$

Proof. Identifying previously calculated integrals, we have

$$\begin{aligned} g_2(x) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda(x-1)} \frac{\left(K(\lambda) - \sqrt{\lambda\pi}\right)}{K(\lambda)^2} d\lambda \\ &= g_1(x) - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda(x-1)} \frac{\sqrt{\lambda\pi}}{K(\lambda)^2} d\lambda. \end{aligned}$$

For the evaluation of the remaining integral, we refer to Lindell and Holst [12]. The transformation theorem finishes the proof of the assertion, for $h_2(x)$.

Corollary 5.3. *The densities of* $\frac{1}{L_3} \stackrel{d}{=} \frac{S}{V_3}$ *and* $L_3 \stackrel{d}{=} \frac{V_3}{S}$ *respectively are:*
For $3 \le x \le \infty$

$$\begin{aligned} g_3(x) &= 2g_2(x) - g_1(x) \\ &- 2\pi \sum_{k=-\infty}^{\infty} \lambda_k^2 e^{\lambda_k(x+2)} \left(2\lambda_k^2(x+x^2) + \lambda_k(2+7x) + 3\right). \end{aligned}$$

For $0 \le x \le 1/3$

$$\begin{aligned} h_3(x) &= 2h_2(x) - h_1(x) \\ &- \frac{2\pi}{x^2} \sum_{k=-\infty}^{\infty} \lambda_k^2 e^{\lambda_k(1/x+2)} \left(2\lambda_k^2(1/x+(1/x)^2) + \lambda_k(2+7/x) + 3\right). \end{aligned}$$

Proof. Identifying previously calculated integrals, we have

$$\begin{aligned} g_3(x) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda(x-1)}\frac{\left(K(\lambda)-\sqrt{\lambda\pi}\right)^2}{K(\lambda)^3}d\lambda \\ &= 2g_2(x)-g_1(x)+\frac{1}{2i}\int_{-i\infty}^{i\infty} e^{\lambda(x-1)}\frac{\lambda}{K(\lambda)^3}d\lambda. \end{aligned}$$

For the evaluation of the remaining integral, we refer to Lindell and Holst [12]. The transformation theorem finishes the proof of the assertion, for $h_3(x)$.

The densities $g_n(x)$ and $h_n(x)$ for arbitrary n can also be obtained by calculus of residues. These calculations are performed in the same way. More details on the application of the Residue theorem are given in Lindell [13]. Appendix A gives values for the 0s that are required in the formulae for the densities. Lindell [13] analyzes the accuracy of the approximations when a limited number of terms are included in the sum. Using only a few terms of the sums, the approximations are accurate for values $0 \le x \le 1/2$.

5.2. Descriptive statistics and graphical illustrations

Using Corollaries 3.1, 5.1, 5.2, 5.3, it is possible to plot approximations of the density functions for the three longest excursions of the Brownian bridge and Brownian motion, in the unit interval. Equations (11), (15), give the exact densities for the longest excursion of the Brownian bridge and Brownian motion respectively, for $1/2 \le x \le 1$.

Approximations of the densities for the longest excursion of Brownian motion, $h_1(x)$, and of the Brownian bridge, $h_1^*(x)$, are plotted in Figure 1. Only the first term of each sum is used for $0 \le x \le 1/2$, and the explicit expressions are used for $1/2 < x \le 1$. We see that $h_1^*(x)$ is shifted to the left relative to $h_1(x)$. Indeed, it is reasonable to expect that, on average, the longest excursion of free Brownian motion exceeds the longest excursion of the Brownian bridge, since the Brownian bridge is tied down at $t = 1$. The density for $h_1(x)$ increases sharply just before $t = 1$ which indicates that it is not unlikely that the longest excursion of free Brownian motion essentially covers the whole interval.

The densities $h_2(x)$, $h_2^*(x)$, $h_3(x)$, $h_3^*(x)$, are plotted in Figures 2, 3. Since there is a probability that the longest excursion, of free Brownian motion, covers the whole interval, there is a probability that the second and third longest excursions are of

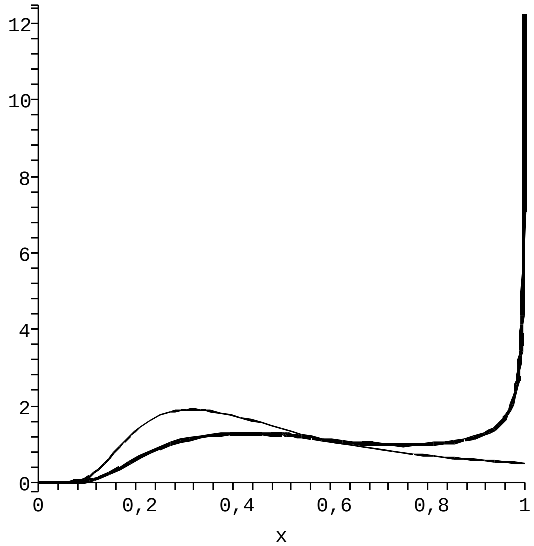

FIGURE 1: Density functions of the longest excursion of free Brownian motion (thick line) and the Brownian bridge (thin line).

length 0. It is interesting that all three densities for the Brownian bridge are smooth and unimodal while the corresponding densities for the free Brownian motion are not as smooth, and for some cases not even unimodal.

The expected values and variances of the variables L_1, L_2, L_1^*, L_2^*, are described by the following table.

Variable	Expected value	Variance
$L_1 \stackrel{d}{=} \frac{V_1}{S}$	0.626...	0.064...
$L_2 \stackrel{d}{=} \frac{V_2}{S}$	0.139...	0.011...
$L_1^* \stackrel{d}{=} V_1^*$	0.483...	0.049...
$L_2^* \stackrel{d}{=} V_2^*$	0.156...	0.007...

6. Concluding remarks

Laplace transforms of the ranked increments of stable subordinators, with characteristic exponent $0 < \alpha < 1$, are presented along with a straightforward proof. The tied down situation is also studied. By inversion of the Laplace transforms, using

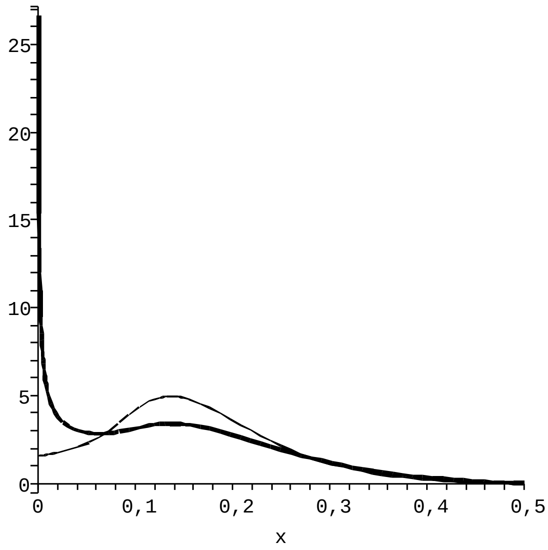

FIGURE 2: Density functions of the second longest excursion of free Brownian motion (thick line) and the Brownian bridge (thin line).

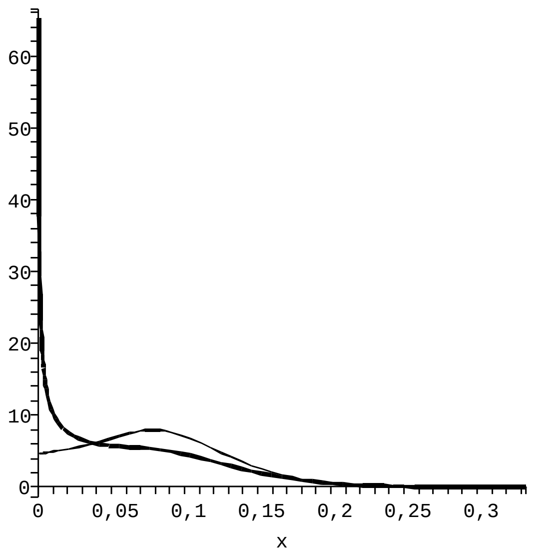

FIGURE 3: Density functions of the third longest excursion of free Brownian motion (thick line) and the Brownian bridge (thin line).

calculus of residues, explicit formulae for the densities are calculated. The expressions for the densities are described by infinite sums involving a certain Kummer function. In particular, the 0s are important. For the special case, $\alpha = 1/2$, we obtain results regarding ranked excursion lengths of free Brownian motion and the Brownian bridge. Marginal distributions as well as joint distributions are explicit for the Brownian bridge. Marginal densities are explicit for free Brownian motion. Thus, it is still an open issue to obtain results concerning *joint* distributions for arbitrary α, including the special case $\alpha = 1/2$ for the free Brownian motion. Since joint densities have been derived for the Brownian bridge, and since marginal densities for arbitrary α should be possible to obtain by similar calculations as for the special case $\alpha = 1/2$, we expect that explicit formulae can be written down also for the general cases. As a final remark, we would like to stress the fact that approximations using only a few terms of the infinite sums are sufficient for accurate results, for $0 \le x \le 1/2$. Since there are exact formulae for $1/2 \le x \le 1$, it is possible to obtain an accurate description of the densities for all values.

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Appendix A. Properties of the Kummer function

The general Kummer (or confluent hypergeometric) function, see Abramovitz and Stegun [1] page 304, is defined as

$$M(a,b,s) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)} \frac{s^k}{k!}.$$

Note that $M(a,b,s) = e^s M(b-a,b,-s)$. We consider in particular

$$K(s) = M\left(-\frac{1}{2},\frac{1}{2},-s\right) = 1 - \sum_{k=1}^{\infty} \frac{1}{2k-1}\frac{(-s)^k}{k!}$$
$$= e^{-s} M\left(1,\frac{1}{2},s\right) = e^{-s}\left(1 + \sum_{k=1}^{\infty} \frac{(2s)^k}{(2k-1)!!}\right)$$

with the derivatives

$$K'(s) = (K(s) - e^{-s})/2s, \quad K''(s) = (e^{-s} - K'(s))/2s.$$

Further derivatives are easy to obtain recursively. All 0s $\{s_k\}_{-\infty}^{\infty}$ of $K(s)$ lie in the left complex plane. In fact

$$\overline{s}_{-k} = s_k = -a_k + ib_k,\ k = 0, 1, 2, \ldots,$$

$$a_0 < a_1 < a_2 < \ldots \nearrow +\infty, \quad b_0 < b_1 < b_2 < \ldots \nearrow +\infty.$$

The first five 0s are given in the following table. The numbers have been calculated using Maple. See also plot of 0s in Figure 4.

k	$a_{\pm k}$	b_k	b_{-k}
0	0.8540326566...	0	0
1	4.248920779...	6.383124294...	-6.383124294...
2	5.184114730...	12.88530517...	-12.88530517...

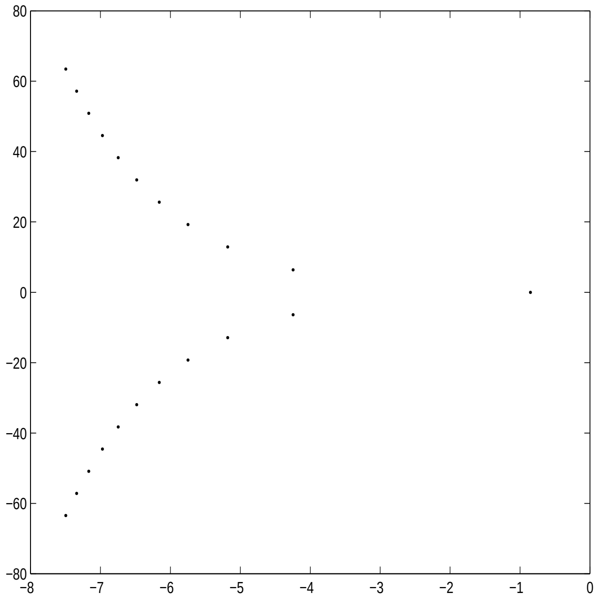

FIGURE 4: The first 21 zeros of $K(s)$ plotted in the complex plane.

According to Wendel [16],

$$a_k \sim 3\ln(k), \quad b_k \sim 2\pi k, \quad k \to \infty.$$

The 0s are further investigated in Lindell [13].

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