

## Rapidly Converging Truncation Scheme of the Exact Renormalization Group

Ken-Ichi AOKI, Keiichi MORIKAWA, Wataru SOUMA,  
Jun-Ichi SUMI and Haruhiko TERAO

*Department of Physics, Kanazawa University  
Kanazawa 920-1192, Japan*

(Received December 8, 1997)

The truncation scheme dependence of the exact renormalization group equations is investigated for scalar field theories in three dimensions. The exponents are numerically estimated to the next-to-leading order of the derivative expansion. It is found that the convergence property in various truncations in the number of powers of the fields is remarkably improved if the expansion is made around the minimum of the effective potential. It is also shown that this truncation scheme is suitable for evaluation of infrared effective potentials. The physical interpretation of this improvement is discussed by considering  $O(N)$  symmetric scalar theories in the large  $N$  limit.

### §1. Introduction

It has been well known for more than two decades that the Wilson renormalization group (RG) offers practical tools as well as profound insights for investigation of non-perturbative phenomena in field theories. The continuum versions of the Wilson RG equation are called the exact renormalization group (ERG) equations, which are given in the form of non-linear functional differential equations. There have been proposed several formulations of the ERG, which are found to be mutually equivalent. These ERG equations give the change of the so-called Wilsonian effective actions<sup>1)-3)</sup> or 1PI cutoff effective actions<sup>4),5)</sup> under scale variation leaving the low energy physics unaltered. The Wilsonian effective action may be regarded as a point in the infinite dimensional space of theories, or the space of coupling constants, and the ERG generates flows of the coupling constants in this space.

However, in practical use, it is inevitable to approximate such an infinite dimensional theory space by a much smaller subspace in order to solve the ERG equations. Needless to say, the non-perturbative nature of the ERG should be maintained in this approximation. The method generally applied is the so-called derivative expansion, which expands the interactions in powers of the derivatives and truncates the series at a certain order.<sup>6)</sup> With this approximation the full equation is reduced to coupled partial differential equations. Recently, the ERG in the derivative expansion approximation for scalar field theories has been extensively studied at the order of the derivative squared and has been found to offer fairly good non-perturbative results even quantitatively.<sup>8)-10)</sup> The lowest order of the derivative expansion, neglecting all corrections to derivative operators, is called the "local potential approximation" (LPA).<sup>11),12)</sup> Although the wave function renormalization

and, therefore, the anomalous dimension is ignored in the LPA, the leading exponent  $\nu$  is known to be estimated rather well by using the ERG. However, the number of the couplings to be incorporated, or the number of the beta functions, is still infinite in the derivative expansion. If we consider application to more complicated systems it would be favorable to approximate it further by a finite number of couplings, as long as this is sufficiently effective. This method is advantageous not only because it simplifies the analysis but also because the effective couplings of physical interest are treated directly. Such an approximation is naively performed by truncated expansion of the Wilsonian effective action, in turn, in powers of the fields. Actually it is found to work well practically without loss of the non-perturbative nature as long as we adopt a special expansion scheme, as is discussed later.

One of the advantageous points of the ERG is certainly that it is able to allow for the systematic improvement of the approximations, as mentioned above. However, the improvement totally relies on the convergence of the results in non-perturbative analysis. We may approve the results obtained by the ERG only after confirming their sufficient convergence, since there is no small parameter which controls the approximation. It should be noted here that the commonly used expansion methods, e.g., the perturbation theory,  $1/N$  expansion and  $\varepsilon$ -expansion etc., generally produce asymptotic series at best, in contrast to the convergence property exhibited by the ERG.<sup>13)</sup>

The main subject of this paper is the convergence of the expansion scheme in terms of the fields. It has been claimed that this convergence is rather poor,<sup>14)</sup> or that the results cease to converge.<sup>15)</sup> If such behavior appears commonly, it would be a fatal defect of the ERG approach.\*<sup>)</sup> However, it has been realized that the expansion around the potential minimum drastically improves this convergence property.<sup>16),17)</sup> Indeed, it is good news for the ERG approach that we may obtain good convergence by adopting the appropriate truncation scheme. However, it has not yet been investigated in detail how effective this scheme is generally, nor has it been determined the origin of this improvement.

In this paper we discuss convergence properties in different expansions schemes by examining  $Z_2$  symmetric scalar theories in three dimensions. As the physical quantities, the critical exponents and also the infrared effective potentials, or the renormalized trajectories, are compared among the different schemes. It is found that the convergence property is significantly improved in the new truncation scheme. We will also discuss the physical reason of the improvement by studying  $O(N)$  symmetric scalar theories in the large  $N$  limit. From this observation it is speculated that good convergence depends on how accurately the relevant operator is covered within the truncated subspace.

---

\*<sup>)</sup> The convergence of the derivative expansion in the non-perturbative calculation remains difficult to see due to complication in the higher orders. Morris examined this problem perturbatively at two loops and found that the ERG with certain cutoff profiles indeed displays convergence.<sup>8)</sup>

### §2. Exact renormalization group equations

First let us briefly review the formulations of the ERG with which we will analyze the scalar theories. The ERG equation widely studied recently is given by considering renormalization of the so-called “cutoff effective action”,  $\Gamma_{\text{eff}}[\phi]$ .<sup>4),5)</sup> The cutoff effective action is the 1PI part of the Wilsonian effective action, namely, the generating functional of the connected and amputated cutoff Green functions. Therefore the ERG equation may be obtained by the Legendre transformation of the Polchinski equation for the Wilsonian effective action.

In this formulation the cutoff is performed by introducing a proper smooth function in terms of the momentum  $q$  and the cutoff scale  $\Lambda$ ;  $C(q, \Lambda)$ , into the partition function as

$$\exp W[J] = \int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \phi \cdot C^{-1} \cdot \phi - S[\phi] + J \cdot \phi \right\}, \quad (1)$$

where dot  $(\cdot)$  denotes matrix contraction in momentum space. From Eq. (1) we may obtain the variation of  $W$  with respect to the cutoff as<sup>3)</sup>

$$\frac{\partial W[J]}{\partial \Lambda} = -\frac{1}{2} \left\{ \frac{\delta W[J]}{\delta J} \cdot \frac{\partial C^{-1}}{\partial \Lambda} \cdot \frac{\delta W[J]}{\delta J} + \text{tr} \left( \frac{\partial C^{-1}}{\partial \Lambda} \cdot \frac{\delta^2 W[J]}{\delta J \delta J} \right) \right\}. \quad (2)$$

The ERG for the cutoff effective action  $\Gamma_{\text{eff}}$  is defined by the Legendre transformation:

$$\Gamma_{\text{eff}}[\phi] + \frac{1}{2} \phi \cdot C^{-1} \cdot \phi = -W[J] + J \cdot \phi. \quad (3)$$

By taking the canonical scaling under the shift of the cutoff scale into consideration, the ERG equation for  $\Gamma_{\text{eff}}$  in  $D$  dimensions may be written down as<sup>6)</sup>

$$\left( \frac{\partial}{\partial t} + d_\phi \phi \cdot \frac{\delta}{\delta \phi} + \Delta_\partial - D \right) \Gamma_{\text{eff}}[\phi] = \frac{1}{2} \text{tr} \left\{ \frac{\partial C^{-1}}{\partial t} \cdot \left( C^{-1} + \frac{\delta^2 \Gamma_{\text{eff}}[\phi]}{\delta \phi \delta \phi} \right)^{-1} \right\}, \quad (4)$$

where  $t = \ln(\Lambda_0/\Lambda)$ , and  $d_\phi$  is the scaling dimension of the scalar field which is given by  $(D - 2 + \eta)/2$  with the anomalous dimension  $\eta$ . The operator  $\Delta_\partial$  counts the number of the derivatives, which is given explicitly by

$$\Delta_\partial \equiv D + \int \frac{d^D q}{(2\pi)^D} \phi(q) q_\mu \frac{\partial}{\partial q_\mu} \frac{\delta}{\delta \phi(q)}. \quad (5)$$

Thus the ERG equation is defined depending on the cutoff functions. The physical quantities such as the exponents are found to be independent of the cutoff scheme, as is expected. However, this does not hold once some approximations have been performed to these equations.\*<sup>)</sup> In this paper we are going to examine the extreme

\*<sup>)</sup> In practice, as far as the scalar theories are concerned, this cutoff scheme dependence is found to be rather weak, and the exponent changes smoothly under variation of the cutoff profiles.<sup>7),18)</sup>

cases, i.e. schemes with a very smooth cutoff and with the sharp cutoff limit. We adopt the cutoff function

$$C(q, \Lambda) = \frac{1}{q^2} \frac{\theta(q, \Lambda)}{1 - \theta(q, \Lambda)}, \quad \theta(q, \Lambda) = \frac{1}{1 + (\Lambda^2/q^2)^2} \quad (6)$$

as the smooth one in the practical calculations, *a la* Ref. 6). The sharp cutoff version of the ERG will be discussed later.

In the derivative expansion, the full ERG equation (4) is reduced to a set of partial differential equations by substituting the effective action  $\Gamma_{\text{eff}}$  of the form

$$\Gamma_{\text{eff}}[\phi] = \int d^D x \left\{ V(\phi; t) + \frac{1}{2} K(\phi; t) (\partial_\mu \phi)^2 + \dots \right\}, \quad (7)$$

where  $V(\phi; t)$  and  $K(\phi; t)$  are cutoff-dependent functions. In the second order of the expansion we may take the variation of the first two terms only,  $\partial V/\partial t$  and  $\partial K/\partial t$ . For the cutoff function given by (6), these ERG equations in three dimensions are found to be

$$\frac{\partial V}{\partial t} = -\frac{1}{2}(1 + \eta)\phi V' + 3V - \frac{1 - \eta/4}{\sqrt{K}\sqrt{V'' + 2\sqrt{K}}}, \quad (8)$$

$$\begin{aligned} \frac{\partial K}{\partial t} = & -\frac{1}{2}(1 + \eta)\phi K' - \eta K + \left(1 - \frac{\eta}{4}\right) \left( \frac{1}{48} \frac{24KK'' - 19(K')^2}{K^{3/2}(V'' + 2\sqrt{K})^{3/2}} \right. \\ & - \frac{1}{48} \frac{58V'''K'\sqrt{K} + 57(K')^2 + (V''')^2 K}{K(V'' + 2\sqrt{K})^{5/2}} \\ & \left. + \frac{5}{12} \frac{(V''')^2 K + 2V'''K'\sqrt{K} + (K')^2}{\sqrt{K}(V'' + 2\sqrt{K})^{7/2}} \right), \quad (9) \end{aligned}$$

where the prime denotes differentiation with respect to  $\phi$ .<sup>6)</sup> The anomalous dimension  $\eta(t)$  is determined by imposing the renormalization condition for the wave function,  $K(\phi = 0; t) = 1$ . In the LPA we may solve only Eq. (8) with respect to the effective potential  $V(\phi; t)$  by reducing  $K = 1$  and, therefore,  $\eta = 0$ . Actually these partial differential equations have been solved and found to give the exponents with fairly good accuracy.<sup>8)-10), 15)</sup> In the next section these equations are examined by expansion in powers of the fields.

In the case of the sharp cutoff scheme, we examine the Wegner-Houghton (WH) equation<sup>1)</sup> instead of Eq. (4) with the sharp cutoff limit. The WH equation is formulated so that the fields are integrated gradually from the modes with higher momentum by introducing the sharp cutoff into the path integral measure. Indeed, the ERG equation for the 1PI effective action as well as the Polchinski equation<sup>3)</sup> turns out to be equivalent to the WH equation in the sharp cutoff limit.<sup>19), 18)</sup> However, the ERG equation is known to exhibit non-analytic dependence on the momentum in this limit. Therefore we examine the sharp cutoff scheme only in the LPA. The WH equation in the LPA is given simply by

$$\frac{\partial V}{\partial t} = 3V - \frac{1}{2}\phi V' + \frac{1}{4\pi^2} \ln(1 + V''), \quad (10)$$

in three dimensions.<sup>11), 12)</sup>

These partial differential equations, of course, may be solved directly. However, this would not be practical for more complicated systems. Indeed, it turns out to be much more economical to solve them by reducing them to coupled ordinary differential equations. Besides, if we are interested in the effective coupling constants of the theories, e.g. masses, self-interactions, gauge couplings and so on, it is natural to solve the ERG equations for these coupling constants by expanding the effective action into a sum of operators. Note that Eqs. (4) and (10) given in the two different cutoff schemes do not produce the same results for physical quantities even if both are analyzed in the LPA. Rather, in this paper we are interested in the convergence properties of the solutions of the equations obtained by truncation of their power expansion in each case. In the next section the convergence behavior is explicitly examined.

### §3. Truncation in the comoving frame

First let us examine the ERG equation (10) for the  $Z_2$  symmetric scalar theory. If we approximate the effective potential  $V(\phi; t)$  by a finite order power series in terms of the  $Z_2$  invariant variable  $\rho = \phi^2/2$  (Scheme I),

$$V(\rho; t) = \sum_{n=1}^M \frac{a_n(t)}{n!} \rho^n, \tag{11}$$

then we obtain  $M$  ordinary differential equations for the running couplings  $a_n$ .\*) We may suppose naively that the results obtained with these truncated equations converge to the solutions of the partial differential equation (10) as the order of the truncation  $M$  is increased. However, this is not the case. In Fig. 1 the truncation dependence of the leading exponent  $\nu$  is shown. It is seen that the solutions cease to converge beyond a certain order and finally to oscillate with 4-fold periodicity around the expected value from the partial differential equation (10). Actually, the non-trivial fixed point itself is also found to oscillate similarly in the truncated approximation. Morris pointed out in Ref. 15) that this oscillatory behavior is related to the singularity of the fixed point solution of Eq. (10) in the complex plane. The singularities  $\rho_* = |\rho_*|e^{i\theta_*}$  closest to the origin are located at  $(|\rho_*|, \theta_*) = (0.123, \pm 0.514\pi)$ . The existence of these singularities implies that the coefficients of the fixed point solution expanded in powers around the origin appear with 4-fold oscillation and that the convergence radius of the expansion series is given by 0.123. From these observations Morris has explained the behavior of the truncated solutions in Scheme I and that the leading exponent cannot converge to a definite value with precision beyond an error of 0.008.

This undesirable behavior, however, is drastically improved if we expand the

---

\*) The  $M$  coupled beta functions lead us to  $M$  distinct fixed point solutions, all of which but two (the trivial fixed point and the so-called Wilson-Fisher fixed point) should be fake in this approximation. However, we may easily identify the true fixed point among these solutions by looking at their stability against the truncation.

potential around its minimum. The effective potential may be approximated in turn by (Scheme II)

$$V(\rho; t) = \sum_{n=2}^M \frac{b_n(t)}{n!} (\rho - \rho_0(t))^n, \quad (12)$$

where  $\rho_0$  is the potential minimum. Therefore here the term linear in  $\rho - \rho_0$  is absent. Indeed, as is seen in Fig. 1 (see also Fig. 4), the results for the exponent calculated in this scheme converge very rapidly to  $0.689459056 \pm 2$ , which is consistent with the results obtained by analysis of the partial differential equation (10),  $0.689459056$ . (The latter will be reported elsewhere.) Thus we may say that the expansion scheme II is a quite effective method in obtaining an accurate answer in a fairly small-dimensional subspace. The fixed point potential obtained with this method (Scheme II) is also shown in Fig. 2 (right). It is rather surprising that the potential of the fixed point solution can be obtained within a certain range of the field variable  $\rho$  quite well with such a simple analysis.

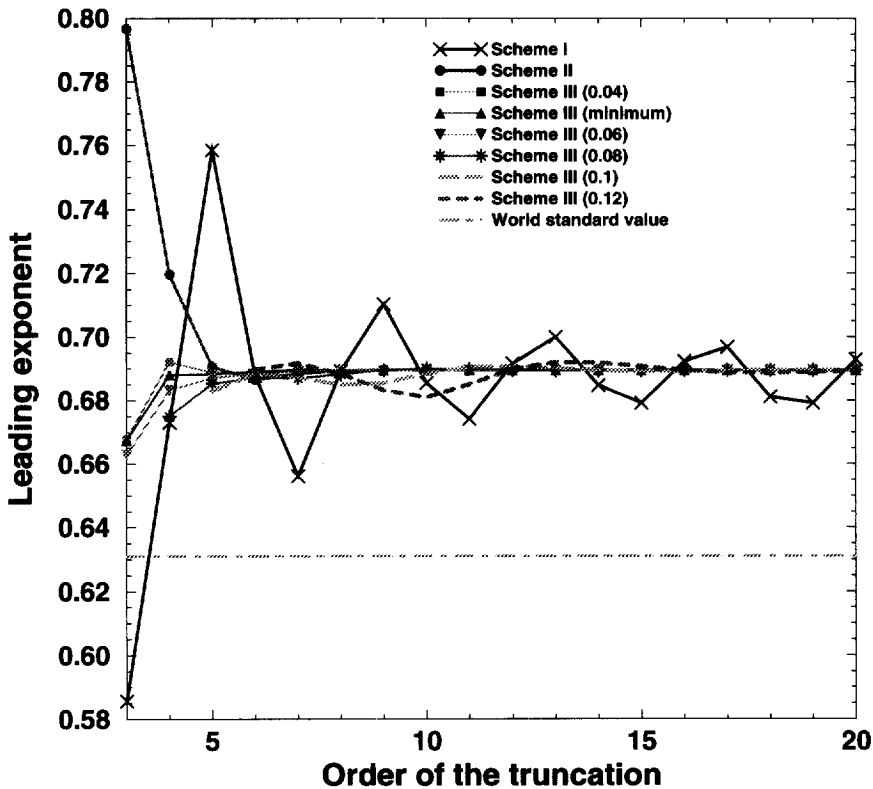


Fig. 1. The truncation dependence of the leading exponent evaluated using various truncation schemes. The solutions not displayed ( $M = 3$  of Scheme III (0.08),  $M = 3, 4$  of Scheme III (0.1) and  $M = 3, 4, 5$  of Scheme III (0.12)) lie outside the range on the vertical axis. The values in the parentheses in the legend correspond to the expansion point  $\rho_0$  in Scheme III. The term "minimum" means the minimum of the fixed point potential in each truncation.

However, the series of the solution obtained in truncated approximation of Scheme II does not converge perfectly. We examined the leading exponent  $\nu$  in this analysis up to the 60th order of truncation. The logarithmic plot of the obtained coefficients  $\rho_0$  and  $b_n$  of the fixed point solution and the leading exponent against the order of truncation is shown up to 60th in Figs. 3 and 4, respectively. It is seen that the truncation dependence does not disappear completely, and the results display oscillatory behavior with 3-fold approximate periodicity, as is shown in Figs. 3 and 4. These types of behavior can be clearly understood by considering the singularities of the untruncated fixed point solution.<sup>20)</sup> Since the minimum of the fixed point potential is at  $\rho_0 = 0.0471$ , we have  $|\rho_* - \rho_0| = 0.134$  and  $\arg(\rho_* - \rho_0) = 0.64\pi$ .

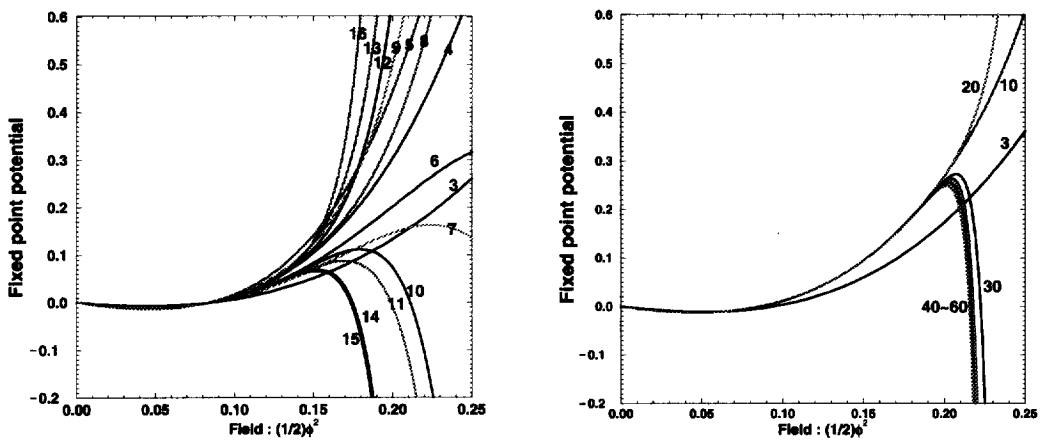


Fig. 2. The truncation dependences of the fixed point potential evaluated using Schemes I and II are shown to the left and right in the figure, respectively. The integers in the figures denote the orders of the truncation.

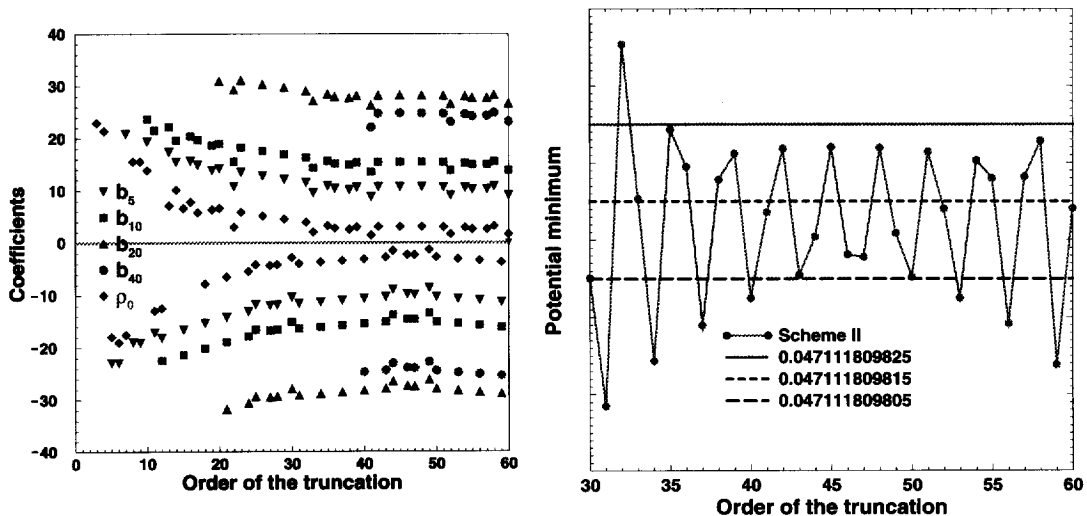


Fig. 3. The large order behavior of the coefficients  $b_n$  and the potential minimum  $\rho_0$  evaluated using Scheme II. The vertical axis of the left figure denotes  $\ln(|b_i - \langle b_i \rangle| 10^{11} + 1) \text{sign}(b_i - \langle b_i \rangle)$ .

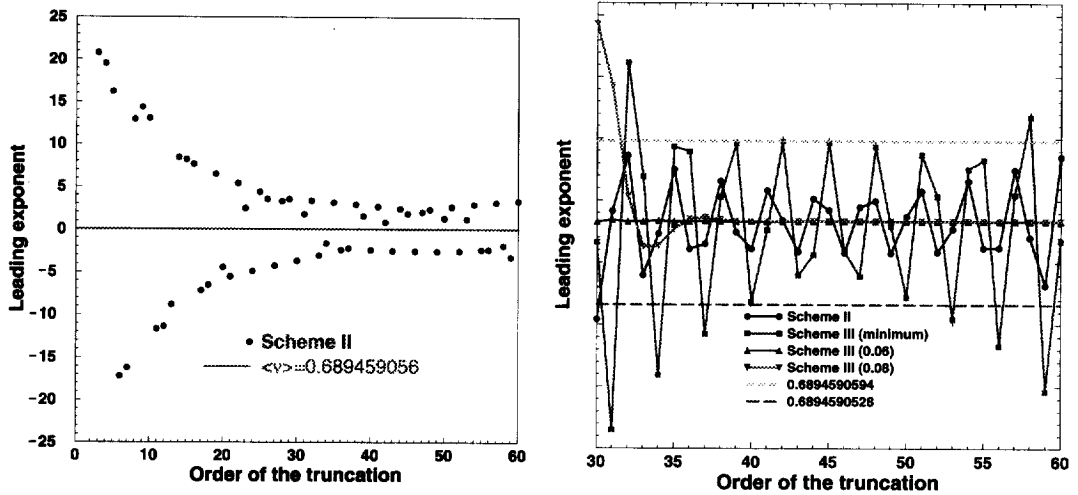


Fig. 4. The large order behavior of the leading exponent evaluated using Schemes II and III. The vertical axis of the left figure denotes  $\ln(|\nu - \langle \nu \rangle| 10^{10} + 1) \text{sign}(\nu - \langle \nu \rangle)$ .

This angle is close to  $2\pi/3$ , which tells us that the truncated solution in Scheme II oscillates with 3-fold periodicity. Also, the expansion should have a finite convergence radius of 0.134. Therefore the boundary of the convergence radius appears around  $\rho = 0.181$ . This is indeed seen in Fig. 2 (right). Although Scheme II extends the convergence radius slightly, the convergence of the leading exponent is much improved up to the error  $10^{-9}$ , as is seen in Fig. 4 (left).\*)

Such truncation dependence in the two schemes is seen in the ERG equations with smooth cutoff, (8) and (9), as well. In Fig. 5 the leading exponent obtained by Schemes I and II in the LPA is shown. We find that Scheme II is again clearly superior to Scheme I, while the oscillation, even in Scheme I, is significantly attenuated in the smooth cutoff scheme. The value to which the leading exponent converges is 0.660.\*\*\*) In the second order of the derivative expansion we examined the truncation dependence of the leading exponent and also of the anomalous dimension in Scheme II. \*\*\*) In this analysis the function  $K(\rho; t)$  is also expanded around  $\rho_0$  and is truncated at the same order as  $V(\rho; t)$ . The results are shown in Figs. 5 and 6. The values so obtained are  $\nu = 0.617476$  and  $\eta = 0.05425$ . The world standard values are  $\nu = 0.6310$  and  $\eta = 0.0375$  from the  $\varepsilon$ -expansion.<sup>21)</sup> It is worth while to mention that the leading exponent indeed approaches the world standard value when going up to the next-to-leading order of the derivative expansion.

In general  $d/dt$  in the ERG defines a vector field of RG flow on the theory space and is given in the coordinate system  $\{g_i\}$  by

$$\frac{d}{dt} = \beta_i(g) \frac{\partial}{\partial g_i}, \quad (13)$$

\*) It is noted that in the analysis of Eq. (10) the minimum expansion in terms of the variable  $\phi$  becomes worse than Scheme I due to the smaller convergence radius.

\*\*) The partial ERG equations given by (8) and (9) are examined in great detail in Refs. 8) and 9). Our results are consistent with those of that analysis.

\*\*\*) The exponents could not even be evaluated in Scheme I.



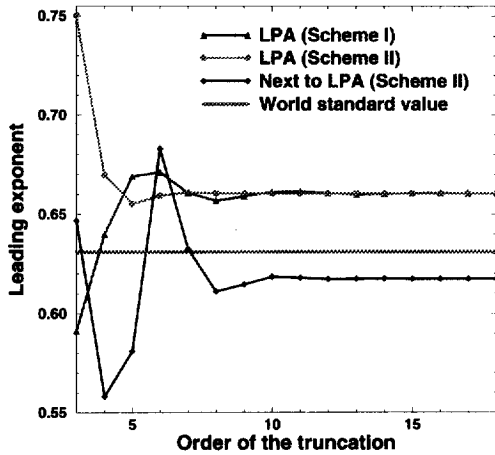


Fig. 5. The leading exponent evaluated using Scheme II.

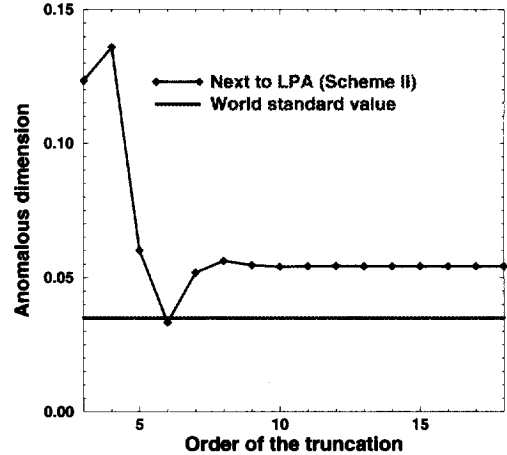


Fig. 6. The anomalous dimension evaluated using Scheme II.

where we have introduced the generalized beta functions  $\beta_i(g) = dg_i/dt$ . Correspondingly, the variation of the effective action  $S$  may be written as

$$\frac{dS}{dt} = \beta_i(g)\xi_i(g), \quad (14)$$

where  $\xi_i(g) = \partial S/\partial g_i$  are the base vectors. In Scheme II the base vectors are dependent on the position in the theory space, while they are fixed in Scheme I. We refer to these coordinate systems as the “comoving frame” and “fixed frame”, respectively.<sup>13)</sup>

Note that Schemes I and II with same maximum powers  $M$  employ the same subspace of the polynomials. We project the flow in the full theory space on the subspace and evaluate the beta functions for the projected flow. The projection depends on the choice of the coordinates, or on the manner of expansion. In fact, the beta functions in the truncation Schemes I and II are evaluated for different projections. This causes deviation of the results between the two schemes discussed here.

From the preceding argument it may be expected that more accurate convergence is achieved if we expand at a point with a larger convergence radius. In order to see this, we examine another type of truncation, similar to Scheme I, but with the expansion point shifted from the origin by some fixed values  $\rho_0$  (Scheme III):

$$V(\rho; t) = \sum_{n=1}^M \frac{c_n(t)}{n!} (\rho - \rho_0)^n. \quad (15)$$

This scheme is an example of the fixed frame. The leading exponent obtained in this approximation is also shown in Fig. 1 and in Fig. 4 (right). Here we employ Eq. (10). It is reasonable that the value will converge to a definite value with high precision if the expansion point  $\rho_0$  is far from singularities. Indeed, we can

obtain a convergent value of leading exponent with precision on the order of  $10^{-16}$ ,  $0.68945905616213484 \pm 1$ , by choosing  $\rho_0 = 0.08$ . However, in order to obtain a highly accurate value, we need a very large order of truncation.

One should employ a truncation method which provides a result with sufficient convergence and precision even with a small order of truncation. As is seen in the next section, in the large  $N$  limit, Scheme II is found to give us the exact solution. Apart from this extreme case of large  $N$  theory, however, it is significant for the practical analysis of complicated systems to be converging with small order of truncation, because it is in general difficult to evaluate at the large order of truncation. It is seen from Fig. 1 that we should set the value of  $\rho_0$  to the minimum of the fixed point potential if we demand better convergence at small order of truncation. If the highly accurate precision is not required, or if it is difficult to analyze with a large order of truncation, the truncated approximation of Scheme II is sufficiently workable owing to its simplicity. The fact that Scheme III with  $\rho_0$  set to the minimum of the fixed point potential also gives good convergence in the small subspace implies that the improvement of the approximation originates in the choice of the base vectors  $\xi_i$  around the fixed point. This seems reasonable, since the exponent is determined solely by the structure of the ERG equation around the non-trivial fixed point. Needless to say, the truncated approximation of the comoving frame is much more advantageous in practical analysis than that of the fixed frame, since the position of the minimum of the fixed point potential cannot be known *a priori*.

#### §4. Large $N$ limit

If we extend the observation examined in the previous section to  $O(N)$  symmetric scalar theories in three dimensions, then the approximation in Scheme II is found to show stronger convergence as  $N$  increases, while the exponents obtained in Scheme I become more fluctuating.<sup>14), 17)</sup> Moreover, the truncation dependence turns out to disappear completely in the large  $N$  limit, as discussed in Ref. 13). Therefore we discuss the physical reason behind this remarkable improvement of convergence by considering  $O(N)$  models in the large  $N$  limit.

The LPA WH equation in  $D$  dimensions is reduced in the large  $N$  limit to

$$\frac{\partial V}{\partial t} = DV + (2 - D)\rho V_\rho + \frac{A_D}{2} \ln(1 + V_\rho), \quad (16)$$

where  $\rho$  denotes  $\sum_{a=1}^N (\phi^a)^2/2$ ,  $V_\rho$  denotes the differentiation of  $V$  with respect to  $\rho$  and  $A_D$  is the surface of the  $D$ -dimensional unit sphere divided by  $(2\pi)^D$ . Here we have rescaled  $\rho$  and  $V$  properly by  $N$  before taking the limit. It is known that Eq. (16) gives the exact effective potential in the large  $N$  limit. If we expand Eq. (16) in Scheme II, the resulting differential equations,

$$\begin{aligned} \frac{d\rho_0}{dt} &\equiv \beta_1 = (D - 2)\rho_0 - \frac{A_D}{2}, \\ \frac{db_2}{dt} &\equiv \beta_2 = (4 - D)b_2 - \frac{A_D}{2}b_2^2, \end{aligned}$$

$$\frac{db_3}{dt} \equiv \beta_3 = (6 - 2D)b_3 - \frac{A_D}{2}(3b_2b_3 - 2b_2^3), \quad \text{etc.}, \quad (17)$$

are found to be analytically soluble order by order. This means that the entire flow diagram in the theory space can be exactly determined within any finite dimensional subspaces. We refer to such a special truncation scheme as the “perfect coordinates”.<sup>13)</sup> Generally, it would be difficult to find out such coordinates that enable us to solve the ERG equations exactly. However, Scheme II turns out to be an example of perfect coordinates in the large  $N$  limit. Such a remarkable simplification does not occur in Scheme I.

Due to the perfectness of the coordinates employed, the exact values for the exponents can be obtained. The exponents are given by the eigenvalues of the matrix  $\Omega_{ij} = \partial\beta_i/\partial b_j$  evaluated at the non-trivial fixed point:

$$\Omega = \begin{pmatrix} D - 2 & 0 & 0 & \cdots \\ 0 & D - 4 & 0 & \cdots \\ 0 & \frac{24}{A_D} \frac{(4 - D)^2}{6 - D} & D - 6 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \quad (18)$$

Thus the exponents are exactly determined from the eigenvalues, or the diagonal elements of this matrix, and  $\nu = 1/(D - 2)$ . Also the corresponding eigenvectors of this matrix give us the so-called relevant and the irrelevant operators. The most characteristic feature of the ERG equations in Scheme II is that the relevant coupling precisely coincides with  $\rho_0$  and is not influenced by the truncation. We can say that this is the direct reason why the leading exponent is calculated in a truncation-independent way.

Thus the relevant operator corresponding to the coupling  $\rho_0$  has been found to be given by  $V_\rho^*$ . In addition to the relevant operator, all eigen-operators can be derived exactly from Eq. (16) as follows. Suppose  $V^*(\rho)$  is the non-trivial fixed point solution of the ERG equation and consider an infinitesimal deviation from this;  $V(\rho; t) = V^*(\rho) + \delta V(\rho; t)$ . Then we obtain the eigenvalue equation with respect to  $\delta V$  as

$$D\delta V - 2\frac{V_\rho^*}{V_{\rho\rho}^*}\delta V_\rho = \lambda\delta V. \quad (19)$$

By solving this equation, all of the eigenvectors are found to be given by

$$\delta V(\rho; t) \propto (V_\rho^*(\rho))^{\frac{D-\lambda}{2}}. \quad (20)$$

If we demand the analyticity of  $\delta V$ , then the eigenvalues are determined to be  $\lambda = D - 2, D - 4, D - 6, \dots$ , as expected.

Moreover, it turns out to be possible to reformulate the ERG so that the effective potential  $V(\rho; t)$  is expanded into a power series of the (ir)relevant operators  $(V_\rho)^n$ . For this purpose, let us first introduce two auxiliary fields  $\chi$  and  $\eta$  to the theory,

$$Z = \int \mathcal{D}\phi^a \mathcal{D}\chi \mathcal{D}\eta \exp \left\{ - \int d^D x \left[ \frac{1}{2}(\partial_\mu \phi^a)^2 + \chi \left( \frac{1}{2}(\phi^a)^2 - N\eta \right) + NV(\eta) \right] \right\}. \quad (21)$$

In the large  $N$  limit, the path integral of these auxiliary fields is evaluated by the saddle point method. Then the effective potential  $V(\bar{\eta}; t)$  is given by solving the coupled equations

$$V(\bar{\eta}; t) = \frac{1}{2} \text{tr} \ln(-\square + \chi) + \chi(\bar{\eta} - \eta) + V(\eta), \quad (22)$$

$$\eta = \bar{\eta} + \frac{1}{2} \text{tr} \frac{1}{-\square + \chi}, \quad (23)$$

$$\chi = \frac{\partial V(\eta)}{\partial \eta}, \quad (24)$$

where  $\bar{\eta}$  denotes  $\bar{\phi}^2/2N$ . Then we change the variable  $\bar{\eta}$  to  $\bar{\chi}$  through the Legendre transformation:

$$\bar{\chi} = \frac{\partial V(\bar{\eta}; t)}{\partial \bar{\eta}}, \quad (25)$$

$$U(\bar{\chi}; t) = -\bar{\chi}\bar{\eta} + V(\bar{\eta}; t). \quad (26)$$

Therefore the WH equation for  $U(\bar{\chi}; t)$  is simply given by

$$\frac{\partial U}{\partial t} = DU - 2\bar{\chi}U_{\bar{\chi}} + \frac{A_D}{2} \ln(1 + \bar{\chi}). \quad (27)$$

It is readily seen that this equation is indeed identical to Eq. (16) owing to the saddle point equation.

This form of the ERG equation, in turn, is exactly solved by expanding  $U$  into an ordinary Taylor series of  $\bar{\chi}$  as

$$U(\bar{\chi}; t) = a_0(t) + a_1(t)\bar{\chi} + \frac{1}{2}a_2(t)\bar{\chi}^2 + \dots, \quad (28)$$

where  $a_1$  is just the potential minimum parametrized previously by  $\rho_0$ . The relevant operator is found to be  $\bar{\chi}$  itself, which has dimension  $D - 2$  at the fixed point. The irrelevant operators are also simply given by  $\bar{\chi}^2, \bar{\chi}^3, \dots$ . Thus we can reformulate the large  $N$  model in terms of the purely (ir)relevant operators by introducing a new variable, which is a composite operator of the original scalar fields. On the renormalized trajectory, we may ignore these irrelevant operators. Once they are eliminated, the theory turns out to be identical to the non-linear  $\sigma$  model.

What do these relations found in the large  $N$  limit imply for the finite  $N$  cases? It would be natural to suppose from the above observation that good convergence in Scheme II for a finite  $N$  may be explained similarly. Actually, if we compare the forms of the eigenvectors of the matrix  $\Omega$  in Schemes I and II, then we will see a clear difference. That is, the eigenvectors are approximated well by the first several components in Scheme II, while this is not the case in Scheme I. Thus we may deduce that Scheme II is able to capture the relevant operator in the small dimensional subspace and, therefore, to make the truncation dependence diminish rapidly. In practice, it is not hard to extend the formulation of ERG so as to incorporate the auxiliary field to finite  $N$  cases. The results of numerical analyses of such ERG equations will be reported elsewhere. To summarize, the physical reason for the good convergence in the comoving frame is speculated to be that the leading operator defined in this scheme covers the relevant operator quite well.

§5. Infrared effective potentials

It is significant to observe the truncation dependence of not only the exponents but also other physical quantities in various approximation schemes. We here discuss the infrared effective potentials for the scalar theory by using the LPA WH equation (10). The infrared effective potentials enable us to calculate physical quantities such as the effective mass and the effective couplings at a low energy scale. The low energy physics is completely described by the one dimensional renormalized trajectory of the relevant operator extending from the non-trivial fixed point on the critical surface. The renormalized trajectory is divided into two parts in the symmetric phase and in the symmetry broken phase. We evaluate the infrared effective potentials by tracing the running coupling constants on the renormalized trajectory. As the cutoff is lowered, the minimum of the effective potential in the symmetric phase shrinks, while it grows in the broken phase.

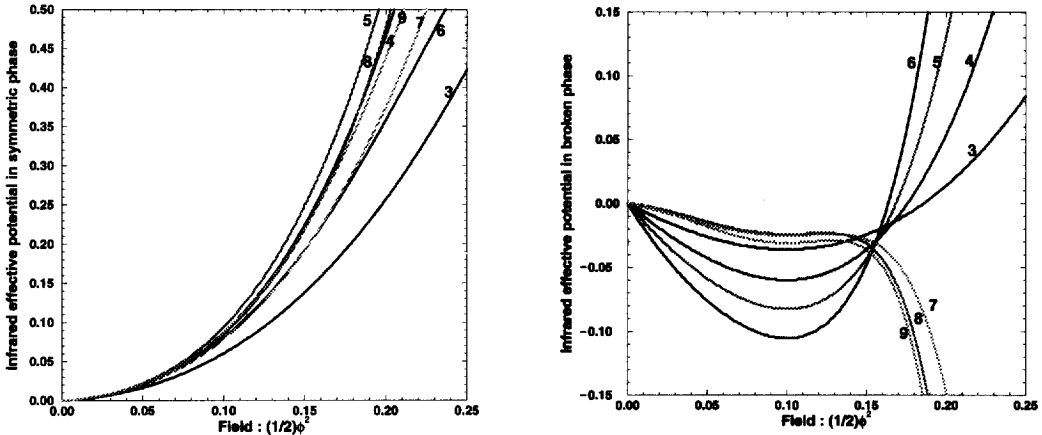


Fig. 7. The truncation dependence of the infrared effective potential in the symmetric and the symmetry broken phases evaluated using Scheme I.

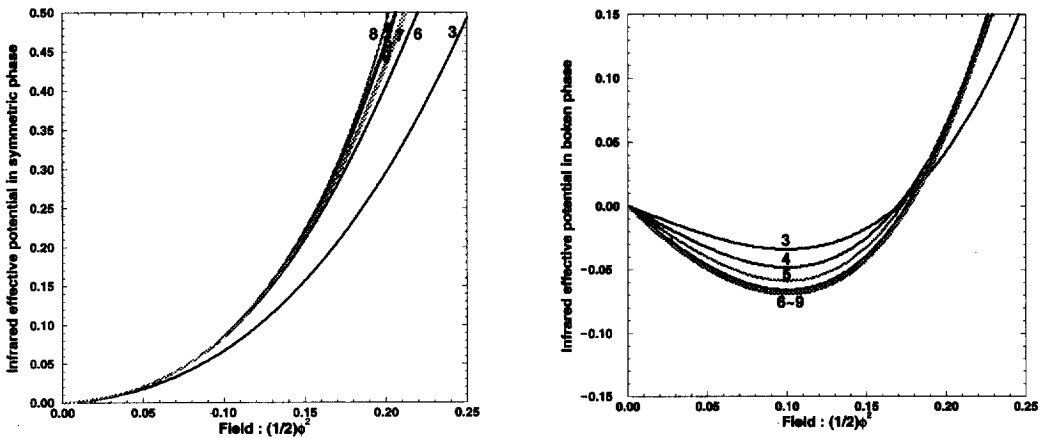


Fig. 8. The truncation dependence of the infrared effective potential in the symmetric and the symmetry broken phases evaluated using Scheme II.

The renormalized trajectories obtained using different truncation approximations do not coincide. We should be careful when examining the renormalized trajectory. We need to impose a common appropriate renormalization condition to compare the infrared effective potentials evaluated in the various approximation schemes. Actually, we evaluate the infrared potentials by employing the point on the renormalized trajectory satisfying the following renormalization condition; the gradient at the origin of the potential is equal to 0.1 in the symmetric phase and the field value at the minimum of the potential is equal to 0.1 in the broken phase. The effective potentials obtained in Schemes I and II are shown in Figs. 7 and 8, respectively. Note that the absolute height of the potential is adjusted so as to vanish at the origin, since it is not taken into account correctly in the ERG equation.\*)

As a result, the truncated approximation in the comoving frame leads to good convergence for the effective potential in both phases as well as for the exponent. It is remarkable property of the comoving frame that it remains so effective after truncation, even away from criticality, and moreover, that this occurs irrespective of the phases. This result would imply that the relevant operator ruling the renormalized trajectory can be approximated well enough in the small dimensional subspace truncated in the comoving frame.

## §6. Summary and discussion

We considered the convergence properties of physical quantities evaluated using the ERG in various truncated approximation schemes in operator expansions. The  $Z_2$  symmetric scalar theory in three dimensions was numerically analyzed, and the approximated solutions for the exponents and the infrared effective potentials were compared in the various schemes. In particular we focused on studies of the difference between the truncation schemes in the expansion at the field origin (Scheme I) and at the minimum of the effective potential (Scheme II).

It was found that Scheme II displays a remarkably strong convergence property with respect to the order of truncation as far as the quantities we examined are concerned; the leading exponent, the anomalous dimension and the infrared effective potentials. Although it is seen that the exponent obtained in Scheme II also ceases to converge eventually at a certain large order, the width of fluctuation is very small, and we can obtain the value with great accuracy.

Indeed we may examine the partial differential equations derived in the derivative expansion scheme directly for such a simple model. However, such analyses would become difficult for more complicated systems, e.g., triviality mass bound for the Higgs particle in the standard model, non-perturbative analysis of dynamical chiral symmetry breaking of strongly coupled fermions, etc.<sup>22) - 25)</sup> Therefore we would like to stress here that the operator expansion scheme is desired if it gives a converging value effectively enough. Actually, Scheme II is found to satisfy such a practical

\*) In the analysis in terms of Scheme II, the minimum of the potential moves from the positive region ( $\rho > 0$ ) to the negative region ( $\rho < 0$ ) in the symmetric phase. Therefore one should switch the evaluation of the potential using Scheme II to Scheme I at the point where the minimum of the potential passes the origin  $\rho = 0$ .

demand for scalar theories. It will be necessary to examine the presence of such a good approximation scheme for the non-perturbative analysis of various models. Naturally, it would be desirable to seek general methods to offer us such effective schemes. Such problems in the ERG approach deserve further study.

We also discussed the physical reason for this rapid convergence in the comoving frame by considering  $O(N)$  symmetric scalar theories in the large  $N$  limit. It was shown that the exponents are derived exactly by operator expansion in the comoving frame. Not only the exponents but also each RG flow of the coupling is exactly derived in every finite order of truncation. Moreover, all of the (ir)relevant operators at the non-trivial fixed point have been given exactly and are found to be highly complicated composite operators in terms of the original scalar fields. We found that the coupling  $\rho_0$ , the potential minimum, defined in the comoving frame corresponds to the exact relevant operator. If such structure remains in the finite  $N$  cases in an approximate sense, it could be regarded as the physical reason for the good convergence of Scheme II. Indeed, the relevant operator is found to be described well within the subspace of the first several operators in Scheme II.

The ERG equation in terms of the composite operators has been proposed. This is equivalent to the ERG equation for scalar theories in the large  $N$  limit. The reformulation is achieved by introducing a composite field to the original theory. Interestingly, this operator turns out to be the exact “relevant” operator after evolution to infrared. Other irrelevant operators are also simply given by the products of this composite. Namely, the naive polynomial expansion leads to the perfect coordinates in turn. Thus this offers an example in which the good expansion scheme in the operators is revealed through a proper change of field variables. A numerical study of the ERG in terms of the composite operators in finite  $N$  cases will be reported elsewhere. Indeed, such a variable change incorporating composite operators has been found to be significant in the RG analysis of the dynamical chiral symmetry breaking.<sup>25)</sup>

### Acknowledgements

We would like to thank T. R. Morris for valuable discussions at YKIS '97. Collaboration in the early stages with K. Sakakibara and Y. Yoshida is also acknowledged. K-I. A. and H. T. are supported in part by Grants-in-Aid for Scientific Research (#08240216 and #08640361) from the Ministry of Education, Science and Culture.

### References

- 1) F. J. Wegner and A. Houghton, *Phys. Rev.* **A8** (1973), 401.
- 2) K. G. Wilson and J. B. Kogut, *Phys. Rep.* **12** (1974), 75.
- 3) J. Polchinski, *Nucl. Phys.* **B231** (1984), 269.
- 4) C. Wetterich, *Phys. Lett.* **B301** (1993), 90.
- 5) M. Bonini, M. D'Attanasio and G. Marchesini, *Nucl. Phys.* **B409** (1993), 441.
- 6) T. R. Morris, *Phys. Lett.* **B329** (1994), 241.
- 7) R. D. Ball, P. E. Haagensen, J. I. Lattore and E. Moreno, *Phys. Lett.* **B347** (1995), 80.
- 8) T. R. Morris, hep-th/9610012 (SHEP 96-25).
- 9) T. R. Morris, *Nucl. Phys.* **B495** [FS] (1997), 477.
- 10) T. R. Morris and M. D. Turner, *Nucl. Phys.* **B509** (1998), 637.

- 11) J. F. Nicoll, T. S. Chang and H. E. Stanley, *Phys. Rev. Lett.* **33** (1974), 540.
- 12) A. Hasenfratz and P. Hasenfratz, *Nucl. Phys.* **B270** [FS16] (1986), 687.
- 13) K-I. Aoki, K. Morikawa, W. Souma, J.-I. Sumi and H. Terao, *Prog. Theor. Phys.* **95** (1996), 409.
- 14) A. Margaritis, G. Ódor and A. Patkós, *Z. Phys.* **C39** (1988), 109.
- 15) T. R. Morris, *Phys. Lett.* **B334** (1994), 355.
- 16) N. Tetradis and C. Wetterich, *Nucl. Phys.* **B422** (1994), 541.
- 17) M. Alford, *Phys. Lett.* **B336** (1994), 237.
- 18) K-I. Aoki, K. Morikawa, W. Souma, J.-I. Sumi and H. Terao, in preparation.
- 19) T. R. Morris, *Int. J. Mod. Phys.* **A9** (1994), 2411.
- 20) T. R. Morris, private communication.
- 21) J. Zinn-Justin, *Quantum field theory and critical phenomena* (Clarendon Press, Oxford, 1996).
- 22) P. Hasenfratz and J. Nager, *Z. Phys.* **C39** (1988), 477.
- 23) T. E. Clark, B. Haeri and S. T. Love, *Nucl. Phys.* **B402** (1993), 628.
- 24) T. E. Clark, B. Haeri, S. T. Love, M. A. Walker and W. T. A. ter Veldhuis, *Phys. Rev.* **D50** (1994), 606.
- 25) K-I. Aoki, hep-ph/9706204 (KANAZAWA-97-09).