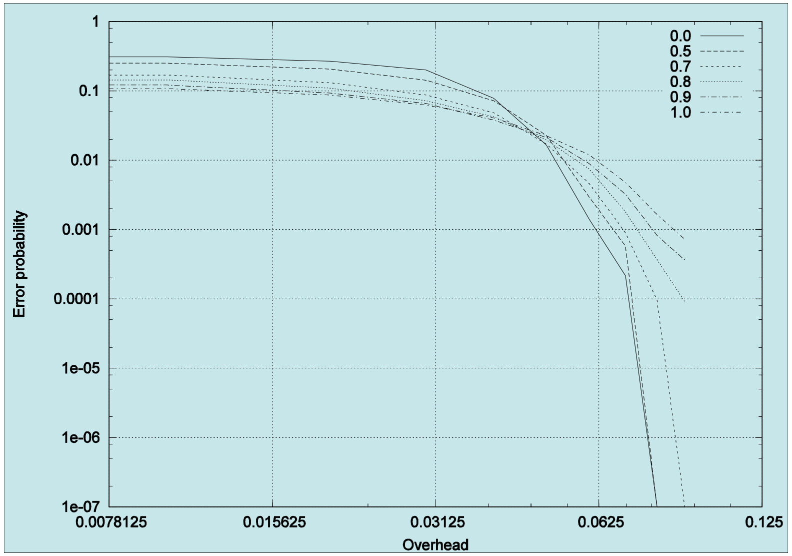
Raptor Codes on Symmetric Channels



Amin Shokrollahi

EPFL
and
Digital Fountain

Joint work with Omid Etesami and Mehdi Molkaraie

Content

- How do Raptor Codes designed for the BEC perform on other symmetric channels (with BP decoding)?
- Gaussian approximation
- Information theoretic bounds, and fraction of nodes of degrees one and two in capacity-achieving Raptor Codes.
- A better Gaussian approximation technique.

Parameters

Raptor Code with parameters $(k, \mathtt{C}, \Omega(x))$.

Channel $\mathcal{C}$.

Overhead ε, if decoding is possible from $\frac{k(1+\varepsilon)}{\mathsf{Cap}(\mathcal{C})}$ many output symbols.

Measure residual error probability as a function of the overhead for a given channel.

Sequences Designed for the BEC

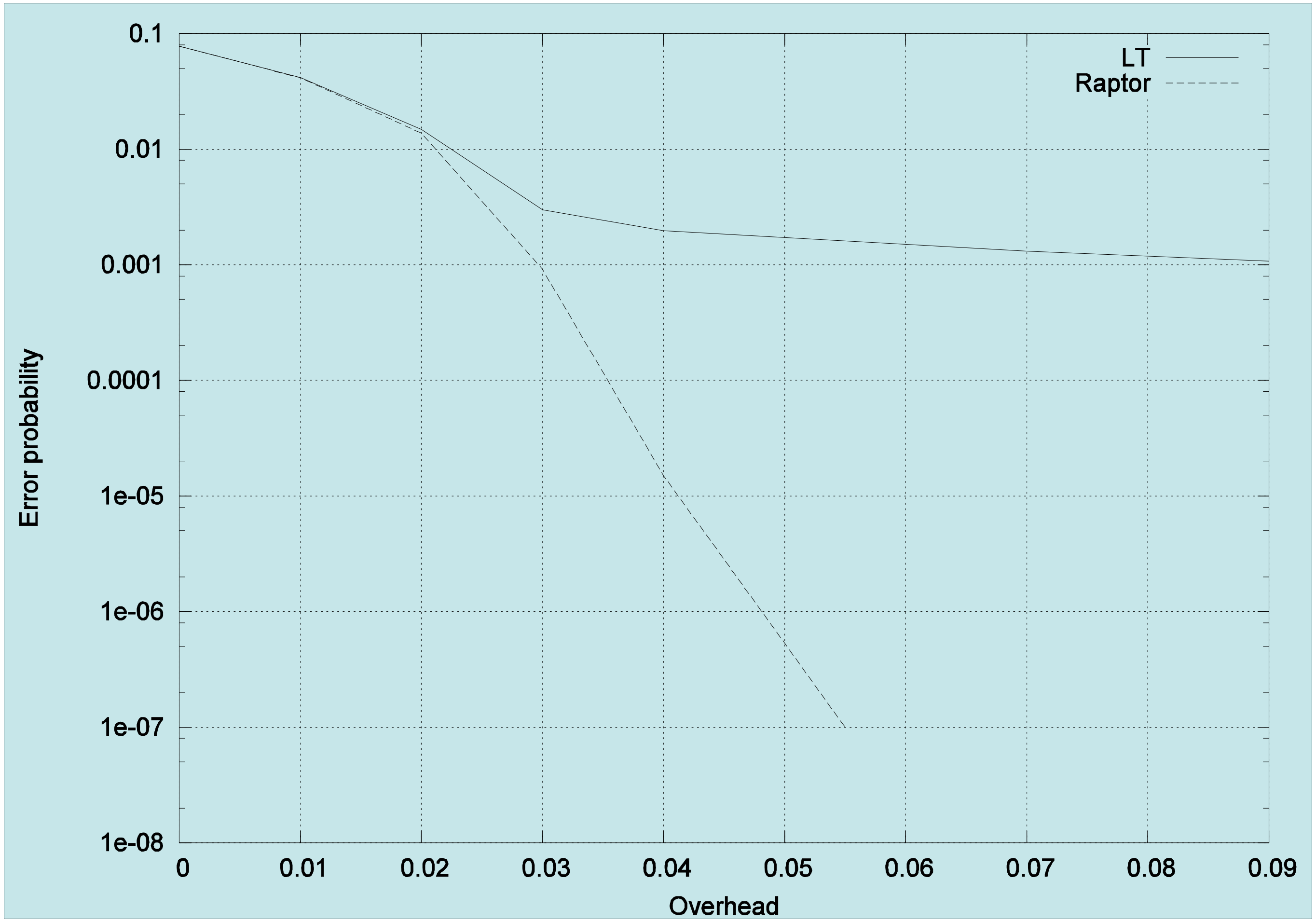
Type: $(65536, \mathtt{C}, \Omega(x))$

$\mathtt{C}$: Left-regular of degree 4, right Poisson, rate 0.98

$$\Omega(x) = 0.008x + 0.049x^2 + 0.166x^3 + 0.072x^4 + 0.083x^5 + 0.056x^8 + 0.037x^9 + 0.056x^{19} + 0.025x^{66} + 0.003x^{67}$$

Simulations done on AWGN(σ) for various σ

Sequences Designed for the BEC



$\sigma = 0.8$

Sequences Designed for the BEC

Not too bad, but quality decreases when the amount of noise on the channel increases.

Need to design better distributions.

Idea: adapt the Gaussian approximation technique of Chung, Richardson, and Urbanke.

Gaussian Approximation

Assume that the messages sent from input to output nodes are Gaussian.

Track the mean of these Gaussians from one round to another.

$$\varphi(\mu_{\ell+1}) \leq 1 - s\omega(1 - \varphi(\alpha\mu_\ell)).$$

Degree distributions can be designed using this approximation.

However, they don't perform that well.

Anything else we can do with this?

Nodes of Degree 2

Use equality, differentiate, and compare values at 0

$$\Omega_2 \geq \frac{\mathrm{Cap}(\mathrm{BIAWGN}(\sigma))}{2\mathrm{E}(\mathrm{BIAWGN}(\sigma))} = \frac{\Pi(\mathcal{C})}{2}.$$

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$$\mathrm{E}(\mathrm{BIAWGN}(\sigma)) = \frac{1}{\sqrt{2\pi m}} \int_{-\infty}^{\infty} \mathrm{tanh}\left(\frac{x}{2}\right) \mathrm{e}^{-\frac{(x-m)^2}{2m}} \mathrm{d}x$$

and $m = \frac{2}{\sigma^2}$.

It can be rigorously proved that above condition is necessary for error probability of BP to converge to zero.

Nodes of Degree 2

What is the fraction of nodes of degree 2 for capacity-achieving Raptor Codes?

Turns out, that in the limit we need to have equality:

$$\Omega_2(\mathcal{C}) := \frac{1}{2}\Pi(\mathcal{C})$$

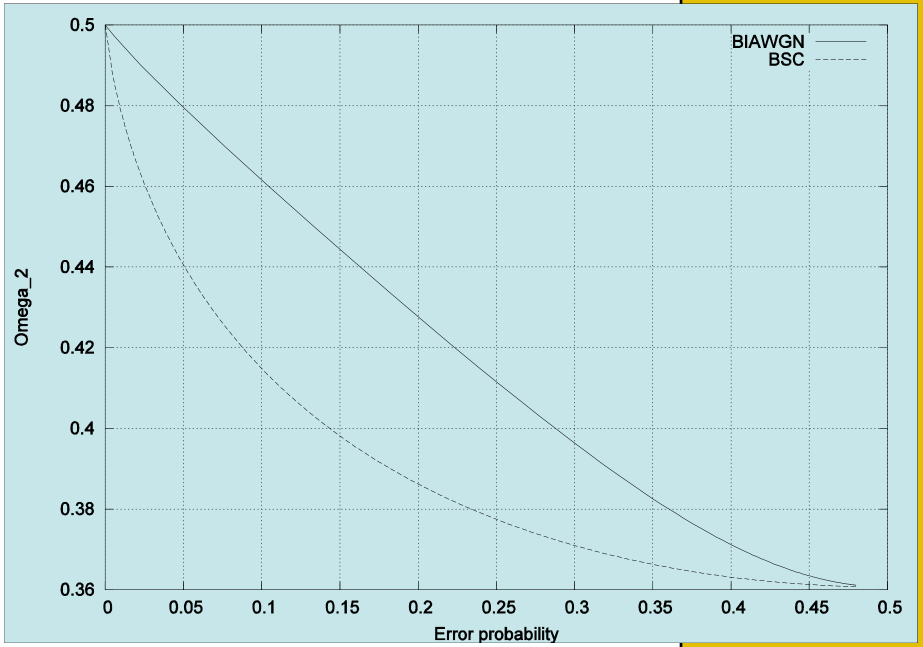
where, in general

$$\Pi(\mathcal{C}) := \frac{\mathrm{Cap}(\mathcal{C})}{\mathrm{E}[\tanh(Z/2)]}$$

and Z is the LLR of the channel.

$\Omega_2(\mathcal{C})$

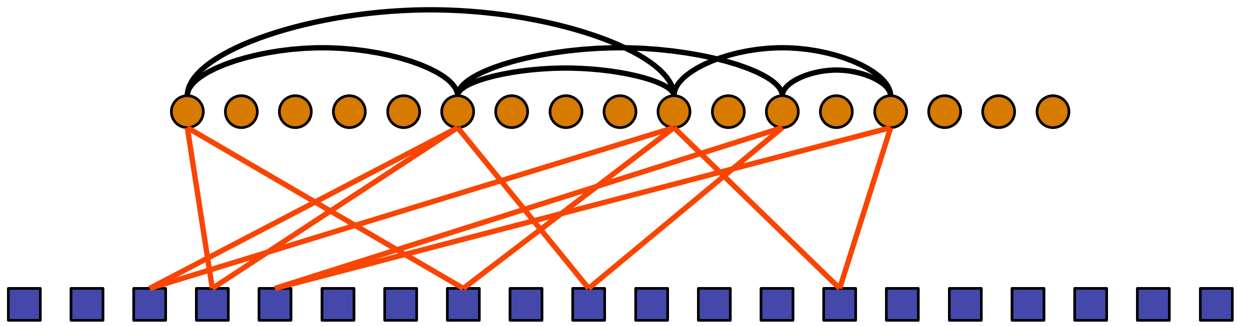
$$\Omega_2(\mathrm{BEC}(p)) = \frac{1}{2}$$



$$\Omega_2(\mathrm{BSC}(p)) = \frac{1 - h(p)}{2(1-2p)^2}$$

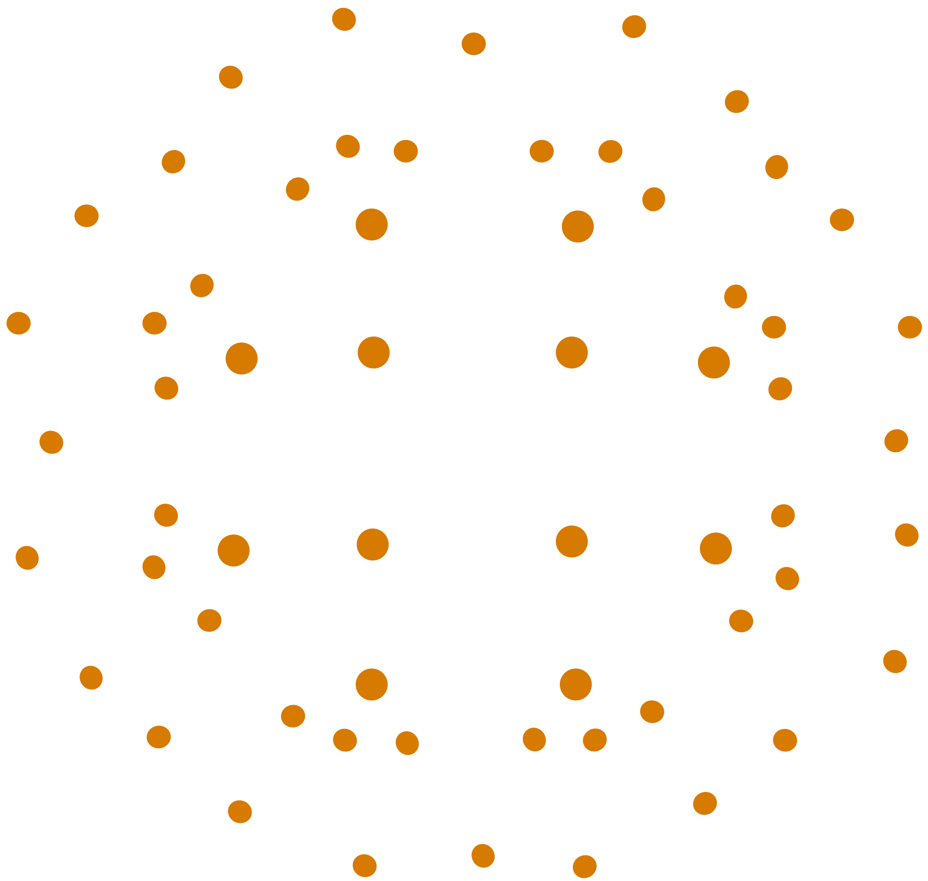
$$\Omega_2(\mathrm{BIAWGN}(\sigma)) \;=\; \frac{1}{4\ln(2)}(1 + O(m)).$$

Nodes of Degree 2



Use graph induced on input symbols by output symbols of degree 2.

Nodes of Degree 2: BEC



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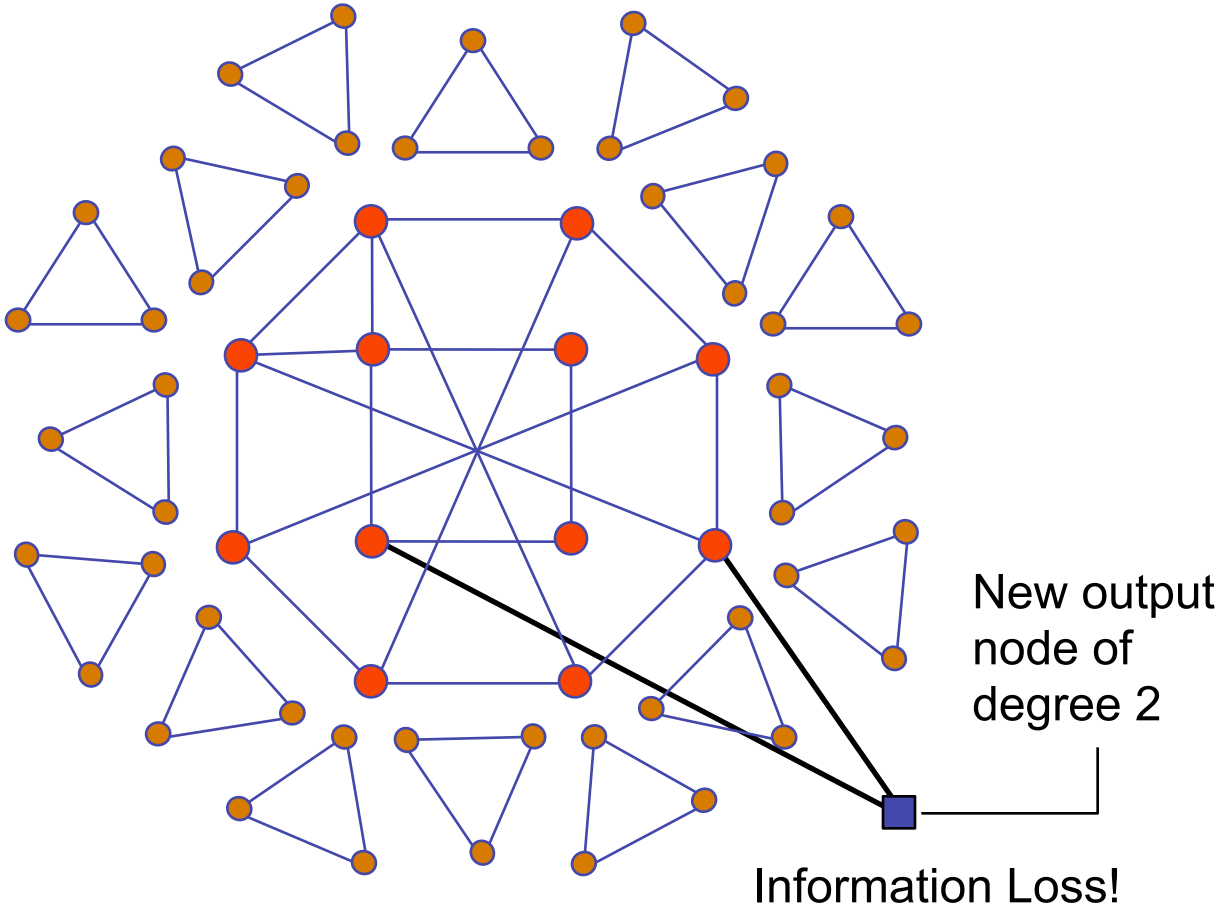
Nodes of Degree 2: BEC



Fraction of Nodes of Degree 2

If there exists component of linear size (i.e., a giant component), then next output node of degree 2 has constant probability of being useless.

Therefore, graph should not have giant component.

This means that for capacity achieving degree distributions we must have: $\Omega_2 \leq \frac{1}{2}$.

On the other hand, if $\Omega_2 < \frac{1}{2}$ then algorithm cannot start successfully.

So, $\Omega_2 = \frac{1}{2}$ for capacity-achieving codes:

General Symmetric Channels: Mimic Proof

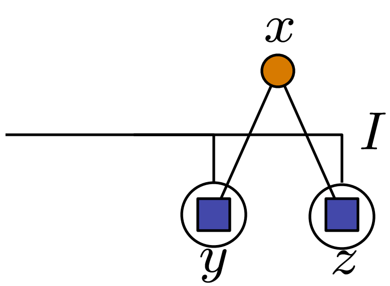
Proof is information theoretic: if fraction of nodes of degree 2 is larger by a constant, then :

- Expectation of the hyperbolic tangent of messages passed from input to output symbols at given round of BP is larger than a constant.
- This shows that $I(x; z_2) < n\Omega_2(\mathrm{Cap}(\mathcal{C}) - \tau)$
- So code cannot achieve capacity.

General Symmetric Channels: Mimic Proof

Fraction of nodes of degree one for capacity-achieving Raptor Codes:

Noisy observations of x



$$I(x; y, z) < I(x; y) + I(x; z)$$

Therefore, if $\Omega_1 > 0$, and if $z_1, \ldots, z_m$ denote output nodes of degree one, then

$$I(x; z_1, \ldots, z_m) < \sum_{i=1}^{m} \mathrm{Cap}(\mathcal{C}) - \eta$$

So $\Omega_1 = 0$

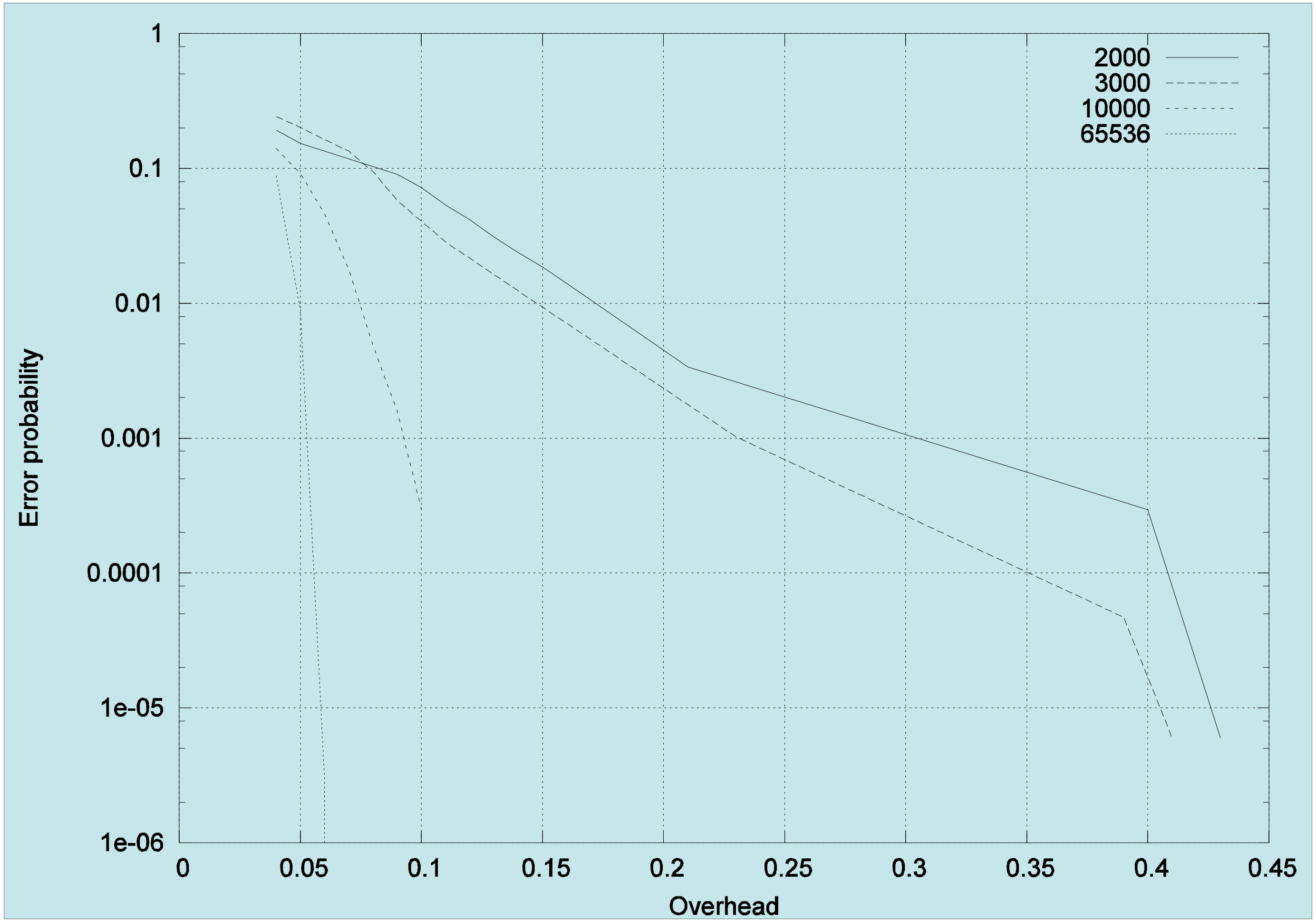
A better Gaussian Approximation

Uses adaptation of a method of Ardakani and Kschischang.

Heuristic: Messages passed from input to output symbols are Gaussian, but not vice-versa.

Find recursion for the means of these Gaussians, and apply linear programming.

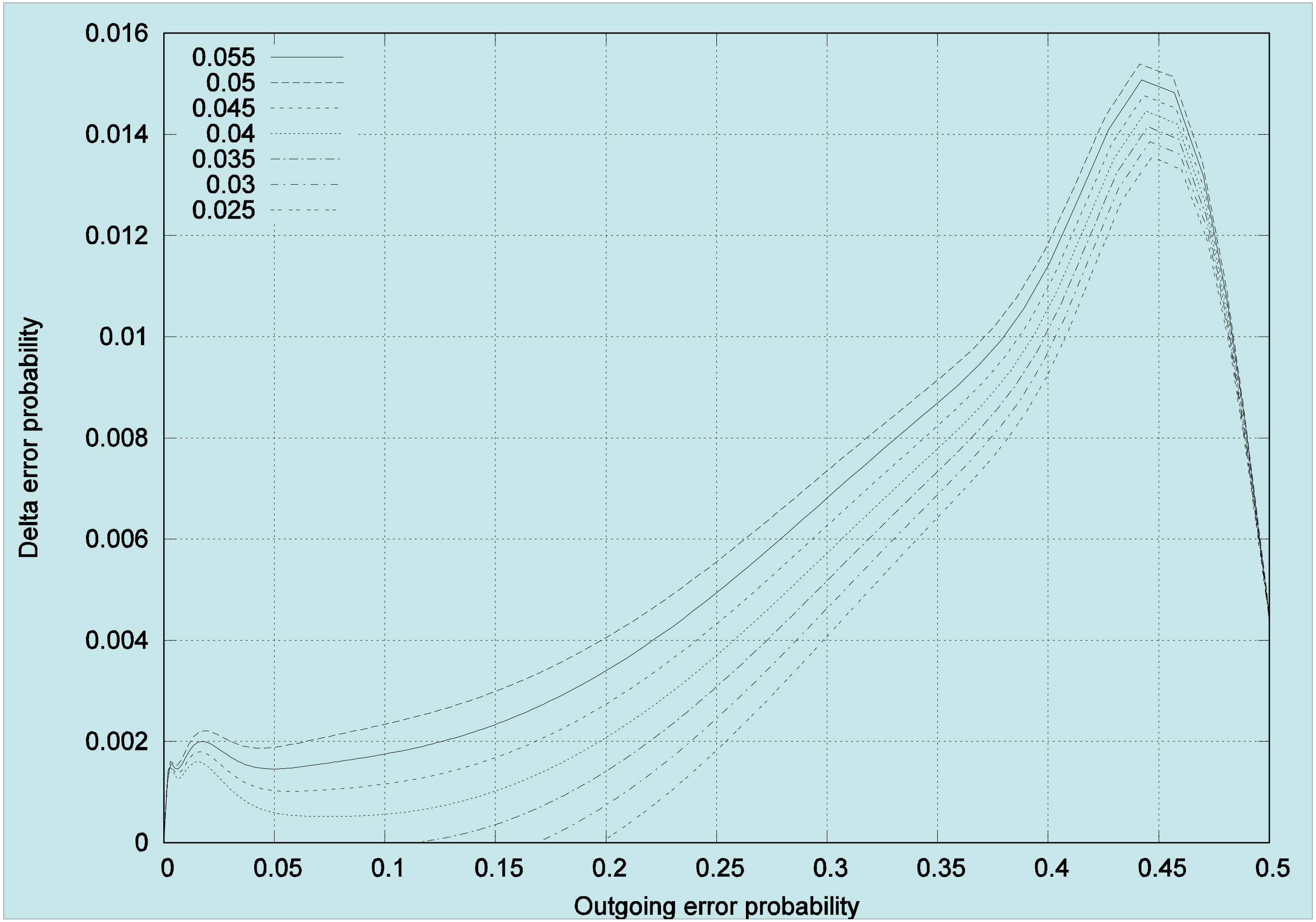
$$\begin{aligned}\Omega(x) &= 0.006x + 0.492x^2 + 0.0339x^3 + 0.2403x^4 + \\ &\quad 0.2403x^5 + 0.095x^8 + 0.049x^{14} + 0.018x^{30} + \\ &\quad 0.036x^{33} + 0.033x^{200}\end{aligned}$$



1
0.1
0.01
0.001
0.0001
1e-05
1e-06
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
Error probability
Overhead
2000
3000
10000
65536

$\sigma = 0.5$

Density Evolution



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Conclusions

- Raptor Codes can be adapted to general symmetric channels using the Belief Propagation algorithm.
- Raptor Codes designed for the BEC are not bad on other channels, but their performance can be improved.
- There are no universal codes on channels other than the BEC, as the fraction of degree 2 nodes in capacity-achieving codes depends on the channel noise.
- General design techniques can be adapted to design good Raptor Codes that perform very well under a variety of channel conditions.