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RAPIDLY DECREASING FUNCTIONS IN REDUCED C^*-ALGEBRAS OF GROUPS

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ABSTRACT. Let Γ be a group. We associate to any length-function L on Γ the space $H_L^\infty(\Gamma)$ of rapidly decreasing functions on Γ (with respect to L), which coincides with the space of smooth functions on the k-dimensional torus when $\Gamma = \mathbf{Z}^k$. We say that Γ has property (RD) if there exists a length-function L on Γ such that $H_L^\infty(\Gamma)$ is contained in the reduced C^*-algebra $C_r^*(\Gamma)$ of Γ. We study the stability of property (RD) with respect to some constructions of groups such as subgroups, over-groups of finite index, semidirect and amalgamated products. Finally, we show that the following groups have property (RD):

(1) Finitely generated groups of polynomial growth;

(2) Discrete cocompact subgroups of the group of all isometries of any hyperbolic space.

INTRODUCTION

Consider the algebra $C(T^k)$ of continuous functions on the k-dimensional torus, which is also the C^*-algebra of the group $\mathbf{Z}^k$. It contains the dense subalgebra $C^\infty(T^k)$ of smooth functions on T^k. Derivation is possible and useful on $C^\infty(T^k)$, but is not allowed on the whole of $C(T^k)$. Our aim is to develop a notion of smooth functions in the reduced C^*-algebras of other groups. The idea is to consider the characterization of $C^\infty(T^k)$ by Fourier series: if $f \in C(T^k)$ and if $\hat{f}$ denotes its Fourier transform, which is a function on $\mathbf{Z}^k$, recall that f belongs to $C^\infty(T^k)$ if and only if $\hat{f}$ is of rapid decay.

Now, if Γ is any group and if L is a length-function on Γ, we denote by $H_L^\infty(\Gamma)$ the space of rapidly decreasing functions on Γ with respect to L. Examples below show that $H_L^\infty(\Gamma)$ is not always contained in the reduced C^*-algebra $C_r^*(\Gamma)$ of Γ. We say that Γ has property (RD) if there exists a length-function L on Γ such that $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$. Some aspects of property (RD) have been already studied in [Haa, FP, and Pi].

This paper is organized as follows: Chapter 1 contains elementary definitions and results concerning length-functions and property (RD). Proposition 1.2.6 is the first important result since it gives a technical condition equivalent to

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property (RD) which will be frequently used later: it is an adaptation of Lemmas 1.3 and 1.4 of [Haa].

We study in Chapter 2 the stability of property (RD) with respect to some classical constructions of groups such as subgroups, over-groups of finite index, semidirect products and amalgamated products. In particular, the following theorem generalizes the case of nonabelian free groups of finite rank due to U. Haagerup [Haa]:

Theorem A. *If Γ_1 and Γ_2 have property* (RD) *then so does their free product* $\Gamma_1 * \Gamma_2$.

Chapter 3 is devoted to examples of groups possessing property (RD). First, we deal with finitely generated amenable groups:

Proposition B. *Let Γ be a finitely generated group and let L denote the word length-function on Γ.*

(1) *If Γ is of polynomial growth then $H_L^\infty(\Gamma)$ is contained in $l^1(\Gamma)$, and thus a fortiori in $C_r^*(\Gamma)$.*

(2) *If Γ is amenable then Γ has property* (RD) *if and only if it is of polynomial growth.*

This result implies that $\mathrm{SL}(n, \mathbf{Z})$ does not have property (RD) when $n \geq 3$.

In the second section of Chapter 3, we study property (RD) for discontinuous groups of hyperbolic spaces (real, complex, quaternionic and exceptional). There, we choose a length-function defined as follows: let x_0 be a point of the hyperbolic space, and define the length $L(g)$ of the isometry g to be the distance between x_0 and $g(x_0)$. We show

Theorem C. *If Γ is a cocompact discrete subgroup of the group of all isometries of a hyperbolic space, then $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$.*

Thus such a group has property (RD). Though $H_L^\infty(\mathrm{SL}(2, \mathbf{Z}))$ is not contained in $C_r^*(\mathrm{SL}(2, \mathbf{Z}))$, a different argument shows that $\mathrm{SL}(2, \mathbf{Z})$ has property (RD) (Corollary 2.1.6).

We end this paper with an Appendix in which we briefly study property (RD) for not necessarily discrete groups. We prove that such a group has property (RD) if it contains a discrete cocompact subgroup with property (RD).

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1. Rapidly decreasing functions and property (RD)

Let Γ be a group. We refer to Chapter 7 of [Ped] for the notations concerning standard spaces and operators algebras associated to Γ.

1.1. Length-functions and rapidly decreasing functions.

1.1.1. **Definition.** A *length-function* on a group Γ is a map $L\colon \Gamma \to \mathbf{R}_+$ satisfying:

(i) $L(gh) \leq L(g) + L(h)$, for all $g, h \in \Gamma$;
(ii) $L(g) = L(g^{-1})$, for every $g \in \Gamma$;
(iii) $L(1) = 0$, where 1 denotes the identity of Γ.

If L_1 and L_2 are length-functions on Γ, we say that L_2 *dominates* L_1 if there exist $a, b \in \mathbf{R}_+$ such that $L_1 \leq aL_2 + b$. If L_1 dominates L_2 and L_2 dominates L_1, then L_1 and L_2 are said to be *equivalent*.

Let L be a length-function on Γ and $r \in \mathbf{R}_+$; the *crown* of radius r is the set

$$C_{r,L} = \{g \in \Gamma; r - 1 < L(g) \leq r\},$$

and $\chi_{r,L}$ denotes the characteristic function of $C_{r,L}$.

The *ball* of radius r is the set

$$B_{r,L} = \{g \in \Gamma; L(g) \leq r\}.$$

Note that $B_{0,L}$ is a subgroup of Γ which may have more than one element.

1.1.2. **Example.** Suppose that Γ is finitely generated and let S be a finite set of generators of Γ. The *algebraic length* of $g \in \Gamma$ with respect to S is the least nonnegative integer n such that g can be written as a product of n elements of $S \cup S^{-1}$. We denote this number by $|g|_S$ and get a map from Γ to $\mathbf{N}$ which is a length-function. If S' is another finite generating set of Γ, it is readily verified that the length-functions associated to S and S' are equivalent. Most of the time we will not specify any finite system of generators and will speak about *the* algebraic length-function on Γ. If Γ_0 is a subgroup of a finitely generated group Γ, the restriction to Γ_0 of the algebraic length-function on Γ is a length-function on Γ_0. It is known that Γ_0 is not finitely generated in general. A simple example is given by the subgroup Γ of $\mathrm{GL}(2, \mathbf{R})$ generated by

$$s_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and its subgroup Γ_0 generated by $\{s_1^k s_2 s_1^{-k}; k \in \mathbf{Z}\}$ which is isomorphic to $\mathbf{Z}[1/2]$.

1.1.3. **Example.** Let X be a metric space with base point $x_0 \in X$ and let Γ be a group of isometries on X. Define for every $g \in \Gamma$

$$L_{x_0}(g) = d(x_0, g(x_0)).$$

Then L_{x_0} is a length-function on Γ since each $g \in \Gamma$ is an isometry. If x_1 is another point of X, L_{x_0} and L_{x_1} satisfy the following inequalities for every $g \in \Gamma$:

$$L_{x_0}(g) - 2d(x_0, x_1) \leq L_{x_1}(g) \leq L_{x_0}(g) + 2d(x_0, x_1).$$

The proof of the following lemma is easy and consequently left to the reader.

1.1.4. **Lemma.** *Let L be a length-function on a group Γ.*

(1) *We have $|L(g) - L(h)| \leq L(gh)$ for every $g, h \in \Gamma$.*

(2) *If Γ is finitely generated then L is dominated by the algebraic length-function.*

(3) (*Peetre's inequality*). *If s is a real number, one has for all $g, h \in \Gamma$*

$$(1 + L(gh))^s \leq (1 + L(g))^{|s|}(1 + L(h))^s.$$

(4) *If L' is a length-function dominated by L, there exists $c \in \mathbf{R}_+$ such that $|B_{r,L}| \leq |B_{cr,L'}|$ for every $r \geq 1$, where $|B_{r,L}|$ denotes the cardinal of $B_{r,L}$.*

1.1.5. *Remark.* The converse of assertion (4) in Lemma 1.1.4 is false in general. In fact, if $\Gamma = \mathbf{Z} \oplus \mathbf{Z}$ and if L_1 and L_2 are defined by

$$L_1(x, y) = |x| + \log(1 + |y|)$$

and

$$L_2(x, y) = L_1(y, x),$$

then $|B_{r,L_1}| = |B_{r,L_2}|$ is finite for every $r \geq 0$, but L_1 does not dominate L_2 which does not dominate L_1.

Let us now give the main definition of this work:

1.1.6. **Definition.** Let L be a length-function on Γ.

(1) If $s \in \mathbf{R}$, the *Sobolev space of order s* (with respect to L) is the set $H^s_L(\Gamma)$ of functions ξ on Γ such that $\xi(1 + L)^s$ belongs to $l^2(\Gamma)$.

(2) The space of *rapidly decreasing functions* on Γ (with respect to L) is the set $H^\infty_L(\Gamma) = \bigcap_{s \in \mathbf{R}} H^s_L(\Gamma)$. If $s \in \mathbf{R}$ and $\xi, \zeta \in H^s_L(\Gamma)$, set

$$(\xi|\zeta)_{2,s,L} = \sum_{g \in \Gamma} \xi(g)\overline{\zeta}(g)(1 + L(g))^{2s}$$

and $\|\xi\|_{2,s,L} = \sqrt{(\xi|\xi)_{2,s,L}}$.

With the above inner product, $H^s_L(\Gamma)$ is a Hilbert space, and $H^\infty_L(\Gamma)$ is a Fréchet space for the projective limit topology induced by the inclusions of $H^\infty_L(\Gamma)$ in $H^s_L(\Gamma)$, for each $s \in \mathbf{R}$. When Γ is finitely generated, we denote by $H^s(\Gamma)$ and $H^\infty(\Gamma)$ the above spaces associated to the algebraic length-function on Γ. We denote by $(\cdot|\cdot)_{2,s}$ and $\|\cdot\|_{2,s}$ the corresponding inner product and norm.

1.1.7. *Remark.* Let L_1 and L_2 be two length-functions on Γ. If L_2 dominates L_1 then $H^s_{L_2}(\Gamma)$ is contained in $H^s_{L_1}(\Gamma)$ for every $s \geq 0$. In particular, if Γ is finitely generated then $H^\infty(\Gamma)$ is contained in $H^\infty_L(\Gamma)$ for every length-function L on Γ. We finally remark that if L is a length-function on Γ and if $s \in \mathbf{R}$, then $H^{-s}_L(\Gamma)$ is the dual of $H^s_L(\Gamma)$ for the bilinear form

$$(\xi, \zeta) \to \sum_{g \in \Gamma} \xi(g)\zeta(g).$$

1.2. Property (RD). Let Γ be a group. The group algebra of Γ is denoted by $\mathbf{C}\Gamma$ and is the set of functions with finite support on Γ. We recall that $\mathbf{C}\Gamma$ acts faithfully by left convolution on $l^2(\Gamma)$ and that the reduced C^*-algebra of Γ, denoted by $C_r^*(\Gamma)$, is the norm closure of $\mathbf{C}\Gamma$ in $B(l^2(\Gamma))$. Furthermore, every element a of $C_r^*(\Gamma)$ can be identified with the image of δ_1 by a, where δ_1 is the characteristic function of $\{1\}$ in Γ. We shall thus consider $C_r^*(\Gamma)$ as a subspace of $l^2(\Gamma)$. If $\varphi \in C_r^*(\Gamma)$, we denote by $\|\varphi\|$ the norm of φ in $C_r^*(\Gamma)$.

1.2.1. **Definition.** A group Γ is said to have *property* (RD) if there exists a length-function L on Γ such that $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$.

(1.2.2.) *Remark.* (1) By the closed graph theorem, Γ has property (RD) if and only if there exists a length-function L on Γ and two positive numbers c and s such that $\|\varphi\| \le c\|\varphi\|_{2,s,L}$ for every $\varphi \in \mathbf{C}\Gamma$.

(2) If Γ is finitely generated and possesses property (RD) then $H^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$ by Remark 1.1.7.

1.2.3. **Example.** (1) If Γ is the infinite cyclic group then it is easily verified that Γ has property (RD); we have for every $\varphi \in \mathbf{C}\Gamma$

$$\|\varphi\| \le \frac{\pi}{\sqrt{3}}\|\varphi\|_{2,1}.$$

(2) Let F_N be the (nonabelian) free group of rank $N \ge 2$. Then F_N has property (RD) since Lemma 1.5 of [Haa] gives the following inequality for each function φ with finite support on F_N:

$$\|\varphi\| \le 2\|\varphi\|_{2,2}.$$

More generally, we will show in Chapter 2 that a free product of two groups with property (RD) has itself property (RD).

1.2.4. **Lemma.** *If L is a length-function on Γ such that $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$, then there exist c and s in $\mathbf{R}_+$ such that one has for every $t \in \mathbf{R}$ and φ, $\psi \in H_L^\infty(\Gamma)$*

$$\|\varphi * \psi\|_{2,t,L} \le \|\varphi\|_{2,s+|t|,L}\|\psi\|_{2,t,L}.$$

In particular, $H_L^\infty(\Gamma)$ is a convolution algebra.

Proof. By Remark 1.2.2, there exist $c, s \in \mathbf{R}_+$ such that

$$\|\varphi * \psi\|_2 \le c\|\varphi\|_{2,s,L}\|\psi\|_2,$$

for φ, $\psi \in H_L^\infty(\Gamma)$. Let us define

$$\varphi_t(h) = |\varphi(h)|(1 + L(h))^{|t|}$$

and

$$\psi_t(h) = |\psi(h)|(1 + L(h))^{t}$$

for $h \in \Gamma$. Then φ_t and ψ_t belong to $H_L^\infty(\Gamma)$ and they satisfy

$$\|\varphi_t\|_{2,s,L} = \|\varphi\|_{2,s+|t|,L} \quad \text{and} \quad \|\psi_t\|_2 = \|\psi\|_{2,t,L}.$$

Then one gets, using Peetre's inequality of Lemma 1.1.4

$$\|\varphi * \psi\|_{2,t,L} \le \|\varphi_t * \psi_t\|_2 \le c\|\varphi\|_{2,s+|t|,L}\|\psi\|_{2,t,L} \quad \text{Q.E.D.}$$

In fact, one of the most important motivations for the study of property (RD) is given by the following result of K-theory which shows that the subalgebra $H_L^\infty(\Gamma)$ is in some sense "large" in $C_r^*(\Gamma)$. The result is due to A. Connes, and we give a proof of it in [J3].

1.2.5. **Theorem.** *If Γ has property* (RD) *and if L is a length-function such that $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$, then the inclusion of $H_L^\infty(\Gamma)$ in $C_r^*(\Gamma)$ induces an isomorphism from $K_i(H_L^\infty(\Gamma))$ onto $K_i(C_r^*(\Gamma))$, for $i = 0, 1$.*

Now we establish some properties which are equivalent to property (RD). Let $A(\Gamma)$ denote the Fourier algebra of Γ (cf. [Ey]) and let $B_\lambda(\Gamma)$ be the set of coefficients of unitary representations of Γ which are weakly contained in the left regular one. $B_\lambda(\Gamma)$ is the dual space of $C_r^*(\Gamma)$ for the bilinear form

$$\langle \varphi, \psi \rangle = \sum_{g\in\Gamma} \varphi(g)\psi(g)$$

for $\varphi \in \mathbf{C}\Gamma$, $\psi \in B_\lambda(\Gamma)$. Since $A(\Gamma)$ is constituted by the coefficients of the left regular representation, $A(\Gamma)$ is a closed subspace of $B_\lambda(\Gamma)$ equipped with the dual norm.

1.2.6. **Proposition.** *Let L be a length-function on a group Γ. The following properties are equivalent*:

(1) *$H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$*;
(2) *there exists $s > 0$ such that $B_\lambda(\Gamma)$ is continuously embedded in $H_L^{-s}(\Gamma)$*;
(3) *there exists $s > 0$ such that $A(\Gamma)$ is continuously embedded in $H_L^{-s}(\Gamma)$*;
(4) *there exists $c > 0$ and $r > 0$ such that if k, l, m belong to $\mathbf{N}$, if φ and ψ belong to $\mathbf{C}\Gamma$ and are supported in $C_{k,L}$ and $C_{l,L}$ respectively, one has*

$$\|(\varphi * \psi)\chi_{m,L}\|_2 \le c\|\varphi\|_{2,r,L}\|\psi\|_2 \quad \text{if } |k-l| \le m \le k+l$$

and $\|(\varphi * \psi)\chi_{m,L}\|_2 = 0$ for the other values of m.

Proof. (1) $\Rightarrow$ (2) By Remark 1.2.2, there exists $s > 0$ such that $H_L^s(\Gamma)$ is continuously embedded in $C_r^*(\Gamma)$. Thus, property (2) is obtained by duality.

(2) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (1) Let us fix $\varphi \in \mathbf{C}\Gamma$, and ξ, $\eta \in l^2(\Gamma)$ such that $\|\xi\|_2 \le 1$, $\|\eta\|_2 \le 1$. Let ψ be defined by

$$\psi(g) = (\xi * \eta^*)(g^{-1}), \quad \text{for every } g \in \Gamma.$$

Then $\psi \in A(\Gamma)$ and $\|\psi\|_{A(\Gamma)} \leq \|\xi\|_2 \|\eta\|_2 \leq 1$ (cf. [Ey]). It is readily verified that $(\varphi * \xi | \eta) = \langle \varphi , \psi \rangle$, which gives

$$\begin{aligned} |(\varphi * \xi|\eta)| &\leq \|\varphi\|_{2,s,L} \|\psi\|_{2,-s,L} \\ &\leq c\|\varphi\|_{2,s,L} \|\psi\|_{A(\Gamma)} \\ &\leq c\|\varphi\|_{2,s,L} . \end{aligned}$$

$(1) \Rightarrow (4)$ is immediate.

$(4) \Rightarrow (1)$ Let $\varphi, \psi \in \mathbf{C}\Gamma$; suppose first that φ is supported on $C_{k,L}$ for some $k \in \mathbf{N}$. Put $\psi_l = \psi \cdot \chi_{l,L}$ for every $l \in \mathbf{N}$. One has by hypothesis

$$\|(\varphi * \psi_l)\chi_{m,L}\|_2 \leq c(1+k)^r \|\varphi\|_2 \|\psi_l\|_2 \quad \text{if } |k-l| \leq m \leq k+l$$

and $(\varphi * \psi_l)\chi_{m,L} = 0$ otherwise.

Then one gets

$$\begin{aligned} \|(\varphi * \psi)\chi_{m,L}\|_2 &\leq \sum_{l \geq 0} \|(\varphi * \psi_l)\chi_{m,L}\|_2 \\ &\leq c(1+k)^r \|\varphi\|_2 \sum_{l=|m-k|}^{m+k} \|\psi_l\|_2 \\ &\leq c(1+k)^r \|\varphi\|_2 \sum_{l=0}^{2\min(k,m)} \|\psi_{m+k-l}\|_2 \\ &\leq c(1+k)^r (2k+1)^{1/2} \|\varphi\|_2 \left(\sum_{l=0}^{2\min(k,m)} \|\psi_{m+k-l}\|_2^2 \right)^{1/2} . \end{aligned}$$

Consequently,

$$\begin{aligned} \|\varphi * \psi\|_2^2 &= \sum_{m \geq 0} \|(\varphi * \psi)\chi_{m,l}\|_2^2 \\ &\leq c_1^2 (1+k)^{2r+1} \|\varphi\|_2^2 \sum_{m \geq 0} \left(\sum_{l=0}^{2\min(k,m)} \|\psi_{m+k-l}\|_2^2 \right) \\ &\leq c_2^2 (1+k)^{2r+2} \|\varphi\|_2^2 \|\psi\|_2^2 . \end{aligned}$$

Finally, if the support of φ is arbitrary, then $\varphi = \sum_{k \geq 0} \varphi_k$, where $\varphi_k = \varphi \cdot \chi_{k,L}$. One has

$$\begin{aligned} \|\varphi\| \leq \sum_{k \geq 0} \|\varphi_k\| &\leq c_2 \sum_{k \geq 0} (1+k)^{r+1} \|\varphi_k\|_2 \\ &\leq c_3 \left(\sum_{k \geq 0} \|\varphi_k\|_2^2 (1+k)^{2(r+2)} \right)^{1/2} \\ &\leq c_4 \|\varphi\|_{2,r+2,L} . \end{aligned}$$

One can take $s = r+2$. Q.E.D.

1.2.7. *Remark.* Let L be a length-function on Γ. If L is bounded then $H_L^{\infty}(\Gamma) = l^2(\Gamma)$. And by [Raj], $l^2(\Gamma)$ is an algebra if and only if Γ is finite. More generally, $H_L^{\infty}(\Gamma)$ is not an algebra in general, even in the case of the algebraic length-function, as the following example shows:

1.2.8. **Example.** Let C denote the infinite cyclic group written multiplicatively with identity 1 and generator σ, and let $A = \mathbf{Z}[C]$ with ε_j the characteristic function of σ^j. Then C acts on A by $xf(y) = f(x^{-1}y)$, so that $\sigma^n(\varepsilon_j) = \varepsilon_{j+n}$.

Let $\Gamma = A \rtimes C$ be the associated semidirect product. Γ is generated by the finite set $S = \{(\varepsilon_0, 1), (0, \sigma)\}$, though A is not finitely generated (compare with Example 1.1.2). From now on we will identify the element a of A with the element $(a, 1)$ of Γ and we will denote by $|a|_\Gamma$ the length of $(a, 1)$ with respect to S. If p is a positive integer, set

$$S_p = \left\{ a \in A\,;\, a = \sum_{j=0}^{p} a_j\varepsilon_j\,, a_j \in \{0, 1\}\,, a_p = 1 \right\}.$$

Let α be a positive real number and let us define the function φ on Γ by

$$\varphi(g) = \begin{cases} 0, & \text{if } g \notin \bigcup_{p\geq 1} S_p\,, \\ \alpha^{p/2}, & \text{if } g \in S_p\,. \end{cases}$$

1.2.9. **Proposition.** *If* $1/\sqrt{5} < \alpha < 1/2$ *then* $\varphi \in H^{\infty}(\Gamma)$ *but* $\varphi * \varphi$ *does not belong to any space* $H^s(\Gamma), s \in \mathbf{R}$.

Proof. If $a = \sum_{j=0}^{p} a_j\varepsilon_j$ belongs to S_p, then $|a|_\Gamma \leq 3p + 1$ since

$$(a, 1) = (\varepsilon_0, 1)^{a_0}(0, \sigma)(\varepsilon_0, 1)^{a_1}(0, \sigma)\cdots(\varepsilon_0, 1)^{a_p}(0, \sigma)^{-p}.$$

If $s \geq 0$, then

$$\|\varphi\|_{2,s}^2 \leq \sum_{p\geq 1}(2\alpha)^p(3p + 1)^{2s} < \infty$$

since $0 < \alpha < 1/2$ and $|S_p| = 2^p$. This shows that φ belongs to $H^{\infty}(\Gamma)$.

Let us now fix a positive integer p. If $a \in A$ is of the form

$$a = \sum_{j=0}^{p-1} a_j\varepsilon_j + 2\varepsilon_p$$

with $a_j \in \{0, 1\}$, let $l(a)$ denote the number of j belonging to $\{0, \dots, p-1\}$ such that $a_j = 1$. One has for such an element a

$$\varphi * \varphi(a) = \sum_{\substack{b,c\in S_p \\ a=b+c}} \varphi(b)\varphi(c).$$

If $l(a) = k$, there are exactly 2^k distinct pairs (b, c) in $S_p \times S_p$ satisfying $a = b + c$. It follows that $\varphi * \varphi(a) = 2^k \alpha^p$ for such an element a. Put

$$S_{p,k} = \left\{ a \in A ; a = \sum_{j=0}^{p-1} a_j \varepsilon_j + 2\varepsilon_p , a_j \in \{0, 1\}, l(a) = k \right\}.$$

The cardinal of $S_{p,k}$ is equal to $\binom{p}{k}$, and $S_{p,k}$ is contained in $B_{3p+2} = \{g \in \Gamma ; |g| \leq 3p + 2\}$.

Let $s \geq 0$ and $a \in S_{p,k}$. One gets $(1 + |a|_\Gamma)^{-2s} \geq 3^{-2s}(1 + p)^{-2s}$. Consequently,

$$\begin{aligned} \|\varphi * \varphi\|_{2,-s}^2 &\geq \sum_{p\geq 1} \sum_{k=0}^{p} \sum_{a\in S_{p,k}} (\varphi * \varphi(a))^2 (1 + |a|_\Gamma)^{-2s} \\ &\geq 3^{-2s} \sum_{p\geq 1} (1 + p)^{-2s} \sum_{k=0}^{p} \binom{p}{k} 4^k \alpha^{2p} \\ &\geq 3^{-2s} \sum_{p\geq 1} \frac{(\alpha\sqrt{5})^{2p}}{(1 + p)^{2s}} = \infty \end{aligned}$$

as soon as $\alpha > 1/\sqrt{5}$. Q.E.D.

2. Property (RD) and some constructions of groups

2.1. Property (RD) and extensions. Let us first remark the following easy fact:

2.1.1. **Proposition.** *Let Γ be a group and let Γ_0 be a subgroup of Γ. If Γ has property* (RD) *then so does Γ_0. More precisely, if L is a length-function on Γ such that $H_L^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$, then $H_{L_0}^\infty(\Gamma_0)$ is contained in $C_r^*(\Gamma_0)$, where L_0 is the restriction of L to Γ_0.*

Now consider two groups G and Γ, and let E be an extension of G by Γ. Let $1 \to G \to E \xrightarrow{\pi} \Gamma \to 1$ be the corresponding exact sequence. Choose a set-theoretic cross-section $\sigma : \Gamma \to E$ of π such that $\sigma(1) = 1$. This determines a function $f : \Gamma \times \Gamma \to G$ measuring the failure of σ to be a homomorphism, namely $f(\gamma_1, \gamma_2) = \sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_1\gamma_2)^{-1}$ for all $\gamma_1, \gamma_2 \in \Gamma$. In addition, let $\rho(\gamma)$ be the conjugation by $\sigma(\gamma)$ in G: $\rho(\gamma)(g) = \sigma(\gamma)g\sigma(\gamma)^{-1}$. For $\alpha \in \Gamma$, let $\mathrm{Ad}(\alpha)$ denote the inner automorphism of Γ associated to α. Then the functions f and ρ are related by

$$\text{(R1)} \qquad \rho(\beta)\rho(\gamma) = \mathrm{Ad}(f(\beta, \gamma))\rho(\beta\gamma)$$

and

$$\text{(R2)} \qquad f(\gamma_1, \gamma_2)f(\gamma_1\gamma_2, \gamma_3) = \rho(\gamma_1)(f(\gamma_2, \gamma_3))f(\gamma_1, \gamma_2\gamma_3).$$

(See [Bro, p. 104].)

We shall identify E with $G \times \Gamma$ equipped with the product

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 \rho(\gamma_1)(g_2) f(\gamma_1, \gamma_2), \gamma_1 \gamma_2).$$

2.1.2. **Lemma.** *Let E, G and Γ be as above. Suppose that there exist length-functions L_0, L_1 and L on G, Γ and E respectively such that*

(i) *$H^\infty_{L_0}(G)$ is contained in $C^*_r(G)$ and $H^\infty_{L_1}(\Gamma)$ is contained in $C^*_r(\Gamma)$;*

(ii) *there exist c and r in $\mathbf{R}_+$ such that*

$$L_0(g) + L_1(\gamma) \leq c \cdot L(g, \gamma)^r \quad \textit{for every } (g, \gamma) \in E.$$

Then $H^\infty_L(E)$ is contained in $C^*_r(E)$.

Proof. Choose first two positive constants d and s such that

$$\|\varphi_0\| \leq d \|\varphi_0\|_{2,s,L_0} \quad \text{for every } \varphi_0 \in \mathbf{C}G$$

and

$$\|\varphi_1\| \leq d \|\varphi_1\|_{2,s,L_1} \quad \text{for every } \varphi_1 \in \mathbf{C}\Gamma.$$

If φ and ψ belong to $\mathbf{C}E$, one gets

$$\begin{aligned}
&\|\varphi * \psi\|_2^2 \\
&= \sum_{(g,\gamma) \in E} \left| \sum_{(h,\beta) \in E} \varphi(h, \beta) \psi(f(\beta^{-1}, \beta)^{-1} \rho(\beta^{-1})(h^{-1} g) f(\beta^{-1}, \gamma), \beta^{-1}\gamma) \right|^2 \\
&= \sum_{\gamma \in \Gamma} \sum_{g \in G} \left| \sum_{\beta \in \Gamma} \sum_{h \in G} \varphi_\beta(h) \psi_{\beta,\gamma}(h^{-1} g) \right|^2 \\
&= \sum_{\gamma \in \Gamma} \left\| \sum_{\beta \in \Gamma} \lambda_G(\varphi_\beta) \psi_{\beta,\gamma} \right\|_2^2,
\end{aligned}$$

where

$$\varphi_\beta(g) = \varphi(g, \beta) \quad \text{and} \quad \psi_{\beta,\gamma}(g) = \psi(f(\beta^{-1}, \beta)^{-1} \rho(\beta^{-1})(g) f(\beta^{-1}, \gamma), \beta^{-1}\gamma).$$

It follows from the triangle inequality that

$$\begin{aligned}
\|\varphi * \psi\|_2^2 &\leq \sum_{\gamma \in \Gamma} \left(\sum_{\beta \in \Gamma} \|\varphi_\beta * \psi_{\beta,\gamma}\|_2 \right)^2 \\
&\leq d^2 \sum_{\gamma \in \Gamma} \left(\sum_{\beta \in \Gamma} \|\varphi_\beta\|_{2,s,L_0} \|\psi_{\beta,\gamma}\|_2 \right)^2 \\
&= d^2 \|\varphi' * \psi'\|_2^2 \leq d^2 \|\varphi'\|_{2,s,L_1}^2 \|\psi'\|_2^2
\end{aligned}$$

where $\varphi'(\beta) = \|\varphi_\beta\|_{2,s,L_0}$ and $\psi'(\beta) = (\sum_{h \in G} |\psi(g, \beta)|^2)^{1/2}$; we have used $\psi'(\beta^{-1}\gamma) = \|\psi_{\beta,\gamma}\|_2$.

Finally, condition (ii) implies that

$$\|\varphi * \psi\|_2 \leq c' \|\varphi\|_{2,2rs,L} \|\psi\|_2 . \quad \text{Q.E.D.}$$

We will first treat the case when G or Γ is finite. Let us give the following lemma which will also be used in the following paragraph:

2.1.3. **Lemma.** *Let E be a group, let G be a finite subgroup of E and L a length-function on E. Then there exists a length-function L' on E with the following properties:*

(1) *L' is equivalent to L and takes integer values;*
(2) *$L'(gxh) = L'(x)$ for all $x \in E$ and $g, h \in G$;*
(3) *$G = \{x \in E ; L'(x) = 0\}$.*

Proof. Let us define successively four length-functions L_1, L_2, L_3 and L_4 on E:

$$L_1(x) = \begin{cases} 0 & \text{if } L(x) = 0, \\ [L(x)] + 1 & \text{if } L(x) \neq 0; \end{cases}$$

then L_1 satisfies condition (1).

Now, the function L_2 defined by

$$L_2(x) = \sum_{g \in G} L_1(gxg^{-1})$$

is a length-function on E satisfying condition (1) and is such that $L_2(gx) = L_2(xg)$ for all $x \in E$ and $g \in G$.

Set $L_3(x) = \min_{g,h \in G} L_2(gxh)$; then L_3 is a length-function on E satisfying conditions (1) and (2).

Finally, if

$$L_4(x) = \begin{cases} 0, & x \in G, \\ 1, & x \notin G, \end{cases}$$

then L_4 is a length-function on E, and $L' = L_3 + L_4$ satisfies conditions (1), (2) and (3), Q.E.D.

2.1.4. **Proposition.** *Let E be an extension of G by Γ as above and suppose that G is finite. Then E has property* (RD) *if and only if Γ does.*

Proof. Suppose first that Γ has property (RD) and let L_1 be a length-function on Γ such that $H^\infty_{L_1}(\Gamma)$ is contained in $C^*_r(\Gamma)$. For each $(g, \gamma) \in E$ set $L(g, \gamma) = L_1(\gamma)$.

Then L is a length-function on E satisfying the conditions of Lemma 2.1.2 which implies that E has property (RD). Conversely, suppose that E has property (RD) and let L be a length-function on E such that

(a) $H^\infty_L(E)$ is contained in $C^*_r(E)$;
(b) $L|_G = 0$ (see Lemma 2.1.3).

Set $L_1(\gamma) = \max_{g \in G} L(g, \gamma)$. Thanks to condition (b), L_1 is a length-function on Γ. If φ and ψ belong to $\mathbf{C}\Gamma$, define functions φ' and ψ' on E by

$$\varphi'(g, \gamma) = \varphi(\gamma) \quad \text{and} \quad \psi'(g, \gamma) = \psi(\gamma).$$

One gets

$$\begin{aligned}\|\varphi * \psi\|_2^2 &= \frac{1}{|G|^3}\|\varphi' * \varphi'\|_2^2 \\ &\leq \frac{c^2}{|G|^3}\|\varphi'\|_{2,s,L}^2\|\psi'\|_2^2 \\ &\leq \frac{c^2}{|G|}\|\varphi\|_{2,s,L}^2\|\psi\|_2^2\end{aligned}$$

for suitable positive constants c and s. Q.E.D.

2.1.5. **Proposition.** *Let E be a group and let E_0 be a subgroup of finite index of E. If E_0 has property* (RD) *then so does E.*

Proof. Define $G = \bigcap_{x\in E} xE_0x^{-1}$; it is a normal subgroup of finite index of E and it is contained in E_0. It follows from Proposition 2.1.1 that G has property (RD). Let then L_0 be a length-function on G such that $H^\infty_{L_0}(G)$ is contained in $C_r^*(G)$. Set $\Gamma = E/G$ and define

$$k(g,\gamma) = \max_{\beta\in\Gamma} L_0(\rho(\beta)(g)f(\beta,\gamma)) \quad \text{for every } (g,\gamma)\in E.$$

Using relations (R1) and (R2) of the beginning of the chapter, it is easily shown that

$$k((g_1,\gamma_1)(g_2,\gamma_2)) \leq k(g_1,\gamma_1) + k(g_2,\gamma_2),$$

and $k|_G \geq L_0$. Set $L = k + \check{k}$, where $\check{k}(x) = k(x^{-1})$. Then L is a length-function on E satisfying the conditions of Lemma 2.1.1. Q.E.D.

2.1.6. **Corollary.** *If Γ is a finitely generated discrete subgroup of* $\mathrm{SL}(2,\mathbf{R})$ *such that* $\mathrm{SL}(2,\mathbf{R})/\Gamma$ *is not compact, then Γ has property* (RD).

Proof. By Lemma 8, p. 154 of [Sel], Γ contains a torsion-free subgroup Γ_0 of finite index. Then the surface $\mathrm{SO}(2)\backslash\mathrm{SL}(2,\mathbf{R})/\Gamma_0$ is noncompact, namely open, and its fundamental group Γ_0 is a free group. By Example 1.2.3(2), Γ_0 has property (RD) and by Proposition 2.1.5, Γ has property (RD), too. Q.E.D.

2.1.7. *Remark.* Let E be a group and let G be a subgroup of finite index of E provided with a length-function L_0. There is generally no length-function L on E whose restriction to G coincides with L_0. In fact, take $G = F_2$, the free group on the two generators x and y, and let α be the automorphism of G defined by

$$\alpha(x) = xy, \quad \alpha(y) = y^{-1}.$$

Then $\alpha^2 = 1$. Let $\Gamma = \{1,\alpha\}$ and let $E = G \rtimes \Gamma$ be the corresponding semidirect product. Suppose that there is a length-function L on E whose restriction to G coincides with the natural algebraic length-function on G. We would get for each $m \in \mathbf{N}$

$$\begin{aligned}2m = |\alpha(x^m)| &= L(\alpha(x^m),1) = L((1,\alpha)(x^m,1)(1,\alpha)) \\ &\leq 2L(1,\alpha) + |x^m| = 2L(1,\alpha) + m,\end{aligned}$$

which gives a contradiction.

Finally we consider extensions of finitely generated groups by finitely generated ones.

Suppose that G and Γ are finitely generated and let $S = S^{-1}$ and $\Sigma = \Sigma^{-1}$ be finite systems of generators of G and Γ respectively. It is easily shown that the finite set $T = \{(s, 1); s \in S\} \cup \{(1, \tau); \tau \in \Sigma\}$ generates E.

2.1.8. **Definition.** (1) Let $\alpha \in \mathrm{Aut}(G)$. The *amplitude* of α (with respect to S) is the number

$$a(\alpha) = \max_{s \in S} |\alpha(s)|.$$

(2) A map $\theta \colon \Gamma \to \mathrm{Aut}(G)$ has *polynomial amplitude* if there exist positive constants c and r such that

$$a(\theta(\gamma)) \leq c(1 + |\gamma|)^r \quad \text{for every } \gamma \in \Gamma.$$

(3) A map $f \colon \Gamma \times \Gamma \to G$ has *polynomial growth* if there exist positive constants c' and r' such that

$$|f(\gamma_1, \gamma_2)| \leq c'(1 + |\gamma_1|)^{r'}(1 + |\gamma_2|)^{r'}$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

2.1.9. **Proposition.** *Suppose that G and Γ are finitely generated and that they have property* (RD). *Let E be an extension of G by Γ and suppose that there is a pair of associated functions (ρ, f) such that ρ has polynomial amplitude and f has polynomial growth. Then E has property* (RD).

Proof. It suffices to show that the algebraic length-functions on G, E and Γ satisfy the conditions of Lemma 2.1.1. There exist positive constants c and r such that

$$a(\rho(\gamma)) \leq c(1 + |\gamma|)^r$$

and

$$|f(\gamma_1, \gamma_2)| \leq c(1 + |\gamma_1|)^r(1 + |\gamma_2|)^r$$

for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$.

Let then $(g, \gamma) \in E$ and suppose that $|(g, \gamma)| = n > 0$. Then there exist $(g_1, \gamma_1), \dots, (g_n, \gamma_n) \in T \cup T^{-1}$ such that

$$(g, \gamma) = \prod_{j=1}^{n} (g_j, \gamma_j) = \left(\prod_{j=1}^{n} \rho(\beta_{j-1})(g_j) f(\beta_{j-1}, \gamma_j), \beta_n \right)$$

where $\beta_0 = 1$ and $\beta_k = \gamma_1 \cdots \gamma_k$ for $k > 0$.

Thus

$$\begin{aligned} |g| &\leq \sum_{j=1}^{n} (a(\rho(\beta_{j-1}))|g_j| + |f(\beta_{j-1}, \gamma_j)|) \\ &\leq c' n^{r+1} = c' |(g, \gamma)|^{r+1} \end{aligned}$$

and $|\gamma| \leq n = |(g, \gamma)|$. Q.E.D.

2.1.10. **Corollary.** *Suppose that* α *is an action of* Γ *on* G *of polynomial amplitude. If* Γ *and* G *have property* (RD) *then so does the semidirect product* $G \times_\alpha \Gamma$.

2.1.11. **Example.** Let F_1 and F_2 be the free groups of rank 1 and 2 respectively. If x and y are the natural generators of F_2, let F_1 act on F_2 in the following way:

$$\alpha(x) = xyx^{-1} \quad \text{and} \quad \alpha(y) = x.$$

It is easily verified that α has polynomial amplitude and consequently $F_2 \times_\alpha F_1$ has property (RD). But this group is a normal subgroup of finite index of the braid group B_3 (see [BZ, Chapter 10]). Therefore B_3 possesses property (RD), too.

2.2. Property (RD) **and amalgamated products.** Let Γ_1, Γ_2 and A be groups such that

(1) Γ_j admits the presentation $\langle \Sigma_j | R_j \rangle$, for $j = 1, 2$;
(2) there exists an injective homomorphism f_j from A to Γ_j, for $j = 1, 2$.

We recall that the amalgamated product of Γ_1 and Γ_2 over A is the group admitting the following presentation:

$$\langle \Sigma_1 \cup \Sigma_2 | R_1 \cup R_2, f_1(a) = f_2(a) \text{ for every } a \in A \rangle.$$

It is denoted by $\Gamma_1 *_A \Gamma_2$. (See [LS].)

Choose a set S_j of representatives of left A-cosets in Γ_j. Then every element g of $\Gamma_1 *_A \Gamma_2$ can be uniquely written as a reduced word $g = s_1 \cdots s_k a$, where $s_j \in S_{i_j} \backslash \{1\}$, $a \in A$ and $i_j \neq i_{j+1}$ for every $j = 1, \dots, k-1$.

Let k and l be nonnegative integers; put

$$\Lambda_k = \{g \in \Gamma_1 *_A \Gamma_2 ; g = s_1 \cdots s_k a \text{ as a reduced word}\}$$

and

$$E_{k,l}(g) = \{(h_1, h_2) \in \Lambda_k \times \Lambda_l ; h_1 h_2 = g\},$$

where g is an element of $\Gamma_1 *_A \Gamma_2$.

The following lemma is taken from [Pi, Lemma 3.1]:

2.2.1. **Lemma.** *Let* k, l, m *and* q *be nonnegative integers such that* $m = k + l - q$, *and let* g *an element of* Λ_m. *If* $g = s_1 \cdots s_m a$ *is its reduced form, one has*

(1) *If* $q = 2p$ *is even, set* $g_1 = s_1 \cdots s_{k-p}$ *and* $g_2 = s_{k-p+1} \cdots s_m a$. *Then* $E_{k,l}(g) = (h_1, h_2) \in \Lambda_k \times \Lambda_l$; *there exists* $w \in \Lambda_p$ *such that* $h_1 = g_1 w$ *and* $h_2 = w^{-1} g_2\}$.

(2) *If* $q = 2p+1$ *is odd, set* $g_1 = s_1 \cdots s_{k-p-1}$ *and* $g_2 = s_{k-p+1} \cdots s_m a$. *Then* $E_{k,l}(g) = \{(h_1, h_2) \in \Lambda_k \times \Lambda_l$; *there exist* $w \in \Lambda_p$ *and* $v_1, v_2 \in \Lambda_1$ *such that* $h_1 = g_1 v_1 w$, $h_2 = w^{-1} v_2 g_2$ *and* $v_1 v_2 = s_{k-p}\}$.

The rest of this paragraph will be devoted to the proof of the following theorem:

2.2.2. **Theorem.** *The amalgamated product* $\Gamma_1 *_A \Gamma_2$ *has property* (RD) *in the following cases*:

(1) Γ_1 *and* Γ_2 *have property* (RD) *and* A *is finite*;
(2) A *has property* (RD) *and is central and of finite index in* Γ_1 *and* Γ_2.

2.2.3. **Corollary.** (1) *If* Γ_1 *and* Γ_2 *possess property* (RD), *then the free product* $\Gamma_1 * \Gamma_2$ *has property* (RD) *too.*

(2) *Let* p *and* q *be positive integers such that* $(p, q) = 1$. *Then the group* $\Gamma = \langle x, y | x^p = y^q \rangle$ *has property* (RD).

In both cases of Theorem 2.2.2, we have first to define a suitable length-function L on $\Gamma_1 *_A \Gamma_2$:

(1) If Γ_1 and Γ_2 have property (RD) and if A is finite, choose a length-function L_j on Γ_j such that

(i) L_j takes integer values;
(ii) $L_j(ag_jb) = L_j(g_j)$ for all $g_j \in \Gamma_j$ and $a, b \in A$;
(iii) $\{g_j \in \Gamma_j; L_j(g_j) = 0\} = A$;
(iv) $H^\infty_{L_j}(\Gamma_j)$ is contained in $C^*_r(\Gamma_j)$.

The existence of such a length-function is proved in Lemma 2.1.3.

If $g \in \Gamma_1 *_A \Gamma_2$, let $g = s_1 \cdots s_m a$ be the corresponding reduced word, with $s_j \in S_{i_j} \backslash \{1\}$. Set

$$L(g) = L_{i_1}(s_1) + \cdots + L_{i_m}(s_m).$$

It is easy to verify that L is a length-function on $\Gamma_1 *_A \Gamma_2$.

(2) If A has property (RD) and is central and of finite index in Γ_1 and Γ_2, choose first a length-function L_0 on A having the following properties:

(i) L_0 takes integer values;
(ii) $H^\infty_{L_0}(A)$ is contained in $C^*_r(A)$;
(iii) for every element $a \in \bigcup_{j=1}^2 \{\alpha \in A;$ there exist $s, t, u \in S_j$ with $st = u\alpha\}$, one has $L_0(a) \le 1$.

The existence of L_0 is ensured by the same arguments as in Lemma 2.1.3.

Set then for each $g \in \Gamma_1 *_A \Gamma_2$

$$K(g) = m + L_0(a)$$

where $g = s_1 \cdots s_m a$ is the reduced word associated to g. Using Lemma 2.2.1, it is easily shown that $K(gh) \le K(g) + K(h)$ for all $g, h \in \Gamma_1 *_A \Gamma_2$. Finally, define $L = K + \check{K}$. Then L is a length-function on $\Gamma_1 *_A \Gamma_2$ such that its restriction L_j to Γ_j satisfies $H^\infty_{L_j}(\Gamma_j) \subset C^*_r(\Gamma_j)$.

Here is the crucial step in the proof of Theorem 2.2.2; it is an adaptation of Lemma 1.3 of [Haa]:

2.2.4. **Lemma.** *Suppose that* Γ_1, Γ_2 *and* A *satisfy one of the conditions of Theorem* 2.2.2. *Then there exist* c *and* $r > 0$ *such that*: *if* $k, l, m \in \mathbf{N}$ *satisfy*

$|k-l| \leq m \leq k+l$, if φ and $\psi \in \mathbf{C}\Gamma_1 *_A \Gamma_2$ are supported in Λ_k and Λ_l respectively, one has

$$\|(\varphi * \psi)\chi_{\Lambda_m}\|_2 \leq c\|\varphi\|_{2,r,L}\|\psi\|_2,$$

where L is the length-function defined above.

Proof. The proof is decomposed into two parts corresponding to the conditions of Theorem 2.2.2.

First part. Case 1.1. Suppose that $m = k+l-2p$. Using Lemma 2.2.1 and similar arguments as in the proof of Lemma 1.3 of [Haa], one verifies that

$$\|(\varphi * \psi)\chi_{\Lambda_m}\|_2 \leq N\|\varphi\|_2\|\psi\|_2$$

where $N = |A|$.

Case 1.2. Suppose now that $m = k+l-1$. If $g = s_1 \cdots s_m a$ belongs to Λ_m, set $g_1 = s_1 \cdots s_{k-1}$ and $g_2 = s_{k+1} \cdots s_m a$. By Lemma 2.2.1,

$$\varphi * \psi(g) = \sum_{b \in A} \sum_{(v_1, v_2 \in E_{1,1}(s_k))} \varphi(g_1 v_1 b)\psi(b^{-1} v_2 g_2)$$

since $E_{1,1}(s_k) = \{(v_1, v_2) \in \Lambda_1 \times \Lambda_1;\ v_1 v_2 = s_k\}$. Let $i_k \in \{1, 2\}$ be such that $s_k \in \Gamma_{i_k}$ and set $\varphi_{g_1,b}(v) = \varphi(g_1 v b)$ and $\psi_{b^{-1},g_2}(v) = \psi(b^{-1} v g_2)$ for every $v \in \Gamma_{i_k}$. Then one gets

$$\varphi * \psi(g) = \sum_{b \in A} (\lambda_{\Gamma_{i_k}}(\varphi_{g_1,b})\psi_{b^{-1},g_2})(s_k).$$

Finally

$$\begin{aligned}\|(\varphi * \psi)\chi_{\Lambda_m}\|_2^2 &\leq c_1^2 \left(\max_j \sum_{(g_1,b) \in \Lambda_{k-1} \times A} \|\varphi_{g_1,b}\|_{2,r,L_j}^2 \right) \\ &\quad \cdot \left(\sum_{(g_2,b) \in \Lambda_{l-1} \times A} \|\psi_{b^{-1},g_2}\|_2^2 \right) \\ &\leq c_1^2 N^2 \|\varphi\|_{2,r,L}^2 \|\psi\|_2^2,\end{aligned}$$

where $r > 0$ is large enough to ensure that $H_{L_j}^{\infty}(\Gamma_j)$ is contained in $C_r^*(\Gamma_j)$ for $j = 1, 2$.

Case 1.3. Suppose finally that $m = k+l-2p-1$ with $p \geq 1$. Define

$$\varphi_1(u) = \begin{cases} \left(\sum_{w \in \Lambda_p} |\varphi(uw)|^2\right)^{1/2} & \text{if } u \in \Lambda_{k-p}, \\ 0 & \text{if } u \notin \Lambda_{k-p}, \end{cases}$$

$$\psi_1(u) = \begin{cases} \left(\sum_{w \in \Lambda_p} |\psi(w^{-1}v)|^2\right)^{1/2} & \text{if } v \in \Lambda_{l-p}, \\ 0 & \text{if } v \notin \Lambda_{k-p}. \end{cases}$$

Then $|\varphi * \psi(g)| \le \varphi_1 * \psi_1(g)$ for each $g \in \Lambda_m$ and $\|\varphi_1\|_{2,r,L} \le N\|\varphi\|_{2,r,L}$, and $\|\psi_1\|_2 \le N\|\psi\|_2$. One gets

$$\|(\varphi * \psi)\chi_{\Lambda_m}\|_2 \le \|(\varphi_1 * \psi_1)\chi_{\Lambda_m}\| \le c_2 N^2 \|\varphi\|_{2,r,L}\|\psi\|_2,$$

using the result established in Case 1.2.

Second part. Suppose now that A has property (RD) and is central and of finite index in Γ_1 and Γ_2.

Let Λ_p^0 denote the set of elements $g = s_1 \cdots s_p$, where $s_j \in S_{i_j}\backslash\{1\}$ and $i_j \ne i_{j+1}$ for $j \le p-1$. If f is a function on $\Gamma_1 *_A \Gamma_2$ and if $g \in \Gamma_1 *_A \Gamma_2$, set $f_g(a) = f(ga)$ for every $a \in A$.

Case 2.1. Suppose that $m = k + l - 2p$. If $g \in \Lambda_m$, it can be uniquely written as $g = g_1 g_2 a$ with $(g_1, g_2, a) \in \Lambda_{k-p}^0 \times \Lambda_{l-p} \times A$, and we have by Lemma 2.2.1

$$\varphi * \psi(g) = \sum_{w \in \Lambda_p^0} (\lambda_A(\varphi_{g_1 w})\psi_{w^{-1}g_2})(a).$$

It gives

$$\begin{aligned}\|(\varphi * \psi)\chi_{\Lambda_m}\|_2^2 &\le c_3^2 \left(\sum_{(g_1, w) \in \Lambda_{k-p}^0 \times \Lambda_p^0} \|\varphi_{g_1 w}\|_{2,r,L_0}^2\right)\left(\sum_{(g_2, w) \in \Lambda_{l-p}^0 \times \Lambda_p^0} \|\psi_{w-1} g_2\|_2^2\right) \\ &\le c_3^2 \|\varphi\|_{2,r,L}^2 \|\psi\|_2^2,\end{aligned}$$

since each element $h \in \Lambda_l$ can be uniquely written as $h = w^{-1}h'a$ with $(w, h', a) \in \Lambda_p^0 \times \Lambda_{l-p}^0 \times A$.

Case 2.2. Consider finally the case where $m = k + l - 2p - 1$ and let $g \in \Lambda_m$, with $g = g_1 s_{k-p} g_2 a$ where $(g_1, s_{k-p}, g_2, a) \in \Lambda_{k-p-1}^0 \times \Lambda_1^0 \times \Lambda_{l-p-1}^0 \times A$. Set $\Lambda(s_{k-p}) = \{(u_1, u_2) \in \Lambda_1^0 \times \Lambda_1^0 ; (u_1 u_2)^{-1} s_{k-p} \in A\}$. Then Lemma 2.2.1 allows us to decompose $E_{k,l}(g)$ as follows:

$$E_{k,l}(g) = \bigcup_{(u_1, u_2) \in \Lambda(s_{k-p})} \{(g_1 u_1 w b, w^{-1} u_2 g_2 b^{-1} \alpha a) ; w \in \Lambda_p^0, b \text{ and }$$

$$\alpha = (u_1 u_2)^{-1} s_{k-p} \in A\}.$$

One gets

$$\begin{aligned}
\|(\varphi * \psi)\chi_{\Lambda_m}\|_2^2 &\leq M^2\left(\sum_{(g_1,g_2)\in\Lambda^0_{k-p-1}\times\Lambda^0_{p-1}} \max_{(u_1,u_2)\in\Lambda_1^0\times\Lambda_1^0}\left\|\sum_{w\in\Lambda_p^0}\lambda_A(\varphi_{g_1u_1w})\psi_{w^{-1}u_2g_2}\right\|_2^2\right)\\
&\leq c_3^2M^2\left(\sum_{(g_1,g_2)\in\Lambda^0_{k-p-1}\times\Lambda^0_{p-1}} \max_{(u_1,u_2)\in\Lambda_1^0\times\Lambda_1^0}\right.\\
&\qquad\left.\cdot\left(\sum_{w\in\Lambda_p^0}\|\varphi_{g_1u_1w}\|^2_{2,r,L_0}\right)\left(\sum_{w\in\Lambda_p^0}\|\varphi_{w^{-1}u_2g_2}\|_2^2\right)\right)\\
&\leq c_3^2M^2\|\varphi\|^2_{2,r,L}\|\psi\|_2^2,
\end{aligned}$$

where $M = |\Lambda_1^0|$. Q.E.D.

Proof of Theorem 2.2.2. It suffices to show that the pair $(\Gamma_1 *_A \Gamma_2, L)$ satisfies property (4) of Proposition 1.2.6. Consider nonnegative integers k, l and m such that $|k-l| \leq m \leq k+l$, and let φ, ψ be elements of $\mathbf{C}\Gamma_1 *_A \Gamma_2$ supported in $C_{k,l}$ and $C_{l,L}$ respectively. Accordingly to the definition of L (in both cases), one has $C_{k,L} \subset \bigcup_{j=0}^k \Lambda_j$.

It follows that $\varphi = \sum_{j=0}^k \varphi_j$ and $\psi = \sum_{i=0}^l \psi_i$, where $\varphi_j = \varphi\cdot\chi_{\Lambda_j}$ and $\psi_i = \psi\cdot\chi_{\Lambda_i}$.

Let us first fix $j \in \{0, \dots, k\}$. Lemma 2.2.4 gives

$$\begin{aligned}
\|(\varphi_j * \psi)\chi_{m,L}\|_2^2 &= \sum_{p=0}^m \|(\varphi_j * \psi)\chi_{\Lambda_p}\|_2^2\\
&\leq c_1^2\|\varphi_j\|^2_{2,r,L}\sum_{p=0}^m\left(\sum_{i=|j-p|}^{j+p}\|\psi_i\|_2\right)^2\\
&\leq 2(j+1)c_1^2\|\varphi_j\|^2_{2,r,L}\sum_{p=0}^m\sum_{i=0}^{2\min(j,p)}\|\psi_{j+p-i}\|_2^2\\
&\leq c_2^2\|\varphi_j\|^2_{2,r+1,L}\|\psi\|_2^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|(\varphi * \psi)\chi_{m,L}\|_2 &\leq \sum_{j=0}^k \|(\varphi_j * \psi)\chi_{m,L}\|_2\\
&\leq c_2\|\psi\|_2\sum_{j=0}^k\|\varphi_j\|_{2,r+1,L}\\
&\leq c_3\|\varphi\|_{2,r+2,L}\|\psi\|_2. \quad \text{Q.E.D.}
\end{aligned}$$

3. Examples

We present two families of examples: groups of polynomial growth on one hand, and discontinuous groups of hyperbolic isometries on the other hand.

3.1. Groups of polynomial growth. In the whole paragraph, Γ is a finitely generated group equipped with its algebraic length-function.

3.1.1. **Definition.** (1) Γ is of *polynomial growth* if there exist c and $r \in \mathbf{R}_+$ such that $|B_k| \leq c(1+k)^r$ for every $k \geq 0$.

(2) If L is a length-function on Γ, then Γ is said to be of *exponential growth with respect to* L if there exist $u > 0$ and $v > 1$ such that $|B_{k,L}| \geq u \cdot v^k$ for every $k \geq 0$.

(3) Γ is of *exponential growth* if it is of exponential growth with respect to its algebraic length-function.

3.1.2. *Remark.* (1) By means of Lemma 1.1.4, (2) and (4), Γ is of polynomial growth as soon as there exists a length-function L on Γ and $c, r > 0$ such that

$$|B_{k,L}| \leq c(1+k)^r, \quad \text{for every } k \geq 0.$$

(2) If Γ is of exponential growth, then it is of exponential growth with respect to any length-function on Γ.

3.1.3. **Example.** If Γ is the abelian free group of rank $N \geq 1$ and if its algebraic length-function is defined with respect to the canonical set of generators of Γ, then we have for each $k \in \mathbf{N}$

$$|B_L| = \sum_{l=0}^{N} 2^l \binom{N}{l} \binom{k}{l},$$

by Proposition 3.6 of [Wo]. Thus Γ is clearly of polynomial growth since there exist $c_1, c_2 > 0$ such that $c_1 k^N \leq |B_k| \leq c_2 k^N$ for every $k \geq 1$. However, in case $\Gamma = \mathbf{Z}$, let $L\colon \mathbf{Z} \to \mathbf{R}_+$ be the length-function defined by $L(n) = \log(1+|n|)$. Then Γ is of exponential growth with respect to L.

3.1.4. **Example.** Let F_N be the nonabelian free group of rank $N \geq 2$, equipped with its natural length-function. Then F_N is of exponential growth since we have for every $k \geq 0$

$$|C_k| = 2N(2N-1)^{k-1},$$

where

$$C_k = \{g \in F_N \,;\, |g| = k\}.$$

3.1.5. *Remark.* (1) If Γ is almost nilpotent (i.e. if it contains a nilpotent subgroup of finite index) then it is of polynomial growth [Wo]. Conversely, a deep theorem of M. Gromov [Gro] asserts that every finitely generated group of polynomial growth is almost nilpotent.

(2) In [Gri], R. Grigorchuk exhibits finitely generated groups which are neither of polynomial nor of exponential growth.

We are going to present some characterizations of groups of polynomial growth in terms of the spaces $H^s(\Gamma)$ and $H^\infty(\Gamma)$. Let us introduce first the following coefficients:

$$e(\Gamma) = \inf\{r \geq 0;\ \text{there exists } c > 0 \text{ with } |B_k| \leq ck^r, k \in \mathbf{N}\}$$

and

$$e'(\Gamma) = \inf\{r > 0;\ \text{there exists } c > 0 \text{ with } |C_k| \leq ck^r, k \in \mathbf{N}\}.$$

Note that Γ is of polynomial growth if and only if $e(\Gamma)$ is finite.

The proof of the following lemma is easy and left to the reader:

3.1.6. **Lemma.** *The coefficient* $e(\Gamma)$ *does not depend on the algebraic length-function. Moreover,* $e(\Gamma)$ *and* $e'(\Gamma)$ *possess the following properties:*

(i) $e(\Gamma) = \lim_n \sup \frac{\log|B_n|}{\log n}$ *and* $e'(\Gamma) = \lim_n \sup \frac{\log|C_n|}{\log n}$;

(ii) $e'(\Gamma) \leq e(\Gamma) \leq e'(\Gamma) + 1$.

Let us now recall briefly the notion of a nuclear space (cf. [Sch or Tr]).

Let E and F be Banach spaces. A linear mapping u from E to F is *nuclear* if there exist bounded sequences $(x'_n)_{n\geq 1} \subset E'$, the dual space of E, and $(y_n)_{n\geq 1} \subset F$, as well as a summable sequence $(c_n)_{n\geq 1} \subset \mathbf{C}$ such that

$$u(x) = \sum_{n\geq 1} c_n \langle x'_n, x\rangle y_n$$

for every $x \in E$.

If E is a locally convex space and if p is a continuous seminorm on E, set $N_p = \{x \in E; p(x) = 0\}$. Then N_p is a closed subspace of E and p induces a norm on the quotient space E_p. Let $\check{E}_p$ denote the completion of E_p. If q is a continuous seminorm on E satisfying $q \geq p$, the identity on E induces a continuous linear mapping $j_{q,p}$ from $\check{E}_q$ to $\check{E}_p$. We say that E is *nuclear* if for every continuous seminorm p on E, there exists a continuous seminorm $q \geq p$ such that $j_{q,p}$ is nuclear [Tr, Definition 50.1].

3.1.7. **Theorem.** *If* Γ *is a finitely generated group, the following properties are equivalent:*

(1) Γ *is of polynomial growth;*

(2) $H^\infty(\Gamma)$ *is contained in* $l^1(\Gamma)$;

(3) *there exists* $p \in (1,2)$ *such that* $H^\infty(\Gamma)$ *is contained in* $l^p(\Gamma)$;

(4) *there exists* $c > 0$ *such that for every pair* (s, s') *which satisfies* $s' < s - c$, *the inclusion of* $H^s(\Gamma)$ *in* $H^{s'}(\Gamma)$ *is of Hilbert-Schmidt class;*

(5) $H^\infty(\Gamma)$ *is nuclear.*

Moreover, if one of the above conditions is fulfilled, then the inclusions in (2) *and* (3) *are continuous, and one can take* $c = \frac{1}{2}(e'(\Gamma) + 1)$ *in* (4).

Proof. $(1) \Rightarrow (2)$ By hypothesis, there exist c and $r > 0$ such that $|C_k| \leq c(1+k)^r$ for each $k \in \mathbf{N}$. Set $s = 1 + r/2$. If $\varphi \in H^s(\Gamma)$, one has

$$\begin{aligned}
\|\varphi\|_1 &= \sum_{k\geq 0}\left(\sum_{g\in C_k}|\varphi(g)|\right) \leq \sum_{k\geq 0}\left(\sum_{g\in C_k}|\varphi(g)|^2\right)^{1/2} |C_k|^{1/2} \\
&\leq c^{1/2}\sum_{k\geq 0}\left(\sum_{g\in C_k}|\varphi(g)|^2(1+k)^r\right)^{1/2} \\
&\leq c'\left(\sum_{k\geq 0}\left\{\sum_{g\in C_k}|\varphi(g)|^2(1+|g|)^{r+2}\right\}\right)^{1/2} \\
&= c'\|\varphi\|_{2,s},
\end{aligned}$$

which shows that $H^s(\Gamma)$ is contained in $l^1(\Gamma)$.

$(2) \Rightarrow (3)$ is immediate.

$(3) \Rightarrow (1)$ If Γ is not of polynomial growth then $e'(\Gamma) = \infty$ by Lemma 3.1.6. Consequently,

$$\sup_{n\geq k}\frac{\log|C_n|}{\log n} = \infty \quad \text{for every } k \geq 1.$$

Thus, one can find a strictly increasing sequence $(n_k)_{k\geq 1}$ in $\mathbf{N}$ such that

$$|C_{n_k}| \geq (n_k)^k \quad \text{for every } k \geq 1.$$

Let $p \in (1,2)$ and set

$$\psi = \sum_{k\geq 1}\frac{\chi_{n_k}}{|C_{n_k}|^{1/p}}.$$

It is readily verified that $\|\psi\|_{2,s}$ is finite for every positive real number s, but that $\|\psi\|_p = \infty$.

$(1) \Rightarrow (4)$ If s and s' satisfy $s - s' > (e'(\Gamma)+1)/2$, let $\varepsilon > 0$ be small enough in order that

$$s - s' > (e'(\Gamma) + 1 + \varepsilon)/2.$$

Then there exists $\alpha_\varepsilon > 0$ such that

$$|C_k| \leq \alpha_\varepsilon(1+k)^{e'+\varepsilon}$$

for every $k \geq 0$.

Set $\delta_g^{(s)} = (1+|g|)^{-s}\delta_g$, for $g \in \Gamma$, where δ_g is the characteristic function of $\{g\}$.

The family $(\delta_g^{(s)})_{g\in\Gamma}$ is the canonical basis of $H^s(\Gamma)$.

If J is the injection of $H^s(\Gamma)$ into $H^{s'}(\Gamma)$, one gets

$$\begin{aligned}
\|J\|_{HS}^2 &= \sum_{g\in\Gamma}\|J(\delta_g^{(s)})\|_{r,s'}^2 = \sum_{k\geq 0}|C_k|(1+k)^{2(s'-s)} \\
&\leq \alpha_\varepsilon\sum_{k\geq 0}(1+k)^{2(s'-s)+e'+\varepsilon} < \infty
\end{aligned}$$

because $2(s'-s) + e'(\Gamma) + \varepsilon < -1$.

$(4) \Rightarrow (5)$ If p is a continuous seminorm on $H^\infty(\Gamma)$, one can find c and $s' > 0$ such that $p(\varphi) \leq c\|\varphi\|_{2,s'}$ for every $\varphi \in H^\infty(\Gamma)$. Since the composition of two Hilbert-Schmidt mappings is nuclear, using (4), there exists $s > s'$ such that the inclusion of $H^s(\Gamma)$ in $H^{s'}(\Gamma)$ is nuclear. Then the mapping induced by the identity from $H^s(\Gamma)$ to $\widehat{H^\infty(\Gamma)}_p$ is nuclear.

$(5) \Rightarrow (1)$ If $H^\infty(\Gamma)$ is nuclear, there is an $s > 0$ such that the inclusion J_0 of $H^s(\Gamma)$ into $H^0(\Gamma) = l^2(\Gamma)$ is nuclear. In particular, J_0 is Hilbert-Schmidt and the series

$$\sum_{g \in \Gamma} \|J_0(\delta_g^{(s)})\|_2^2 = \sum_{k \geq 0} |C_k|(1+k)^{-2s}$$

converges, which shows that Γ is of polynomial growth. Q.E.D.

3.1.8. **Corollary.** *If* Γ *is amenable then* Γ *has property* (RD) *if and only if* Γ *is of polynomial growth.*

Proof. If Γ is of polynomial growth then it has property (RD) by Theorem 3.1.7(2).

If Γ is not of polynomial growth, there exists $\psi \in H^\infty(\Gamma)$ which takes nonnegative values, such that $\psi \notin l^1(\Gamma)$. As Γ is amenable, one has for every function $\varphi \in l^1(\Gamma)$, with $\varphi(g) \geq 0$ for every $g \in \Gamma$

$$\|\varphi\|_1 \leq \|t(\varphi)\| \leq \|\lambda(\varphi)\| \leq \|\varphi\|_1,$$

where t is the trivial representation of Γ. It follows that Γ cannot have property (RD). Q.E.D.

3.1.9. **Corollary.** *Consider an integer* $n \geq 3$. *Then* $\mathrm{SL}(n, \mathbf{Z})$ *does not possess property* (RD).

Proof. Let α denote the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ (which belongs to $\mathrm{SL}(2, \mathbf{Z})$) and let $\Gamma = \mathbf{Z}^2 \times_\alpha \mathbf{Z}$ be the corresponding semidirect product. Γ is embedded in the standard semidirect product $\mathbf{Z}^2 \rtimes \mathrm{SL}(2, \mathbf{Z})$, which may be identified with the subgroup of $\mathrm{SL}(3, \mathbf{Z})$ constituted by matrices of the form $\left(\begin{smallmatrix} A & u \\ 0 & 1 \end{smallmatrix}\right)$, where $u \in \mathbf{Z}^2$ and $A \in \mathrm{SL}(2, \mathbf{Z})$. Thus Γ is a subgroup of $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 3$. By Proposition 2.1.1, it suffices to verify that Γ does not have property (RD). But Γ is solvable and of exponential growth (cf. J. Tits, Appendix to [Gro, Lemma 3]). By the above corollary, Γ does not have property (RD). Q.E.D.

3.2. Discontinuous subgroups of hyperbolic isometries. In this paragraph, X is a complete noncompact Riemannian manifold with bounded strictly negative sectional curvature K. More precisely, there exist $K_1 < K_2 < 0$ such that $K_1 \leq K(x) \leq K_2$ for every $x \in X$ (cf. [BGS, Kl]).

For example, the hyperbolic spaces $\mathbf{H}^n(\mathbf{R})$, $\mathbf{H}^n(\mathbf{C})$, $\mathbf{H}^n(K)$ and $\mathbf{H}^2(O)$ satisfy the above conditions [Mo, Chapter 19] where K is the field of quaternions and where O denotes Cayley numbers.

Now we describe simply connected Riemannian spaces with constant negative sectional curvature [Kl, 1.11.8]: Let $\rho > 0$ and $n \geq 2$; set $\mathscr{B}_\rho^n = \{x \in \mathbf{R}^n ; |x| < \rho\}$. Then $\mathscr{B}_\rho^n$ is a Riemannian space with sectional curvature $K = \rho^{-2}$ with respect to the metric

$$ds = \frac{2|dx|}{1 - |x|^2/\rho^2}.$$

Let us consider a geodesic triangle in $\mathscr{B}_\rho^n$ with angles α, β and γ, let a (resp. b, c) be the length of the opposite side of α (resp. β, γ). Then (α, β, γ) and (a, b, c) satisfy the *sine rule* [Kl, 2.7.5]

$$\frac{\sin\alpha}{\sinh(a/\rho)} = \frac{\sin\beta}{\sinh(b/\rho)} = \frac{\sin\gamma}{\sinh(c/\rho)}.$$

Let Γ be a discontinuous group of isometries of X and let 0 be a point of X. Set $L(g) = d(0, g(0))$ for every $g \in \Gamma$. As it is remarked in Lemma 1.1.4, L is a length-function on Γ satisfying for every $x \in X$

$$d(x, g(x)) - 2d(0, x) \leq L(g) \leq d(x, g(x)) + 2d(0, x).$$

Though L depends on the chosen point 0, the spaces $H_L^s(\Gamma)$ and $H_L^\infty(\Gamma)$ do not.

Here is the main result of the paragraph:

3.2.1. **Theorem.** *If Γ is cocompact (i.e. X/Γ is compact), then Γ has property* (RD). *More precisely, there exists a positive constant c which depends only on the action of Γ on X such that $\|\varphi\| \leq c\|\varphi\|_{2,2,L}$ for every finitely supported function φ on Γ.*

Note that, by Theorem 6.15 of [Rag], such a group is always finitely presented. Thus $H^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$. When Γ is finitely generated but not cocompact, we do not know whether $H^\infty(\Gamma)$ is contained in $C_r^*(\Gamma)$ except in dimension 2 (Corollary 2.1.6). However, $H_L^\infty(\Gamma)$ is not contained in $C_r^*(\Gamma)$ in general: a simple example is supplied with the action of $\Gamma = \{\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \in M_2(\mathbf{R}) ; m \in \mathbf{Z}\}$ by homographic transformations on the Poincaré half-plane $\mathbf{H}^2 = \{z \in \mathbf{C} ; \operatorname{Im}(z) > 0\}$. In this case, one can set

$$L(g) = d(i, g(i)) \quad \text{for } g \in \Gamma.$$

If $g = \left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$, one has

$$2\log(1 + |m|) - 2\log 2 \leq L(g) \leq 2\log(1 + |m|)$$

since the distance between two points z and w in $\mathbf{H}^2$ is given by

$$d(z, w) = \log\left(\frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}\right).$$

Note that Γ is of exponential growth with respect to L.

Set

$$\psi\left(\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\right) = \begin{cases} 0, & \text{if } m \leq 0, \\ m^{-2/3}, & \text{if } m > 0. \end{cases}$$

One verifies easily that ψ belongs to $H_L^\infty(\Gamma)$, but that

$$\left(\psi * \psi\begin{pmatrix}1 & 2q+1\\ 0 & 1\end{pmatrix}\right)^2 \geq c \cdot q^{-2/3}$$

for every $q \geq 1$. Thus $\psi * \psi$ does not belong to $l^2(\Gamma)$. Finally, Γ is a subgroup of $\mathrm{SL}(2, \mathbf{Z})$ which implies that $H_L^\infty(\mathrm{SL}(2, \mathbf{Z}))$ is not a convolution algebra and consequently not contained in $C_r^*(\mathrm{SL}(2, \mathbf{Z}))$.

Let us now give the proof of Theorem 3.2.1. From now on we suppose that X/Γ is compact and, without loss of generality, that the stabilizer of 0 in Γ is trivial. There is thus a $\delta > 0$ such that

$$d(u(0), v(0)) \geq 2\delta$$

for each pair of distinct elements u and v of Γ. Since the length-function L is henceforth fixed, we write

$$C_r = \{g \in \Gamma; r - 1 < L(g) \leq r\},$$

and χ_r the characteristic function of C_r. Set also

$$C_{r,a} = \{g \in \Gamma; r - a \leq L(g) \leq r + a\},$$

if r and a belong to $\mathbf{R}_+$.

3.2.2. **Lemma.** *Let c, k, l and p be nonnegative real numbers such that $p \leq \min(k, l)$. Consider in X a triangle whose vertices x, y and z satisfy*

$$d(x, y) = k + l - 2p - c, \quad d(x, z) = k \quad \textit{and} \quad d(y, z) = l.$$

Let x' denote the unique point on the geodesic segment $[x, y]$ which satisfies $d(x, x') = k - p$.

Then there exists $a > 0$ independent of k, l and p such that $p \leq d(x', z) \leq p + a$.

Proof. The sectional curvature K of X satisfies $K \leq -\rho^{-2}$ for some positive ρ. Consider then a triangle in $\mathscr{B}_\rho^n$ with vertices x_ρ, y_ρ, z_ρ, and let $x'_\rho \in [x_\rho, y_\rho]$ with the following properties:

(1) $d(x_\rho, y_\rho) = k + l - 2p - c$, $d(x_\rho, x'_\rho) = k - p$ and $d(x'_\rho, z_\rho) = d(x', z)$;

(2) the angle θ at x'_ρ between $[x'_\rho, x_\rho]$ and $[x'_\rho, z_\rho]$ is equal to the angle at x' between $[x', x]$ and $[x', z]$.

We adopt the following notations:

$$k' = d(x_\rho, z_\rho), \quad l' = d(y_\rho, z_\rho), \quad r = d(x', z) = d(x'_\rho, z_\rho);$$

α is the angle at x_ρ between $[x_\rho, y_\rho]$ and $[x_\rho, z_\rho]$; β is the angle at y_ρ between $[y_\rho, x_\rho]$ and $[y_\rho, z_\rho]$; γ_1 and γ_2 are the angles at z_ρ between $[z_\rho, x_\rho]$ and $[z_\rho, x'_\rho]$ and between $[z_\rho, x'_\rho]$ and $[z_\rho, y_\rho]$ respectively.

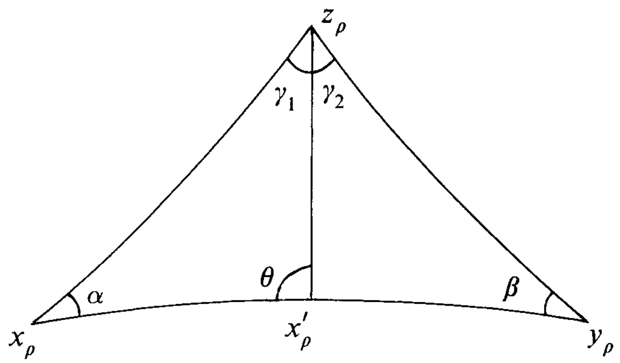

FIGURE 1

By the sine rule in $\mathscr{B}_\rho^n$ one gets

$$\begin{aligned}
\sinh\left(\frac{r}{\rho}\right) &= \frac{\sin\alpha}{\sin\theta}\cdot\sinh\left(\frac{k'}{\rho}\right)\\
&= \frac{\sin\alpha}{\sin(\gamma_1+\gamma_2)}\cdot\frac{\sin(\gamma_1+\gamma_2)}{\sin\theta}\cdot\sinh\left(\frac{k'}{\rho}\right)\\
&= \left(\frac{\sin\gamma_1}{\sin\theta}\cos\gamma_2+\cos\gamma_1\frac{\sin\gamma_2}{\sin\theta}\right)\cdot\frac{\sin\left(\frac{k'}{\rho}\right)\sinh\left(\frac{l'}{\rho}\right)}{\sinh\left(\frac{k+l-2p-c}{\rho}\right)}\\
&\le \frac{\sinh((k-p)/\rho)\sinh(l'/\rho)+\sinh(k'/\rho)\sinh((l-p-c)/\rho))}{\sinh((k+l-2p-c)/\rho)}
\end{aligned}$$

since

$$\frac{\sin\gamma_1}{\sin\theta}=\frac{\sinh((k-p)/\rho)}{\sin(k'/\rho)}\quad\text{and}\quad\frac{\sin\gamma_2}{\sin\theta}=\frac{\sinh((l-p-c)/\rho)}{\sinh(l'/\rho)}.$$

By comparison of the sinh function with the exponential one, one has

$$e^{r/\rho}\le c_1e^{p/\rho}$$

because the inequality $K\le-\rho^{-2}$ implies $k'\le k$ and $l'\le l$ [Kl, 2.7.6]. Thus, $r\le p+a$ where $a>0$ and independent of k, l and p. Moreover, $r+k-p\ge k$ and then $r\ge p$. Q.E.D.

3.2.3. **Lemma.** *Let a and b be positive constants. There exists a positive number N, depending only on a, b and on the action of Γ on X, with the following property: For every pair of nonnegative real numbers k and l and for each $g\in C_{k+l}$, one has*

$$|\{h\in C_{k,a};h^{-1}g\in C_{l,b}\}|\le N.$$

Proof. Let $(g,h)\in C_{k+l}\times C_{k,a}$ be such that $h^{-1}g\in C_{l,b}$. Let us consider the geodesic triangle in X with vertices $0, g(0)$ and $h(0)$. If $x\in X$ is the unique point on $[0,g(0)]$ such that $d(0,x)=k$, Lemma 3.2.2 (with $p=0$) ensures the existence of a positive constant A, independent of k, l, g and h, such that $d(x, h(0)) \leq A$. This implies that the ball $B(h(0), \delta)$ with center $h(0)$ and radius δ is contained in $B(x, A+\delta)$. Since the different balls $B(h(0), \delta)$ are disjoint, and since the sectional curvature is bounded below, the number of h's in $C_{k,a}$ such that $h^{-1}g \in C_{l,b}$ is bounded by a constant independent of k, l and g. Q.E.D.

Now we expose the main technical result for the proof of Theorem 3.2.1 (by Proposition 1.2.6); it is a generalization of Lemma 1.3 of [Haa]:

3.2.4. **Proposition.** *There exists a positive constant c, depending only on the action of Γ on X, with the following property: If $k, l, m \in \mathbf{N}$ satisfy $|k-l| \leq m \leq k+l$, if $\varphi, \psi \in \mathbf{C}\Gamma$ are supported in C_k and C_l respectively, then*

$$\|(\varphi * \psi)\chi_m\|_2 \leq c\|\varphi\|_2\|\psi\|_2 .$$

Proof. We distinguish two cases: (i) $m = k+l$, and (ii) $|k-l| \leq m < k+l$. Let us first show the following more general assertion:

(A) If a and b are positive constants, there exists a positive number c', depending only on a, b and on the action of Γ on X, with the following property: If $k', l' \in \mathbf{R}_+$, if $\varphi', \psi' \in \mathbf{C}\Gamma$ are supported in $C_{k',a}$ and $C_{l',b}$ respectively, then

$$\|(\varphi' * \psi')\chi_{k'+l'}\|_2 \leq c'\|\varphi'\|_2\|\psi'\|_2 .$$

In fact, using Lemma 3.2.3, there exists $N > 0$ such that for every $g \in C_{k'+l'}$, the number of h's in $C_{k',a}$ such that $h^{-1}g \in C_{l',b}$ is bounded by N. Then, using the Cauchy-Schwarz inequality, we get

$$|\varphi' * \psi'(g)| \leq N \left(\sum_{\substack{(h_1, h_2) \in C_{k',a} \times C_{l',b} \\ h_1 h_2 = g}} |\varphi'(h_1)|^2 |\psi'(h_2)|^2 \right),$$

hence

$$\begin{aligned} \|(\varphi' * \psi)\chi_{k'+l'}\|_2 &\leq N \left(\sum_{g \in C_{k'+l'}}, \left(\sum_{\substack{(h_1, h_2) \in C_{k',a} \times C_{l',b} \\ h_1 h_2 \in C_{k'+l'}}} |\varphi'(h_1)|^2 |\psi'(h_2)|^2 \right)\right) \\ &\leq N\|\varphi'\|_2^2\|\psi'\|_2^2 . \end{aligned}$$

Thus, Case (i) is a direct consequence of (A), with $a = b = 1$.

Suppose now that $|k-l| \leq m < k+l$; then $m = k+l-2p$ for some half-integer $p \leq \min(k, l)$. Let $g \in C_m$.

Let $x'(g)$ be the unique point of $[0, g(0)]$ such that $d(0, x'(g)) = k-p$. Since X/Γ is compact, there exists $r > 0$ such that the distance from $x'(g)$ to the orbit of 0 is at most equal to r. There then exists $u_g \in \Gamma$ which satisfies

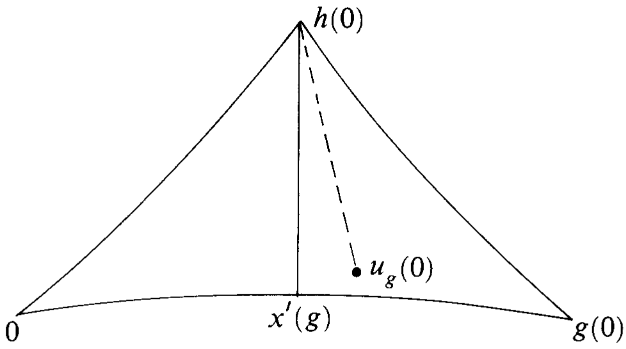

FIGURE 2

$d(u_g(0), x'(g)) \leq r$. Set $v_g = u_g^{-1} g$. Then g is written in "reduced form" $g = u_g v_g$ since the following inequalities hold:

$$k - p - r \leq L(u_g) \leq k - p - r$$

and

$$l - p - r - 1 \leq L(v_g) \leq l - p + r.$$

Thanks to Lemma 3.2.2, if $(h_1, h_2) \in C_k \times C_l$ and $h_1 h_2 = g$, then

$$p - a \leq L(u_g^{-1} h_1) \leq p + a$$

where $a > 0$ depends only on the action of Γ on X. Then we get that if $g \in C_m$ can be written as $g = h_1 h_2$ with $(h_1, h_2) \in C_k \times C_l$, there exists necessarily $w \in C_{p,a}$ such that $h_1 = u_g w$ and $h_2 = w^{-1} v_g$. Set

$$\varphi_p(u) = \begin{cases} \left(\sum_{w \in C_{p,a}} |\varphi(uw)|^2 \right)^{1/2} & \text{if } k - p - r \leq L(u) \leq k - p + r, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\varphi_p(v) = \begin{cases} \left(\sum_{w \in C_{p,a}} |\varphi(w^{-1} v)|^2 \right)^{1/2} & \text{if } l - p - r \leq L(v) \leq l - p + r, \\ 0 & \text{otherwise}. \end{cases}$$

One has

$$\begin{aligned} \|(\varphi * \psi)\chi_m\|_2^2 &= \sum_{g \in C_m} \left| \sum_{w \in C_{p,a}} \varphi(u_g w) \psi(w^{-1} v_g) \right|^2 \\ &\leq \sum_{g \in C_m} \varphi_p(u_g)^2 \psi_p(v_g)^2 \leq \sum_{g \in C_m} (\varphi_p * \psi_p(g))^2 \\ &= \|(\varphi_p * \psi_p)\chi_m\|_2^2 \leq N(a, r)^2 N(a, r+1) \|\varphi\|_2^2 \|\psi\|_2^2, \end{aligned}$$

using Assertion (A), where $N(a, r)$ and $N(a, r+1)$ come from Lemma 3.2.3. Q.E.D.

Finally, using the same arguments as in Lemma 4 of [Fl, p. 213], we can adapt our proof of Theorem 3.2.1 and get

3.2.5. **Proposition.** *If Γ is a geometrically finite Kleinian group without parabolic elements, then there exists a positive constant c depending only on the action of Γ on $\mathbf{H}^3(\mathbf{R})$ such that $\|\varphi\| \leq c\|\varphi\|_{2,2,L}$ for every finitely supported function φ on Γ.*

Appendix. Property (RD) for locally compact groups

Let G be a locally compact second countable group. Let $C_c(G)$ denote the space of compactly supported continuous functions on G and $L^2_{\mathrm{loc}}(G)$ the space of (classes of) measurable functions which are square-summable on every compact subset of G.

If s is a real number and if L is a continuous length-function on G, the space $H^s_L(G)$ is the set of (classes of) functions $\varphi \in L^2_{\mathrm{loc}}(G)$ such that

$$\|\varphi\|_{2,s,L} = \left(\int_G |\varphi(g)|^2(1+L(g))^{2s}\,dg\right)^{1/2}$$

is finite.

The space of *rapidly decreasing functions* on G (with respect to L) is the intersection of the spaces $H^s_L(G)$. We denote it by $H^\infty_L(G)$. It is a Fréchet space for the topology induced by the family of norms $(\|\cdot\|_{2,s,L})_{s\in\mathbf{R}}$.

A.1. **Definition.** G is said to have *property* (*RD*) if there exist a continuous length-function L on G and c and $s > 0$ such that $\|\lambda(\varphi)\| \leq c\|\varphi\|_{2,s,L}$ for every function $\varphi \in C_c(G)$.

A.2. **Proposition.** *If G has property* (RD) *and if L is a continuous length-function on G which satisfies the conditions of Definition* A.1, *then $H^\infty_L(G)$ is an involutive algebra* (*with respect to natural convolution and involution*) *which is identified by the left regular representation with a dense subalgebra of $C^*_r(G)$.*

The proof of Proposition A.2 is the same as that of Lemma 1.2.4.

We are going to show

A.3. **Proposition.** *Let G be a second countable locally compact unimodular group and let L be a continuous length-function on G. Suppose that G contains a discrete cocompact subgroup Γ such that $H^\infty_{L_0}(\Gamma)$ is contained in $C^*_r(\Gamma)$, where L_0 is the restriction of L to Γ. Then G has property* (RD).

A.4. **Corollary.** *If G is a connected noncompact semisimple Lie group of real rank one and finite center, then G possesses property* (RD). (*Compare with* [He].)

Proof of Corollary. Let K be a maximal compact subgroup of G and L a K-invariant length-function on G coming from a Riemannian metric on the homogeneous space G/K, i.e. $L(g) = d(x_0, g(x_0))$ for every $g \in G$, where $x_0 \in G/K$. If Γ is a cocompact subgroup of G then $H^\infty_{L_0}(\Gamma)$ is contained in $C^*_r(\Gamma)$ by Theorem 3.2.1. Q.E.D.

Proof of Proposition A.3. If $X = \Gamma\backslash G$ denotes the set of right cosets of G modulo Γ then X is compact and possesses a G-invariant probability measure μ. By Theorem 8.11 of [Var], there exists a regular Borel section σ for X; more precisely, σ is a Borel mapping from X to G satisfying

(1) $\pi \circ \sigma = \mathrm{id}_X$, where π is the canonical projection of G onto X;

(2) $\sigma(X)$ is relatively compact.

Note that the map $\varphi \to \int_X(\sum_{\gamma\in\Gamma}\varphi(\gamma\sigma(x)))\,d\mu(x)$, defined on $C_c(G)$, is a Haar measure on G.

By hypothesis, there exist c and $s > 0$ such that

$$\|f_1 * f_2\|_2 \le c\|f_1\|_{2,s,L_0}\|f_2\|_2$$

for all $f_1, f_2 \in \mathbf{C}\Gamma$.

Let φ, ψ be elements of $C_c(G)$. We have

$$\begin{aligned}
\|\lambda(\varphi)\psi\|_2^2 &= \int_X\left(\sum_{\gamma\in\Gamma}\left|\int_X\left\{\sum_{\beta\in\Gamma}\varphi(\beta\sigma(y))\psi(\sigma(y)^{-1}\beta^{-1}\gamma\sigma(x))\right\}d\mu(y)\right|^2\right)d\mu(x)\\
&\le \int_X\left(\sum_{\gamma\in\Gamma}\int_X\left|\left\{\sum_{\beta\in\Gamma}\varphi(\beta\sigma(y))\psi(\sigma(y)^{-1}\beta^{-1}\gamma\sigma(x))\right|^2 d\mu(y)\right)d\mu(x)\\
&= \int_X\int_X\left(\sum_{\gamma\in\Gamma}\left|\sum_{\beta\in\Gamma}\varphi_y(\beta)\psi_{y,x}(\beta^{-1}\gamma)\right|^2\right)d\mu(y)\,d\mu(x)\\
&= \int_X\int_X\|\lambda_\Gamma(\varphi_y)\psi_{y,x}\|_2^2\,d\mu(y)\,d\mu(x)\\
&\le c^2\int_X\int_X\|\varphi_y\|_{2,s,L_0}^2\|\psi_{y,x}\|_2^2\,d\mu(y)\,d\mu(x)
\end{aligned}$$

where $\varphi_y(\gamma) = \varphi(\gamma\sigma(y))$ and $\psi_{y,x}(\gamma) = \psi(\sigma(y)^{-1}\gamma\sigma(x))$ for every $\gamma \in \Gamma$. Thus we get

$$\begin{aligned}
\|\lambda(\varphi)\psi\|_2^2 &\le c^2\int_X\|\varphi_y\|_{2,s,L_0}^2\left(\int_X\|\psi_{y,x}\|_2^2\,d\mu(x)\right)d\mu(y)\\
&\le c^2\|\psi\|_2^2\int_X\left(\sum_{\gamma\in\Gamma}|\varphi(\gamma\sigma(y))|^2(1+L(\gamma))^{2s}\right)d\mu(y)\\
&\le c^2M^2\|\varphi\|_{2,s,L}^2\|\psi\|_2^2,
\end{aligned}$$

where $M = \sup_{x\in X}(1 + L(\sigma(x)))^s$ is finite since $\sigma(X)$ is relatively compact and L is continuous. Q.E.D.

We do not know whether the converse of Proposition A.3 holds.

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