RANKIN-SELBERG L FUNCTIONS IN THE LEVEL ASPECT

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Abstract. In this paper we calculate the asymptotics of various moments of the central values of Rankin-Selberg convolution L functions of large level, thus generalizing the results and methods of Duke-Friedlander-Iwaniec and the authors. Consequences include convexity-breaking bounds, non-vanishing of a positive proportion of central values, and linear independence results for certain Hecke operators.

Contents

1. Introduction

In this paper, we continue the program sketched in [KMV2] on the central values of L-functions in the level aspect, building on the method developed by Duke, Friedlander, and Iwaniec in [DFI II] and then further refined in [KMV2].

Let D be square-free, and fix g a modular form (not necessarily cuspidal or holomorphic) on $\Gamma_0(D)$. Let $k \geqslant 2$ be even and let q be coprime to D. We let $S_k^*(q)$ denote the set of primitive cuspidal newforms on $\Gamma_0(q)$ with trivial nebentypus and weight k. In this paper we investigate the distribution of the values of the Rankin-Selberg convolution L-functions $\{L(f \otimes g, s)\}_{f \in S_k^*(q)}$ for s on the critical line ($\Re e s = \frac{1}{2}$) as q grows. Recall that $L(f \otimes g, s)$ admits an analytic continuation

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to all of $\mathbf{C}$ and a functional equation of the form

$$L_\infty(f\otimes g,s)L(f\otimes g,s)=\varepsilon(f\otimes g)(\frac{qD}{4\pi})^s L_\infty(f\otimes g,1-s)L(f\otimes \overline{g},1-s)$$

where $L_\infty(f\otimes g,s)$ is an explicit product of Γ factors (see Section 4) and $\varepsilon(f\otimes g)$ is called the root number. When $\varepsilon(f\otimes g)$ depends only on g and q we shall denote it as $\varepsilon(g)$.

One important challenge in the theory of L functions is to provide upper bounds for their values on the critical line, in particular to improve, by a positive exponent, the "convexity" bound arising from the Phragmen-Lindelöf principle (see [IS2] for a motivating introduction to these questions). For our first result, we do so, in the q aspect (see [DFI II, DFI I, CI] for some other examples):

Theorem 1.1. *Let g be a primitive cuspidal holomorphic newform of integral weight, or a non-exceptional weight zero Maass form, on $\Gamma_1(D)$ with D square-free. Then for all $\epsilon>0$, all integers $j\geqslant 0$, and all $f\in S_k^*(q)$ with $(q,D)=1$,*

$$|L^{(j)}(f\otimes g,\tfrac{1}{2}+it)|\ll_{\epsilon,k,j,g}(1+|t|)^B q^{\epsilon+1/2-1/80},$$

where the exponent B is absolute.

As one can check from the proof, the constant involved in $\ll_{\epsilon,k,j,g}$ depends polynomially on the parameters of g (the level, the weight, or the eigenvalue).

Remark. In the above Theorem, a Maass form is called "non-exceptional" if its eigenvalue under $-\Delta$ is greater than $1/4$. According to a conjecture of Selberg, exceptional Maass forms do not exist. For g exceptional, a weaker but still convexity-breaking bound can be obtained but we have preferred to limit ourselves to the simplest (and presumably only) case.

Theorem 1.1 can be seen as the generalization to the cuspidal case of a famous convexity-breaking result of Duke-Friedlander-Iwaniec [DFI II]:

Theorem 1.2. *Let χ_D be a primitive character of conductor D. Then for all $\epsilon>0$, all integers $j\geqslant 0$, and for all $f\in S_k^*(q)$ one has*

$$|L^{(j)}(f\otimes\chi_D,\tfrac{1}{2}+it)|^2\ll_{\epsilon,k,j}D^B(1+|t|)^B q^{\epsilon+1/2-1/96},$$

where the exponent B is absolute.

To see the analogy between Theorem 1.1 and this one, note that for χ_D primitive there is a (non-holomorphic, weight zero) Eisenstein series E_{χ_D} on $\Gamma_0(D^2)$ with nebentypus χ_D^2 such that $L(f\otimes E_{\chi_D},s)=L(f\otimes\chi_D,s)^2$. The slightly better exponent $1/80$ of Theorem 1.1 also applies in Theorem 1.2: it is the result of exploiting some extra averaging in the large sieve inequality of Section 7.1.

Remark. In fact our result is a little bit more general: one can also break convexity for f's with non-trivial nebentypus as long as the conductor of the nebentypus is relatively small with respect to q, we refer the reader to Theorem 7.2 for the general statement. Moreover, by using slightly more sophisticated arguments, the same method yields the convexity-breaking bound without any assumption on D [Mi].

As a corollary, we obtain the first unconditional improvement of the "trivial bound" for the problem of distinguishing modular forms by their first Fourier coefficients. This analogue of the smallest quadratic non-residue problem was suggested to us by P. Sarnak.

Corollary 1.3. *Let g be a primitive cusp form as in Theorem 1.1 with $D \geqslant 1$, $k \geqslant 2$ an even integer and $\varepsilon > 0$. There exists a constant $C = C(g, k, \varepsilon)$ depending on g, k and ε only such that for any primitive holomorphic form $f \in S_k^*(q)$, there exists $n \leqslant Cq^{1-1/40+\varepsilon}$ such that*

$$\lambda_f(n) \neq \lambda_g(n). \tag{1.1}$$

Here the $\lambda_f(n)$, $\lambda_g(n)$ are the Fourier coefficients of f and g.

The "obvious" bound is $n \leqslant Cq^{1+\varepsilon}$ for any $\varepsilon > 0$. Under the generalized Riemann Hypothesis for Rankin-Selberg L-functions, (1.1) would be true for $n = C(\log qD)^2$, with an absolute C. Using modularity, this applies in particular to elliptic curves. See [DK] for a related result "on average".

Let us mention another possible application of Theorem 1.1: P. Sarnak has recently obtained a convexity-breaking bound for Ranking-Selberg convolution L-functions in the weight aspect, using somewhat different techniques [Sa]. His main motivation was the Quantum Unique Ergodicity problem: for $f \in S_k^*(q)$, let μ_f be the probability measure on $X_0(q)$ given by $\mu_f = \frac{3}{\pi}\frac{|f(z)|^2}{(f,f)}y^k\frac{dxdy}{y^2}$ where (f, f) is the Petersson inner product.

Conjecture 1.4. [RuSa] *Let $q \geqslant 1$. Let $\{f_j\}_{j\geqslant 1}$ be any sequence of primitive holomorphic forms on $X_0(q)$ with increasing weights k_j. As $j \to +\infty$ the sequence of probability measure μ_{f_j}, $j \geqslant 1$ converge weakly to the Poincaré measure on $X_0(q)$, $\mu = \frac{1}{vol(X_0(q))}\frac{dxdy}{y^2}$.*

The main application of the convexity-breaking bound of [Sa] is the proof of this conjecture when the primitive forms are of CM type (in the sense of [Ri]). The proof uses a formula of T. Watson [W], relating the central value of the triple product L function of 3 modular forms to the square of the integral of the product of the 3 forms. This formula shows that proving a non-trivial estimate for the Weyl sums corresponding to this equidistribution problem is tantamount to proving a convexity-breaking bound (in the weight aspect) for the central value of the triple product L function of $f_j \otimes f_j \otimes g$ where g is a weight 0 Maass form (or even an Eisenstein series). For f_j a CM form this reduces to a convexity-breaking bound for a Rankin-Selberg L function.

A possible analog of conjecture 1.4 in the level aspect is the following: for any $q \geqslant 1$ let $\pi_q : X_0(q) \to X_0(1)$ be the canonical projection

Conjecture 1.5. *For $k \geqslant 2$ even and fixed, let $\{f_j\}_{j\geqslant 1}$ be any sequence of primitive holomorphic forms of weight k with increasing levels q_j. As $j \to +\infty$ the sequence of probability measure $\pi_{q_j,*}(\mu_{f_j})$, $j \geqslant 1$ converge weakly to the Poincaré measure on $X_0(1)$, $\mu = \frac{3}{\pi}\frac{dxdy}{y^2}$.*

Although it has not yet been fully established in this context, it is likely that Watson's formula continues to hold, so that, combined with Theorem 1.1, it should give a proof of Conjecture 1.5, when the f_j are CM forms.

Theorem 1.1 follows from the *amplification* method invented by Friedlander and Iwaniec, applied to bounds of the form

$$\mathcal{M}_{g,\overline{g}}(\tfrac{1}{2} + \mu; \ell) = \frac{1}{|S_k^*(q)|}\sum_{f\in S_k^*(q)} |L(f \otimes g, \tfrac{1}{2} + \mu)|^2\lambda_f(\ell) \ll_{\epsilon,g} (1 + |t|)^B\frac{q^\epsilon}{\ell^{1/2}}$$

for $\mu = \delta + it$, $\delta, t \in \mathbf{R}$, $|\delta| \leqslant 1/\log q$, and $0 < \ell < q^\alpha$ for some small $\alpha > 0$. We prove these bounds in the process of obtaining an explicit asymptotic expansion of this second moment. While we prove the bounds in the general case, we calculate the asymptotics only for the more restrictive case when q is prime and $k < 12$, so that the space of cusps forms is spanned by the newforms[1]. An asymptotic formula for $M_{g,\overline{g}}$ appears in Theorem 7.3, which is a generalization of the main result of [KMV2], where an asymptotic formula is established for the second moment $\mathcal{M}_{g,\overline{g}}(\frac{1}{2}+\mu;\ell)$ for g the level-one Eisenstein series

$$E'(z,\tfrac{1}{2}) := \frac{\partial}{\partial s}E(z,s)_{|s=1/2} = y^{1/2}\log y + 4y^{1/2}\sum_{n\geqslant 1}\tau(n)\cos(2\pi nx)K_0(2\pi ny). \tag{1.2}$$

Here $E(z,s)$ is the Eisenstein series for the full modular group (see [I1] Chap. 3 p.68) and $\tau(n)$ is the divisor function.In the later case, the second moment is the fourth moment of $L(f,\frac{1}{2}+\mu)$. However there are several differences: since g is cuspidal, a very complicated term from our previous work (so called "off-off-diagonal" in [KMV2]), vanishes here; on the other hand, we have to take care of the perturbations created by the (possibily non-trivial) level and nebentypus of g using the theory of Atkin-Lehner-Li operators, showing a nice matching between the "diagonal" and "off-diagonal" terms.

An immediate application of Theorem 7.3 is to take the specific values $\mu = 0$, $\ell = 1$ in these asymptotics, and assume for example that χ_D is trivial, and g is holomorphic. We have the following result

Proposition 1.6. *For q prime, $k < 12$, χ_q and χ_D trivial,*

$$\frac{1}{|S_k^*(q)|}\sum_{f\in S_k^*(q)}|L(f\otimes g,\tfrac{1}{2})|^2 = P(\log q) + O_{g,k,\varepsilon}(q^{-1/12+\varepsilon})$$

for all $\varepsilon > 0$, where $P(x)$ is a cubic polynomial, depending on g. The leading coefficient of P is $\frac{1}{3\zeta(2)^2}L(\mathrm{sym}^2 g,1)\prod_{p|D}\frac{(1-p^{-1})^2}{1+p^{-1}}$, where $L(\mathrm{sym}^2 g,s)$ is the symmetric square L-function of g.

The asymptotics for small ℓ, combined with those for the first twisted moment

$$\mathcal{M}_g(\tfrac{1}{2}+\mu;\ell) := \frac{1}{|S_k^*(q)|}\sum_f L(f\otimes g,\tfrac{1}{2}+\mu)\lambda_f(\ell)$$

at $\mu = 0$, allow us to apply the older *mollification* technique to infer various nonvanishing results for the critical values $L(f\otimes g,\frac{1}{2})$.

Theorem 1.7. *There exists a positive constant c such that, given g as in Theorem 1.1, $k < 12$ and for any sufficiently large prime q,*

- *if either g has real Fourier coefficients and $\varepsilon(g) = 1$ or g has non-real Fourier coefficients,*

$$\frac{|\{f\in S_k^*(q),\ L(f\otimes g,\frac{1}{2})\neq 0\}|}{|S_k^*(q)|} \geqslant c + o_g(1), \tag{1.3}$$

- *if g has real Fourier coefficients and $\varepsilon(g) = -1$, then*

$$\frac{|\{f\in S_k^*(q),\ L'(f\otimes g,\frac{1}{2})\neq 0\}|}{|S_k^*(q)|} \geqslant c + o_g(1). \tag{1.4}$$

[1]This hypothesis is rather technical and comes from the fact that we are using, rather crudely, Petersson's trace formula to average over newforms. Recently Iwaniec provided a very convenient variant of Petersson's formula in the square-free level case, in order to average over the set of primitive newforms (rather than over a full orthogonal basis) and we suspect that, using this, one can prove the results of this paper valid for any square-free level with no small prime divisor (see [IS1] for a striking application of this formula).

This second case is needed because $L(f \otimes g, \frac{1}{2})$ is identically zero when $\varepsilon(g) = -1$. The dependence on D is much weaker here than in Theorem 1.2, since we use a zero-free region for $L(g \otimes \overline{g}, s)$.

Finally, the precise evaluation of the second moment for $\mu \neq 0$ allows one, following the methods of [KM1, HM], to study the order of vanishing of $L(f \otimes g, s)$ at the critical point $s = \frac{1}{2}$, which we denote by $r(f \otimes g)$, in order to infer that there exists an absolute constant $A > 0$ such that, for all q prime,

$$\sum_{f \in S_k^*(q)} \exp(Ar(f \otimes g)) \ll_g |S_k^*(q)|. \tag{1.5}$$

Since the methods are essentially identical to those of the cited papers, we do not pursue this proof in detail here.

Our paper is organized as follows. First, we discuss some arithmetic interpretations of these results. In the next section we recall some basic facts about various types of modular functions. Next we recall the definition of the Rankin-Selberg convolution L function and its functional equation. In Sections 5 and 6 we compute the first moment in two ways, the second of which introduces tools necessary to attack the second moment. We evaluate the second moment in Section 7, while proving the convexity-breaking bound (Theorem 1.1) along the way. In Section 8 we use our estimates of the first moment to deduce various linear independence lemmas in the spirit of Theorem 2.1. We include the mollification in Section 9. In the first of a series of appendices we review the Atkin-Lehner theory of newforms and deduce from it several Poisson-type summation formulae which are our basic tools. Then we prove a bound on shifted convolutions in a manner based on [DFI II]. Finally, we review some basic identities and bounds for Bessel functions.

In several places, we use the following notational convenience: given $L(s) = \prod_p L_p(s)$ be an Euler product, we will write $L_D(s) := \prod_{p|D} L_p(s)$ and $L^{(D)}(s) := \prod_{(p,D)=1} L_p(s)$.

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2. Arithmetic interpretations of the results

This project was motivated by the fact that in many situations the central value $L(f \otimes g, \frac{1}{2})$ is known (or, more often, conjectured) to have deep arithmetical interpretations. One of these is found in the work of Rohrlich [Ro], based on a remark of Kazhdan: let A be an abelian variety defined over $\mathbf{Q}$, and $G_{\mathbf{Q}} = Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ the absolute Galois group. A central object of study is the Galois module defined by the algebraic points on A, $A(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{C}$ ($G_{\mathbf{Q}}$ acting trivially on the second factor). By the Mordell-Weil theorem, this Galois representation decomposes as an algebraic direct sum of finite dimensional complex irreducible representations of $G_{\mathbf{Q}}$, each of them occuring with finite multiplicity. Given $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_n(\mathbf{C})$ an irreducible continuous complex Galois representation (automaticaly the image of ρ is finite, and ρ factorizes through a finite quotient of $G_{\mathbf{Q}} := Gal(\overline{\mathbf{Q}}/\mathbf{Q})$), the remark of Kazhdan gives a conjectural formula for the multiplicity of ρ in $A(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{C}$: under various standard conjectures (including the Birch and Swinnerton-Dyer conjecture for A over the subfields of $\overline{\mathbf{Q}}^{\ker \rho}$) one has the following formula:

$$\text{multiplicity of } \rho \text{ in } A(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{C} = \mathrm{ord}_{s=1} L(A_{\mathbf{Q}} \otimes \rho, s)$$

where A in $L(A_{\mathbf{Q}}\otimes\rho, s)$ refers to the Galois representation on an ℓ-adic Tate module of $A_{\mathbf{Q}}$ (see [Ro] for the definition of the L function). In our applications we will take $A = Jac^{new}(X_0(q)) = J_0^{new}(q)$ the new part of the Jacobian of the modular curve. Up to a finite number of Euler factors,

$$L(J_0^{new}(q)_{\mathbf{Q}} \otimes \rho, s + \tfrac{1}{2}) = \prod_{f\in S_2^*(q)} L(f \otimes \pi_\rho, s)$$

where π_g is the conjectural GL_n automorphic representation associated to ρ by the Langlands correspondance and $L(f \otimes \pi_\rho, s)$ is the Rankin-Selberg convolution of f (more accurately, π_f, the GL_2 automorphic representation canonically associated with f) and π_ρ. Hence, conjecturally,

$$\text{multiplicity of } \rho \text{ in } J_0^{new}(q)(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{C} = \sum_{f\in S_2^*(q)} \mathrm{ord}_{s=1/2} L(f \otimes \pi_\rho, s),$$

so it seems worthwhile to study the distribution of the "analytic ranks"

$$r(f \otimes \pi_\rho) := \mathrm{ord}_{s=1/2} L(f \otimes \pi_\rho, s)$$

when f varies over the family $S_2^*(q)$.

When ρ is one dimensional π_ρ corresponds to a Dirichlet character. When this character is trivial, the multiplicity is simply the rank of $J_0^{new}(q)(\mathbf{Q})$. This was investigated first in [Br, Mu] and then more thoroughly in [IS1, KM1, KM2, KMV2, V1] when q is prime (so that $J_0(q) = J_0^{new}(q)$); the latter analytical works, combined with the partial progress of Gross-Zagier, Kolyvagin and Kolyvagin-Logachev towards the Birch–Swinnerton-Dyer conjecture, imply both conditional and unconditional results regarding the rank of $J_0(q)(\mathbf{Q})$ for q prime. We address the question of non-trivial characters in the paper [MV].

In this paper we focus on the case $n = 2$. Here the very existence of π_ρ is a deep question, which has now been solved in many cases: if ρ is of dihedral, tetrahedral or octahedral type it follows from work of Hecke, Maass, Langlands and Tunnel; while for certain infinite families of icosahedral type it follows from recent work of Buzzard, Dickinson, Shepherd-Baron and Taylor. In cases where the existence is known (now using classical terminology) one may associate to an irreducible ρ a cuspidal modular form g_ρ which is either an holomorphic form of weight one (if $\det\rho$ is odd) or a Maass form with eigenvalue $1/4$ (if $\det\rho$ is even). In particular, assuming the Birch–Swinnerton-Dyer conjecture, Theorem 1.7 has some bearing on the multiplicity of ρ in $J_0(q)(\overline{\mathbf{Q}})$.

In [Ro], the main concern is to produce non-trivial explicit examples of vanishing of $L(f\otimes g_\rho, s)$ at $s = \frac{1}{2}$; here, our result is in the other direction, giving a large collection of f's for which $L(f \otimes g_\rho, s)$ does not vanish at this point (or at least vanishes to the minimal possible order). Moreover, inequality (1.5) suggests that f's with a high order of vanishing are very rare.

Concerning the conjecture of Birch and Swinnerton-Dyer, one can say much more when ρ is dihedral. We recall briefly the theory of Heegner points and the Gross-Zagier formula [GZ]: let $K = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field with ring of integers O_K, let H_K be the Hilbert class field, and let $\tilde{\chi}$ be a character of $Gal(H_K/K)$. Let $\rho_{\tilde{\chi}} = Ind_{G_K}^{G_{\mathbf{Q}}}$ denote the 2-dimensional representation induced by $\tilde{\chi}$. It is irreducible if and only if $\tilde{\chi}$ is not quadratic. The associated modular form $g_{\rho_{\tilde{\chi}}}$ is the theta series on $\Gamma_0(D)$ of weight one and nebentypus χ_D (the Kronecker symbol of K) given by

$$g_{\tilde{\chi}}(z) = \delta_{\tilde{\chi}=\mathbf{1}} L(\chi_D, 1) + \sum_{0\neq\mathfrak{a}\subset O_K} \tilde{\chi}((\mathfrak{a}))e(N_{K/\mathbf{Q}}(\mathfrak{a})z),$$

where we have identified $\tilde{\chi}$ with a character of the ideal class group of O_K. If $\tilde{\chi}$ is non-quadratic, $g_{\tilde{\chi}}$ is cuspidal, otherwise $g_{\tilde{\chi}}$ is an Eisenstein series. When every prime factor of q splits in K, one finds in $X_0(q)(H_K)$ the set of Heegner points corresponding to pairs of elliptic curves, linked by a cyclic isogeny of degree q, with complex multiplication by O_K. Letting $e_K \in J_0(q)(H_K)$ denote the image of one of these points in the Jacobian, the Gross-Zagier formula (for D odd at least) interprets the central value of the derivative $L'(f \otimes g_{\tilde{\chi}}, \frac{1}{2})$ in terms of the Neron-Tate height of the $(\tilde{\chi}, f)$-eigencomponent of e_K. In particular, for q prime, we obtain from (1.4) a lower bound for the dimension of the $\tilde{\chi}$-isotypical component of the $Gal(\overline{\mathbf{Q}}/K)$-module $J_0(q)(H_K)$: let $J_0(q)(H_K)^{\tilde{\chi}} \subset J_0(q)(H_K) \otimes_{\mathbf{Z}} \mathbf{Z}[\tilde{\chi}]$:

$$\text{rank}_{\mathbf{Z}} J_0(q)(H_K)^{\tilde{\chi}} \geqslant (c + o_K(1)) \dim J_0(q). \tag{2.1}$$

This can also be shown for $\tilde{\chi}$ real by more elementary methods (it amounts to proving a double non-vanishing result for two real characters) by combining a slight variant of the methods of [IS1, KM2, KMV1]); in particular we have

$$\text{rank}_{\mathbf{Z}} J_0(q)(H_K) \geqslant (c + o_K(1))|H_K| \dim J_0(q).$$

Note also that Theorem 1.1 provides a non-trivial upper bound for the Neron-Tate height of the $(f, \tilde{\chi})$-eigencomponent $e_{K,\tilde{\chi},f}$ in the q aspect.

Concerning the original question of estimating the multiplicity of $\rho_{\tilde{\chi}}$, observe that if $\tilde{\chi}$ is not real and $e_{K,\tilde{\chi},f} \neq 0$, then the vector space spanned by $\{e_{K,\tilde{\chi},f}, \sigma(e_{K,\tilde{\chi},f})\}$ (σ the complex conjugation) is 2-dimensional and realizes $\rho_{\tilde{\chi}}$: in fact $\sigma(e_{K,\tilde{\chi},f}) = e'_{K,\overline{\tilde{\chi}},f}$ where $e'_K = \sigma(e_K)$, so the two vectors $e_{K,\tilde{\chi},f}$ and $\sigma(e_{K,\tilde{\chi},f})$ cannot be colinear. In conclusion, we have for $\tilde{\chi}$ non real

$$\text{multiplicity of } \rho_{\tilde{\chi}} \text{ in } J_0(q)(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{C} \geqslant (c + o_K(1)) \dim J_0(q).$$

Following the methods of [V2], the techniques used to prove Theorem 1.7 can also be used to give the following linear independence result, which, along with some variants given in Section 8, may have interesting arithmetic applications:

Theorem 2.1. *Let K be an imaginary quadratic field of odd discriminant $-D$, let H_K be the Hilbert-class field of K, and let $\tilde{\chi}$ be a character of $Gal(H_K/K)$. Let q be a prime number which splits in K and let e_K be a Heegner divisor in $J_0(q)(H_K)$. For $\varepsilon > 0$ fixed and $L \ll_\varepsilon (q/D^5)^{\frac{1}{2}-\epsilon}$ an integer, the images of the $\tilde{\chi}$-eigencomponent of e_K by the first Hecke operators*

$$T_1 e_{K,\tilde{\chi}}, T_2 e_{K,\tilde{\chi}}, \ldots, T_L e_{K,\tilde{\chi}}$$

are linearly independent in $J_0(q)(H_K) \otimes_{\mathbf{Z}} \mathbf{C}$.

If q is inert in K, or if K is real and q splits, one has very similar constructions of Hecke modules generated by Heegner points (or cycles) endowed with a proper height pairing, whose values at the Heegner eigencomponents are expressed in terms of the central values $L(f \otimes g_{\tilde{\chi}}, \frac{1}{2})$; we refer the interested reader to [G, GKZ, D, BD]. In particular, when K is imaginary, q is inert and $\tilde{\chi}$ is a character of order $\leqslant 4$, (1.3), combined with the results of Bertolini-Darmon [BD], proves the existence, for q a sufficiently large prime, of a quotient $J_{\tilde{\chi}}$ defined over $\mathbf{Q}$ of dimension $\gg \dim J_0(q)$ such that the $\tilde{\chi}$-eigencomponent of the $Gal(\overline{\mathbf{Q}}/K)$-module $J_{\tilde{\chi}}(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{C}$ is zero-dimensional.

3. Review of automorphic forms

In this section we review the types of automorphic forms to considered in the rest of the paper.

3.1. **Holomorphic cusp forms.** For k and q two integers, $k \geqslant 2$, and χ_q a Dirichlet character of modulus q and conductor $\hat{q}$, let $S_k(q,\chi_q)$ denote the complex vector space of weight k holomorphic cusp forms with level q and nebentypus χ_q. These are the bounded holomorphic functions on the upper half plane which satisfy the automorphy relation

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q),\ f_{|\gamma}(z) := \frac{f(\gamma z)}{(cz+d)^k} = \chi_q(d) f(z) \tag{3.1}$$

We represent these elements by their Fourier series:

$$f(z) = \sum_{n\geqslant 1} \hat{f}(m) e(mz) = \sum_{n\geqslant 1} \psi_f(m) m^{(k-1)/2} e(mz).$$

This space is equipped with Petersson's inner product,

$$(f,g)_k = \int_{X_0(q)} f(z)\overline{g}(z) y^{k-2} dxdy.$$

The Hecke operators T_n, with $(n,q)=1$, are normal with respect to the inner product, more precisely, $T^* = \overline{\chi_q}(n) T_n$ for $(n,q)=1$, where T^* denotes the adjoint. One can thus find an orthogonal basis of $S_k(q,\chi_q)$, $\mathcal{B}_k(q)$, formed of eigenvectors of all the $\{T_n,\ (n,q)=1\}$. For f an Hecke-eigenvector, let $\lambda_f(n) n^{\frac{k-1}{2}}$ denote the eigenvalue of T_n (sometimes one speaks of $\lambda_f(n)$ of the *normalized* eigenvalue). We have the adjointness relation

$$\overline{\lambda_f(n)} = \overline{\chi_q(n)} \lambda_f(n),\quad \overline{\psi_f(n)} = \overline{\chi_q(n)} \psi_f(n)\ for\ (n,q)=1. \tag{3.2}$$

and for $(n,q)=1$ the recursion formulae

$$\psi_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi_q(d) \psi_f(\frac{m}{d}\frac{n}{d}) \tag{3.3}$$

$$\psi_f(mn) = \sum_{d|(m,n)} \mu(d)\chi_q(d)\psi_f(\frac{m}{d})\lambda_f(\frac{n}{d}). \tag{3.4}$$

The space of newforms $S_k(q,\chi_q)^{new}$ is the orthogonal complement of the (old) subspace generated by the forms $f(dz)$ with $f \in S_k(q',\chi_{q'})$, $dq'|q\ q' \neq q$ and $\chi_{q'}$ inducing χ_q. This space is stable under all the T_n, and in [ALi] it is shown that the T_n can be simultaneously diagonalized. For f such a newform, the equalities (3.3), (3.4) hold for all m,n, and in particular $\psi_f(1) \neq 0$. We say that such an f is primitive if $\psi_f(1)=1$; in that case, for all n, $\psi_f(n) = \lambda_f(n)$. We let $S_k^*(q,\chi_q)$ denote the set of primitive new forms, it forms an orthogonal basis of $S_k(q,\chi_q)^{new}$.

3.1.1. *Petersson's trace formula.* Let $\mathcal{B}_k(q)$ be an orthogonal basis of $S_k(q,\chi_q)$. Petersson's formula states that

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{B}_k(q)} \frac{\psi_f(m)\overline{\psi_f}(n)}{(f,f)} = \delta_{m,n} + \Delta(m,n) \tag{3.5}$$

with

$$\Delta(m,n) := 2\pi i^{-k} \sum_{\substack{c\equiv 0(q)\\ c>0}} \frac{S(m,n;c)}{c} J_{k-1}(\frac{4\pi\sqrt{mn}}{c})$$

and $S(m,n;c)$ the (twisted) Kloostermann sum

$$S(m,n;c) = \sum_{x(c),(x,c)=1} \overline{\chi_q}(x) e(\frac{mx+n\overline{x}}{c}).$$

Note that (3.5) is independent of the choice of basis; in what follows, we assume that $\mathcal{B}_k(q)$ is an Hecke-eigenbasis and that it contains $S_k^*(q,\chi_q)$. For $\mathcal{B}_k(q)$ such a basis we use the following notation:

$$\sum_{f\in\mathcal{B}_k(q)}^h \alpha_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{f\in\mathcal{B}_k(q)}\frac{\alpha_f}{(f,f)},$$

and refer to it as the harmonic average.

Remark. Sometimes we will make the stronger hypothesis that there are no old forms: $\mathcal{B}_k(q)=S_k^*(q,\chi_q)$. This is the case when χ_q is primitive or if $k<12$ and χ_q is trivial.

3.2. **Maass forms.** Let D be a positive integer, χ_D a character of modulus D and conductor $\hat{D}$, and λ a positive real number. Let $M_\lambda(D,\chi_D)$ denote the (finite dimensional) space of weight zero Maass forms of level D, nebentypus χ_D and eigenvalue λ. In other words, $M_\lambda(D,\chi_D)$ consists of functions g, satisfying relation (3.1) for $k=0$, which satisfy $(\Delta+\lambda)g=0$ for the hyperbolic Laplacian Δ, with $\lambda=\frac{1}{4}+r^2$ and either $r\in\mathbf{R}^+$ or $ir\in[-1/4,1/4]$.[2] Let $S_\lambda(D,\chi_D)\subset M_\lambda(D,\chi_D)$ denote the subspace of Maass cusp forms. Any $g\in M_\lambda(D,\chi_D)$ has a Fourier expansion near infinity of the form

$$g(z)=\sum_{n\in\mathbf{Z}}\psi_g(n)e(nx)2|y|^{1/2}K_{ir}(2\pi|ny|),$$

where $z=x+iy$, K_s is the K Bessel function, and $\psi_g(0)=0$ for $g\in S_\lambda(D,\chi_D)$. The Hecke operators act on the Hilbert space $S_\lambda(D,\chi_D)$ (equipped with the Petersson inner product), and the theory of old/new forms is identical to that in the holomorphic case. We again call a new form f primitive if $\psi_f(1)=1$; in that case, $\psi_f(n)=\lambda_f(n)$ for all n, where $\lambda_f(n)$ is the eigenvalue of T_n associated to f and the equalities (3.3) and (3.4) are valid (replacing q by D) for all m and n. Let $S_\lambda(D,\chi_D)^*$ denote the set of primitive new forms. There is another operator acting on $M_\lambda(D,\chi_D)$, namely the reflection operator $Rf(z)=f(-\overline{z})$. Since it commutes with the T_n, a primitive cusp form g satisfies $Rg=\varepsilon_g g$ with $\varepsilon_g=\pm1$. We call g even or odd according to the value of ε_g.

3.3. **Bounds on Fourier coefficients.** Given g a primitive form of one of the types presented above, we often need upper bounds for the normalized Fourier coefficients $\lambda_g(n)$. For p prime, let $\alpha_{g,1}(p),\alpha_{g,2}(p)$ be the complex roots of the quadratic polynomial $X^2-\lambda_g(p)X+\chi_D(p)$, when g is holomorphic (by work of Eichler, Shimura, Ihara and Deligne) the Ramanujan-Petersson bounds are valid, namely

$$|\alpha_{g,1}(p)|,|\alpha_{g,2}(p)|\leqslant 1,\ \text{so that}\ \forall|n\geqslant 1,\ \lambda_g(n)|\leqslant\tau(n). \tag{3.6}$$

When g is a Maass form this bound is not known in general (althought it holds for the forms of type g_ρ given in the introduction). Nevertheless, the following bound of Serre [Sh] will be sufficient for our purposes:

$$|\alpha_{g,1}(p)|,|\alpha_{g,2}(p)|\leqslant p^{1/5},\ \text{so that}\ \forall n\geqslant 1,|\lambda_g(n)|\leqslant\tau(n)n^{1/5}. \tag{3.7}$$

More often we will use the results of Rankin-Selberg theory, which gives the Ramanujan-Petersson bound "on average":

$$\forall X>0,\ \sum_{n\leqslant X}|\lambda_g(n)|^2\ll_g X. \tag{3.8}$$

[2]The latter comes from the "Selberg bound" for exceptional eigenvalues on congruence subgroups. We use it only in section A.4, everywhere else we will assume that $r\in\mathbf{R}$.

Finally we will need to introduce the following function $\sigma_g(n) := \sum_{d|n} |\lambda_g(d)|$, first this function is almost multiplicative: by (3.3), and (3.4)

$$(mn)^{-\varepsilon}\sigma_g(mn) \ll \sigma_g(m)\sigma_g(n) \ll (mn)^{\varepsilon}\sigma_g(mn) \tag{3.9}$$

forall $\varepsilon > 0$, the implied constant depending on ε only, and from (3.8) we have

$$\forall X > 0, \sum_{n \leqslant X} \sigma_g(n)^2 \ll X^{1+\varepsilon}, \tag{3.10}$$

for all $\varepsilon > 0$, the implied constant depending on ε, and g.

4. Review of Rankin-Selberg convolution L-functions

Let χ_D and χ_q be Dirichlet characters modulo q and D, respectively, and let $\chi := \chi_q\chi_D$ denote the character modulo qD. Let $f \in S_k(q, \chi_q)^*$ and $g \in S_*(D, \chi_D)^*$ be normalized newforms, g either a holomorphic form of weight k' or a Maass form with eigenvalue $\lambda = 1/4 + r^2 \geqslant 1/4$ (so that $r \in \mathbf{R}$). The Rankin-Selberg convolution L function is

$$L(f \otimes g, s) := L(\chi, 2s) \sum_{n \geqslant 1} \frac{\lambda_f(n)\lambda_g(n)}{n^s} = \prod_p \prod_{i=1}^{2} \prod_{j=1}^{2} \Bigl(1 - \frac{\alpha_{f,i}(p)\alpha_{g,j}(p)}{p^s}\Bigr)^{-1} \tag{4.1}$$

where for each prime p, $\alpha_{f,1}(p), \alpha_{f,2}(p)$ and $\alpha_{g,1}(p), \alpha_{g,2}(p)$ are the roots of the quadratic equations

$$X^2 - \lambda_f(p)X + \chi_q(p) = 0, \; X^2 - \lambda_g(p)X + \chi_D(p) = 0.$$

Rankin, Selberg, and others proved that $L(f \otimes g, s)$ admits an analytic continuation over the whole complex plane except when $f = \overline{g}$, in which case there are simple poles at $s = 0, 1$. Moreover, this L function admits a functional equation linking s to $1 - s$. When $(q, D) = 1$ we set (see [Li2][3], Theorem 2.2 and Example 2)

$$\Lambda(f \otimes g, s) := (\frac{qD}{4\pi^2})^s \Gamma_g(s) L(f \otimes g, s), \tag{4.2}$$

with

$$\Gamma_g(s) = \Gamma(s + \frac{|k - k'|}{2})\Gamma(s + \frac{k + k'}{2} - 1) \text{ for } g \text{ holomorphic}$$
$$\Gamma_g(s) = \Gamma(s + \frac{k + 2ir - 1}{2})\Gamma(s + \frac{k - 2ir - 1}{2}) \text{ for } g \text{ a Maass form.}$$

We then have

$$\Lambda(f \otimes g, s) = \varepsilon(f \otimes g)\Lambda(\overline{f} \otimes \overline{g}, 1 - s), \tag{4.3}$$

with

$$\varepsilon(f \otimes g) = \begin{cases} \chi_D(-q)\chi_q(D)\eta_f(q)^2\eta_g(D)^2 & \text{if } g \text{ is holomorphic and } k' \geqslant k, \\ \chi_D(q)\chi_q(-D)\eta_f(q)^2\eta_g(D)^2 & \text{else.} \end{cases} \tag{4.4}$$

Here $\eta_f(q)$, $\eta_g(D)$ are the pseudo-eigenvalues of f, g for the Atkin-Lehner-Li operators W_q, W_D and $\overline{g}$ is the primitive form proportional to $W_D g$, that is $W_D g = \eta_g(D)\overline{g}$ (see Section A.1). A particularly important case arises when $\eta_f(q)^2 = 1$ (for example, if χ_q is real), so we define

$$\varepsilon(g) = \begin{cases} \chi_D(q)\eta_g(D)^2 & \text{if } g \text{ is holomorphic and } k' \geqslant k \\ \chi_D(-q)\eta_g(D)^2 & \text{else.} \end{cases} \tag{4.5}$$

[3]This gives the holomorphic case, but the proof also works when g is a Maass form.

4.1. **The twisted moments.** In what follows we will use μ to denote a complex number of the form $\mu = \delta + it$ with $|\delta| \leqslant 1/\log q$. Let g and g' be primitive cusp forms with weights $k_g, k_{g'}$ or eigenvalues λ_g, $\lambda_{g'}$, square-free levels D and D', and nebentypus χ_D and $\chi_{D'}$, respectively, which are not exceptional. Following the methods of [KMV2], we wish to compute the twisted moments

$$\mathcal{M}_g(\mu;\ell) := \sum_{f \in S_k^*(q,\chi_q)}^h L(f \otimes g, \tfrac{1}{2} + \mu)\overline{\lambda_f}(\ell) \tag{4.6}$$

and

$$\mathcal{M}_{g,g'}(\mu,\mu';\ell) := \sum_{f \in S_k^*(q,\chi_q)}^h L(f \otimes g, \tfrac{1}{2} + \mu) L(\overline{f} \otimes g', \tfrac{1}{2} + \mu')\overline{\lambda_f}(\ell). \tag{4.7}$$

We use the functional equation (4.3) to represent $L(f \otimes g, 1/2 + \mu)$ as a rapidly converging series. To this end, we set

$$G_{g,\mu}(s) := \frac{(4\pi^2)^\mu}{\Gamma_g(\frac{1}{2} + \mu)} (\frac{\xi(\frac{1}{2} + s - \mu)}{\xi(\frac{1}{2})})^5 \frac{P_g(s)}{P_g(\mu)}$$

Here $\xi(s) = s(1-s)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is the completed ζ function of Riemann[4] and $P_g(s)$ is an even polynomial with real coefficients, depending on k, k_g only, such that $P_g(s)\Gamma_g(\frac{1}{2}+s)$ is holomorphic in the region $-A < \Re e s$ for some $A > 1/2$. The functional equation of $\xi(s)$ implies that

$$G_{g,\mu}(-s) = \frac{(4\pi^2)^\mu \Gamma_g(\frac{1}{2} - \mu)}{(4\pi^2)^{-\mu}\Gamma_g(\frac{1}{2} + \mu)} G_{\overline{g},-\mu}(s). \tag{4.8}$$

With this in mind, we define

$$H_{g,\mu}(s) := (4\pi^2)^{-s}\Gamma_g(\tfrac{1}{2} + s) G_{g,\mu}(s)$$

Note that $H_{g,\mu}(\mu) = 1$.

Using contour shifts and the functional equations of $L(f \otimes g, s)$ and (4.8), one shows that

$$(qD)^\mu L(f \otimes g, 1/2 + \mu) = \sum_{n \geqslant 1} \frac{\lambda_f(n)\lambda_g(n)}{n^{1/2}} V_{g,\mu}(\frac{n}{qD}) + \varepsilon_\mu(f \otimes g) \sum_{n \geqslant 1} \frac{\overline{\lambda_f}(n)\overline{\lambda_g(n)}}{n^{1/2}} V_{\overline{g},-\mu}(\frac{n}{qD}) \tag{4.9}$$

with

$$V_{g,\mu}(y) := \frac{1}{2\pi i}\int_{(3)} H_{g,\mu}(s) L(\chi, 1 + 2s) y^{-s} \frac{ds}{s - \mu}, \tag{4.10}$$

$$V_{\overline{g},-\mu}(y) := \frac{1}{2\pi i}\int_{(3)} H_{\overline{g},-\mu}(s) L(\overline{\chi}, 1 + 2s) y^{-s} \frac{ds}{s + \mu},$$

and

$$\varepsilon_\mu(f \otimes g) = \frac{(4\pi^2)^\mu \Gamma_g(\frac{1}{2} - \mu)}{(4\pi^2)^{-\mu}\Gamma_g(\frac{1}{2} + \mu)} \varepsilon(f \otimes g).$$

[4]The arithmetic nature of this function is not needed for the proof, it is here merely to force polynomial growth in the t variable in the forthcoming computations.

In the particular case where the $\lambda_f(n)$, $\lambda_g(n)$ are real , $\varepsilon(g) = -1$, and $\mu = 0$, the functional equation then automatically forces $L(f\otimes g, 1/2)$ to vanish, so any analysis must focus on the first derivative, which is given in much the same way by

$$L'(f\otimes g, 1/2) = 2\sum_{n\geqslant 1}\frac{\lambda_f(n)\lambda_g(n)}{n^{1/2}}W_g(\frac{n}{qD})$$

with

$$W_g(y) := \frac{1}{2\pi i}\int_{(3)} H_g(s)L(\chi, 1+2s)y^{-s}\frac{ds}{s^2}. \tag{4.11}$$

In what follows, for notational simplicity, we will suppress dependence on μ and μ'. However to be consistent, we make the following convention explicit:

Convention 4.1. *An expression* $\mathrm{Exp}_{g,\mu}$ *depending on* g *and* μ*, will usually be writen* Exp_g*; accordingly, by* $\mathrm{Exp}_{g'}$ *we mean implicitly* $\mathrm{Exp}_{g',\mu'}$*, and by* $\mathrm{Exp}_{\overline{g}}$ *(resp.* $\mathrm{Exp}_{\overline{g}'}$*) we always mean* $\mathrm{Exp}_{\overline{g},-\mu}$ *(resp.* $\mathrm{Exp}_{\overline{g}',-\mu'}$*).*

In this paper, these notations will never conflict.

By a contour shift to the right, the definition of ξ, and Stirling's formula,

$$V_g(y) \ll_A (1+|t|)^B y^{-A} \tag{4.12}$$

for all $A > 0$, and by shifting the contour to the left to $\Re s = -1/\log(qD)$ we pass a pole at $s = \mu$ and get

$$V_g(y) \ll_g (1+|t|)^B \tau(qD)(\log qD + |\log y|) \tag{4.13}$$

while by shifting the contour to $\Re s = -1/4$ we obtain, using Burgess's bound on $L(\chi, s)$,

$$V_g(y) = (\mathrm{res}_{s=0} + \mathrm{res}_{s=\mu})H_g(s)L(\chi, 1+2s)\frac{y^{-s}}{s-\mu} + O((1+|t|)^B(qD)^\varepsilon(\hat{q}\hat{D})^{3/16}y^{1/4}) \tag{4.14}$$

for all $\varepsilon > 0$, the implied constant depending on ε and g.

We compute the moments (4.6) and (4.7). Closely related to these are the more general, "spectrally complete" moments defined by extending the averaging over the whole Hecke-eigenbasis $\mathcal{B}_k(q)$

$$M_g(\ell) := \sum_n \frac{\lambda_g(n)}{n^{1/2}}V_g(\frac{n}{qD})\sum_{f\in\mathcal{B}_k(q)}^h \psi_f(n)\overline{\psi_f(\ell)}. \tag{4.15}$$

$$M_{g,g'}(\ell) = \sum_{m,n\geqslant 1}\frac{\lambda_g(m)\lambda_{g'}(n)}{m^{1/2}n^{1/2}}V_g(\frac{m}{qD})V_{g'}(\frac{n}{qD'})\sum_{f\in\mathcal{B}_k(q)}^h \psi_f(m)\overline{\psi_f}(n)\overline{\lambda_f}(\ell), \tag{4.16}$$

where we use Convention 4.1 but also take as a definition that the χ' appearing in $V_{g'} = V_{g',\mu'}$ is $\chi' = \overline{\chi}_q\chi_{D'}$. Note that, to have $\lambda_f(\ell)$ well-defined for all $f\in\mathcal{B}_k(q)$, we must assume either that $\mathcal{B}_k(q) = S_k^*(q,\chi_q)$ or that $(\ell, q) = 1$.

For simplicity we give the expression of (4.6) and (4.7) only when χ_q is trivial and $\mathcal{B}_k(q) = S_k^*(q,\chi_q)$: then $\varepsilon(f\otimes g) = \varepsilon(g)$ is independent of f (see (4.5)). From (4.9) we have

$$(qD)^\mu\mathcal{M}_g(\ell) = M_g(\ell) + \varepsilon_\mu(g)M_{\overline{g}}(\ell) \tag{4.17}$$

with

$$\varepsilon_\mu(g) = \frac{(4\pi^2)^\mu \Gamma_g(\frac{1}{2}-\mu)}{(4\pi^2)^{-\mu}\Gamma_g(\frac{1}{2}+\mu)}\varepsilon(g). \tag{4.18}$$

Similarly, (4.7) is a sum of four terms:

$$q^{\mu+\mu'} D^\mu D'^{\mu'} \mathcal{M}_{g,g'}(\ell) = \tag{4.19}$$
$$M_{g,g'}(\ell) + \varepsilon_\mu(g) M_{\overline{g},g'}(\ell) + \varepsilon_{\mu'}(g') M_{g,\overline{g}'}(\ell) + \varepsilon_\mu(g)\varepsilon_{\mu'}(g') M_{\overline{g},\overline{g}'}(\ell)$$

5. Evaluation of the first partial moment

We now evaluate the first partial moment $M_g(\ell)$ in two ways.

- The first approach to evaluating this sum, given in the present section, is largely axiomatic, and uses very little information about the λ_g's. This very robust method is based on large sieve inequalities for Kloosterman sums developed by Deshouillers and Iwaniec [DI]. In particular, we shall use the form given in Proposition 1 of [DFI II].
- The second approach, given in Section 6, uses the modularity properties of the λ_g, and rests on the summation formulae of Section A.5. While this approach is less general, it is more powerful, and we will use its techniques to compute the second moments as well.

5.1. **The first approach: the large sieve.** We compute the linear form

$$L_g(\vec{x}) := \sum_{\ell \leqslant L} x_\ell M_g(\ell)$$

for $\vec{x} = (x_1, x_2, \ldots, x_L) \in \mathbf{C}^L$. In what follows L will be smaller than a fixed power of qD.

5.1.1. *Treatment of $L_g(\mu;\vec{x})$.* Applying (3.5), we obtain

$$L_g(\vec{x}) = L_g^D(\vec{x}) + L_g^{ND}(\vec{x})$$

with

$$L_g^D(\vec{x}) = \sum_{\ell\leqslant L} x_\ell \frac{\lambda_g(\ell)}{\ell^{1/2}} V_g(\frac{\ell}{qD})$$

$$L_g^{ND}(\vec{x}) = \sum_{\ell\leqslant L} x_\ell \sum_n \frac{\lambda_g(n)}{n^{1/2}} V_g(\frac{n}{qD})\Delta(n,\ell).$$

Note that, by Weil's bound for Kloosterman sums, $\Delta(\ell,n) \ll \sqrt{\ell n} q^{-3/2}$. Using the rapid decay of $V_g(y)$ for $y>1$, we may assume that $n \leqslant (qD)^{1+\epsilon}$ for any small positive ϵ, since the contribution of the remaining terms will be a $O_{\epsilon,A,g}((1+|t|)^B ||\vec{x}||_1 (qD)^{-A})$ for all A. Now we can use the large sieve for Kloosterman sums ([DFI II] Prop. 1 and the remark following it):

Proposition 5.1. *Let $k\geqslant 2$ be an integer. For η a smooth function supported in $[C,2C]$ such that $\eta^{(i)} \ll_i C^{-i}$ for all $i\geqslant 0$, set*

$$\Delta_\eta(n,\ell) := 2\pi i^{-k} \sum_{\substack{c\equiv 0(q)\\ c>0}} \frac{S(n,\ell;c)}{c} J_{k-1}(\frac{4\pi\sqrt{\ell n}}{c})\eta(c).$$

Then for any sequences of complex numbers x_ℓ, y_n,

$$\sum_{\ell\leqslant L}\sum_{n\leqslant N} x_\ell y_n \Delta_\eta(\ell,n) \ll_{\epsilon,k} C^\epsilon (\frac{\sqrt{LN}}{C})^{k-3/2} (1+\frac{L}{q})^{1/2} (1+\frac{N}{q})^{1/2} ||\mathbf{x}||_2 ||\mathbf{y}||_2$$

with any $\epsilon > 0$. Moreover the exponent $k - 3/2$ can be replaced by $1/2$.

Proofs of this proposition only appear in the literature for χ_q trivial and k even, but one can show, following the methods of [DFI II] and [DeI] (using Proskurin's generalization of the Kuznetzov trace formula for integral weight), that it holds in our more general case as well. Thus for all $\epsilon > 0$

$$\begin{aligned} L_g^{ND}(\vec{x}) &\ll_\epsilon (1+|t|)^B q^\epsilon \left(\frac{DL}{q}\right)^{1/4}\left(1+\frac{L}{q}\right)^{1/2} D^{1/2}\left(\sum_{n\leqslant (qD)^{1+\epsilon}} \frac{|\lambda_g(n)|^2}{n}\right)^{1/2} ||\vec{x}||_2 \\ &\ll_{\epsilon,g} (1+|t|)^B q^\epsilon \left(\frac{L}{q}\right)^{1/4} D^{3/4}||\vec{x}||_2, \end{aligned} \tag{5.1}$$

so long as $L \leqslant q$ (here we have used (3.8)).

Next we apply formula (4.14) to evaluate $\mathcal{L}^D(\vec{x})$. The resulting error term is

$$O_{\epsilon,g}((1+|t|)^B (qD)^\epsilon (\hat{q}\hat{D})^{3/16}\left(\frac{L}{qD}\right)^{1/4}||\vec{x}||_2).$$

Thus we obtain

Proposition 5.2. *For $L \leqslant q$ and $\vec{x} \in \mathbf{C}^L$,*

$$L_g(\vec{x}) = \sum_{\ell\leqslant L} \frac{x_\ell}{\ell^{1/2}}\lambda_g(\ell)\mathrm{res}_{s=\mu}\frac{H_g(s)L(\chi,1+2s)}{s-\mu}(\frac{\ell}{qD})^{-s} + O_{\epsilon,g}((1+|t|)^B(qD)^\epsilon\left(\frac{L}{q}\right)^{1/4}(\hat{q}^{3/16}+D^{3/4})||\vec{x}||_2).$$

If $\chi = \chi_D\chi_q$ is non-trivial and $\mu = 0$

$$\mathrm{res}_{s=0}\frac{H_g(s)L(\chi_D,1+2s)}{s}(\frac{\ell}{qD})^{-s} = L(\chi_D,1);$$

while if $\chi = \chi_D\chi_q$ is the trivial character modulo qD and $\mu = 0$

$$\mathrm{res}_{s=0}\frac{H_g(s)L(\chi_D,1+2s)}{s}(\frac{\ell}{qD})^{-s} = \tfrac{1}{2}\frac{\varphi(D)}{D}\log(\frac{qD}{\ell}) + \frac{\varphi(D)}{D}\Psi(qD) + C_{qD},$$

with $\Psi(D) := \sum_{p|D}\frac{\log p}{p-1}$, and $C_{qD} = O_g(1)$.

The evaluation of the original first moment $M_g(\ell)$ then follows quickly by restricting the sums to one term.

6. Another approach for the first partial moment

In this section we give another approach, which stems from the techniques of Iwaniec and Sarnak (see [IS1]) and is based on the summation formulae of Appendix A, thus relying on the automorphic nature of the $\lambda_g(n)$. We assume that g is a modular form of *square-free* level. We again compute $M_g(\ell)$, providing a slight improvement on Proposition 5.2 (although this has no significant contribution to applications) and also allowing us to attack the second partial moment.

Applying (3.5), we obtain $M_g(\ell) = M_g^D(\ell) + M_g^{ND}(\ell)$, with

$$M_g^D(\ell) = \frac{\lambda_g(\ell)}{\ell^{1/2}}V_g(\frac{\ell}{qD}) \tag{6.1}$$

(6.2)
$$M_g^{ND}(\ell) = \sum_n \frac{\lambda_g(n)}{n^{1/2}} V_g(\frac{n}{qD})\Delta(n,\ell)$$
$$= 2\pi i^{-k} \sum_{c\equiv 0(q), c>0} \frac{1}{c^2} c \sum_n \lambda_g(n) S(n,\ell;c) \frac{1}{n^{1/2}} V_g(\frac{n}{qD}) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\ell n}\right).$$

We partition the n sum using a smooth function $\eta(x)$ which is zero for $x \leqslant 1/2$, one for $x \geqslant 1$, and partitions further into smooth functions by

$$\eta(x) = \sum_{M\geqslant 1} \eta_M(x)$$

with η_M compactly supported in $[M/2, 2M]$ such that $x^i \eta_M(x)^{(i)} \ll_i 1$ for any $i \geqslant 0$. We also require that $\sum_{M\leqslant X} 1 \ll \log X$. We set

$$F(x) := \frac{1}{x^{1/2}} V_g(\frac{x}{qD})\eta(x) = \sum_{M\geqslant 1} \frac{1}{n^{1/2}} V_g(\frac{n}{qD})\eta_M(x) := \sum_{M\geqslant 1} F_M(x)$$

so that for all $i, A \geqslant 0$ (using (4.13) and (4.12))

$$x^i \frac{\partial^i}{\partial^i x} F_M(x) \ll_{g,A,i} (1+|t|)^B \tau(q) M^{-1/2} (\log q)^{1+i} (\frac{qD}{x})^A.$$

We also define

$$T_M(c) := c \sum_n \lambda_g(n) S(n,\ell;c) F_M(n) J_{k-1}(\frac{4\pi}{c}\sqrt{\ell n})$$

so that

$$M_g^{ND}(\mu;\ell) = 2\pi i^{-k} \sum_{c\equiv 0(q), c>0} \sum_M \frac{1}{c^2} T_M(c).$$

For $C \geqslant q^2$, it is enough to bound the sum on $c > C$ through use of Weil's bound on Kloosterman sums, getting (by (3.8))

$$O_{\epsilon,g}((1+|t|)^B q^\epsilon \sqrt{\frac{\ell}{C}}) = O_{\epsilon,g}((1+|t|)^B q^\epsilon \frac{\sqrt{\ell}}{q}).$$

Thus we may assume that $c \leqslant C$. We may also assume that $M \leqslant q^{1+\epsilon}$ since the contribution of the $M's$ such that $M \geqslant q^{1+\epsilon}$ is a $O_g((1+|t|)^B \sqrt{\ell} M^{-A})$ for some large $A > 0$.

We open the Kloosterman sum and apply Proposition A.5 to the n sum (which is possible since D is square free) to get

(6.3)
$$T_M(c) = \chi_{D_2}(-c) \frac{\eta_g(D_2)}{\sqrt{D_2}} \sum_n \lambda_{g_{D_2}}(n) G_{\chi_{qD_1}}(\ell - n\overline{D_2}; c) \int_0^\infty F_M(x) J_{k-1}(\frac{4\pi}{c}\sqrt{\ell x}) J_g(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}}) dx$$
$$+ \chi_{D_2}(c) \frac{\eta_g(D_2)}{\sqrt{D_2}} \sum_n \lambda_{g_{D_2}}(n) G_{\chi_{qD_1}}(\ell + n\overline{D_2}; c) \int_0^\infty F(x)_M J_{k-1}(\frac{4\pi}{c}\sqrt{\ell x}) K_g(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}}) dx$$

where

$$G_{\chi_{qD_1}}(a;c) = \sum_{\substack{x(c)\\(x,c)=1}} \chi_{qD_1}(x) e(ax/c),$$

$D_1 = (c, D)$, and $D_2 = D/D_1$.

We split this sum into two parts: the term corresponding to $n = \ell D_2$, which we denote

$$(6.4)\quad T_M^{OD}(c) = \chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\lambda_{g_{D_2}}(\ell D_2)G_{\chi_{qD_1}}(0;c)\int_0^\infty F_M(x)J_{k-1}(\frac{4\pi\sqrt{\ell x}}{c})J_g(\frac{4\pi\sqrt{\ell x}}{c})dx,$$

and the remaining terms for which $n \neq \ell D_2$, which we denote $T_M^{\neq}(c)$. In the next section we show that (under the Ramanujan-Petersson for the Fourier coefficients of g) the $T_M^{\neq}(c)$ terms are negligibly small:

$$\sum_{M\leqslant q^{1+\epsilon}}\sum_{\substack{c\leqslant C\\ q|c}}\frac{1}{c^2}T_M^{\neq}(c) = O_{\epsilon,g}((1+|t|)^B q^\epsilon\frac{\sqrt{\hat{q}}\sqrt{\ell}}{q}).$$

6.1. **The off-diagonal remainder term.** The term $T_M^{\neq}(c)$ splits naturally as the sum of two terms corresponding to $\ell - \overline{D}_2 n$ and $\ell + \overline{D}_2 n$.

We consider the case of $\ell - \overline{D}_2 n$, the case of $\ell + \overline{D}_2 n$ being treated in the same way. The full sum is then

$$\sum_{\substack{c\leqslant C\\ q|c}}\frac{1}{c^2}\chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{n\neq D_2\ell}\lambda_{g_{D_2}}(n)G_{\chi_{qD_1}}(\ell - n\overline{D_2};c)\int_0^\infty F_M(x)J_{k-1}(\frac{4\pi}{c}\sqrt{\ell x})J_g(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}})dx.$$

We exploit the oscillations of $J_g(x)$, which is either $2\pi i^{k'}J_{k'-1}(x)$ (with $k' \geqslant 1$ an integer), $-2\pi Y_0(x)$, or $-\pi(J_{2ir}(x) - J_{-2ir}(x))/\sin(\pi ir)$ (with $r \in \mathbf{R}^*$ by our assumptions on g).

Lemma 6.1. *Let $h(x)$ be a smooth function supported in $[M, 2M]$ which satisfies*

$$|x^i h^{(i)}(x)| \ll_i a^i(1 + |\log x|)$$

for some $a \geqslant 1$, and all $i \geqslant 0$, $x > 0$. For ν complex and $j \geqslant 0$,

$$\int_0^\infty J_\nu(x)h(x)dx \ll_{\nu,j} \frac{a^j(1+|\log M|)}{M^{j-1}}\frac{M^{\Re e\nu+j+1}}{(1+M)^{\Re e\nu+j+1/2}}.$$

Proof. We integrate by parts, using (C.1) and (C.4):

$$\int_0^\infty J_\nu(x)h(x)dx = -\int_0^\infty \frac{x^{\nu+1}}{M^{\nu+1}}J_{\nu+1}(x)[(\frac{M}{x})^{\nu+1}h(x)]'dx = M^{-1}\int_0^\infty J_{\nu+1}(x)h_1(x)dx$$

with $h_1(x) := -(\frac{x}{M})^{\nu+1}M[(\frac{M}{x})^{\nu+1}h(x)]'$, so that

$$|x^i g_1^{(i)}(x)| \ll_{i,\nu} a^{i+1}(1 + |\log x|)$$

for all $i \geqslant 0$, $x > 0$. Iterating, we have

$$\int_0^\infty J_\nu(x)h(x)dx = M^{-j}\int_0^\infty J_{\nu+j}(x)h_j(x)dx$$

with h_j a smooth function supported on $[M, 2M]$ with

$$|x^i h_j^{(i)}(x)| \ll_{i,j,\nu} a^{i+j}(1 + |\log x|)$$

for all $i \geqslant 0$, $x > 0$. The lemma then follows from (C.4) applied to $J_{\nu+j}$. □

Remark. The lemma also holds for Y_0, up to an extra factor of $(1 + |\log M|)$.

We apply this to the integral

$$\int_0^\infty F_M(x)J_{k-1}(\frac{4\pi}{c}\sqrt{\ell x})J_\nu(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}})dx = 2\frac{c}{4\pi}\sqrt{\frac{D_2}{n}}\int_0^\infty h(y)J_\nu(y)dx$$

with

$$h(y) = \frac{c}{4\pi}\sqrt{\frac{D_2}{n}}yF_M((\frac{c}{4\pi}\sqrt{\frac{D_2}{n}}y)^2)J_{k-1}(\sqrt{\frac{\ell D_2}{n}}y).$$

The function h satisfies the conditions of Lemma 6.1 with $M' = \frac{4\pi}{c}\sqrt{\frac{Mn}{D_2}}$ and $a = \log q$. In particular, if $n \geqslant q^\epsilon c^2/M$ the integral is very small. For $n \leqslant q^\epsilon c^2/M$ we use (C.4), $k \geqslant 2$, and $\Re e\nu \geqslant 0$ to bound the integral by

$$O_{\epsilon,g}(q^\epsilon(1+|t|)^B\frac{M\sqrt{\ell}}{c}).$$

To finish bounding this term with minimal effort, we assume that the Ramanujan-Petersson conjecture holds for g: so that for $n \geqslant 1, D_2|D, \epsilon > 0$ we have $|\lambda_{g_{D_2}}(n)| \ll_\epsilon n^\epsilon$. Using the trivial bound

$$|G_{\chi_{qD_1}}(\ell \pm \overline{D}_2 n; c)| \leqslant \sqrt{\hat{q}D_1}(n \pm D_2\ell, c),$$

we find that these terms are then bounded (using $\sum_{M\leqslant X} 1 \leqslant \log X$) by

$$\ll_{g,\epsilon} q^\epsilon(1+|t|)^B\sqrt{\hat{q}}\sqrt{\ell}\sum_{M\leqslant q^{1+\epsilon}} M\sum_{\substack{c\leqslant C\\ q|c}}\frac{1}{c^3}\sum_{n\leqslant q^\epsilon c^2/M}(n - D_2\ell, c) \ll_{g,\epsilon} q^\epsilon(1+|t|)^B\frac{\sqrt{\hat{q}}\sqrt{\ell}}{q}.$$

Remark. The Ramanujan-Petersson conjecture is not vital here, and it could be avoided with more work, but since this section is more to demonstrate techniques we used the shorter proof. The main point here is the next subsection where the off-diagonal term is evaluated quite precisely (without any hypothesis). It will only contribute a remainder term (as we already knew by the results of the previous section), but the identity we obtain will be essential for the computation of the second moment.

6.2. **The off-diagonal main term.** Now we evaluate $T_M^{OD}(c)$ (the $n = \ell D_2$ term of (6.3)), which will also contribute as a remainder term to the first moment. However, since terms very much like it contribute to the second moment, we evaluate it in a little more detail. Convergence issues are less of a worry here, so we can sum $T_M^{OD}(c)$ over all M (including those with $M > q^{1+\epsilon}$, which we know to be negligible) to get $T^{OD}(c)$, defined like $T_M^{OD}(c)$ with F_M replaced by F.

If χ_{qD_1} is non-trivial, $G_{\chi_{qD_1}}(0;c) = 0$, while if χ_{qD_1} is trivial $G_{\chi_{D_1}}(0;c) = \varphi(c)$. So (6.4) vanishes identically if χ_q is non-trivial, in which case we are done. Thus we may assume that χ_{qD_1} is trivial (and thus the $\lambda_f(n)$ are real). Since $(c, D_2) = 1$, c must be coprime to the primitive conductor of χ_D, which we call $\hat{D}$. From (A.2) and the first part of Proposition A.1 we have $g_{D_2} = g_D = \overline{g}$. Thus we may replace c with cqD_1, where $D_1D_2 = D$, $\hat{D}|D_2$, and $(c, D_2) = 1$. Summing over the new c variable, we obtain the main term of $M_g^{ND}(\ell)$, which we call the "off-diagonal" term:

$$M_{g,C}^{OD}(\ell) = 2\pi\frac{i^{-k}}{q^2}\chi_D(-q)\sum_{\substack{D_1D_2=D\\(D_1,\hat{D})=1}}\frac{\chi_{\hat{D}}(D_1)}{D_1^2}\frac{\eta_g(D_2)}{\sqrt{D}_2}\overline{\lambda_g(\ell D_2)}$$

$$\times\sum_{c\leqslant C/qD_1}\frac{\varphi(cqD_1)}{c^2}\chi_{D_2}(c)\int_0^\infty F(x)J_{k-1}(\frac{4\pi\sqrt{\ell x}}{cqD_1})J_g(\frac{4\pi\sqrt{\ell x}}{cqD_1})dx.$$

We can now replace $\eta(x)$ by 1 in the formula for $F(x)$ at a cost of at msot $O_{g,\epsilon}(q^{\epsilon-1})$, by Lemma C.2. We can also remove the constraint $c \leqslant C/qD_1$ with an error term of $O_{g,\epsilon}(q^{\epsilon}\sqrt{\ell}/C)$ (using Lemma C.2 and $k \geqslant 2$). Setting $y = 4\pi\sqrt{\ell x}/(cqD_1)$, the main term of $M_g^{ND}(\ell)$ becomes

$$M_g^{OD}(\ell) = \frac{i^{-k}}{q}\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\chi_D(-q)\sum_{\substack{D_1D_2=D\\(D_1,\hat{D})=1}}\frac{\chi_{\hat{D}}(D_1)}{D_1}\frac{\eta_g(D_2)}{\sqrt{D_2}}\overline{\lambda_g(D_2)} \tag{6.5}$$

$$\times\frac{1}{2\pi i}\int_{(1)} H_g(s)L(\chi,1+2s)Z(s)H(s)(\frac{(4\pi)^2D_2\ell}{D_1q})^s\frac{ds}{s-\mu}$$

with

$$Z(s) := \sum_{c\geqslant 1}\frac{\varphi(cqD_1)\chi_{D_2}(c)}{c^{1+2s}} \text{ and } H(s) := \int_0^\infty J_{k-1}(y)J_g(y)y^{-2s}dy.$$

Since φ and χ_{D_2} are multiplicative,

$$Z(s) = \varphi(qD_1)L_{D_2}(\chi_{\hat{D}},2s)^{-1}\frac{L(\chi_{\hat{D}},2s)}{L(\chi,1+2s)},$$

where

$$L_n(\chi,s)^{-1} := \prod_{p|n}(1-\frac{\chi(p)}{p^s}).$$

Shifting the contour in (6.5) to $\Re es = 1/2+\epsilon$ and using the standard bounds (see Lemma (C.2)) for the Bessel functions, we have

$$M_g^{OD}(\ell) = O_{\epsilon,g}((1+|t|)^Bq^{\epsilon}\frac{\sqrt{\ell}\sigma_g(\ell)}{q}), \tag{6.6}$$

while if we shift to $\Re es = \epsilon$ we have

$$M_g^{OD}(\ell) = O_{\epsilon,g}((1+|t|)^Bq^{\epsilon}\frac{\sigma_g(\ell)}{\ell^{1/2}}). \tag{6.7}$$

This completes our bound for the non-diagonal term.

6.3. **A more precise computation of the off-diagonal term.** We now calculate the off-diagonal term more precisely. For this we use the precise expression for $H(s)$.

In the holomorphic case, from [EMOT] 6.8 (33),

$$H(s) = \frac{2\pi i^{k_g}2^{-2s}\Gamma(2s)\Gamma(\frac{k+k_g}{2}-\frac{1}{2}-s)}{\Gamma(\frac{k+k_g}{2}-\frac{1}{2}+s)\Gamma(\frac{|k-k_g|}{2}+\frac{1}{2}+s)\Gamma(-\frac{|k-k_g|}{2}+\frac{1}{2}+s)}. \tag{6.8}$$

Using (A.15) and (A.16), we have

$$H(s) = i^{k_g}\sqrt{\pi}\frac{\Gamma(s)\Gamma(s+\frac{1}{2})}{\pi\cos(\pi(s-\frac{|k-k_g|}{2}))}\frac{\Gamma_g(\frac{1}{2}-s)}{\Gamma_g(\frac{1}{2}+s)},$$

so the off-diagonal term is

$$M_g^{OD}(\ell) = \frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\chi_D(-q)\sum_{\substack{D_1D_2=D\\(D_1,\hat{D})=1}}\frac{\varphi(D_1)}{D_1}\frac{\eta_g(D_2)}{\sqrt{D_2}}\overline{\lambda_g(D_2)}\frac{1}{2\pi i}\int_{(3)}(4\pi)^s\Gamma_g(\tfrac{1}{2}-s)G_g(s) \tag{6.9}$$

$$\times\sqrt{\pi}\frac{i^{k_g-k}\Gamma(s)\Gamma(s+\frac{1}{2})}{\pi\cos(\pi(s-\frac{|k-k_g|}{2}))}(\frac{\hat{D}}{\pi})^s L(\chi_{\hat{D}},2s)L_{D_2/\hat{D}}(\chi_{\hat{D}},2s)^{-1}(\frac{D_2\ell}{\hat{D}D_1q})^s\frac{ds}{s-\mu}.$$

The functional equation for $L(\chi_{\hat{D}},s)$ is (see [D])

$$(\frac{\hat{D}}{\pi})^s\Gamma(s+\frac{a}{2})L(\chi_{\hat{D}},2s)=\varepsilon_{\chi_{\hat{D}}}(\frac{\hat{D}}{\pi})^{\frac{1}{2}-s}\Gamma(\tfrac{1}{2}+\frac{a}{2}-s)L(\overline{\chi_{\hat{D}}},1-2s),\ \varepsilon_{\chi_{\hat{D}}}=(-i)^a\frac{G(\chi_{\hat{D}})}{\hat{D}^{1/2}},\tag{6.10}$$

where $a=0$ if $\chi_{\hat{D}}$ is even and $a=1$ if it is odd. We consider these cases separately.

6.3.1. χ *even.* In this case χ_D (hence $\chi_{\hat{D}}$) is even, and k_g too, so using (A.15) we have

$$\frac{i^{k_g-k}\Gamma(s)\Gamma(s+\frac{1}{2})}{\pi\cos(\pi(s-\frac{|k-k_g|}{2}))}=\frac{\Gamma(s)}{\Gamma(\frac{1}{2}-s)}.\tag{6.11}$$

Applying (6.10), the integral in (6.9) simplifies and, after making the change of variable $s\leftrightarrow -s$ and using (4.8), we obtain

$$M_g^{OD}(\ell)=-\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\chi_D(-q)\frac{G(\chi_{\hat{D}})\overline{\lambda_g(\hat{D})}}{\sqrt{\hat{D}}}\sum_{D_1D_2=D/\hat{D}}\chi_{\hat{D}}(D_1)\frac{\varphi(D_1)}{D_1}\eta_g(\frac{D}{D_1})\frac{\overline{\lambda_g(D_2)}}{\sqrt{D_2}}$$

$$\times\frac{1}{2\pi i}\int_{(-3)}\frac{(4\pi^2)^{\mu}\Gamma_g(\frac{1}{2}-\mu)}{(4\pi^2)^{-\mu}\Gamma_g(\frac{1}{2}+\mu)}H_{\overline{g}}(s)L(\overline{\chi_{\hat{D}}},1+2s)L_{D_2}(\chi_{\hat{D}},-2s)^{-1}(\frac{D_2\ell}{\hat{D}D_1q})^{-s}\frac{ds}{s+\mu}.\tag{6.12}$$

6.3.2. χ *odd.* In this case χ_D (hence $\chi_{\hat{D}}$) is odd, and k_g too, so using (A.15) we have

$$\frac{i^{k_g-k}\Gamma(s+\frac{1}{2})\Gamma(s)}{\pi\cos(\pi(s-\frac{|k-k_g|}{2}))}=i^{k_g-k+|k-k_g|-1}\frac{\Gamma(s+\frac{1}{2})\Gamma(s)}{\pi\sin(\pi s)}=i^{|k-k_g|-1}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(1-s)}.$$

Again using (6.10), the integral in (6.9) simplifies and we find that $M_g^{OD}(\ell)$ equals the right hand side of (6.12) multiplied by $-i^{k_g-k+|k-k_g|}=\pm 1$.

6.4. **When g is a Maass form.** Here $\chi_{\hat{D}}$ is even and $k_g=0$. Using [EMOT] 6.8 (33), we obtain (using $0<\Re e s\leqslant 3/4<\frac{1}{2}+\frac{k-1}{2}-|\Re e ir|$)

$$H(s)=\frac{-\pi 2^{-2s}}{\sin(\pi ir)}[\frac{B(2s,\frac{1}{2}-s+\frac{k-1+2ir}{2})}{\Gamma(\frac{1}{2}+s+\frac{k-1-2ir}{2})\Gamma(\frac{1}{2}+s-\frac{k-1-2ir}{2})}-\frac{B(2s,\frac{1}{2}-s+\frac{k-1-2ir}{2})}{\Gamma(\frac{1}{2}+s+\frac{k-1+2ir}{2})\Gamma(\frac{1}{2}+s-\frac{k-1+2ir}{2})}],$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$. Taking a common denominator, applying (A.15) to the numerator, and transforming some cosines in the denominator into products of Gamma functions through (A.15), we get

$$H(s)=2^{1-2s}\Gamma(2s)\cos(\pi(s-\frac{k}{2}))\frac{\Gamma_g(\frac{1}{2}-s)}{\Gamma_g(\frac{1}{2}+s)}=i^k\sqrt{\pi}\frac{\Gamma(s)}{\Gamma(\frac{1}{2}-s)}\frac{\Gamma_g(\frac{1}{2}-s)}{\Gamma_g(\frac{1}{2}+s)}.$$

This leads to (6.12) after applying the procedure given after (6.11).

6.5. **End of the computations.** We use the following equalities which follow from Proposition A.1: recall that we have the decomposition $D = D_1D_2\hat{D}$ where $\hat{D}$ is the conductor of χ_D,

$$|\lambda_g(\hat{D})| = 1 \textit{ so } \overline{\lambda_g(\hat{D})} = \lambda_g(\hat{D})^{-1}, |\lambda_g(D_2)\sqrt{D_2}| = 1 \textit{ so } \overline{\lambda_g(D_2)\sqrt{D_2}} = (\lambda_g(D_2)\sqrt{D_2})^{-1}$$

$$\frac{G(\chi_{\hat{D}})\overline{\lambda_g(\hat{D})}}{\sqrt{\hat{D}}} = \eta_g(\hat{D}),\ 1 = \chi_{\hat{D}}(D_1)\eta_g(D_1)^2, \eta_g(\frac{D}{D_1}) = \chi_{\hat{D}}(D_2)\eta_g(D_2)\eta_g(\hat{D})$$

$$\frac{\overline{\lambda_g(D_2)}}{\sqrt{D_2}} = \frac{\mu(D_2)\eta_g(D_2)}{D_2},\ (\chi_{\hat{D}}(D_1D_2)\eta_g(D_1)\eta_g(D_2)\eta_g(\hat{D}))^2 = \eta_g(D)^2.$$

We plug these formulas into (6.12); after some straightforward computations, we obtain using (4.5) and (4.18):

$$M^{OD}(\ell) = -\varepsilon_\mu(g)\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\frac{1}{2\pi i}\int_{(-3)} H_{\overline{g}}(s)L(\overline{\chi_{\hat{D}}}, 1+2s)$$

$$\times \sum_{D_1D_2=D/\hat{D}} \frac{\varphi(D_1)}{D_1^{1-s}}\frac{\mu(D_2)\overline{\chi_{\hat{D}}(D_2)}}{D_2^{1+s}}\prod_{p|D_2}(1-\chi_{\hat{D}}(p)p^{2s})(\frac{\ell}{\hat{D}q})^{-s}\frac{ds}{s+\mu}.$$

The arithmetic function inside the integral equals

$$\prod_{p|D/\hat{D}}\left(\frac{p-1}{p^{1-s}} - \frac{\overline{\chi_{\hat{D}}(p)}}{p^{1+s}}(1-\chi_{\hat{D}}(p)p^{2s})\right) = (D/\hat{D})^s\prod_{p|D/\hat{D}}(1-\frac{\overline{\chi_{\hat{D}}(p)}}{p^{1+2s}})$$

so we have

$$M^{OD}(\ell) = -\varepsilon_\mu(g)\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\frac{1}{2\pi i}\int_{(-3)} H_{\overline{g}}(s)L(\overline{\chi}_D, 1+2s)(\frac{\ell}{qD})^{-s}\frac{ds}{s+\mu}$$

$$= \mathrm{res}_{s=0,-\mu}\left(\varepsilon_\mu(g)\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\frac{1}{s+\mu}H_{\overline{g}}(s)L(\overline{\chi_D}, 1+2s)(\frac{\ell}{Dq})^{-s}\right) \tag{6.13}$$

$$-\varepsilon_\mu(g)\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\frac{1}{2\pi i}\int_{(3)} H_{\overline{g}}(s)L(\overline{\chi_D}, 1+2s)(\frac{\ell}{Dq})^{-s}\frac{ds}{s+\mu}.$$

Note that there may be another pole at $s=0$ if χ_D is trivial. Note also that the second term of (6.13) is very similar to the diagonal term

$$-\varepsilon_\mu(g)M^D_{\overline{g}}(\ell) = -\varepsilon_\mu(g)\frac{1}{2\pi i}\int_{(3)} H_{\overline{g}}(s)L^{(q)}(\overline{\chi_D}, 1+2s)(\frac{\ell}{Dq})^{-s}\frac{ds}{s+\mu}. \tag{6.14}$$

More precisely, since

$$L^{(q)}(\overline{\chi}_D, 1+2s) - \frac{\varphi(q)}{q}L(\overline{\chi_D}, 1+2s) = \left(\prod_{p|q}(1-\frac{\overline{\chi_D}(q)}{q^{1+2s}}) - \prod_{p|q}(1-\frac{1}{q})\right)L(\overline{\chi_D}, 1+2s),$$

the difference between the second term of (6.13) and (6.14), after shifting the contour in the integrals for the V's to $\Re es = \epsilon > 0$, is at most $O((1+|t|)^B q^\epsilon \frac{|\lambda_g(\ell)|}{q^-\sqrt{\ell}})$ where q^- is the smallest prime divisor of q.

Proposition 6.2. *Let $\hat{q}$ be the conductor of χ_q and let q^- be the smallest prime factor of q. If $\hat{q}\neq 1$, then $M_g^{OD}(\ell)=0$. Otherwise, we have*

$$\begin{aligned} M_g^{OD}(\ell) &= \varepsilon_\mu(g)\frac{\overline{\lambda_g(\ell)}}{\sqrt{\ell}}\frac{\varphi(q)}{q}\mathrm{res}_{s=0,-\mu}H_{\overline{g}}(s)L(\overline{\chi_D},1+2s)(\frac{\ell}{Dq})^{-s}\frac{1}{s+\mu}\\ &\quad -\varepsilon_\mu(g)M_{\overline{g}}^D(\ell)+O_{\epsilon,g}((1+|t|)^B q^\epsilon\frac{\sigma_g(\ell)}{q^-\sqrt{\ell}}). \end{aligned}$$

Moreover, assuming that the $\lambda_g(n)$ satisfies the Ramanujan-Petersson bound,

$$M_g(\ell)=M_g^D(\ell)+M_g^{OD}(\ell)+O_{\epsilon,g}((1+|t|)^B q^\epsilon\frac{\sqrt{\hat{q}}\ell}{q}).$$

7. The second moment

In this section we compute the twisted second partial moment $M_{g,g'}(\mu,\mu';\ell)$ defined by (4.16). Recall that we have to assume either that ℓ is coprime with q or that $S_k^*(q,\chi_q)=\mathcal{B}_k(q)$ so that $\lambda_f(\ell)$ is well defined for all $f\in\mathcal{B}_k(q)$. As with the first moment, it will be useful to consider for $\vec{x}=(x_1,\dots,x_\ell,\dots,x_L)\in\mathbf{C}^L$ the linear form

$$L_{g,g'}(\vec{x})=\sum_{\substack{\ell\leqslant L\\(\ell,q)=1}}x_\ell M_{g,g'}(\ell). \tag{7.1}$$

¿From our hypothesis on ℓ we may apply the Hecke recursion formula (3.3) for f, followed by (3.4) for g' (since g' is primitive, (3.4) holds without restriction on m,n), and obtain

$$\begin{aligned} M_{g,g'}(\ell) &= \sum_{de=\ell}\frac{\overline{\chi_q}(d)}{d^{1/2}}\sum_{ab=d}\frac{\mu(a)\chi_{D'}(a)}{a^{1/2}}\lambda_{g'}(b)\\ &\times\sum_{m,n\geqslant 1}\frac{\lambda_g(m)\lambda_{g'}(n)}{(mn)^{1/2}}V_g(\frac{m}{qD})V_{g'}(\frac{adn}{qD'})\sum_f^h\psi_f(m)\overline{\psi_f}(aen). \end{aligned} \tag{7.2}$$

We apply Petersson's formula (3.5) to obtain $M_{g,g'}(\ell):=M_{g,g'}^D(\ell)+M_{g,g'}^{ND}(\ell)$ with

$$M_{g,g'}^D(\ell)=\sum_{de=\ell}\frac{\overline{\chi_q}(d)}{d^{1/2}}\sum_{ab=d}\frac{\mu(a)\chi_{D'}(a)}{a^{1/2}}\lambda_{g'}(b)\sum_n\frac{\lambda_g(aen)\lambda_{g'}(n)}{(ae)^{1/2}n}V_g(\frac{aen}{qD})V_{g'}(\frac{adn}{qD'}),$$

$$\begin{aligned} M_{g,g'}^{ND}(\ell) &= 2\pi i^{-k}\sum_{de=\ell}\frac{1}{d^{1/2}}\sum_{ab=d}\frac{\mu(a)\chi_{D'}(a)}{\sqrt{a}}\lambda_{g'}(b)\\ &\times\sum_{c\equiv 0(q)}\frac{1}{c}\sum_{m,n}\frac{\lambda_g(m)\lambda_{g'}(n)}{\sqrt{mn}}V_g(\frac{m}{qD})V_{g'}(\frac{adn}{qD'})S(m,aen;c)J_{k-1}(\frac{4\pi\sqrt{aemn}}{c}). \end{aligned}$$

Applying (3.4) in the reverse direction we have (by (4.13), (3.9) and (3.10))

$$\begin{aligned} M_{g,g'}^D(\ell) &= \frac{1}{\ell^{1/2}}\sum_{de=\ell}\sum_n\frac{\lambda_g(en)\lambda_{g'}(dn)}{n}V_g(\frac{en}{qD})V_{g'}(\frac{dn}{qD'})\\ &= O_{\epsilon,g,g'}\left((1+|t|+|t'|)^B q^\epsilon\frac{\sigma_g(\ell)}{\ell^{1/2}}\right). \end{aligned} \tag{7.3}$$

To save notation we now assume that $|t|\geqslant|t'|$.

7.1. **The non-diagonal main term.** We next evaluate $M^{ND}(\ell)$, following the methods used in [DFI II, KMV2]. We will evaluate the sum precisely for restricted ranges of the variable m, n, c, and the remaining ranges will be bounded using the large sieve inequality of Section 5.1. As in Section 6, we define

$$F_{M,N}(m,n) := \frac{1}{(mn)^{1/2}} V_g(\frac{m}{qD}) V_{g'}(\frac{dan}{qD'}) \eta_M(m)\eta_N(n)$$

and

$$F(m,n) := \frac{1}{(mn)^{1/2}} V_g(\frac{m}{qD}) V_{g'}(\frac{dan}{qD'}) = \sum_{M,N\geqslant 1} F_{M,N}(m,n).$$

We also define

$$T_{M,N}(c) = c\sum_{m,n} \lambda_g(m)\lambda_{g'}(n) S(m, aen; c) F_{M,N}(m,n) J_{k-1}(\frac{4\pi\sqrt{aemn}}{c}), \tag{7.4}$$

$$T_{M,N} := \sum_{q|c} \frac{1}{c^2} T_{M,N}(c),$$

so that

$$M^{ND}(\ell) = 2\pi i^{-k} \sum_{de=\ell} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)\chi_{D'}(a)}{a^{1/2}} \lambda_{g'}(b) \sum_{M,N} T_{M,N}. \tag{7.5}$$

Note that the derivatives of $F_{M,N}$ satisfy the bounds

$$x^i y^j \frac{\partial^i}{\partial^i x}\frac{\partial^j}{\partial^j y} F_{M,N}(x,y) \ll (1+|t|)^B (MN)^{-1/2} (\log q)^{i+j} (\frac{qD}{x})^A (\frac{qD'}{day})^{A'} \tag{7.6}$$

for all $i, j, A, A' \geqslant 0$. By taking $i = j = 0$ and either A or A' large in (7.6), we have

$$\sum_{M+N\gg q^{1+\varepsilon}} M^{ND}_{g,g',M,N} \ll_{\varepsilon,A,g,g'} (1+|t|)^B q^{-A},$$

for any $\varepsilon > 0$ and any $A > 0$. Thus we may assume that

$$M \leqslant (qD)^{1+\varepsilon},\ N \leqslant (qD')^{1+\varepsilon}. \tag{7.7}$$

It will also prove convenient to remove large values of c through the large sieve inequality (Proposition 5.1) together with (3.8), which implies that, for $M, N \ll q^{1+\varepsilon}$,

$$\sum_{q|c, c\geqslant C} \frac{1}{c^2} T_{M,N}(c) \ll_\varepsilon (1+|t|)^B q^\varepsilon \frac{\ell^{3/4}(DD')^{1/2}(MN)^{1/4}}{C^{1/2}}. \tag{7.8}$$

Note also that one can do slightly better when averaging over ℓ. Namely, using also (3.7) to bound $\lambda_{g'}(b)$, the corresponding term for $L_{g,g'}(\vec{x})$ (see (7.1)) is bounded by

$$\sum_{\substack{\ell\leqslant L\\ (\ell,q)=1}} x_\ell \sum_{de=\ell} \frac{1}{d^{1/2}} \sum_{ab=d} \frac{\mu(a)\chi_{D'}(a)}{a^{1/2}} \lambda_{g'}(b) \sum_{q|c,c\geqslant C} \frac{1}{c^2} T_{M,N}(c) \tag{7.9}$$

$$\ll_\varepsilon (1+|t|)^B q^\varepsilon \frac{L^{3/4}(DD')^{1/2}(MN)^{1/4}}{C^{1/2}} ||\vec{x}||_2.$$

Accordingly, in the next two sections we attach a smooth compact function $\eta_C(c)$ vanishing for $c \geqslant 2C$ and equal to one for $c \leqslant C$.

7.2. **Applying the summation formula.** We apply the summation formula (A.5) on the m variable with the effect of splitting

$$T_{M,N}(c) = T^{OD}_{M,N}(c) + T^{-}_{M,N}(c) + T^{+}_{M,N}(c) \tag{7.10}$$

with (see (6.3))

$$T^{OD}_{M,N}(c) = \delta_{\chi_q=\varepsilon_q}|\chi_{\hat{D}}(c)|\varphi(c)\chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_n \lambda_{g_{D_2}}(aenD_2)\lambda_{g'}(n)G^-(aen,n) \tag{7.11}$$

$$T^{\pm}_{M,N}(c) = \chi_{D_2}(\pm c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{\substack{m,n\\ m\neq\pm aenD_2}} \lambda_{g_{D_2}}(m)\lambda_{g'}(n)G_{\chi_q D_1}(aen\pm m\overline{D_2};c)G^{\pm}(\frac{m}{D_2},n),$$

$$G^-(z,y) := \int_0^\infty J_g(\frac{4\pi\sqrt{zx}}{c})J_{k-1}(\frac{4\pi\sqrt{aexy}}{c})F_{M,N}(x,y)dx \tag{7.12}$$

$$G^+(z,y) := \int_0^\infty K_g(\frac{4\pi\sqrt{zx}}{c})J_{k-1}(\frac{4\pi\sqrt{aexy}}{c})F_{M,N}(x,y)dx.$$

7.2.1. *Treatment of* $T^{\pm}_{M,N}(c)$*.* We rewrite $T^{\pm}_{M,N}(c)$ as

$$T^{\pm}_{M,N}(c) := \chi_{D_2}(\pm c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{h\neq 0} G_{\chi_q D_1}(\pm h\overline{D_2};c)T^{\pm}_h(c)$$

with

$$T^{\pm}_h(c) = \sum_{\substack{m,n\\ m\pm aeD_2 n = h}} \lambda_{g_{D_2}}(m)\lambda_{g'}(n)G^{\pm}(\frac{m}{D_2},n).$$

Following [DFI II], we set $P := 1 + c^{-1}(aeMN)^{1/2}$, $Y = aeN$ and $Z := c^2P^2M^{-1}$ and $f(z,y) = G^{\pm}(m/D_2, y/ae)$ by integrating by part (cf. [DFI II] p. 229) one has for all $A, i, j \geqslant 0$

$$z^i y^j f^{(ij)} f(z,y) \ll_{A,i,j} (1+\frac{z}{Z})^{-A}(1+\frac{y}{Y})^{-A}\frac{(ae)^{1/2}M}{c}P^{i+j-3/2}.$$

We evaluate expressions like $T^{\pm}_h(c)$ in Proposition B.1, which gives

$$\begin{aligned} T^{+}_{M,N}(c) + T^{-}_{M,N}(c) &= O_{\epsilon,g,g'}((1+|t|)^B\sqrt{\hat{q}}Z(aeP)^{3/4}N^{1/4}M^{1/2}c^{\epsilon})) \\ &= O_{\epsilon,g,g'}((1+|t|)^B\sqrt{\hat{q}}(ae)^{3/4}P^{11/4}N^{1/4}M^{-1/2}c^{2+\epsilon})). \end{aligned}$$

Remark. Note that at this point there is a major simplification compared with the treatment given in [DFI II] or [KMV2] (and also the forthcoming [MV]): in these papers, g is an Eisenstein series, with the effect that $T^{+}_{M,N}(c) + T^{-}_{M,N}(c)$ contains a main term (coming from the constant coefficient of g) thus contributing to the second moment.

Setting $\hat{q} = q^\beta$ and summing over c, we bound the contribution of these terms by

$$\begin{aligned} &\ll_{\varepsilon,g,g'} (1+|t|)^B q^{\varepsilon}\left(\sqrt{\hat{q}}\ell^{3/4}\frac{N^{1/4}}{M^{1/2}}\frac{C}{q} + \sqrt{\hat{q}}\ell^{17/8}\frac{M^{7/8}N^{13/8}}{q^{11/4}}\right) \\ &\ll_{\varepsilon} (1+|t|)^B q^{\varepsilon}\left(\ell^{3/4}q^{-(1-2\beta)/12} + \ell^{17/8}q^{-(1-2\beta)/4}\right) \end{aligned} \tag{7.13}$$

on choosing $C = M^{1/2}q^{2/3(1-\beta/2)}$. The first error term agrees with the one in (7.8). If now we consider the corresponding terms for the average $L_{g,g'}(\vec{x})$ then their contribution is bounded by

$$\ll_\varepsilon (1+|t|)^B q^\varepsilon \left(\sqrt{\hat{q}}L^{3/4}\frac{N^{1/4}}{M^{1/2}}\frac{C}{q}||\vec{x}||_1 + \sqrt{\hat{q}}L^{17/8}\frac{M^{7/8}N^{13/8}}{q^{11/4}}||\vec{x}||_1\right)$$

$$\ll_\varepsilon (1+|t|)^B q^\varepsilon \left(L^{3/4}q^{-(1-2\beta)/12}||\vec{x}||_2^{2/3}||\vec{x}||_1^{1/3} + L^{17/8}q^{-(1-2\beta)/4}||\vec{x}||_1\right) \tag{7.14}$$

on choosing $C = M^{1/2}q^{2/3(1-\beta/2)}(||\vec{x}||_2/||\vec{x}||_1)^{2/3}$ so that the first error terms equals that of refsclargelinear.

Note that, to beat the convexity bound by a positive power with our current methods, the conductor of χ_q has to be smaller than $q^{1/2-\delta}$ for some positive δ. We suspect that this can be improved through more careful analysis of the cancellation between various Gauss sums, but for the purposes of this paper such a restriction is acceptible.

This finishes our estimate if χ_q is non-trivial, since in that case the $T^{OD}_{M,N}(c)$ are zero: from (7.14) we obtain for non-trivial χ_q the equality

$$M_{g,g'}(\ell) = M^D_{g,g'}(\ell) + O_{\varepsilon,g,g'}((1+|t|)^B q^\varepsilon \left(\ell^{3/4}q^{-(1-2\beta)/12} + \ell^{17/8}q^{-(1-2\beta)/4}\right)$$

where $M^D_{g,g'}(\ell)$ is defined in (7.3), while from (7.14) we have

$$\begin{aligned} L_{g,g'}(\vec{x}) = &\sum_{\substack{\ell\leqslant L\\(\ell,q)=1}} x_\ell M^D_{g,g'}(\ell) \\ &+ O_{\varepsilon,g,g'}\left((1+|t|)^B q^\varepsilon \left(L^{3/4}q^{-(1-2\beta)/12}||\vec{x}||_2^{2/3}||\vec{x}||_1^{1/3} + L^{17/8}q^{-(1-2\beta)/4}||\vec{x}||_1\right)\right). \end{aligned} \tag{7.15}$$

7.3. **Expanding the c sum.** Assume now that χ_q is the trivial character. We concentrate on

$$T^{OD}_{M,N} := \sum_{\substack{q|c\\(c,\hat{D})=1}} \frac{1}{c^2} T^{OD}_{M,N}(c),$$

where the sum is performed over $c \leqslant C$. We first add back the terms from large c at an admissible cost. To be specific, the bounds $J_g(x), K_g(x) \ll \log(x), J_{k-1}(x) \ll x$ imply that $T^{OD}_{M,N}(c) \ll_\epsilon (1+|t|)^B q^\epsilon \ell^{1/2}MN$, so

$$\sum_{q|c,c\geqslant C} \frac{1}{c^2}T^{OD}_{M,N}(c) \ll_{\epsilon,g,g'} (1+|t|)^B q^\epsilon \frac{\ell^{1/2}MN}{Cq} \ll_{\epsilon,g,g'} (1+|t|)^B q^\epsilon \frac{\ell^{1/2}}{q^{1/6}}.$$

Next we consider

$$T^{OD} := \sum_{M,N} T^{OD}_{M,N}$$

where the sum is over the indices such that $M \leqslant (qD)^{1+\epsilon}, N \leqslant (qD')^{1+\epsilon}$. Again, we may reinsert large values of M and N at a cost which is $O_A((1+|t|)^B q^{-A})$ for all $A > 0$. Thus up to an admissible term T^{OD} can be rewritten like (7.11) but with $F_{M,N}(x,y)$ replaced by $\eta(x)\eta(y)F(x,y)$ in (7.12). We can replace $\eta(y)$ by 1 since it leaves the n sum unchanged. We can also replace $\eta(x)$

by one at an admissible cost, namely the size of the term obtained by replacing $\eta(x)$ by $1-\eta(x)$, which is $O_\epsilon((1+|t|)^B q^\epsilon \ell^{1/2}\frac{D'}{q})$. Thus we have, up to admissible error,

$$2\pi i^{-k} T^{OD} = \sum_n \frac{\lambda_{g'}(n)}{\sqrt{n}} V_{g'}(\frac{adn}{qD'}) M_g^{OD}(aen),$$

where $M_g^{OD}(\ell)$ is the non-diagonal term defined in (6.5). From (7.5), this equality, (7.13) and the Hecke recursion formulae (3.4) we obtain that the non-diagonal term is given by

$$M^{ND}(\ell) = M_{g,g'}^{OD}(\ell) + O_{\varepsilon,g,g'}\bigg((1+|t|)^B q^\varepsilon \big(\ell^{3/4} q^{-(1-2\beta)/12} + \ell^{17/8} q^{-(1-2\beta)/4}\big)\bigg)$$

where

$$M_{g,g'}^{OD}(\ell) = \sum_{de=\ell} \frac{1}{d^{1/2}} \sum_n \frac{\lambda_{g'}(dn)}{\sqrt{n}} V_{g'}(\frac{dn}{qD'}) M_g^{OD}(en). \tag{7.16}$$

Applying the bound (6.7) gives

$$M_{g,g'}^{OD}(\ell) = O_{\epsilon,g,g'}((1+|t|)^B q^\epsilon \frac{\sigma_g(\ell)}{\sqrt{\ell}}),$$

and adding this to (7.3) we obtain, in view of (7.13), (7.14), and (7.15) the following proposition.

Proposition 7.1. *Let D be a square-free integer, denote by $\hat{q} = q^\beta$ the conductor of χ_q. For $L \geqslant 1$ an integer and $\vec{x} = (x_1, \dots x_L) \in \mathbf{C}^L$ let $L_{g,g'}(\vec{x})$ the linear form defined in (7.1). We have*

$$L_{g,g'}(\vec{x}) \ll q^\varepsilon (1+|t|+|t'|)^B \left(\sum_{\ell \leqslant L} |x_\ell| \frac{\sigma_g(\ell)}{\ell^{1/2}} + L^{3/4} q^{-(1-2\beta)/12} ||\vec{x}||_2^{2/3} ||\vec{x}||_1^{1/3} + L^{17/8} q^{-(1-2\beta)/4} ||\vec{x}||_1\right), \tag{7.17}$$

for all $\varepsilon > 0$ the implied constants depending only on ε, k, g, g'. In particular for $\ell \leqslant q^{(1-2\beta)/15}$ coprime with q, we have

$$M_{g,g'}(\ell) \ll (1+|t|+|t'|)^B q^\varepsilon \frac{\sigma_g(\ell)}{\sqrt{\ell}}.$$

Before continuing our evaluation of the second moment under somewhat restrictive conditions we derive a generalization of the main result of [DFI II].

7.4. **Breaking convexity for Rankin-Selberg convolutions.** We now use Proposition 7.1 to bound $L(f \otimes g, s)$ and its derivatives along the critical line.

Theorem 7.2. *Let g be cuspidal with square-free level D, and let $\hat{q} = q^\beta$ be the conductor of χ_q. Then for all $f \in S_k^*(q, \chi_q)$, $t \in \mathbf{R}$, and $j \geqslant 0$,*

$$|L^{(j)}(f \otimes g, \tfrac{1}{2} + it)| \ll_\varepsilon (1+|t|)^B q^{1/2-(1-2\beta)/80+\varepsilon}.$$

for all $\varepsilon > 0$, the exponent B being absolute, and the implied constant depending on k, g, ε.

Proof. We use the amplification technique of [DFI II]: consider a sequence of real numbers $\vec{x} := (x_\ell)_{\ell \leqslant L}$ indexed by the integers coprime with q. For $\mu = \delta + it$ with $|\delta| \leqslant 1/\log q$ we consider the quadratic form

$$\mathcal{Q}(\vec{x}) := \sum_{f \in S_k^*(q,\chi_q)}^h |\sum_{\substack{\ell \leqslant L \\ (\ell,q)=1}} x_\ell \lambda_f(\ell)|^2 |L(f \otimes g, \tfrac{1}{2} + \mu)|^2$$

We want an upper bound for this quadratic form. Using the functional equation, we have

$$|L(f\otimes g,\tfrac{1}{2}+\mu)|^2 \ll |\sum_m \frac{\lambda_g(m)\lambda_f(m)}{m^{1/2}}V_g(\frac{m}{qD})|^2 + |\sum_m \frac{\overline{\lambda}_g(m)\overline{\lambda_f}(m)}{m^{1/2}}V_{\overline{g}}(\frac{m}{qD'})|^2.$$

Since we only need an upper bound, we can extend the sum to all $f\in\mathcal{B}_k(q)$:

$$\mathcal{Q}(\vec{x}) \ll \sum_{f\in\mathcal{B}_k(q)}^h |\sum_{\substack{\ell\leqslant L\\(\ell,q)=1}} x_\ell\lambda_f(\ell)|^2\Big\{|\sum_m \frac{\lambda_g(m)\psi_f(m)}{m^{1/2}}V_g(\frac{m}{qD'})|^2 + |\sum_m \frac{\overline{\lambda}_g(m)\overline{\psi_f}(m)}{m^{1/2}}V_{\overline{g}}(\frac{m}{qD'})|^2\Big\}.$$

Using $(\ell,q)=1$ along with (3.3), we have

$$|\sum_{\substack{\ell\leqslant L\\(\ell,q)=1}} x_\ell\lambda_f(\ell)|^2 = \sum_d \sum_{\substack{\ell_1,\ell_2\leqslant L/d\\(d\ell_1\ell_2,q)=1}} \overline{x}_{d\ell_1}\overline{\chi}_q(\ell_2)x_{d\ell_2}\overline{\lambda_f}(\ell_1\ell_2).$$

Now for $\ell\leqslant L^2$ we define $X_\ell=\sum_d\sum_{\ell_1\ell_2=\ell}\overline{x}_{d\ell_1}\overline{\chi}_q(\ell_2)x_{d\ell_2}$ so that

$$\mathcal{Q}(\vec{x}) \ll L_{g,\overline{g}}(\vec{X}) + L_{\overline{g},g}(\vec{X}).$$

Hence from Proposition 7.1 applied to (g,μ), $(g',\mu')=(\overline{g},\overline{\mu})$ and from the estimates

$$\sum_{\ell\leqslant L^2}|X_\ell|\frac{\sigma_g(\ell)}{\ell^{1/2}} \ll L^\varepsilon\sum_{\ell\leqslant L}|x_\ell|^2,\ \sum_{\ell\leqslant L^2}|X_\ell| \ll L^\varepsilon(\sum_{\ell\leqslant L}|x_\ell|)^2,\ \sum_{\ell\leqslant L^2}|X_\ell|^2 \ll L^\varepsilon(\sum_{\ell\leqslant L}|x_\ell|^2)^2,$$

we obtain for each $f\in S_k^*(q,\chi_q)$

$$(f,f)^{-1}|\sum_{\substack{\ell\leqslant L\\(\ell,q)=1}} x_\ell\lambda_f(\ell)|^2|L(f\otimes g,\tfrac{1}{2}+\mu)|^2 \ll \mathcal{Q}(\vec{x})$$

$$\ll (1+|t|)^Bq^\varepsilon\Big(||\vec{x}||_2^2 + L^{3/2}q^{-(1-2\beta)/12}||\vec{x}||_2^{4/3}||\vec{x}||_1^{2/3} + L^{17/4}q^{-(1-2\beta)/4}||\vec{x}||_1^2\Big).$$

We finish the proof by choosing for (x_ℓ) similarly to [DFI II], page 236: we take

$$x_\ell = \begin{cases} -\overline{\chi}_q(p) & \text{if } \ell=p^2,\ p \text{ a prime} \leqslant L^{1/2}\\ \overline{\chi}_q(p)\lambda_f(p) & \text{if } \ell=p \text{ a prime} \leqslant L^{1/2}\\ 0 & \text{else}\end{cases}.$$

The x_ℓ satisfy

$$|\sum_{\substack{\ell\leqslant L\\(\ell,q)=1}} x_\ell\lambda_f(\ell)|^2 \gg q^{-\varepsilon}L, \text{ and } ||\vec{x}||_1+||\vec{x}||_2^2 \ll L^{1/2}.$$

Using this for $L=q^{1/20}$ and the upper bound $(f,f)\ll_k q\log^3 q$, we obtain the theorem in the case $j=0$ and for the variable $s=\frac{1}{2}+\mu$ in a $1/\log q$-neigborhood of the critical line. Cauchy's formula then provides the bound for $\frac{1}{2}+\mu$ on the critical line for higher derivatives. □

7.4.1. *Proof of Corollary 1.3.* Let φ be a smooth test function on $[0,+\infty[$, equal to 1 for $0\leqslant x\leqslant 1/2$ and equal to 0 for $x\geqslant 1$. Let $f\in S_k^*(q)$ as in the statement. Let $X=X(f)$ be the largest integer such that $\lambda_f(n)=\lambda_g(n)$ for all $n\leqslant X$. If $X<D$, we are finished. Otherwise we have

$$X\ll_g\sum_{n\leqslant X/2}|\lambda_g(n)|^2=\sum_{n\leqslant X/2}\lambda_f(n)\lambda_g(n)\leqslant\sum_{n\leqslant X}\lambda_f(n)\lambda_g(n)\varphi(n/X),$$

where the first inequality follows by the fact that the Rankin-Selberg convolution $L(g\otimes g,s)$ has a simple pole at $s=1$, and the others by the non-negativity of of φ and of $\lambda_f(n)\lambda_g(n)$ for $n\leqslant X$. The last sum is estimated by Mellin transform and a contour shift to $\mathrm{Re}(s)=1/2+\delta$ for any $\delta>0$: applying the Phragmen-Lindelöf principle with the bound of Theorem 1.1, one finds that

$$\sum_{n\leqslant X}\lambda_f(n)\lambda_g(n)\varphi(n/X)\ll_{\varphi,\delta}X^{1/2+\delta}q^{(1/2-\delta)(1-2\gamma)}$$

(where $\gamma=1/80-\varepsilon$ is the exponent improving on the convexity bound for the Rankin-Selberg convolution). It follows that

$$X\ll_{g,\delta}q^{1-2\gamma+\delta}$$

for any $\delta>0$, which is Corollary 1.3.

7.5. **Asymptotic evaluation.** To prove Theorem 1.7, we need the asymptotics of the second moment, not just an upper bound. Our techniques only accomplish this when $S_k^*(q)=\mathcal{B}_k(q)$, so we add the assumptions that χ_q is trivial, q is prime, and $k<12$. At this point essentially all of the work has been done, it is a matter to combine all the preceeding estimates: first by (7.16) along with Proposition 6.2 we have

$$\begin{aligned}M_{g,g'}^{OD}(\ell)\quad&=\varepsilon_\mu(g)\frac{\varphi(q)}{q\ell^{1/2}}\mathrm{res}_{s=0,-\mu}\sum_{de=\ell}e^{-s}\sum_n\frac{\lambda_{\overline{g}}(en)\lambda_{g'}(dn)}{n^{1+s}}L(\overline{\chi_D},1+2s)H_{\overline{g}}(s)V_{g'}(\frac{dn}{qD'})\frac{(Dq)^s}{s+\mu}\\ &-\varepsilon_\mu(g)\sum_{de=\ell}d^{-1/2}\sum_n\frac{\lambda_{g'}(dn)}{n^{1/2}}V_{g'}(\frac{dn}{qD'})M_{\overline{g}}^D(en)+O_{\varepsilon,g,g'}((1+|t|)^Bq^{\varepsilon-1}),\end{aligned}\tag{7.18}$$

and from (7.3) (applied to the pairs $(g_1,\mu_1)=(\overline{g},-\mu)$, $(g_2,\mu_2)=(g',\mu')$) and (6.1) we know that the second term in (7.18) equals

$$-\varepsilon_\mu(g)M_{\overline{g},g'}^D(\ell)+O_{\varepsilon,g,g'}((1+|t|)^Bq^{\varepsilon-1}).$$

Combining (4.19) with the equality

$$M_{g,g'}(\ell)=M_{g,g'}^D(\ell)+M_{g,g'}^{OD}(\ell)+O_{\varepsilon,g,g'}\biggl((1+|t|+|t'|)^Bq^\varepsilon\Bigl(\ell^{3/4}q^{-1/12}+\ell^{17/8}q^{-1/4}\Bigr)\biggr),$$

we see that the various M^D terms cancel each other as well as the possible extra residu at $s=0$ for $\mu\neq 0$ (since in that case χ_D is trivial and $g=\overline{g}$); hence we obtain the following:

Theorem 7.3. *Let g and g' be primitive (non exceptional) cusp forms of square free level D,D' and nebentypus $\chi_D,\chi_{D'}$, respectively. Assume that q is prime, coprime with DD', that χ_q is the trivial character and $S_k^*(q)=\mathcal{B}_k(q)$. Let $\mu:=\delta+it,\mu'=\delta'+it'$ with $|\delta|,|\delta'|\leqslant 1/\log q$ and $t,t'\in\mathbf{R}$. For any $\ell<q$ and any $\varepsilon>0$,*

$$\begin{aligned}(qD)^\mu(qD')^{\mu'}\mathcal{M}_{g,g'}(\ell)\quad=\quad&M_{g,g'}^{main}(\ell)+\varepsilon_\mu(g)\varepsilon_{\mu'}(g')M_{\overline{g},\overline{g}'}^{main}(\ell)\\ &+\varepsilon_{\mu'}(g')M_{g,\overline{g}'}^{main}(\ell)+\varepsilon_\mu(g)M_{\overline{g},g'}^{main}(\ell)\\ &+O_\varepsilon\biggl((1+|t|+|t'|)^Bq^\varepsilon\Bigl(\ell^{3/4}q^{-1/12}+\ell^{17/8}q^{-1/4}\Bigr)\biggr),\end{aligned}$$

the implied constant depending on ε, k, g, g' with $\varepsilon_\mu(g)$ defined in (4.18),

$$M^{main}_{g,g'}(\ell) := \frac{\varphi(q)}{q\ell^{1/2}}\mathrm{res}_{s=\mu}\frac{1}{2\pi i}\int_{(3)} J_{g,g'}(s,t)q^{s+t}D^sD'^t\frac{dt}{(s-\mu)(t-\mu')},$$

$$J_{g,g'}(s,t) := H_g(s)L(\chi_D, 1+2s)H_{g'}(t)L^{(q)}(\chi_{D'}, 1+2t)L(g\otimes g'; \ell; s,t),$$

$$L(g\otimes g'; \ell; s,t) = \nu_{g,g'}(\ell; s,t)\frac{L(g\otimes g', 1+s+t)}{L(\chi_D\chi_{D'}, 2+2s+2t)},$$

where $\nu_{g,g'}(\ell; s,t)$ is the multiplicative function of ℓ defined by

$$\nu_{g,g'}(\ell; s,t) := \sum_{de=\ell}\frac{1}{d^se^t}\prod_{p|de}\Big(\sum_{k=0}^{\infty}\frac{\lambda_g(p^{k+v_p(d)})\lambda_{g'}(p^{k+v_p(e)})}{p^{k(1+s+t)}}\Big)\Big(\sum_{k=0}^{\infty}\frac{\lambda_g(p^k)\lambda_{g'}(p^k)}{p^{k(1+s+t)}}\Big)^{-1};$$

and the 3 other mains terms are defined following Convention 4.1

Now we prove Proposition 1.6. We suppose that g has trivial nebentypus (so g is self-dual) and holomorphic. Applying Theorem 7.3 for $\mu = \mu' = 0$ and $\ell = 1$, we find that, up to remainder terms of order $q^{-1/12+\varepsilon}$, the second moment equals

$$\mathcal{M}_{g,g}(1) = 4\frac{\phi(q)}{q}\mathrm{res}_{s=0}\ \frac{1}{2\pi i}\int_{(3)} H_g(s)H_g(t)$$

$$\times(qD)^{s+t}\zeta^{(D)}(1+2s)\zeta^{(D)}(1+2t)\frac{L(g\otimes g, 1+s+t)}{\zeta^{(D)}(2+2s+2t)}\frac{dt}{st}.$$

We can shift the t contour to $\Re e t = -1/2$ without hitting any poles other than at $t = 0$, so it remains to calculate the asymptotics at that point since the remaining contour contributes as a $O(q^{-1/2+\varepsilon})$. The residue at $s = t = 0$ clearly give a polynomial in $\log q$. The leading term comes from replacing $\zeta^{(D)}(1+2z)$ by $\frac{\phi(D)}{D}(2z)^{-1}$ and $L(g\otimes g, 1+s+t)$ by $R_g(s+t)^{-1}$ where

$$R_g = \mathrm{res}_{s=1}L(g\otimes g, s) = \mathrm{res}_{s=1}\frac{\zeta(s)}{\zeta_D(s)}\frac{L(\mathrm{sym}^2 g, s)}{\zeta(2s)} = \frac{\prod_{p|D}(1-p^{-1})}{\zeta(2)}L(\mathrm{sym}^2 g, 1).$$

Since $\mathrm{res}_{s,t=0}\frac{q^{s+t}}{s^2t^2(s+t)} = \frac{(\log q)^3}{3}$, we can conclude the proof of Proposition 1.6.

8. Quadratic forms and linear independence

In this section, we restrict to the case when χ_q is trivial, q is prime, and $k < 12$, so $\mathcal{B}_k(q) = S_k^*(q)$ and $\varepsilon(f\otimes g) = \varepsilon(g)$. We restrict to the case when $\mu = 0$ and g has real coefficients (thus $\varepsilon(g) = \pm 1$ and χ_D is real), and we assume that the coefficients $\lambda_g(n)$ satisfy the Ramanujan-Petersson bound (3.6). We will use the results of Sections 5 and 6 to analyze the behavior of certain quadratic forms related to the values $L(f\otimes g, 1/2)$. This will in turn allow us to prove a series of linear independence results similar to those of [V2].

8.1. When $\varepsilon(g) = 1$.

We compute the quadratic form

$$\mathcal{Q}(\vec{x}) := \sum_{f\in S_k^*(q)}^{h} L(f\otimes g, 1/2)|\sum_{\ell\leqslant L} x_\ell\lambda_f(\ell)|^2.$$

We will show that, under certain restrictions, this is positive definite.

Theorem 8.1. *For any fixed $1/2 > \epsilon > 0$, for $L \leqslant (q/D^5)^{1/2-\epsilon}$, and for q sufficiently large (depending on ε, g) the quadratic form $\mathcal{Q}(\vec{x})$ is positive definite.*

Proof. For the moment, we assume that χ is non-trivial. We have

$$\lambda_f(\ell_1)\lambda_f(\ell_2) = \sum_{d|(\ell_1,\ell_2)} \varepsilon_q(d)\lambda_f(\ell_1\ell_2/d^2),$$

so that

$$\mathcal{Q}(\vec{x}) = \sum_d \varepsilon_q(d) \sum_{\ell_1,\ell_2} x_{d\ell_1}x_{d\ell_2}M_1(\ell_1\ell_2).$$

Let $L' = L^2$ and define, for $\ell \leqslant L'$,

$$x_\ell := \sum_d \varepsilon_q(d) \sum_{\ell_1\ell_2=\ell} x_{d\ell_1}x_{d\ell_2},$$

so that

$$\mathcal{Q}(\vec{x}) = \mathcal{L}(\vec{X}),\ \ \vec{X} \in \mathbf{R}^{L'}.$$

This is now a linear form in $\vec{X}$, so we can use the results of Section 5.1 to show that

$$\mathcal{Q}(\vec{x}) = \mathcal{Q}^M(\vec{x}) + Err(\vec{x})$$

with

$$Err(\vec{x}) \ll_\epsilon q^\epsilon \left(\frac{L'^{1/2}}{q^{1/2}D^{1/2}} + \left(\frac{L'}{q}\right)^{1/4} D^{3/4} \right) ||\vec{X}||_2, \tag{8.1}$$

the implied constant depending on ε, g and

$$\mathcal{Q}^M(\vec{x}) = 2L(\chi_D, 1) \sum_d \varepsilon_q(d) \sum_{\ell_1,\ell_2} \frac{x_{d\ell_1}x_{d\ell_2}}{\ell_1^{1/2}\ell_2^{1/2}} \lambda_g(\ell_1\ell_2).$$

By the Hecke recursion,

$$\lambda_g(\ell_1\ell_2) = \sum_{b|(\ell_1,\ell_2)} \chi_D(b)\mu(b)\lambda_g(\ell_1/b)\lambda_g(\ell_2/b),$$

so

$$\mathcal{Q}^M(\vec{x}) = 2L(\chi_D, 1) \sum_k \nu(k)(\sum_\ell x_{k\ell}\frac{\lambda_g(\ell)}{\ell^{1/2}})^2 := 2L(\chi_D, 1)\sum_k \nu(k)y_k^2$$

with

$$\nu(k) = \sum_{bd=k} \varepsilon_q(d)\mu(b)\frac{\varepsilon_2(b)}{b} = \prod_{p|k}(1 - \frac{1}{p}) \text{ if } k < q \text{ (since } q \text{ is prime)},$$

and

$$y_k = \sum_\ell x_{k\ell}\frac{\lambda_g(\ell)}{\ell^{1/2}}.$$

In other words, we have diagonalized $\mathcal{Q}^M(\vec{x})$, which is now clearly positive definite since $L(\chi_D, 1)\nu(k)$ is positive for all k. It remains to bound $Err(\vec{x})$ in terms of $\mathcal{Q}^M(\vec{x})$ to prove that $\mathcal{Q}(\vec{x})$ is positive definite. We have

$$\sum_{\ell\leqslant L^2} |x_\ell|^2 \leqslant \sum_\ell (\sum_{\ell_1\ell_2=\ell}\sum_d |x_{d\ell_1}||x_{d\ell_2}|)^2 \ll_\epsilon L^\epsilon \sum_{\ell_1,\ell_2}(\sum_d |x_{d\ell_1}||x_{d\ell_2}|)^2$$

$$\ll_\epsilon L^\epsilon \sum_{\ell_1,\ell_2}\sum_{d_1} |x_{d_1\ell_1}|^2 \sum_{d_2} |x_{d_2\ell_2}|^2 \ll_\epsilon L^\epsilon (\sum_\ell |x_\ell|^2)^2.$$

By Moebius inversion,

$$x_\ell = \sum_k y_{k\ell} \frac{(\lambda_g^{-1})(k)}{k^{1/2}}$$

where λ_g^{-1} is the convolution inverse of λ_g:

$$(\lambda_g^{-1})(k) = \sum_{mn^2=k} \mu(m)\lambda_g(m)\chi_D(n)\mu^2(n).$$

Thus

$$\sum_{\ell\leqslant L} |x_\ell|^2 \leqslant \sum_\ell (\sum_{k\leqslant L/l} |y_{k\ell}| \frac{(\lambda_g^{-1})(k)}{k^{1/2}})^2$$

$$\leqslant \sum_\ell (\sum_{k\leqslant L/\ell} |y_{k\ell}|^2)(\sum_{k\leqslant L/\ell} \frac{|(\lambda_g^{-1})(k)|^2}{k}) \ll_\epsilon q^\epsilon \sum_k |y_k|^2,$$

by (3.8). Combining this with (8.1) we have

$$Err(\vec{x}) \ll_\epsilon q^\epsilon (\frac{L}{q^{1/2}D^{1/2}} + \left(\frac{L^2}{q}\right)^{1/4} D^{3/4}) \sum_k |y_k|^2.$$

Returning to $\mathcal{Q}^M(\vec{x})$, since χ_D is real we have $L(\chi_D, 1) \gg D^{-1/2}$ (of course, we could use $L(\chi_D, 1) \geqslant c(\epsilon)D^{-\epsilon}$ but we prefer the results to be effective) so that

$$\mathcal{Q}^D(\vec{x}) \gg_\epsilon q^\epsilon D^{-1/2} \sum_k |y_k|^2$$

which finishes the proof of Theorem 8.1 if χ_D is non-trivial.

If χ_D is trivial, the analysis of the quadratic form is essentially that of [V2] and we do not repeat the details here. □

8.2. **When** $\varepsilon(g) = -1$**.** Now we consider the case where $\varepsilon(g) = -1$ and χ_D is non-trivial. The functional equation then implies that $L(f \otimes g, 1/2) = 0$, so we must consider the first derivative to get interesting results. The corresponding quadratic form is then

$$\mathcal{Q}'(\vec{x}) := \sum_{f\in S_k^*(q)}^h L'(f\otimes g, 1/2)(\sum_{\ell\leqslant L} x_\ell \lambda_f(\ell))^2.$$

Theorem 8.2. *For any fixed* $1/2 > \epsilon > 0$*, for* $L \leqslant (q/D^5)^{1/2-\epsilon}$*, and for* q *large enough (depending on* ε, g*) the quadratic form* $\mathcal{Q}'(\vec{x})$ *is positive definite.*

We can again use the functional equation to get

$$L'(f\otimes g, 1/2) = 2\sum_{n\geqslant 1} \frac{\lambda_f(n)\lambda_g(n)}{n^{1/2}} W_g(\frac{n}{qD}) \tag{8.2}$$

with

$$W_g(y) := \frac{1}{2\pi i}\int_{(3)} H_g(s)L(\chi, 1+2s)y^{-s}\frac{ds}{s^2}.$$

Note that, by the usual argument, for all $A > 0$ we have

$$W_g(y) \ll_A y^{-A}$$

and

$$W_g(y) = \mathrm{res}_{s=0} H_g(s) L(\chi, 1+2s) \frac{y^{-s}}{s^2} + O(\tau(D)\hat{D}^{1/2} y^{1/2}). \tag{8.3}$$

Following the methods of the previous section, we can again write

$$\mathcal{Q}'(\vec{x}) = \mathcal{Q}'^M(\vec{x}) + Err(\vec{x})$$

with $Err(\vec{x})$ satisfying the same bound as before and

$$\mathcal{Q}'^M(\vec{x}) = (L(\chi,1)\log qD + 2L'(\chi,1) + H_g'(0)L(\chi,1))\mathcal{R}(\vec{x}) - L(\chi,1)\mathcal{S}(\vec{x})$$

where $\mathcal{R}(\vec{x})$, $\mathcal{S}(\vec{x})$ are as above. Thus

$$\mathcal{Q}'^M(\vec{x}) \geqslant (L(\chi,1)\log(qD/L^2)(1+o(1)) + 2L'(\chi,1)) \sum_k \nu(k) y_k^2$$

From the Hadamard factorization theorem [D] for $L(\chi, s)$, we have

$$\frac{L'}{L}(\chi,1) \geqslant -\tfrac{1}{2}\log D + O(1)$$

so that

$$\mathcal{Q}'^M(\vec{x}) \geqslant L(\chi,1)\log(q/L^2)(1+o(1)) \sum_k \nu(k) y_k^2,$$

and we finish as before.

Remark. There are stronger variants of Theorems 8.1 and 8.2 which can be obtained along the same lines. It is possible to replace $L(f\otimes g, \frac{1}{2})$ by $L(f, \frac{1}{2})L(f\otimes g, \frac{1}{2})$ to get the same result at the cost of sharpening the condition $L \ll_\epsilon (q/D^5)^{1/2-\epsilon}$ to $L \ll_\epsilon (q/D^5)^{1/4-\epsilon}$. Using Theorem 7.3, it is also possible to replace $L(f\otimes g, \frac{1}{2})$ by $L(f\otimes g, \frac{1}{2})L(f\otimes g', \frac{1}{2})$ with the stronger condition that $L \ll_{\epsilon,g,g'} q^{1/30-\epsilon}$.

8.3. **Linear independence results.** We now use Theorems 8.1 and 8.2 to prove the linear independence of the actions of Hecke operators in various settings. The general situation is the following: we have a complex hermitian vector space $(V, (,))$ equipped with a linear action of the Hecke algebra $\mathcal{T}$, generated by $\{T_1, \dots T_\ell \dots\}_{(\ell,q)=1}$, which is symmetric with respect to the inner product. Suppose there is an element $e_g \in V$ with f-eigencomponents $e_{g,f}$ satisfying the orthogonality relations

$$(e_{g,f}, e_{g,f'}) = \delta_{f=f'} c_g L(f\otimes g, 1/2) \ \text{ if } \varepsilon(g) = 1 \tag{8.4}$$

$$(e_{g,f}, e_{g,f'}) = \delta_{f=f'} c_g L'(f\otimes g, 1/2)) \text{if } \varepsilon(g) = -1 \tag{8.5}$$

for some positive $c_g > 0$. In the settings we consider, the fact that $(e_{g,f}, e_{g,f'}) = 0$ for $f \neq f'$ follows immediately from the symmetry of the action of the T_ℓ and the strong multiplicity one theorem.

Theorem 8.3. *For any fixed $1/2 > \epsilon > 0$, for $L \leqslant (qD^{-5})^{1/2-\epsilon}$, and for q large enough the vectors $T_1 e_g, T_2 e_g, \dots, T_L e_g$ are linearly independent.*

Proof. Suppose that there exists $\vec{x} = (x_1, \dots x_L) \in \mathbf{C}^L$ such that

$$\sum_\ell x_\ell T_\ell e_g = 0_V.$$

Then for each $f \in S_k^*(q,\chi_q)$

$$\sum_\ell x_\ell T_\ell e_{g,f} = 0_V = \sum_\ell x_\ell \lambda_f(\ell) e_{g,f},$$

and taking the inner product of this last vector with itself, and summing over f, gives

$$\mathcal{Q}(\vec{x}) = 0 = \sum_f L(f\otimes g, 1/2)|\sum_{\ell\leqslant L} x_\ell \lambda_f(\ell)|^2.$$

Since the $\lambda_f(\ell)$ and $L(f\otimes g,1/2)$ are real numbers Theorem 8.1 implies that $\vec{x}=0$ if L is small enough. □

Theorem 8.3 may have interesting arithmetic interpretations. Let $K := \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field of discriminant D, $\chi_D = (\frac{-D}{*})$ the associated Kronecker symbol, and H_K the Hilbert class field of K. As discussed in the introduction, given a character $\tilde{\chi}$ of $Gal(H_K/K)$, we can associate to it a theta function $g_{\tilde{\chi}}(z)$ which is a modular form of weight one on $\Gamma_0(D)$ with nebentypus χ_K. The sign of the functional equation of $L(f\otimes g_{\tilde{\chi}}, s)$ is $\varepsilon(g) = \chi_K(-q) = -\chi_K(q)$. When every prime factor of q splits in K, the above axiomatic applies to $e_g = e_{K,\tilde{\chi}}$, the $\tilde{\chi}$-eigencomponent of a Heegner divisor in $J_0(q)(H_K)$. In this case formula (8.5) is the celebrated formula of Gross and Zagier [GZ]. Theorem 8.3 thus implies Theorem 2.1.

If q is inert in K (see [G, BD]), the vector space V is defined to be $M\otimes\mathbf{C}$, where M is the free finite $\mathbf{Z}$-module of degree zero divisors supported on the set of supersingular points in the fiber (of bad reduction) of $X_0(q)(\mathcal{F}_q)$ (this fiber is formed by the union of two projective curves intersecting transversally at the super-singular points). The action of the Hecke algebra extends to this fiber, and the Hecke module M is equipped with an nondegenerate inner product which gives $(,)$ on V. Alternatively, M can be described as the character group of the toric part of the fiber at q of the Jacobian $J_0(q)$, the inner product becoming the monodromy pairing. Associated to K is a finite set of Heegner divisors $\{e_K\}$ living on M which are acted on by $Gal(H_K/K)$. For any e_K, the $\tilde{\chi}$-eigencomponent $e_{K,\tilde{\chi}}$ can be taken as our e_g of Theorem 8.3. In this case the formula (8.4) was proven by Gross in [G].

In higher weight ($k>2$) there is a similar theory and similar formulae which involve higher dimensional Sato-Kuga varieties instead of $X_0(q)$. The analogue of the Gross-Zagier formula is due to Zhang (see [Z1],[Z2]). In this setting, the inner product is not known to be positive (and $L'(f\otimes g,1/2)$ is not known to be non-negative), but we do not use the individual positivity in our argument, so the linear independence of the image of the corresponding Heegner cycles still holds.

9. Mollification

In this section, we compute the mollified moments and thus prove Theorem 1.7. We assume for the moment that $L(f\otimes g,\frac{1}{2})$ is not identically zero (that is, if g is real then $\varepsilon(g)=1$), so we wish to evaluate

$$\mathcal{M}_g := \sum_{f\in S_k^*(q)}^h L(f\otimes g,1/2) M(f\otimes g)$$

and

$$\mathcal{M}_{g,\overline{g}} := \sum_{f\in S_k^*(q)}^h |L(f\otimes g,1/2)|^2 |M(f\otimes g)|^2$$

for a particular mollifier. Since the exact proportion of non-vanishing for $L(f\otimes g,\frac{1}{2})$ (or its derivatives), is not terribly important, we will take a very coarse mollifier. Moreover, since we do not want to assume the Ramanujan-Petersson conjecture in general, we are also going to sieve a certain number of "small" primes: let $C > 1$ be a constant to be chosen later, and set $P = \prod_{p\leqslant C} p$ the product of the primes less than C. For $L > 1$, not an integer, we define the mollifier

$$M^P(f\otimes g) = \sum_{\ell\leqslant L,(\ell,P)=1} \frac{\mu(\ell)\lambda_f(\ell)\lambda_g(\ell)}{\ell^{1/2}}\frac{1}{2\pi i}\int_{(3)}(\frac{L}{\ell})^z\frac{dz}{z^3} = \sum_{\ell<L,(\ell,P)=1}\frac{\lambda_f(\ell)}{\ell^{1/2}}x_\ell. \tag{9.1}$$

The summation condition $\ell \leqslant L$ is redundant, since when $\ell > L$ the integral defining x_ℓ vanishes, as is easily seen by shifting the contour to the right. Since our focus in this section will usually be on poles of holomorphic functions, rather than their values, we denote by $\nu(z_1, z_2, \dots, z_n)$ any function, holomorphic and non-vanishing in the domain $\Re e z_i \geqslant -1/10$ $i = 1\dots n$, which is uniformly bounded in this domain, as are its inverse and all of its low partial derivatives. In particular, we can use this notation for

$$\nu(z) = L_P(g\otimes g, 1+z) := \prod_{p|P} L_p(g\otimes g, 1+z),$$

the product over small primes of the local factors of the Rankin-Selberg convolution L-function. By (3.6) and (3.7), this above function satisfies the required properties in every domain of the form $\Re e z \geqslant -4/5+\epsilon$ for $\epsilon > 0$. In the sequel, the value of the ν function may change from one line to another.

9.1. **The first mollified moment.** Shifting the z contour in (9.1) to $\Re e z = \epsilon$, we find by (3.8) that

$$\sum_{\ell\leqslant L,(\ell,P)=1}\frac{|x_\ell|^2}{\ell} \ll_{g,\epsilon} L^\epsilon.$$

By (4.6) and Proposition 5.2 (for $\mu = 0$),

$$\mathcal{M}_g = M_g^{main} + \varepsilon(g)M_{\overline{g}}^{main} + O_{\varepsilon,g}(L^{1/4}q^{-1/4+\epsilon}),$$

where

$$M_g^{main} = \mathrm{res}_{s=0}\frac{H_g(s)L(\chi_D, 1+2s)(qD)^s}{s}\sum_\ell \frac{x_\ell\lambda_g(\ell)}{\ell^{1+s}}$$

$$= \mathrm{res}_{s=0}\frac{H_g(s)L(\chi_D,1+2s)(qD)^s}{s}\frac{1}{2\pi i}\int_{(3)}\sum_{(\ell,P)=1}\frac{\mu(\ell)\lambda_g^2(\ell)}{\ell^{1+s+z}}\frac{L^z dz}{z^3}. \tag{9.2}$$

The error term is admissible when $L = q^\Delta$ for some fixed $\Delta < 1$. We have the factorization

$$\sum_{(\ell,P)=1}\frac{\mu(\ell)\lambda_g^2(\ell)}{\ell^{1+s+z}} = \nu_1(s+z)\frac{L^{(P)}(\chi_D^2, 2+2s+2z)}{L^{(P)}(g\otimes g, 1+s+z)} = \nu_1(s,z)\frac{L(\chi_D^2,2+2s+2z)}{L(g\otimes g,1+s+z)},$$

where for P large enough $\nu_1(s,z)$ is of the type described above. If P is choosen large enough, then $|\nu_1(0,0)|$ is bounded from below by a positive constant depending on P only.

9.1.1. *Shifting the z contour.* As in [KMV1] Section 4., we evaluate M_g^{main} by calculating the s residue and deforming the z contour to the left of the line $\Re ez = 0$, so that the main term comes from the residue of the resulting expression at $z = 0$, the integral along the new contour being negligible. We use the following lemma which can be proved by combining the modularity of the symetric square L function established by Gelbart-Jacquet, the modularity of $GL_2 \times GL_2$ Rankin-Selberg L functions established recently by Ramakrisnan [Ra] with the method of Hadamard–de la Vallée-Poussin and the general theory of Rankin-Selberg L functions for automorphic forms on GL_3 and GL_4 (see [Mo]).

Lemma 9.1. *Given g as above there exists $c_g > 0$ such that the functions $L(g \otimes g, 1 + z)$, $L(g \otimes \overline{g}, 1 + z)$, and $L(\overline{g} \otimes \overline{g}, 1 + z)$ have no zeros in the domain*

$$\{z,\ \Re ez \geqslant \frac{-c_g}{\log(|\Im mz| + 2)}\}.$$

Moreover, on the border $\gamma := \{z,\ \Re ez = \frac{-c_g}{\log(|\Im mz|+2)}\}$, these functions, their inverses and all their derivatives up to any order α are bounded in modulus by $C_{g,\alpha,\delta}(1 + |\Im mz|)^\delta$ for any $\delta > 0$.

Let γ denote the contour $\{z,\ \Re ez = \frac{-c_g}{\log(|\Im mz|+2)}\}$.[5] By shifting the z contour of (9.2) to γ we encounter a pole at $z = 0$ and a contour integral which, by Lemma 9.1, is bounded by $O_g(e^{-c'_g\sqrt{\log L}})$. This will turn out to be a negligible term since the residue will be a power of $\log q$.

So the main term of (9.2) is

$$\text{res}_{s,z=0} \frac{\nu_1(0,0)H_g(s)L(\chi_D, 1 + 2s)(qD)^s}{s} \frac{L(\chi_D^2, 2 + 2s + 2z)L^z}{L(g \otimes g, 1 + s + z)z^3}. \tag{9.3}$$

What is important to us are the highest powers of $\log q$ or $\log L$ coming from this expression, so in the forthcoming computation we will replace any function holomorphic and smaller than any power of $\log q$ near $(0, 0)$ by its value at that point. Evaluating this term depends on the number of poles at the origin, which in turn depends on whether χ_D is trivial or not. We thus break into two cases.

9.1.2. *χ_D non-trivial.* Suppose first that g has complex coefficients. The only poles in (9.3) come from the $1/sz^3$ factor, so the main term is

$$\nu_1(0,0)\frac{L(\chi_D, 1)L(\chi_D^2, 2)}{L(g \otimes g, 1)}\text{res}_{s,z=0}\frac{(qD)^sL^z}{sz^3} = \frac{\nu_1(0,0)}{2}\frac{L(\chi_D, 1)L(\chi_D^2, 2)}{L(g \otimes g, 1)}\log^2 L, \tag{9.4}$$

the remaining terms being smaller by a power of $\log q$ (note also the importance of $\eta_1(0, 0)$ being non-zero). Putting this into $M_{\overline{g}}^{main}$, we have

$$\sum_\ell \frac{\mu(\ell)|\lambda_g(\ell)|^2}{\ell^{1+s+z}} = \nu_1(s,z)\frac{L(\chi_D\overline{\chi}_D, 2 + 2s + 2z)}{L(g \otimes \overline{g}, 1 + s + z)},$$

so that, setting $R_g = \text{res}_{s=1}L(g \otimes \overline{g}, s)$,

$$\nu_1(0,0)\frac{L(\chi_D, 1)L(\chi_D\overline{\chi}_D, 2)}{R_g}\text{res}_{s,z=0}\frac{(qD)^sL^z(s+z)}{sz^3} = \nu_1(0,0)\frac{L(\chi_D, 1)L(\chi_D^2, 2)}{R_g}\log L,$$

which is smaller than (9.4) by a factor of $\log L$. Thus this term can be ignored, and the main term for the first mollified moment is given by (9.4).

[5] We take $c_g < 1/10$ so that we may bound any function of the form ν_1 on γ.

Similarly, if $g=\overline{g}$, the main term of the mollified first moment is given by

$$2\nu_1(0,0)\frac{L(\chi_D,1)L(\chi_D^2,2)}{R_g}\mathrm{res}_{s,z=0}\frac{(qD)^sL^z(s+z)}{sz^3}=2\nu_1(0,0)\frac{L(\chi_D,1)L(\chi_D^2,2)}{R_g}\log L. \tag{9.5}$$

9.1.3. χ_D *trivial.* If χ_D is the trivial character, then $g=\overline{g}$ and $L(\chi_D,1+2s)$ has a simple pole at $s=0$. The same sort of calculations as above lead to a main term of

$$2\nu_1(0,0)\frac{\varphi(D)}{2D}\frac{L(\chi_D^2,2)}{R_g}\mathrm{res}_{s,z=0}\frac{(qD)^sL^z(s+z)}{s^2z^3}$$

$$=\nu_1(0,0)\frac{\varphi(D)}{D}\frac{L(\chi_D^2,2)}{R_g}(\log qD\log L+\tfrac{1}{2}\log^2 L). \tag{9.6}$$

9.2. **The second mollified moment.** Next we turn to the second moment. We start with the square of the mollifier: by (3.3) we obtain

$$|M^P(f\otimes g)|^2=\sum_{\substack{\ell<L^2\\(\ell,P)=1}}\frac{\lambda_f(\ell)}{\ell^{1/2}}=\sum_{\ell<L^2,(\ell,P)=1}\frac{\lambda_f(\ell)}{\ell^{1/2}}x_\ell \tag{9.7}$$

where

$$x_\ell:=\frac{1}{(2i\pi)^2}\iint_{(3)^2}\sum_{d,\ (d,P\ell)=1}\sum_{\ell_1\ell_2=\ell}\frac{\mu^2(d)|\lambda_g(d)|^2}{d^{1+z_1+z_2}}\frac{\mu(\ell_1)\lambda_g(\ell_1)\mu(\ell_2)\overline{\lambda}_g(\ell_2)}{\ell_1^{z_1}\ell_2^{z_2}}L^{z_1+z_2}\frac{dz_1dz_2}{z_1^3z_2^3}.$$

Shifting the z_1 and z_2 contours to $\Re e z_i=\varepsilon$ and using (3.8), we obtain

$$|x_\ell|\ll_{\varepsilon,g}q^\varepsilon\sum_{\ell_1|\ell}|\lambda_g(\ell_1)|^2. \tag{9.8}$$

Now we have

$$\mathcal{M}_{g,\overline{g}}=\sum_{\substack{\ell\leqslant L^2\\(\ell,P)=1}}\frac{x_\ell}{\ell^{1/2}}\mathcal{M}_{g,\overline{g}}(0,0;\ell),$$

where $\mathcal{M}_{g,\overline{g}}(\ell)$ is defined in (4.7). Using Theorem 7.3 for $\mu=\mu'=0$, $g'=\overline{g}$, together with (9.8), we obtain

$$\mathcal{M}_{g,\overline{g}}=\mathcal{M}_{g,\overline{g}}^{main}+O_{\varepsilon,g}(q^\varepsilon(L^{5/2}q^{-1/12}+L^{21/4}q^{-1/4})), \tag{9.9}$$

with

$$\mathcal{M}_{g,\overline{g}}^{main}=M_{g,\overline{g}}^{main}(L)+M_{\overline{g},g}^{main}(L)+\varepsilon(g)M_{\overline{g},\overline{g}}^{main}(L)+\overline{\varepsilon}(g)M_{g,g}^{main}(L),$$

$$M_{g,\overline{g}}^{main}(L):=\frac{\varphi(q)}{q}\mathrm{res}_{s=0}\frac{1}{2\pi i}\int_{(3)}J_{g,\overline{g}}(s,t)\sum_{\substack{\ell<L^2\\(\ell,P)=1}}\frac{x_\ell\eta_{g,\overline{g}}(\ell;s,t)}{\ell}(qD)^{s+t}\frac{dt}{st}, \tag{9.10}$$

and

$$J_{g,g'}(s,t):=H_g(s)H_{\overline{g}}(t)L(\chi_D,1+2s)L^{(q)}(\overline{\chi}_D,1+2t)\frac{L(g\otimes\overline{g},1+s+t)}{L(\chi_D\overline{\chi}_D,2+2s+2t)}, \tag{9.11}$$

where

$$\eta_{g,\overline{g}}(\ell;s,t)=\sum_{de=\ell}\frac{1}{d^se^t}\prod_{p|de}\Big(\sum_{k\geqslant 0}\frac{\lambda_g(p^{k+v_p(d)})\overline{\lambda}_g(p^{k+v_p(e)})}{p^{k(1+s+t)}}\Big)\Big(\sum_{k\geqslant 0}\frac{\lambda_g(p^k)\overline{\lambda}_g(p^k)}{p^{k(1+s+t)}}\Big)^{-1}. \tag{9.12}$$

The other three terms, $M^{main}_{\overline{g},g}(L)$, $M^{main}_{\overline{g},\overline{g}}(L)$, and $M^{main}_{g,g}(L)$ are defined similarly. As long as $L = q^{\Delta}$ for some fixed $0 < \Delta < 1/30$ the error term in (9.9) is $o(1)$ which is admissible since the main term will be a power of $\log q$. We also replace $\varphi(q)/q$ by 1 at an admissible cost.

We will concentrate on the first term, $M^{main}_{g,\overline{g}}(L)$, but we will also mention where the other terms differ.

9.3. **The main term of the second moment.** First we get rid of the t integral in (9.10): we evaluate the residue at $s = 0$ then shift the t contour to $\Re e t = -1/2+\varepsilon$, picking up a pole at $t = 0$. By (9.8), together with (3.8), the integral along the new contour is bounded by $O_{\varepsilon,g}(q^{-1/2+\varepsilon})$, which we can ignore. Thus, up to an admissible error term,

$$M^{main}_{g,\overline{g}}(L) = \mathrm{res}_{s,t=0} J_{g,\overline{g}}(s,t) \sum_{(\ell,P)=1} \frac{x_l \eta_{g,\overline{g}}(\ell; s,t)}{\ell} \frac{(qD)^{s+t}}{st}$$

$$= \frac{1}{(2i\pi)^2} \iint_{(3)^2} \mathrm{res}_{s,t=0} J_{g,\overline{g}}(s,t) \frac{(qD)^{s+t}}{st} K_{g,\overline{g}}(s,t,z_1,z_2) L^{z_1+z_2} \frac{dz_1 dz_2}{z_1^3 z_2^3}, \tag{9.13}$$

where

$$K_{g,\overline{g}}(s,t,z_1,z_2) = \sum_{(\ell,P)=1} \frac{\eta_{g,\overline{g}}(\ell; s,t)}{\ell} \Big(\sum_{\ell_1 \ell_2 = \ell} \frac{\mu(\ell_1)\lambda_g(\ell_1)\mu(\ell_2)\overline{\lambda}_g(\ell_2)}{\ell_1^{z_1} \ell_2^{z_2}} \Big) \sum_{d,(d,P\ell)=1} \frac{\mu^2(d)|\lambda_g(d)|^2}{d^{1+z_1+z_2}}.$$

Note that

$$\sum_{d,(d,P\ell)=1} \frac{\mu^2(d)|\lambda_g(d)|^2}{d^{1+z_1+z_2}} = \prod_{p|\ell} (1 + \frac{|\lambda_g(p)|^2}{p^{1+z_1+z_2}}) \nu_2(z_1+z_2) \frac{L(g \otimes \overline{g}, 1+z_1+z_2)}{L(\chi_D \overline{\chi}_D, 2(1+z_1+z_2))}.$$

We may choose P large enough (but independently of g) so that by (3.7), $K_{g,\overline{g}}(s,t,z_1,z_2)$ admits a factorization of the form

$$K_{g,\overline{g}}(s,t,z_1,z_2) = \tag{9.14}$$

$$\frac{\nu_2(s,t,z_1,z_2) L(\chi_D \overline{\chi}_D, 2) |L(\chi_D^2, 2)|^2 L(g \otimes \overline{g}, 1+z_1+z_2)}{L(g \otimes \overline{g}, 1+s+z_2) L(g \otimes \overline{g}, 1+t+z_1) L(g \otimes g, 1+s+z_1) L(\overline{g} \otimes \overline{g}, 1+t+z_2)}$$

Once again we extract the main term by shifting the contours of z_1 and z_2 one at a time to γ. The difference from the first moment is that we will have to deal with an extra pole at $z_1 + z_2 = 0$. We will handle this extra contribution in a way similar [KMV2] Section 5.3 (except with only two z variables instead of the four found there). Here we merely sketch the argument: first we shift the z_j contour to $-\frac{\gamma}{4-j}$, then we shift the z_1 contour to γ encountering poles at $z_1 = 0$ and $z_1 = -z_2$. The resulting integral of z_1 along γ is negligible: shifting z_2 to $\Re z_2 = 1/\log q$ lets us bound the resulting z_2 integral by $(\log q(1+|z_1|))^B$ for some absolute constant B using Lemma 9.1, so the z_1 integral along γ is bounded by

$$\int_{\mathbf{R}} (\log q)^{B'} (\log(2+|u|))^{B'} L^{-c_g/2 \log(2+|u|)} \frac{du}{(1+|u|)^4}$$

$$\ll_g (\log q)^{B'} e^{B' \log(\log q)) - b_1 (\log L)^{1/2}} \ll_g e^{-\delta \log^{1/2} q}$$

for some absolute $\delta = \delta(g, B', \Delta) > 0$. This is certainly sufficient, so we may ignore the resulting z_1 contour and deal only with the residues passed during the shift.

9.3.1. *The pole at $z_1 = -z_2$.* ¿From (9.14) this pole is simple, so to evaluate it we replace $L(g \otimes \overline{g}, 1 + z_1 + z_2)$ by R_g and all other z_1's by $-z_2$. At this point there are no powers of L remaining, so we are left with the residues at $s = t = 0$ and for these the main contribution contribution come only from the $J_{g,\overline{g}}(s,t)(qD)^{s+t}$ portion (recall that z_2 is along $-\gamma/2$ which is faraway from 0). We find that the contribution of this pole is

$$O_g(\log q^3) \qquad \text{if } \chi_D \text{ is trivial,} \tag{9.15}$$

$$O_g(\log q) \qquad \text{if } \chi_D \text{ is non-trivial.} \tag{9.16}$$

In each case these will turn out to be smaller than the poles encountered at $z_1 = z_2 = 0$. Note that the powers of $\log q$ in (9.15) and (9.16) are independent of the power of z, we have taken for the mollifier (here 3). Had we choosen that power to be 2, these terms would have been the main contribution, which is very inefficient for the mollification.

9.3.2. *The pole at $z_1 = 0$.* We are left with the contribution of the pole at $z_1 = 0$. We evaluate the residue at $z_1 = 0$ then shift the z_2 contour to γ. In the process, we meet a pole at $z_2 = 0$ and the resulting z_2 integral along γ contributes a negligible error term as above. So the main term of $M^{main}_{g,\overline{g}}(L)$ is given by the residue of (9.13) at $s = t = z_1 = z_2 = 0$. The term with the highest power of $\log q$ and $\log L$ is

$$\nu_2(0,0,0,0)|L(\chi_D^2, 2)|^2$$

$$\mathrm{res}_{s,t,z_1,z_2=0} \frac{(qD)^{s+t} L^{z_1+z_2} L(\chi_D, 1+2s) L(\overline{\chi}_D, 1+2t) L(g \otimes \overline{g}, 1+z_1+z_2) L(g \otimes \overline{g}, 1+s+t)}{st z_1^3 z_2^3 L(g \otimes \overline{g}, 1+s+z_2) L(g \otimes \overline{g}, 1+t+z_1) L(g \otimes g, 1+s+z_1) L(\overline{g} \otimes \overline{g}, 1+t+z_2)}.$$

Again, it makes a considerable difference whether χ_D is trivial or not, and whether $g = \overline{g}$ or not.

9.3.3. *χ_D non-trivial and $g \neq \overline{g}$.* When χ_D is non-trivial we may replace $L(\chi_D, 1+2s)L(\overline{\chi}_D, 1+2t)$ by $|L(\chi_D, 1)|^2$, since they contribute no poles to the expression. We may also replace $L(g \otimes g, 1 + s + z_1)L(\overline{g} \otimes \overline{g}, 1 + t + z_2)$ by $|L(g \otimes g, 1)|^2$, so it is enough to compute

$$\mathrm{res}_{s,t,z_1,z_2=0} \frac{(qD)^{s+t} L^{z_1+z_2}}{st z_1^3 z_2^3} \frac{(s+z_2)(t+z_1)}{(s+t)(z_1+z_2)}.$$

The pole at $s = 0$ is simple, so this equals

$$\mathrm{res}_{t,z_1,z_2=0} \frac{(qD)^t L^{z_1+z_2}}{t^2 z_1^3 z_2^2} \frac{(t+z_1)}{(z_1+z_2)} = \frac{\log^4 L}{8} + \log qD \frac{\log^3 L}{3}.$$

Note that from (9.16) the $z_1 + z_2 = 0$ pole does not contribute as a main term.

This is the first time that the evaluation of $M^{main}_{g,g}(L)$ differs. In that case, $\eta_{g,\overline{g}}$ is replaced by $\eta_{g,g}$ and $J_{g,\overline{g}}$ is transformed into $J_{g,g}$, so arguing as before we have a main term of

$$\nu_2(0,0,0,0)|L(\chi_D^2, 2)|^2 \mathrm{res}_{s,t,z_1,z_2=0}$$

$$\frac{(qD)^{s+t} L^{z_1+z_2} L(\chi_D, 1+2s) L(\overline{\chi}_D, 1+2t) L(g \otimes g, 1+s+t) L(g \otimes \overline{g}, 1+z_1+z_2)}{st z_1^3 z_2^3 L(g \otimes g, 1+s+z_1) L(g \otimes g, 1+t+z_1) L(g \otimes \overline{g}, 1+s+z_2) L(g \otimes \overline{g}, 1+t+z_2)}.$$

This time $L(g \otimes g, 1 + s + t)$ has no pole at $s + t = 0$ so the residue is smaller by a factor $\log L$, thus this term and that coming from $M^{main}_{\overline{g},\overline{g}}(L)$ do not contribute as main terms.

Adding in the contribution from $M^{main}_{\overline{g},g}(L)$, the main term for the second mollified moment is

$$2\nu_2(0,0,0,0) \frac{|L(\chi_D^2, 2)|^2 |L(\chi_D, 1)|^2}{|L(g \otimes g, 1)|^2} \left(\frac{\log^4 L}{8} + \log qD \frac{\log^3 L}{3}\right). \tag{9.17}$$

Comparing with the square of the modulus of (9.4) we see that the powers of $\log L$ and $\log q$ match, so the fraction of $L(f\otimes g,\frac{1}{2})$ which do not vanish is at least

$$\sum_{\substack{f\in S_k^*(q)\\ L(f\otimes g,\frac{1}{2})\neq 0}}^h 1 \geqslant \frac{|M_1(g)|^2}{M_2(g)} \gg \Delta\frac{|\nu_1(0,0)|^2}{\nu_2(0,0,0,0)}(1+o_g(1)).$$

In particular, the positive constant $\Delta\frac{|\nu_1(0,0)|^2}{\nu_2(0,0,0,0)}$ is bounded from below independent of g.

9.3.4. χ_D *non-trivial and* $g=\overline{g}$. In this case all four $M^{main}_{\overline{g},g}(L)$ terms are equal, so the main term is

$$4\nu_2(0,0,0,0)|L(\chi_D^2,2)|^2\mathrm{res}_{s,t,z_1,z_2=0}$$
$$\frac{(qD)^{s+t}L^{z_1+z_2}|L(\chi_D,1)|^2L(g\otimes g,1+z_1+z_2)L(g\otimes g,1+s+t)}{stz_1^3z_2^3L(g\otimes g,1+s+z_1)L(g\otimes g,1+s+z_2)L(g\otimes g,1+t+z_1)L(g\otimes g,1+t+z_2)},$$

which reduces to

$$4\nu_2(0,0,0,0)\frac{|L(\chi_D^2,2)|^2|L(\chi_D,1)|^2}{R_g^2}\mathrm{res}_{s,t,z_1,z_2=0}\frac{(qD)^{s+t}L^{z_1+z_2}}{stz_1^3z_2^3}\frac{(s+z_1)(s+z_2)(t+z_1)(t+z_2)}{(s+t)(z_1+z_2)}$$

$$= 4\nu_2(0,0,0,0)\frac{|L(\chi_D^2,2)|^2|L(\chi_D,1)|^2}{R_g^2}(\log^2 L+\log L\log qD) \tag{9.18}$$

Again, from (9.16), the $z_1+z_2=0$ pole does not contribute as a main term and the ratio of the square of the modulus of (9.5) to the modulus of (9.18) is bounded from below by an absolute positive constant.

9.3.5. χ_D *trivial.* In this case $g=\overline{g}$ and we are reduced to

$$4\nu_2(0,0,0,0)(\frac{\varphi(D)}{2D})^2\frac{L(\chi_D^2,2)^2}{R_g^2}$$
$$\times\mathrm{res}_{s,t,z_1,z_2=0}\frac{(qD)^{s+t}L^{z_1+z_2}}{s^2t^2z_1^3z_2^3}\frac{(s+z_1)(s+z_2)(t+z_1)(t+z_2)}{(s+t)(z_1+z_2)}$$

$$= \nu_2(0,0,0,0)(\frac{\varphi(D)}{D})^2\frac{L(\chi_D^2,2)^2}{R_g^2}(\frac{\log^4 L}{4}+\log^3 L\log qD+\log^2 L\log^2 qD+\frac{\log L(\log qD)^3}{3}) \tag{9.19}$$

and again this matches with the square of (9.6) in magnitude.

9.4. **Non-vanishing of the first derivative.** Finally, if $g=\overline{g}$ and $\varepsilon(g)=-1$ we have $L(f\otimes g,\frac{1}{2})=0$ identically, so we turn to computing the first and second mollified moments of the first derivative:

$$\sum_{f\in S_k^*(q)}^h L'(f\otimes g)M(f\otimes g),\ \text{and}\ \sum_{f\in S_k^*(q)}^h L'(f\otimes g)^2M(f\otimes g)^2$$

The case is completely analoguous to the above one, starting from (4.11), all that we do is replace $\frac{ds}{s}$ and $\frac{dt}{t}$ by $\frac{ds}{s^2}$ and $\frac{dt}{t^2}$, which affects the orders of magnitude of the first and second moments in the same way (increasing order of magnitude of the former by one factor of $\log q$, and increasing the order of the latter by two factors of $\log q$, while changing the constants slightly).

9.5. **Removing the harmonic weight.** So far we have proved (1.3) with

$$\frac{|\{f\in S_k^*(q),\ L(f\otimes g,\frac12)\neq 0\}|}{|S_k^*(q)|}\text{ replaced by }\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{\substack{f\in S_k^*(q)\\ L(f\otimes g,\frac12)\neq 0}}\frac{1}{(f,f)}$$

and (1.4) with

$$\frac{|\{f\in S_k^*(q),\ L'(f\otimes g,\frac12)\neq 0\}|}{|S_k^*(q)|}\text{ replaced by }\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{\substack{f\in S_k^*(q)\\ L'(f\otimes g,\frac12)\neq 0}}\frac{1}{(f,f)}.$$

In [KMV2], Section 6, a procedure is described for passing from harmonic weights to natural weights. This same procedure applies in our case, the key point being the convexity-breaking bound of Theorem 1.1 (in [KMV2] the necessary breaking-convexity bound was provided by Theorem 1.2). From this one can conclude the proof of Theorem 1.7.

Appendix A. Summation formulae

In this appendix we derive formulas for expressions of the type

$$\sum_n \lambda_g(n)e(\frac{an}{c})F(n),$$

where g is one of the forms discussed in Section 3, with nebentypus χ_D, a and c are relatively prime, and F is a smooth function, decaying rapidly at infinity, which vanishes in a neighborhood of the origin.

Such formulae do not seem to be in the literature for non-trivial levels (although they are certainly known to several people). We mix the methods of Jutila [J] and Duke-Iwaniec [DI], starting first by establishing the analytic continuation and functional equation of the Dirichlet series $L(g,\frac{a}{c},s)=\sum_{n\geqslant 1}\lambda_g(n)e(n\frac{a}{c})n^{-s}$, using the automorphic properties of g. The summation formulae then follow through Mellin inversion.

Recall that the action (of weight k) of $GL_2^+(\mathbf{R})$ on the space of functions on the upper half is

$$f_{|\gamma}(z):=\frac{\det^{k/2}\gamma}{j(\gamma,z)^k}f(\gamma.z)$$

where $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $j(\gamma,z)=cz+d$, and $\gamma.z=\frac{az+b}{cz+d}$. The form g satisfies the automorphy equation

$$g_{|\gamma}(z)=\chi_D(\gamma)g(z)$$

where $\chi_D(\gamma)=\chi_D(d)$.

A.1. **A review of Atkin-Lehner theory.** The theory of holomorphic newforms in the classical setting is primarily due to Atkin and Lehner [AL] in the case of trivial nebentypus, and was extended by Li in [Li1, ALi] to the general case. We review these results in this section, noting that they apply equally well to Maass forms. In what follows let $S_*(D,\chi_D)$ denote either a space of holomorphic forms (in which case $*$ may be replaced by a k) or Maass forms.

An important role in this theory is played by the Atkin-Lehner operators, which are defined as follows. Given a factorization $D=D_1D_2$ with $(D_1,D_2)=1$, χ_D factors uniquely into $\chi_D=$

$\chi_{D_1}\chi_{D_2}$ where χ_{D_i} is a Dirichlet character of modulus D_i. For any $y \equiv 1(D_1)$, $x \equiv 1(D_2)$ and integers z, w such that $D_1^2 xw - Dyz = D_1$, the matrix

$$\begin{pmatrix} xD_1 & y \\ zD & wD_1 \end{pmatrix},$$

defines a linear map

$$W_{D_1} : S_*(D, \chi_{D_1}\chi_{D_2}) \Longrightarrow S_*(D, \overline{\chi}_{D_1}\chi_{D_2})$$

which is independent of the choice of w, x, y, and z. In addition, if W'_{D_1} comes from another matrix of the same form but with no assumption on congruence classes x' (mod D_2) and y' (mod D_1), then

$$W'_{D_1} = \overline{\chi}_{D_1}(y')\overline{\chi}_{D_2}(x')W_{D_1}. \tag{A.1}$$

For $(D_1, D_2) = 1$ with $D_1D_2|D$ such that $(D_1D_2, D/D_1D_2) = 1$, we have

$$W_{D_2} \circ W_{D_1} = \overline{\chi}_{D_2}(D_1)W_{D_1D_2}, \quad W_{D_1} \circ W_{D_1} = \chi_{D_1}(-1)\overline{\chi}_{D/D_1}(D_1)I, \tag{A.2}$$

with I the identity operator. Note that if $D_1 = D$, W_D is the Atkin-Lehner involution represented by the matrix

$$W_D = \begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix}.$$

These operators act in a convenient way on the space of newforms. To be more precise, we enclose in the two propositions below the properties of these operators that we will use in the sequel:

Proposition A.1. [Li1, ALi] *If $D = D_1D_2$ with $(D_1, D_2) = 1$, the operator W_{D_1} sends newforms to newforms:*

- *For $g \in S_*(D, \chi_{D_1}\chi_{D_2})^*$,*

$$g_{|W_{D_1}} =: \eta_g(D_1)g_{D_1}$$

  with $g_{D_1} \in S_*(D, \overline{\chi}_{D_1}\chi_{D_2})^*$ and $|\eta_g(D_1)| = 1$. The constant $\eta_g(D_1)$ is called the pseudo-eigenvalue of the operator W_{D_1}. If χ_{D_1} is trivial, then W_{D_1} is an endomorphism of $S_*(D, \chi)$ and $\eta_g(D_1)$ is a true *eigenvalue: $g_{D_1} = g$.*
- *If $\lambda_g(D_1) \neq 0$,*

$$\eta_g(D_1) = \frac{G(\chi_{D_1})}{\lambda_g(D_1)\sqrt{D_1}} \tag{A.3}$$

  where $G(\chi_{D_1})$ is the Gauss sum associated with the character χ_{D_1}. Moreover for $D_1D_2|D$, such that D_1 and D_2 are coprimes, coprimes with D/D_1D_2 and $\lambda_g(D_1D_2) \neq 0$ we have

$$\eta_g(D_1D_2) = \chi_{D_1}(D_2)\chi_{D_2}(D_1)\eta_g(D_1)\eta_g(D_2)$$

- *The Fourier coefficients of g_{D_1} are*

$$\lambda_{g_{D_1}}(n) = \begin{cases} \overline{\chi_{D_1}}(n)\lambda_g(n) & \text{if } (n, D_1) = 1 \\ \chi_{D_2}(n)\overline{\lambda_g(n)} & \text{if } n|D_1^\infty \end{cases} \tag{A.4}$$

- *If $D_1 = D$ then $g_{|W_D} = \eta_g(D)\overline{g}$ with $\overline{g}(z) = \sum_{n\geqslant 1} \overline{\hat{g}(n)}e(nz)$, and*

$$\overline{\lambda_g(n)} = \overline{\chi_D}(n)\lambda_g(n) \text{ if } (n, D) = 1. \tag{A.5}$$

In [ALi] the formula (A.3) is given only for D_1 a prime power but it extends to composite D_1 by (A.2) and by the multiplicative relation for the Gauss sums:

$$G(\chi_{D_1}\chi_{D_2}) = \chi_{D_1}(D_2)G(\chi_{D_1})\chi_{D_2}(D_1)G(\chi_{D_2}).$$

Proposition A.2. *Let $D = D_0\hat{D}$ where $\hat{D}$ is the conductor of χ_D. For $g \in S_*(D, \chi_{D_1}\chi_{D_2})^*$ and p a prime factor of D one has*

- *if $p|D_0$,*

$$\lambda_g(p)^2 = \begin{cases} \chi_{\hat{D}}(p)p^{-1} & \text{if } (p, D/p) = 1, \\ 0 & \text{if } p^2 \ |D \end{cases} \tag{A.6}$$

- *if $p|\hat{D}/(D_0,\hat{D})$, $|\lambda_g(p)| = 1$.*

A.2. **A factorization lemma.**

Lemma A.3. *Let $D \geqslant 1$ be an integer, and let a, c two coprimes integers. We suppose that $D_1 := (c, D)$ is coprime with $D_2 := D/D_1$ then there exists a matrix of the form $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$, such that the following identity holds (as operators acting on weight k modular forms, not as matrices)*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c'D & d' \end{pmatrix} \circ W_{D_2} \circ \begin{pmatrix} 1 & 0 \\ 0 & D_2 \end{pmatrix}$$

for some $\gamma_1 = \begin{pmatrix} a' & b' \\ c'D & d' \end{pmatrix} \in \Gamma_0(D)$. The integer d' satisfies the congruences

$$d' \equiv \overline{a}(D_1),\ d' \equiv -c(D_2) \tag{A.7}$$

Proof. Since $(a, c) = 1$ and moreover (c, D) is coprime with D_2, we may select of the form $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $D_2|d$. Now pick any matrix $W_{D_2} = \begin{pmatrix} xD_2 & y \\ zD & wD_2 \end{pmatrix}$ representing the Atkin-Lehner operator (in particular $y \equiv 1(D_2),\ x \equiv 1(D_1)$) then form

$$\gamma_1 := D_2 Id.\gamma.\begin{pmatrix} 1 & 0 \\ 0 & D_2^{-1} \end{pmatrix}.W_{D_2}^{-1} = \begin{pmatrix} awD_2 - zbD_1 & -ay + xb \\ cwD_2 - zdD_1 & -cy + xd \end{pmatrix} \in \Gamma_0(D)$$

by our choice of d. Moreover, $d' = -cy + xd \equiv d \equiv \overline{a}(D_1)$ and $d' = -cy + xd \equiv -c(D_2)$, hence the lemma follows since scalar matrices act as the identity. □

A.3. **The summation formula: holomorphic cusp forms.** Given a, c coprimes integers such that $D_1 = (c, D)$ is coprime with $D_2 = D/D_1$, in the next three sections we return to showing that

$$L(g, \frac{a}{c}, s) := \sum_{n\geqslant 1} \lambda_g(n)e(n\frac{a}{c})n^{-s}$$

admits an analytic continuation over $\mathbf{C}$ with a functional equation which we describe below. We then obtain the desired summation formula by Mellin inversion.

We start with g a holomorphic cusp form of weight k. Take $\gamma \in SL_2(Z)$ as in Lemma A.3. For any z with $\Re e z > 0$ we have, in view of Lemma A.3,

$$g(\gamma z) = (cz + d)^k g_{|\gamma}(z) = (cz + d)^k D_2^{-k/2}\chi(\gamma_1)\eta_g(D_2)g_{D_2}(\frac{z}{D_2}).$$

From (A.7) we have

$$\chi(\gamma_1) = \chi_{D_1}(d')\chi_{D_2}(d') = \overline{\chi_{D_1}}(a)\chi_{D_2}(-c),$$

so that

$$g(\gamma z) = \chi_{D_1}(\overline{a})\chi_{D_2}(-c)\eta_g(D_2)D_2^{-k/2}(cz + d)^k g_{D_2}(z/D_2) := \kappa D_2^{-k/2}(cz + d)^k g_{D_2}(z/D_2). \tag{A.8}$$

For $t \in \mathbf{R}^{*+}$ set $z_t := -\frac{d}{c} + \frac{i}{ct}$, so that $cz_t + d = i/t,\ \gamma z_t = it/c$. By Lemma (A.3), D_2 divides d, so

$$e(-n\frac{d}{cD_2}) = e(-n\frac{d\overline{D_2}}{c} - n\frac{d\bar{c}}{D_2}) = e(-n\frac{d\overline{D_2}}{c}) = e(-n\frac{\overline{aD_2}}{c}),$$

which in turn implies that

$$g(\gamma z_t) = \sum_{n\geqslant 1} \lambda_g(n) e(n\frac{a}{c}) n^{\frac{k-1}{2}} \exp(-\frac{2\pi n t}{c}) =$$

$$\text{(A.9)} \qquad i^k \frac{\kappa}{\sqrt{D}_2} \sum_{n\geqslant 1} \lambda_{g_{D_2}}(n) e(-n\frac{\overline{aD_2}}{c})(\frac{n}{D_2})^{\frac{k-1}{2}} \exp(-\frac{2\pi n}{cD_2 t}) t^{-k}.$$

Since $t \to g(\gamma z_t)$ has exponential decay at zero and infinity, the Mellin transform $\int_0^\infty g(\gamma z_t) t^{s+\frac{k-1}{2}} \frac{dt}{t}$ is analytic on $\mathbf{C}$. For $\Re es$ sufficiently large, by the first equality in (A.9) we have

$$\int_0^\infty g(\gamma z_t) t^{s+\frac{k-1}{2}} \frac{dt}{t} = (\frac{c}{2\pi})^{s+\frac{k-1}{2}} \Gamma(s + \frac{k-1}{2}) L(g, \frac{a}{c}, s),$$

which provides the analytic continuation of $\Gamma(s + \frac{k-1}{2}) L(g, \frac{a}{c}, s)$ over $\mathbf{C}$. Multipliying both sides of (A.9) by $t^{s+(k-1)/2-1}$ with $-\Re es$ sufficiently large, then integrating over t, we have

$$\text{(A.10)} \quad (\frac{c\sqrt{D}_2}{2\pi})^s \Gamma(s + \frac{k-1}{2}) L(g, \frac{a}{c}, s) = i^k \kappa (\frac{c\sqrt{D}_2}{2\pi})^{1-s} \Gamma(1 - s + \frac{k-1}{2}) L(g_{D_2}, -\frac{\overline{D_2 a}}{c}, 1 - s),$$

the functional equation. Note in particular that since the left side is holomorphic for $s \geqslant (k+1)/2$, the poles of the Γ function on the right must be cancelled by zeros of the L function.

Now let $F : \mathbf{R}^+ \to \mathbf{R}$ be smooth, vanish near the origin, and decay (along with all its derivatives) rapidly at infinity. Let $\hat{F}(s)$ denote its Mellin transform. Equation (A.10) then gives

$$\frac{1}{2\pi i} \int_{(1)} \hat{F}(s) L(g, \frac{a}{c}, s) ds = i^k \kappa \frac{1}{2\pi i} \int_{(1)} \hat{F}(s) (\frac{c\sqrt{D}_2}{2\pi})^{1-2s} \frac{\Gamma(1 - s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})} L(g_{D_2}, -\frac{\overline{D_2 a}}{c}, 1 - s) ds.$$

By Mellin inversion (and a contour shift slightly to the right), the left-hand side equals

$$\sum_{n\geqslant 1} \lambda_g(n) e(n\frac{a}{c}) F(n).$$

To evaluate the right-hand side, we shift the contour to $\Re es = -1$ (crossing no poles, by the above argument) and change variables, replacing $2(1-s)$ with s. The sum on n is then absolutely convergent, and we can write our sum as

$$\frac{2\pi i^k \kappa}{c\sqrt{D_2}} \sum_{n\geqslant 1} \lambda_{D_2}(n) e(-\frac{n\overline{D_2 a}}{c})^{\frac{1}{2}} \frac{1}{2\pi i} \int_{(4)} \hat{F}(1 - \frac{s}{2}) \frac{\Gamma(\frac{s}{2} + \frac{k-1}{2})}{\Gamma(1 - \frac{s}{2} + \frac{k-1}{2})} (\frac{2\pi\sqrt{n}}{c\sqrt{D_2}})^{-s} ds$$

Next we open the Mellin transform, with the goal of writing this expression as the integral of a single function against F. There is a slight problem of absolute convergence when $k = 1$, to avoid it we deform the line $\Re es = 4$ to the contour joining the points $-1/4 + i\infty, -1/4 + i, 3/4 + i, 3/4 - i, -1/4 - i, -1/4 - i\infty$, replace $\hat{F}(1 - s/2)$ with the integral of F against $x^{-s/2}$, and shift the s contour back to $\Re es = 3/4$ (note that the shift is justified since we perform a shift in the domain $\Re es<1$ for which the residual horizontal integral go to 0 as $|t|\to+\infty$). Finally, we use the identity ([EMOT])

$$J_{k-1}(x)=\tfrac{1}{2}\frac{1}{2\pi i}\int_{(\sigma)}(x/2)^{-s}\frac{\Gamma(\frac{s}{2}+\frac{k-1}{2})}{\Gamma(1-\frac{s}{2}+\frac{k-1}{2})}ds,\ \ for\ 0<\sigma<1$$

to obtain

$$c\sum_{n\geqslant 1}\lambda_g(n)e(n\frac{a}{c})F(n)=$$
$$\chi_{D_1}(\overline{a})\chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)e(-n\frac{\overline{aD_2}}{c})\int_0^\infty F(x)\left[2\pi i^k J_{k-1}(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}})\right]dx.$$

A.4. **The summation formula: Maass cusp forms.** Suppose now that g is a primitive Maass form with nebentypus χ_D, with a Fourier development at infinity of the form

$$g(z)=\sum_{n\geqslant 1}\lambda_g(n)c^{\varepsilon_g}(nx)2|y|^{1/2}K_{ir}(2\pi|y|).$$

where

$$c^+(x)=2\cos(2\pi x),\ \ c^-(x)=2i\sin(2\pi x).$$

Since these forms satisfy the same sorts of modular relations as do the holomorphic forms, Atkin-Lehner theory applies to them as well. Note also that since $R(f_{|\gamma})=(R(f))_{|\tilde{\gamma}}$, where $\tilde{\gamma}$ is the matrix γ with the antidiagonal multiplied by -1, we can use (A.1) to show that, for $D_2|D$, $W_{D_2}g$ has parity $\chi_{D_2}(-1)\varepsilon_g$.

For $(a,c)=1$, we first evaluate the sums

$$\sum_{n\geqslant 1}\lambda_g(n)c^{\pm}(n\frac{a}{c})F(n),$$

of which our desired sum is a simple linear combination. As before we start by establishing the analytic continuation and functional equation of

$$L^+(g,\frac{a}{c},s):=\sum_{n\geqslant 1}\lambda_g(n)c^{\varepsilon_g}(n\frac{a}{c})n^{-s},\ \ L^-(g,\frac{a}{c},s):=\sum_{n\geqslant 1}\lambda_g(n)c^{-\varepsilon_g}(n\frac{a}{c})n^{-s}$$

Once again, relation (A.8) is valid (with $k=1$), where g_{D_2} is some primitive Maass form of nebentypus $\chi_{D_1}\overline{\chi}_{D_2}$. Using the same z_t as before, we get

$$\sum_{n\geqslant 1}\lambda_g(n)c^{\varepsilon_g}(n\frac{a}{c})2\sqrt{\frac{t}{c}}K_{ir}(2\pi n\frac{t}{c})=\kappa\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)c^{\varepsilon_{g_{D_2}}}(-n\frac{\overline{D_2a}}{c})2\sqrt{\frac{1}{cD_2t}}K_{ir}(2\pi n\frac{1}{cD_2t}).\tag{A.11}$$

For $\Re es$ sufficiently large, we use the identity (see [EMOT] 6.8 (26))

$$\int_0^\infty\sqrt{\frac{t}{c}}K_{ir}(2\pi n\frac{t}{c})t^s\frac{dt}{t}=\frac{1}{\sqrt{c}}(\frac{c}{2\pi n})^{s+\frac{1}{2}}2^{s-3/2}\Gamma(\frac{s+\frac{1}{2}+ir}{2})\Gamma(\frac{s+\frac{1}{2}-ir}{2})$$

to show that

$$\int_0^\infty g(\gamma z_t)t^s\frac{dt}{t}=\tfrac{1}{2}c^s\pi^{-s-\frac{1}{2}}\Gamma(\frac{s+\frac{1}{2}+ir}{2})\Gamma(\frac{s+\frac{1}{2}-ir}{2})L^+(g,\frac{a}{c},\tfrac{1}{2}+s).$$

The exponential decay of $g(\gamma z_t)$ at zero and infinity implies the analytic continuation of

$$\Gamma(\frac{s+\frac{1}{2}+ir}{2})\Gamma(\frac{s+\frac{1}{2}-ir}{2})L^+(g,\frac{a}{c},s).$$

Arguing as before, we infer the functional equation

$$\Lambda^+(g,\frac{a}{c},s)=\kappa\Lambda^+(g_{D_2},-\frac{\overline{D_2a}}{c},1-s) \tag{A.12}$$

$$\Lambda^+(g,\frac{a}{c},s)=(\frac{c\sqrt{D}_2}{\pi})^s\Gamma(\frac{s+ir}{2})\Gamma(\frac{s-ir}{2})L^+(g,\frac{a}{c},s).$$

To get L^-,we apply the differential operator $\frac{1}{2\pi i}\frac{\partial}{\partial x}$ on both sides of (A.8) getting (since $\gamma\in SL_2(\mathbf{Z})$)

$$\frac{1}{(cz+d)^2}\sum_{n\geqslant 1}\lambda_g(n)nc^{-\varepsilon_g}(n\Re e\gamma z)2|\Im m\gamma z|^{1/2}K_{ir}(2\pi n\Im m\gamma z)=$$

$$\frac{\kappa}{D_2}\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)nc^{-\varepsilon_{g_{D_2}}}(nx/D_2)2\sqrt{\frac{y}{D_2}}K_{ir}(2\pi ny/D_2).$$

The same argument as above leads to the functional equation

$$\Lambda^-(g,\frac{a}{c},s)=-\kappa\Lambda^-(g_{D_2},-\frac{\overline{D_2a}}{c},1-s), \tag{A.13}$$

$$\Lambda^-(g,\frac{a}{c},s)=(\frac{c\sqrt{D}_2}{\pi})^s\Gamma(\frac{1+s+ir}{2})\Gamma(\frac{1+s-ir}{2})L^-(g,\frac{a}{c},s).$$

Taking a linear combination, we have

$$2L(g,\frac{a}{c},s)=L^+(g,\frac{a}{c},s)+L^-(g,\frac{a}{c},s)$$

$$=\kappa(\frac{c\sqrt{D}_2}{\pi})^{1-2s}[(C^+(s)-C^-(s))L(g_{D_2},-\frac{\overline{D_2a}}{c},s)+\varepsilon_{g_{D_2}}(C^+(s)+C^-(s))L(g_{D_2},\frac{\overline{D_2a}}{c},s)]$$

with

$$C^+(s)=\frac{\Gamma(\frac{1-s+ir}{2})\Gamma(\frac{1-s-ir}{2})}{\Gamma(\frac{s+ir}{2})\Gamma(\frac{s-ir}{2})},\ C^-(s)=\frac{\Gamma(\frac{2-s+ir}{2})\Gamma(\frac{2-s-ir}{2})}{\Gamma(\frac{1+s+ir}{2})\Gamma(\frac{1+s-ir}{2})}.$$

This in turn leads to the identity

$$2\sum_{n\geqslant 1}\lambda_g(n)e(n\frac{a}{c})F(n)=$$

$$\frac{2\pi\kappa}{c\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{D_2}(n)e(-\frac{n\overline{D_2a}}{c})\frac{1}{2}\frac{1}{2\pi i}\int_{(4)}\hat{F}(1-\frac{s}{2})(C^+(1-\frac{s}{2})-C^-(1-\frac{s}{2}))(\frac{2\pi\sqrt{n}}{c\sqrt{D_2}})^{-s}ds \tag{A.14}$$

$$+\varepsilon_{g_{D_2}}\frac{2\pi\kappa}{c\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{D_2}(n)e(\frac{n\overline{D_2a}}{c})\frac{1}{2}\frac{1}{2\pi i}\int_{(4)}\hat{F}(1-\frac{s}{2})(C^+(1-\frac{s}{2})+C^-(1-\frac{s}{2}))(\frac{2\pi\sqrt{n}}{c\sqrt{D_2}})^{-s}ds.$$

A.4.1. *Computation of $C^+(s) \pm C^-(s)$.* We compute $C^+(s) \pm C^-(s)$ using the basic recursion properties of Γ (the functional equation and the duplication formula):

$$\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s), \Gamma(\tfrac{1}{2}+s)\Gamma(\tfrac{1}{2}-s) = \pi/\cos(\pi s), \tag{A.15}$$

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\pi^{1/2}\Gamma(2s). \tag{A.16}$$

Some straightforward trigonometric calculations give

$$C^+(s) + C^-(s) = \frac{\cos(\pi ir)}{2^{-2s}\pi}\Gamma(1-s+ir)\Gamma(1-s-ir), \tag{A.17}$$

$$C^+(s) - C^-(s) = \frac{2^{2s-1}}{\sin(\pi ir)}\Big[\frac{\Gamma(1-s-ir)}{\Gamma(s-ir)} - \frac{\Gamma(1-s+ir)}{\Gamma(s+ir)}\Big]. \tag{A.18}$$

At this point, we repeat the arguments of the preceeding section, arriving at integrals of the form

$$\frac{1}{2\pi i}\int_{(3/4)} \hat{F}(1-\frac{s}{2})\Gamma(\frac{s}{2}+ir)\Gamma(\frac{s}{2}-ir)(\frac{4\pi\sqrt{n}}{c\sqrt{D_2}})^{-s}ds$$

or

$$\frac{1}{2\pi i}\int_{(3/4)} \hat{F}(1-\frac{s}{2})\Big[\frac{\Gamma(\frac{s}{2}-ir)}{\Gamma(1-\frac{s}{2}-ir)} - \frac{\Gamma(\frac{s}{2}+ir)}{\Gamma(1-\frac{s}{2}+ir)}\Big](\frac{4\pi\sqrt{n}}{c\sqrt{D_2}})^{-s}ds.$$

We again replace $\hat{F}(1-s/2)$ with the integral of F against $x^{-s/2}$ and exchange orders of integration. In order to maintain convergence, and to avoid the poles at $s = \pm 2ir$, instead of staying at $\Re es = 3/4$ we use the s-contour linking the points $-1/4+i\infty, -1/4+2i|r|, 3/4+2i|r|, 3/4-2i|r|, -1/4-2i|r|, -1/4-i\infty$. In particular, if $ir \in \mathbf{R}^*$ (the case of exceptional eigenvalues) we still have from Selberg's bound that $3/4 > 1/2 \geqslant |2ir|$. From this and the identities [EMOT]

$$K_{2ir}(x) = \tfrac{1}{2}\frac{1}{2\pi i}\int_{(\sigma)} (x/2)^{-s}\Gamma(\frac{s}{2}+ir)\Gamma(\frac{s}{2}-ir)ds, \ -\Re e 2ir < \sigma$$

$$J_{2ir}(x) = \tfrac{1}{2}\frac{1}{2\pi i}\int_{(\sigma)} (x/2)^{-s}\frac{\Gamma(\frac{s}{2}+ir)}{\Gamma(1-\frac{s}{2}+ir)}ds, \ -\Re e 2ir < \sigma < 1.$$

we have

$$c\sum_{n\geqslant 1}\lambda_g(n)e(n\frac{a}{c})F(n) = \chi_{D_1}(\overline{a})\chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)e(-n\frac{\overline{aD_2}}{c})\int_0^\infty F(x)J_g(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}})dx$$

$$+\varepsilon_{g_{D_2}}\chi_{D_1}(\overline{a})\chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)e(n\frac{\overline{aD_2}}{c})\int_0^\infty F(x)K_g(\frac{4\pi\sqrt{nx}}{c\sqrt{D_2}})dx,$$

where $\varepsilon_{g_{D_2}} = \chi_{D_2}(-1)\varepsilon_g$ and

$$J_g(x) = -\frac{\pi}{\sin(\pi ir)}(J_{2ir}(x) - J_{-2ir}(x)), \ K_g(x) = 4\cos(\pi ir)K_{2ir}(x).$$

When $r=0$, the case of greatest interest, the functions reduces to $-2\pi Y_0$ and $4K_0$, respectively.

A.5. **The summation formula recapitulated.** We put everything into one formula for convenient reference.

Theorem A.4. *Let D be positive integer, χ_D a character of modulus D, and g be one of the forms presented in Section 3. For $(a,c)=1$, set $D_1=(c,D)$, $D_2=D/D_1$, and assume that $(D_1,D_2)=1$, so that $\chi_D=\chi_{D_1}\chi_{D_2}$ is the unique factorization of χ_D into characters of modulus D_1 and D_2. For $F\in C^\infty(\mathbf{R}^{*+})$ a smooth function vanishing in a neighborhood of zero and rapidly decreasing*

$$c\sum_{n\geqslant 1}\lambda_g(n)e(n\frac{a}{c})F(n)=$$

$$\chi_{D_1}(\overline{a})\chi_{D_2}(-c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)e(-n\frac{\overline{aD_2}}{c})\int_0^\infty F(x)J_g(\frac{4\pi}{c}\sqrt{\frac{nx}{D_2}})dx$$

$$+\chi_{D_1}(\overline{a})\chi_{D_2}(c)\frac{\eta_g(D_2)}{\sqrt{D_2}}\sum_{n\geqslant 1}\lambda_{g_{D_2}}(n)e(n\frac{\overline{aD_2}}{c})\int_0^\infty F(x)K_g(\frac{4\pi}{c}\sqrt{\frac{nx}{D_2}})dx.$$

In this formula

- *$\eta_g(D_2)$ is the pseudo-eigenvalue of the Atkin-Lehner operator W_{D_2}. If $\lambda_g(D_2)\neq 0$ it equals value*
$$\eta_g(D_2)=\frac{G(\chi_{D_2})}{\lambda_g(D_2)\sqrt{D_2}};$$
- *If g is holomorphic of weight k_g, then*
$$J_g(x)=2\pi i^{k_g}J_{k_g-1}(x),\ \ K_g(x)=0;$$
- *if g is a Maass form with $(\Delta+\lambda)g=0$, then let r satisfy $\lambda=(\frac{1}{2}+ir)(\frac{1}{2}-ir)$ and let ε_g be the eigenvalue of g under the reflection operator:*
$$J_g(x)=\frac{-\pi}{\sin(\pi ir)}(J_{2ir}(x)-J_{-2ir}(x)),\ \ K_g(x)=\varepsilon_g 4\cosh(\pi r)K_{2ir}(x).$$
- *if $r=0$,*
$$J_g(x)=-2\pi Y_0(x),\ \ K_g(x)=\varepsilon_g 4K_0(x).$$

Finally, for completeness, we recall the summation formula of [J]:

$$c\sum_{n\geqslant 1}\tau(n)e\Big(\frac{an}{c}\Big)F(n)=2\int_0^{+\infty}(\log\frac{\sqrt{x}}{c}+\gamma)F(x)dx$$

$$+\sum_{n\geqslant 1}\tau(n)\int_0^{+\infty}\Big(-2\pi e\Big(-\frac{\overline{a}n}{c}\Big)Y_0\Big(\frac{4\pi\sqrt{nx}}{c}\Big)+4e\Big(\frac{\overline{a}n}{c}\Big)K_0\Big(\frac{4\pi\sqrt{nx}}{c}\Big)\Big)F(x)dx.$$

This corresponds to the summation formula for the Eisenstein series $E'(z,1/2)$ given in (1.2).

Appendix B. Shifted convolutions of modular forms

In this section we establish the bound cited in Section 7.2.1; we follow [DFI I] almost exactly. We work in a more general setting: consider two primitive forms g, g' of squarefree level D, D', and nebentypus $\chi_D, \chi'_{D'}$ which are not exceptional. We are given a smooth test function $f(z,y)$ on $\mathbf{R}^{*+}\times\mathbf{R}^{*+}$ satisfying

$$z^iy^jf^{(ij)}(z,y)\ll_{i,j}(1+\frac{z}{Z})(1+\frac{y}{Y})P^{i+j}$$

for all $i,j\geqslant 0$ with some $P,Z,Y\geqslant 1$. For $(a,b)=1$, we wish to evaluate the sum

$$D^{\pm}_f(a,b;h):=\sum_{am\pm bn=h}\lambda_g(m)\lambda_{g'}(n)f(am,bn). \tag{B.1}$$

Proposition B.1. *Suppose $(a,b)=1$ and $h\neq 0$, and suppose that f satisfies the above conditions. Then*

$$D^{\pm}_f(a,b;h)=O_{\epsilon,g,g'}(P^{5/4}(Z+Y)^{1/4}(YZ)^{1/4+\epsilon}) \tag{B.2}$$

This is a direct generalization of Theorem 1 of [DFI I], which corresponds to the case $g=g'=E'(z,\frac{1}{2})$.

We borrow the notation of [DFI I] (22) p.214 and set $U=R^2=\min(Y,Z)P^{-1}$. Following that proof, we have

$$D^{\pm}_f(a,b;h):=\sum_{1\leqslant r<2R}D^{\pm}_f(a,b,r;h)$$
$$=\sum_{1\leqslant r<2R}\sideset{}{^*}\sum_{d(r)}e(\frac{-dh}{r})\sum_{m,n}\lambda_g(m)\lambda_{g'}(n)e(\frac{dam\pm dbn}{r})E(m,n),$$

where $E(z,y):=F(az,by)\Delta_r(az-by-h)$ with $\Delta_r(u)$ defined by equation (11) of [DFI I] (with r replaced by q) and $F(z,y)=f(z,y)\varphi(x-y-h)$ where $\varphi(u)$ is a smooth function supported on $|u|<U$ such that $\varphi(0)=1$ and $\varphi^{(i)}(u)\ll_i U^{-i}$.

We apply the formulae of Section A in each variable, setting $r_a=r/(a,r)$, $a_r=a/(a,r)$ $r_b=r/(b,r)$, $b_r=b/(b,r)$, $D_1=(D,r_a)$ and $D'_1=(D',r_b)$, $D_1D_2=D$, $D'_1D'_2=D'$, so that

$$\frac{r^2}{(ab,r)}\sum_{m,n}=\chi'_{D'_1}(\pm1)\kappa\kappa'\chi_{D_1}\chi_{D'_1}(\bar{d})\sum_{m,n}\lambda_{g_{D_2}}(m)\lambda_{g'_{D'_2}}(n)e\left(-m\frac{(a,r)\overline{a_rdD_2}}{r}\mp n\frac{(b,r)\overline{b_rdD'_2}}{r}\right)I_{ab}(m,n)$$
$$+***$$

with

$$\kappa(a,r,D)=\chi_{D_1}(\overline{a}_r)\chi_{D_2}(-r_a)\eta_g(D_2)/\sqrt{D_2},$$
$$\kappa(b,r,D')=\chi'_{D'_1}(\overline{b}_r)\chi'_{D'_2}(-r_b)\eta_{g'}(D'_2)/\sqrt{D'_2},$$

and

$$I_{ab}(m,n,h)=\iint_0^\infty J_g(\frac{4\pi(a,r)\sqrt{mz}}{r\sqrt{D}_2})J_{g'}(\frac{4\pi(b,r)\sqrt{ny}}{r\sqrt{D}'_2})E(z,y)dzdy$$

and three more terms involving the $K_g, K_{g'}$ functions implicit in $+***$.

Summing over d, we obtain

$$D^{\pm}_f(a,b;h)=\sum_{1\leqslant r<2R}\frac{(ab,r)}{r^2}\chi'_{D'_1}(\pm1)\kappa\kappa'\Bigg\{***$$

$$+\sum_{m,n\geqslant 1}\lambda_{g_{D_2}}(m)\lambda_{g_{D'_2}}(n)S_{\overline{\chi}_{D_1}\overline{\chi}'_{D'_1}}(h,\mp n(b,r)\overline{b_r dD'_2}-m(a,r)\overline{a_r dD_2};r)I_{ab}(m,n)\Bigg\},$$

where $S_{\overline{\chi}_{D_1}\overline{\chi}'_{D'_1}}(x,y;r)$ is the Kloosterman sum of modulus r, twisted by the caracter $\overline{\chi}_{D_1}\overline{\chi}'_{D'_1}$.

We proceed as in [DFI I], using that the modular forms g, g' are not exceptional and (3.8)). to find that all the terms above contribute to the error term of (B.2).

Appendix C. Properties of Bessel functions

In this section we collect some useful properties of the Bessel functions encountered in Section A. They can be obtained from [EMOT2] Chap. VII.

Lemma C.1. *For $z>0$, and for all ν*

$$(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z) \tag{C.1}$$

$$(z^\nu K_\nu(z))' = -z^\nu K_{\nu-1}(z) \tag{C.2}$$

$$(z^\nu Y_\nu(z))' = z^\nu Y_{\nu-1}(z). \tag{C.3}$$

Lemma C.2. *For $z>0$ and $k\geqslant 0$ we have*

$$|z^k J_\nu^{(k)}(z)| \ll_{k,\nu} \frac{z^{\Re e\nu}}{(1+z)^{\Re e\nu+1/2}} \left(\ll_{k,\nu} \frac{1}{(1+z)^{1/2}} \text{ if } \Re e\nu\geqslant 0\right) \tag{C.4}$$

$$\Re e\nu = 0,\ |z^k K_\nu^{(k)}(z)| \ll_{k,\nu} \frac{e^{-z}(1+|\log z|)}{(1+z)^{1/2}} \tag{C.5}$$

$$|z^k Y_0^{(k)}(z)| \ll_k \frac{(1+|\log z|)}{(1+z)^{1/2}}. \tag{C.6}$$

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