Rapidly switched random links enhance spatiotemporal regularity

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(Received 15 August 2008; published 15 December 2008)

We investigate the spatiotemporal properties of a lattice of chaotic maps whose coupling connections are rewired to random sites with probability p. Keeping p constant, we change the random links at different frequencies in order to discern the effect (if any) of the time dependence of the links. We observe two different regimes in this network: (i) when the network is rewired slowly, namely, when the random connections are quite static, the dynamics of the network is spatiotemporally chaotic and (ii) when these random links are switched around fast, namely, the network is rewired frequently, one obtains a spatiotemporal fixed point over a large range of coupling strengths. We provide evidence of a sharp transition from a globally attracting spatiotemporal fixed point to spatiotemporal chaos as the rewiring frequency is decreased. Thus, in addition to geometrical properties such as the fraction of random links in the network, dynamical information on the time dependence of these links is crucial in determining the spatiotemporal properties of complex dynamical networks.

DOI: 10.1103/PhysRevE.78.066209 PACS number(s): 05.45.Gg, 89.75.Hc, 89.75.Fb, 05.45.Xt

I. INTRODUCTION

Coupled map lattices (CMLs) were introduced as a simple model capturing essential features of nonlinear dynamics of extended systems [1]. A very well-studied coupling form in CMLs is nearest-neighbor coupling. While this regular network is the chosen topology of innumerable studies, there are strong reasons to revisit this fundamental issue in the light of the fact that some degree of randomness in spatial coupling can be closer to physical reality than strict nearest-neighbor scenarios [2–4]. In fact many systems of biological, technological, and physical significance are better described by randomizing some fraction of the regular links. So here we will study the spatiotemporal dynamics of CMLs with some of its coupling connections rewired randomly [5,6].

Now these random links in the network could be static or dynamic. Static links imply that the connectivity is invariant throughout the evolution of the system, i.e., the coupling connections are constant in time. Dynamic links on the other hand imply that the random links are switched around. So at any instant of time, both kinds of rewiring have the same fraction of random links. However, for static connections the random links are unchanged in time, while for dynamic rewiring the random links are time varying. Dynamic rewiring is relevant, for instance, in a network of neurons or a socioeconomic network, where the connectivity matrix can change over time.

Here we study effects of rewiring the network with different frequencies, interpolating between fast dynamic rewiring and static rewiring limits. So we investigate spatiotemporal effects of random rewiring, where the rewirings are updated at the time scale of the nodal dynamics, to rewirings that are much slower than the nodal dynamics and approach the static limit. We will especially demonstrate how quick changes in the connections enhances spatiotemporal regularity, as compared with slow network changes.

The important point is that at any particular instant of time the connectivity properties of the network with quenched randomness looks identical to that of the dynamically rewired network. However the crucial feature is the change in the connections. Namely, while the average number and type of links remain the same, the dynamically rewired network evolves under rapidly changing local connectivity environments. So simply knowing the rewiring fraction p is not enough to capture the spatiotemporal dynamics of large interactive systems. It is very crucial in some situations to know whether the randomness in the coupling connections of the network is quenched or dynamic. The purpose of this study is to underscore this important issue that has not been discussed adequately in the literature [7].

In Sec. II we introduce our model and describe all the parameters. In Sec. III we present our numerical results demonstrating and characterizing the enhancement of stability due to fast rewiring. Section IV introduces an approximate analytical method to understand the basic mechanisms behind the observed phenomena. Finally, we draw our conclusions in Sec. V.

II. MODEL

We consider here a one-dimensional ring of coupled strongly chaotic logistic maps. The sites are denoted by $i = 1, \ldots, N$, where N is the linear size of the lattice. On each site is defined a continuous state variable denoted by $x_n(i)$, which corresponds to the physical variable of interest. The evolution of this lattice, under standard nearest-neighbor interactions, in discrete time n is given by

$$x_{n+1}(i) = (1-\epsilon)f[x_n(i)] + \frac{\epsilon}{2}\{x_n(i+1) + x_n(i-1)\}. \quad (1)$$

ϵ is the strength of coupling. The local on-site map is chosen to be the fully chaotic logistic map $f(x) = 4x(1-x)$. This map

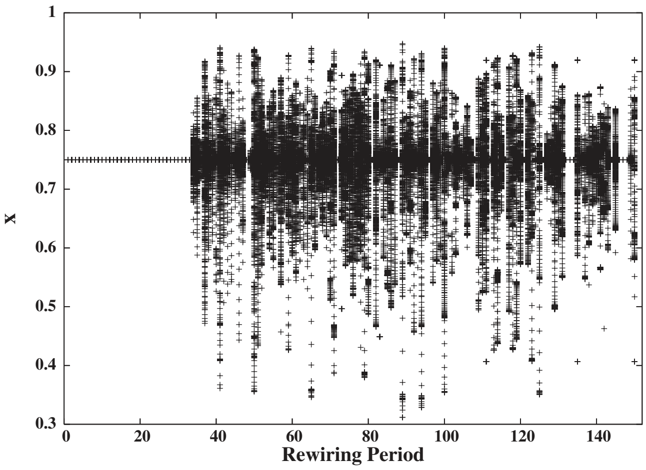

FIG. 1. Bifurcation diagram displaying the dynamics of a representative site vs rewiring period r for $\epsilon=0.85$, $p=0.65$. Here $N=50$.

has widespread relevance as a prototype of low-dimensional chaos.

We study the above system with its coupling connections rewired randomly in varying degrees. Namely, a fraction p of randomly chosen sites in the lattice will be connected to 2 other random sites, instead of their nearest neighbors. That is, a fraction p of nearest-neighbor links are replaced by random links. The case of $p=0$ corresponds to the usual nearest-neighbor interaction, i.e., a regular network, while $p=1$, corresponds to a completely random coupling, i.e., a random network.

In this work, we introduce a time scale for the random rewiring. We rewire the network after r dynamical updates of the nodal maps, namely the rewiring time period of the network is r. Thus a new connectivity matrix is formed, with the same fraction p of random links, every r time steps. Alternately, we can consider that the random links persist for r time dynamical updates of the local maps.

We investigate the asymptotic dynamics of this network, evolving from random initial conditions of $x(i)$, when the following parameters are varied: (i) fraction of random links p, (ii) coupling strength ϵ, and (iii) the time period for switching the random links r which gives all the cases from very fast rewiring for low r to very slow rewiring for high r.

III. RESULTS

First, we study the stability of the spatiotemporal fixed point, namely, the state where all elements are steady at x^*, i.e., $x_n(i)=x^*$ for all i and n (after transience). Here $x^*=3/4$ is the fixed point solution of the local map, which is strongly unstable for the isolated map.

Figure 1 displays the state of a representative site in the lattice, as the rewiring time period r is varied. It is evident that there exists a sharp transition, as the rewiring time period r increases, from simple spatiotemporal order, a fixed point, to spatiotemporal chaos.

We denote as r_c the largest rewiring time period that allows the spatiotemporal fixed point to be stable. For the parameters chosen in Fig. 1, we find that $r_c=33$. For slower rewiring $r>r_c$, the system becomes essentially chaotic. So networks rewired at time scales comparable to the nodal dynamics yield spatiotemporal order, while slow changing networks and static networks are spatiotemporally chaotic.

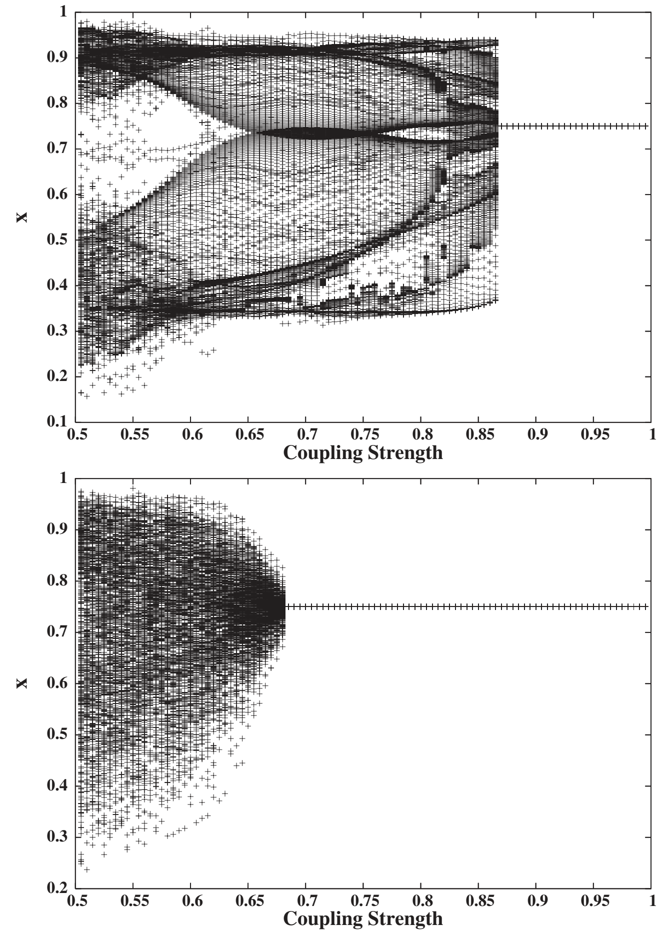

FIG. 2. Bifurcation diagram displaying the dynamics of a representative site vs coupling strength ϵ, for the rewiring time periods $r=100$ (top) and 1 (bottom). Here the rewiring probability $p=0.65$.

Next we analyze the state of a representative site in the lattice, as the coupling strength ϵ is varied, for two different values of r (fast and slow, respectively) (Fig. 2). It is evident that the system is stabilized at a fixed point for a much larger range of coupling strengths for faster network rewiring, compared to slower ones.

Figure 3 displays the state of a representative site in the lattice, as the fraction of random links p is varied, again for two different values of r. Again it is clear that the system is stabilized at a fixed point for a much larger range of p for faster network rewiring.

Now we study the critical value of r above which there is no spatiotemporal regularity. Recall that for all $r<r_c$ the system stabilizes to a spatiotemporal fixed point, while for $r>r_c$ this simple spatiotemporal order is lost. So r_c indicates how slowly the connections can be rewired in order to still achieve spatiotemporal regularity. We analyze r_c as a function of both the fraction of random links p and the coupling strength ϵ (Fig. 4).

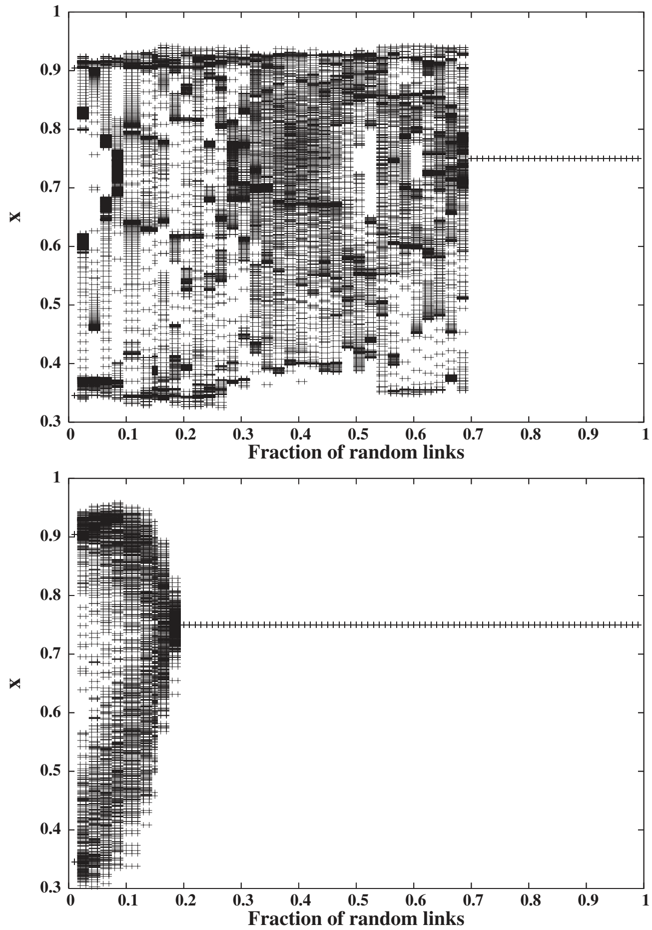

FIG. 3. Bifurcation diagram displaying the dynamics of a representative site vs fraction of random links p, for rewiring time period (top) $r=100$ and (bottom) $r=1$. Here the coupling strength $\epsilon=0.85$.

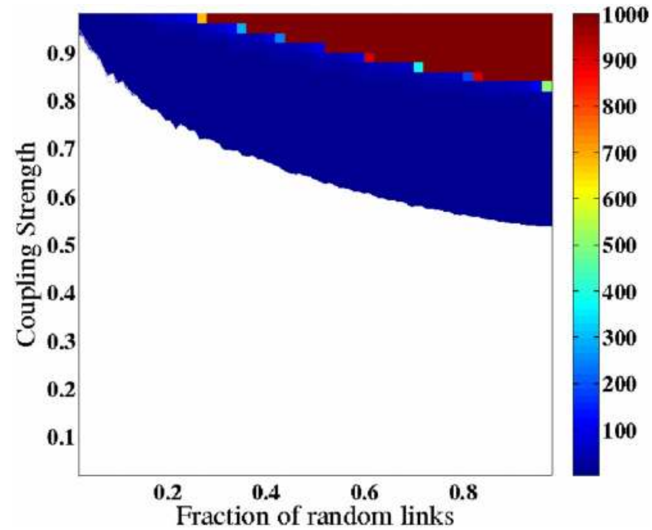

FIG. 4. (Color online) Density plot of r_c in the ϵ-p plane, where r_c is the largest rewiring time period which still yields spatiotemporal fixed points. The blue (dark gray) end of the spectrum corresponds to $r_c=1$, namely, the case where very fast rewiring is required in order to obtain a spatiotemporal fixed point. The red (light gray) end of the spectrum corresponds to $r_c=1000$, namely, the case where very slow rewiring is sufficient to obtain a spatiotemporal fixed point. The intermediate scenarios (namely $1<r_c<1000$) correspond to the colors (gray shades) in between, with increasingly static networks approaching the red end (light gray) of the color bar (grayscale). The white region covers the set of ϵ-p values which do not yield spatiotemporal order even at the fastest physically significant rewiring rate, namely, for no value of $r(r\geqslant 1)$ does one obtain a fixed point here. Observe that the transition from small r_c (blue/dark gray) to very large r_c (red/light gray) is sharp.

From Fig. 4 it is evident, especially, while there is no spatiotemporal regularity beyond $r=4$ for $\epsilon=0.72$, we have spatiotemporal order for rewiring as slow as $r=500$ when $\epsilon=0.92$. So r_c increases significantly with increasing coupling strengths. Namely, for strongly coupled systems the network need not be rewired that frequently, in order to obtain a spatiotemporal steady state. So there is a sharp transition in the ϵ space, from a situation where spatiotemporal order is obtained only in networks dynamically rewired at the timescale of the local dynamics, to a situation where spatiotemporal order emerges even in (almost) static networks.

IV. ANALYSIS

We now analyze system (1) to account for the much enhanced stability of the homogeneous phase under fast changing random connections. The only possible solution for a spatiotemporally synchronized state here is the one where all $x_n(i)=x^*$ and $x^*=f(x^*)$ is the fixed point solution of the local map. For the case of the logistic map $x^*=4x^*(1-x^*)=3/4$.

To calculate the stability of the lattice with all sites at x^*, we construct an average probabilistic evolution rule for the sites, which becomes a sort of mean-field version of the dynamics. In our formulation, the average influence of the random connections on the evolution of the local maps is given by p_{eff}, and the influence of the nearest neighbors is given by $(1-p_{\text{eff}})$. So p_{eff} provides the "weight" for the random time-varying coupling and $(1-p_{\text{eff}})$ provides the "weight" for the regular static coupling. Clearly p_{eff} is determined by the rewiring probability p and the rewiring time period r.

In terms of p_{eff} the averaged evolution equation of a site i then reads

$$x_{n+1}(i) = (1-\epsilon)f[x_n(i)] + (1-p_{\text{eff}})\frac{\epsilon}{2}[x_n(i+1)+x_n(i-1)] + p_{\text{eff}}\frac{\epsilon}{2}[x_n(\zeta)+x_n(\eta)], \tag{2}$$

where ζ and η are two random numbers between 1 and N. Now in order to calculate the stability of the synchronized spatiotemporal fixed point, we linearize Eq. (2). Replacing $x_n(j)=x^*+h_n(j)$, and expanding to first order gives

$$h_{n+1}(j) = (1-\epsilon)f'(x^*)h_n(j) + (1-p_{\text{eff}}) \times \frac{\epsilon}{2}\{h_n(j+1)+h_n(j-1)\} + p_{\text{eff}}\frac{\epsilon}{2}\{h_n(\zeta)+h_n(\eta)\}. \tag{3}$$

As a first approximation one can consider the sum over the fluctuations of the uncorrelated random neighbors to be equal to zero. This gives the approximate evolution equation

$$h_{n+1}(j) = (1-\epsilon)f'(x^*)h_n(j) + (1-p_{\text{eff}}) \times \frac{\epsilon}{2}\{h_n(j+1)+h_n(j-1)\}. \tag{4}$$

This approximation is clearly more valid for small p_{eff}.

For stability considerations one can diagonalize the above expression using a Fourier transform $[h_n(j) = \Sigma_q \phi_n(q)\exp(ijq)$, where q is the wave number and j is the site index$]$, which finally leads us to the following growth equation:

$$\frac{\phi_{n+1}(q)}{\phi_n(q)} = f'(x^*)(1-\epsilon) + \epsilon(1-p_{\text{eff}})\cos q \tag{5}$$

with q going from 0 to π. The condition for stability depends on the nature of the local map $f(x)$ through the term $f'(x)$. Considering the fully chaotic logistic map with $f'(x^*)=-2$, one finds that the growth coefficient that appears in this formula is smaller than one in magnitude if and only if

$$\frac{1}{1+p_{\text{eff}}} < \epsilon < 1. \tag{6}$$

This inequality implies that the coupling strength ϵ^* after which the spatiotemporal fixed point is stabilized, is given by

$$\epsilon^* = \frac{1}{1+p_{\text{eff}}} \tag{7}$$

and the range of the spatiotemporal fixed point R is

$$R = 1 - \epsilon^* = \frac{p_{\text{eff}}}{1+p_{\text{eff}}}. \tag{8}$$

For small p_{eff} ($p_{\text{eff}} \ll 1$) the standard expansion yields

$$R \sim p_{\text{eff}}. \tag{9}$$

Now, since p_{eff} is the effective probability that a coupling link is random, p_{eff} should be directly proportional to the probability of random rewiring p. In addition, the probability of having a random connection is an increasing function of rewiring frequency f, where $f=r^{-1}$.

So we can start with the ansatz that $p_{\text{eff}}=pg(f)$, where the function g, for consistency with the fully dynamic and static limits, should have value 1 when $f=1$ and value 0 when $f=0$. So $g(f)$ can be assumed to be power law, giving the ansatz

$$p_{\text{eff}} = pf^{\nu}. \tag{10}$$

Figure 5 displays the dependence of the numerically obtained values of the range of the spatiotemporal fixed point R on f. Fitting this to Eq. (10) yields $\nu \sim 0.42$.

In conclusion, our numerics appear to be consistent with the ansatz that the effective "strength" of the random links is given by pf^{ν}, where p is the fraction of random links in the system at any instant of time and f is the frequency of random rewiring. Some effects due to fluctuations are lost, but as a first approximation we have found this approach qualitatively correct, and quantitatively close to the numerical results as well [5].

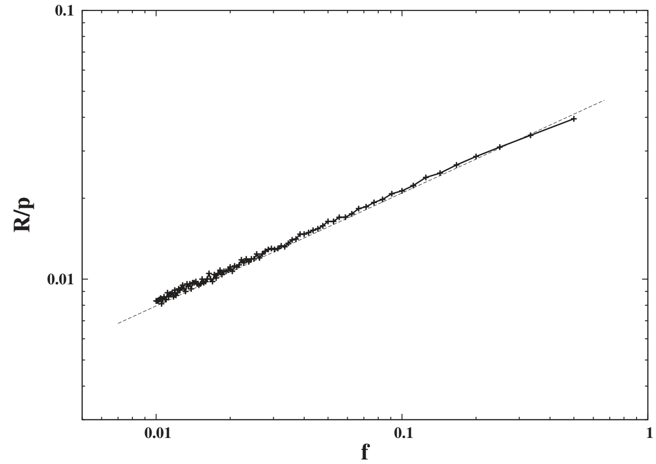

FIG. 5. R/p vs f, where R is the range of the spatiotemporal fixed point, p is the rewiring probability, and $f=1/r$ is the rewiring frequency. Here $p=0.05$.

V. CONCLUSIONS

We have investigated spatiotemporal properties of a lattice of coupled strongly chaotic maps whose coupling connections are rewired to random sites with probability p. Keeping p constant, we change the random links at different frequencies, in order to discern the effect (if any) of the time dependence of the links.

Our main findings are when the network is rewired slowly, namely, when the random connections are quite static, the dynamics of the network is spatiotemporally chaotic. However, when these random links are switched around fast, namely, the network is rewired frequently, one obtains a spatiotemporal fixed point over a large range of coupling strengths.

We provide evidence of a sharp transition from a globally attracting spatiotemporal fixed point to spatiotemporal chaos as the rewiring frequency is decreased. So the system behaves effectively as a static network with quenched randomness after a certain critical rewiring time period.

We also analyzed the stability of the spatiotemporal fixed point of this network of strongly chaotic maps. Our analysis is consistent with the ansatz that the effective "strength" of the random coupling is given by pf^{ν}, where p is the fraction of random links in the system at any instant of time and f is the frequency of random rewiring. In summary, in addition to geometrical properties such as the fraction of random links in the network at any instant of time, dynamical information on the time dependence of these links is crucial in determining the spatiotemporal properties of complex dynamical networks.

ACKNOWLEDGMENTS

A.M. and J.K. would like to thank the Federal Ministry of Education and Research of the Federal Republic of Germany for financially supporting parts of this work through the research project GoFORSYS.

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