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Rate Constitutive Theories of Orders n and ^1n for Internal Polar Non-Classical Thermofluids without Memory

Karan S. Surana[1], Stephen W. Long[1], J. N. Reddy[2]

[1]Department of Mechanical Engineering, University of Kansas, Lawrence, USA
[2]Department of Mechanical Engineering, Texas A & M University, College Station, USA
Email: kssurana@ku.edu

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Abstract

In recent papers, Surana *et al.* presented internal polar non-classical Continuum theory in which velocity gradient tensor in its entirety was incorporated in the conservation and balance laws. Thus, this theory incorporated symmetric part of the velocity gradient tensor (as done in classical theories) as well as skew symmetric part representing varying internal rotation rates between material points which when resisted by deforming continua result in dissipation (and/or storage) of mechanical work. This physics referred as internal polar physics is neglected in classical continuum theories but can be quite significant for some materials. In another recent paper Surana *et al.* presented ordered rate constitutive theories for internal polar non-classical fluent continua without memory derived using deviatoric Cauchy stress tensor and conjugate strain rate tensors of up to orders n and Cauchy moment tensor and its conjugate symmetric part of the first convected derivative of the rotation gradient tensor. In this constitutive theory higher order convected derivatives of the symmetric part of the rotation gradient tensor are assumed not to contribute to dissipation. Secondly, the skew symmetric part of the velocity gradient tensor is used as rotation rates to determine rate of rotation gradient tensor. This is an approximation to true convected time derivatives of the rotation gradient tensor. The resulting constitutive theory: (1) is incomplete as it neglects the second and higher order convected time derivatives of the symmetric part of the rotation gradient tensor; (2) first convected derivative of the symmetric part of the rotation gradient tensor as used by Surana *et al.* is only approximate; (3) has inconsistent treatment of dissipation due to Cauchy moment tensor when compared with the dissipation mechanism due to deviatoric part of symmetric Cauchy stress tensor in which convected time derivatives of up to order n are considered in the theory. The purpose of this paper is to present ordered rate constitutive theories for deviatoric Cauchy strain tensor, moment tensor and heat vector for thermofluids without memory in which convected time derivatives of strain tensors up to order n are conjugate with the Cauchy stress tensor and the convected time derivatives of the symmetric part of the rotation gradient tensor up to orders ${}^1n$ are conjugate with the moment tensor. Conservation and balance laws are used to determine the choice of dependent variables in the constitutive theories: Helmholtz free energy density Φ, entropy density η, Cauchy stress tensor, moment tensor and heat vector. Stress tensor is decomposed into symmetric and skew symmetric parts and the symmetric part of the stress tensor and the moment tensor are further decomposed into equilibrium and deviatoric tensors. It is established through conjugate pairs in entropy inequality that the constitutive theories only need to be derived for symmetric stress tensor, moment tensor and heat vector. Density in the current configuration, convected time derivatives of the strain tensor up to order n, convected time derivatives of the symmetric part of the rotation gradient tensor up to orders ${}^1n$, temperature gradient tensor and temperature are considered as argument tensors of all dependent variables in the constitutive theories based on entropy inequality and principle of equipresence. The constitutive theories are derived in contravariant and covariant bases as well as using Jaumann rates. The n^{th} and ${}^1n^{\text{th}}$ order rate constitutive theories for internal polar non-classical thermofluids without memory are specialized for $n = 1$ and ${}^1n = 1$ to demonstrate fundamental differences in the constitutive theories presented here and those used presently for classical thermofluids without memory and those published by Surana *et al.* for internal polar non-classical incompressible thermofluids.

Keywords

Rate Constitutive Theories, Non-Classical Thermofluids, Without Memory, Convected Time Derivatives, Internal Rotation Gradient Tensor, Generators and Invariants, Cauchy Moment Tensor

1. Introduction

Conservation and balance laws: conservation of mass, balance of linear momenta, balance of angular momenta, balance of moments of moments (or couples), first law of thermodynamics (energy equation) and second law of thermodynamics (entropy inequality) for internal polar non-classical fluent continua were presented in references [1] [2]. A summary of these was also presented in reference [3] in which Surana *et al.* also presented constitutive theories for internal polar non-classical thermofluids without memory that incorporated convected time derivatives of strain tensor up to order n, density, rate of the symmetric part of the rotation gradient tensor, temperature gradient tensor and temperature as argument tensors of the dependent variables in the constitutive theories at the onset of the derivation. In references [1] [2] [3] comprehensive literature was presented regarding various aspects of non-classical theories that were pertinent in context with internal polar non-classical continuum theory used here for fluent continua. For the sake of brevity these are not repeated here instead interested readers can see references [1] [2] [3].

Another significant discussion in references [1] [2] [3] is the discussion of mathematical description for fluent continua. It was established that in fluent continua one monitors the state of the matter at fixed locations, hence mathematical models describing such processes do not contain information regarding displacements therefore these descriptions can neither be Lagrangian nor Eulerian. Nonetheless since the fixed locations are occupied by different material particles during evolution, the fixed location can be viewed as current positions of some material particle during evolution. This thinking persuades one to believe that the mathematical descriptions used for fluent continua are Eulerian descriptions. This thinking is not contested in this paper, but is rather used as this approach is what is used for mathematical descriptions of fluent continua.

The notations used in this paper have been used by the authors in the current literature, nonetheless some description and their use in deriving conservation and balance laws are presented in the following. Over bar is used on quantities to express quantities in the current configuration in Eulerian description, that is, all quantities with over bars are functions of current coordinates $\bar{x}_i$ and time *t*. $\bar{\rho}$ is the density of the fluid in the current configuration and is a function of $\bar{x}$ and $\bar{\Phi}$, $\bar{\theta}$, and $\bar{\eta}$ denote the Helmholtz free-energy density, temperature, and entropy density, respectively in the current configuration and are also functions of $\bar{x}$. $\bar{\boldsymbol{\sigma}}^{(0)}$ is the Cauchy stress tensor (in Eulerian description in contravariant basis). The superscript "0" is used to signify that it is rate of order zero and the lowercase parenthesis destinguish it from the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ used in Lagrangian description. Dot on any quantity refers to the material derivative. As explained above undeformed and deformed configurations can be used in the derivatives as long as the final equations from the conservation and balance laws contain $\bar{x}_i$ and *t* and do not have displacements and strains in them as these are not available for fluent continua. In the following a brief explanation of notations is necessary as some of the notations are new. x_i and $\bar{x}_i$ denote the position coordinates of a material point in the reference and current configurations, respectively, in a fixed frame (*x*-frame)

$$\bar{x}_i = \bar{x}_i\left(x_1, x_2, x_3, t\right) \tag{1}$$

or

$$x_i = x_i\left(\bar{x}_1, \bar{x}_2, \bar{x}_3, t\right) \tag{2}$$

If $\{dx\} = \left[dx_1, dx_2, dx_3\right]^{\mathrm{T}}$ and $\{d\bar{x}\} = \left[d\bar{x}_1, d\bar{x}_2, d\bar{x}_3\right]^{\mathrm{T}}$ are the components of length ds and $d\bar{s}$ in the reference and current configurations, and if one neglects the infinitesimals of orders two and higher in both configurations, then one obtains

$$\{d\bar{x}\} = \left[J_{(0)}\right]\{dx\} \tag{3}$$

$$\{dx\} = \left[\bar{J}^{(0)}\right]\{d\bar{x}\} \tag{4}$$

with

$$\left[J_{(0)}\right]=\left[\bar{J}^{(0)}\right]^{-1};\quad \left[\bar{J}^{(0)}\right]=\left[J_{(0)}\right]^{-1};\quad \left[J_{(0)}\right]\left[\bar{J}^{(0)}\right]=\left[\bar{J}^{(0)}\right]\left[J_{(0)}\right]=[I] \tag{5}$$

In Murnaghan's notation

$$\left[J_{(0)}\right]=\left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right]=\left[\frac{\bar{x}_1,\bar{x}_2,\bar{x}_3}{x_1,x_2,x_3}\right];\quad \left[\bar{J}^{(0)}\right]=\left[\frac{\partial\{x\}}{\partial\{\bar{x}\}}\right]=\left[\frac{x_1,x_2,x_3}{\bar{x}_1,\bar{x}_2,\bar{x}_3}\right] \tag{6}$$

in which *the columns of* $\left[J_{(0)}\right]$ *are covariant base vectors* $\tilde{\boldsymbol{g}}_i$ *, whereas the rows of* $\left[\bar{J}^{(0)}\right]$ *are contravariant base vectors* $\tilde{\boldsymbol{g}}^i$ [4]. $\left[J_{(0)}\right]$ and $\left[\bar{J}^{(0)}\right]$ are Jacobians of deformation in covariant and contravariant bases. Furthermore $\left[J_{(0)}\right]$ is Lagrangian description while $\left[\bar{J}^{(0)}\right]$ is Eulerian description. The basis defined by $\left[\bar{J}^{(0)}\right]$ is reciprocal to the basis defined by $\left[J_{(0)}\right]$. The following relations are useful in the paper:

$$\frac{D}{Dt}\left[J_{(0)}\right]=\left[\bar{L}\right]\left[J_{(0)}\right] \tag{7}$$

$$\frac{D}{Dt}\left[\bar{J}^{(0)}\right]=-\left[\bar{J}^{(0)}\right]\left[\bar{L}\right] \tag{8}$$

where

$$\bar{L}_{ij}=\frac{\partial\bar{v}_i}{\partial\bar{x}_j} \tag{9}$$

in which $\frac{D}{Dt}$ stands for material time derivative, $\left[\bar{L}\right]$ is the spatial velocity gradient tensor, and $\bar{v}_i$ are velocity components of a material point $\bar{x}_i$ in the current configuration in the x-frame. Over bar on all dependent quantities refers to their Eulerian description, *i.e.* they are functions of $\bar{x}_i$ and t whereas the quantities without over bar are their Lagrangian description *i.e.* they are functions of x_i and t. Thus $\bar{Q}(\bar{\boldsymbol{x}},t)$ and $Q(\boldsymbol{x},t)$ are Eulerian and Lagrangian description of a quantity Q in the current configuration.

The purpose of this paper is to present ordered rate constitutive theories for deviatoric Cauchy strain tensor, Cauchy moment tensor and heat vector for thermofluids without memory in which convected time derivatives of strain tensors up to order n are conjugate with the Cauchy stress tensor and the convected time derivatives of the symmetric part of the rotation gradient tensor up to orders ${}^1n$ are conjugate with the Cauchy moment tensor.

2. Rotation Gradients, Their Convected Time Derivatives and Conservation and Balance Laws

In reference [1] [2] conservation and balance laws were derived for internal polar (non-classical) fluent continua. The derivations were presented using contravariant and covariant measures of stress, moment tensors as well as using Jaumann rates. Measures of stress, moment and strain tensors and their convected time derivatives in the respective bases can be considered. Following references [3] [4] for example $\left(\bar{\boldsymbol{\sigma}}^{(0)},\bar{\boldsymbol{\sigma}}_{(0)},{}^{(0)}\bar{\boldsymbol{\sigma}}^{J}\right)$, $\left(\bar{\boldsymbol{m}}^{(0)},\bar{\boldsymbol{m}}_{(0)},{}^{(0)}\bar{\boldsymbol{m}}^{J}\right)$ can be considered as measures of Cauchy stress and Cauchy moment tensors in contravariant and covariant bases and corresponding to Jaumann rates.

Likewise one can let $\left[\gamma^{(k)}\right]$, $\left[\gamma_{(k)}\right]$, $\left[{}^{(k)}\gamma^{J}\right]$; $k=0,1,\cdots,n$ be the convected time derivatives of Almansi, Green's strain tensors and the Jaumann rates. Where, $\left[\gamma^{(0)}\right]=\left[\gamma^{(1)}\right]=\left[\gamma_{(0)}\right]=\left[\gamma_{(1)}\right]=\left[{}^{(0)}\gamma^{J}\right]=\left[{}^{(1)}\gamma^{J}\right]=\left[\bar{D}\right]$, symmetric part of the velocity gradient tensor. Let ${}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, ${}^{(k)}\boldsymbol{\gamma}$; $k=0,1,\cdots,n$ define Cauchy stress tensor, Cauchy moment tensor, and convected time derivatives of the conjugate strain tensor in a chosen basis. Derivations of the constitutive theories is presented using this notation so that the resulting derivations are basis independent. By replacing

$${}^{(0)}\bar{\boldsymbol{\sigma}},\ {}^{(0)}\bar{\boldsymbol{m}},\ {}^{(k)}\boldsymbol{\gamma};\ k=0,1,\cdots,n \ \text{ with } (\bar{\boldsymbol{\sigma}}^{(0)},\bar{\boldsymbol{m}}^{(0)},\boldsymbol{\gamma}_{(k)};\ k=0,1,\cdots,n),$$

$$(\bar{\boldsymbol{\sigma}}_{(0)},\bar{\boldsymbol{m}}_{(0)},\boldsymbol{\gamma}^{(k)};\ k=0,1,\cdots,n),\ \text{and } ({}^{(0)}\bar{\boldsymbol{\sigma}}^{J},\ {}^{(0)}\bar{\boldsymbol{m}}^{J},\ {}^{(k)}\boldsymbol{\gamma}^{J};\ k=0,1,\cdots,n),$$

the constitutive theories in contravariant basis, covariant basis and in Jaumann rates can be obtained. In addition to the convected derivatives of the strain tensors one must also consider convected derivatives of the rotation gradient tensor that are also basis dependent. In reference [3] the authors show that Cauchy moment tensor and symmetric part of the gradient of the rate of rotation tensor are conjugate. In reference [3], the authors considered symmetric part of the gradients of rates of rotation obtained using skew symmetric part of the velocity gradient tensor. One notes that the Cauchy moment tensor is basis dependent: $\bar{\boldsymbol{m}}^{(0)},\bar{\boldsymbol{m}}_{(0)},{}^{(0)}\bar{\boldsymbol{m}}^{J}$ being moment tensors in contravariant basis, covariant basis, and in Jaumann rates. Thus, the convected time derivatives of the symmetric part of the rotation gradient tensor in general must also be basis dependent. Let

$$(\left[{}_{\Theta}\gamma^{(k)}\right];\ k=1,2,\cdots,{}^{1}n),\ (\left[{}_{\Theta}\gamma_{(k)}\right];\ k=1,2,\cdots,{}^{1}n)\ \text{and}\ (\left[{}^{(k)}_{\Theta}\gamma^{J}\right];\ k=1,2,\cdots,{}^{1}n)$$

be the convected time derivatives of the rotation gradient tensors in contravariant basis, covariant basis, and Jaumann rates. With these convected time derivatives, the conjugate pairs are $(\bar{\boldsymbol{m}}^{(0)},\left[{}_{\Theta}\gamma_{(k)}\right];\ k=1,2,\cdots,{}^{1}n)$, $(\bar{\boldsymbol{m}}_{(0)},\left[{}_{\Theta}\gamma^{(k)}\right];\ k=1,2,\cdots,{}^{1}n)$ and $({}^{(0)}\bar{\boldsymbol{m}}^{J},\left[{}^{(k)}_{\Theta}\gamma^{J}\right];\ k=1,2,\cdots,{}^{1}n)$ in contravariant and covariant bases and in Jaumann rates. Covariant and contravariant bases are important in conservation and balance laws as well as constitutive theories. Jacobian of deformation $\boldsymbol{J}_{(0)}=\boldsymbol{e}_i\otimes\boldsymbol{e}_j\dfrac{\partial\bar{x}_j}{\partial x_i}$ or $\left[J_{(0)}\right]=\left[\dfrac{\partial\{\bar{x}\}}{\partial\{x\}}\right]$ is Lagrangian description. Columns of $\left[J_{(0)}\right]$ are covariant base vectors [4]. Thus, quantities derived using $\boldsymbol{J}_{(0)}$ are in covariant basis and are Lagrangian descriptions. Likewise $\bar{\boldsymbol{J}}^{(0)}=\boldsymbol{e}_i\otimes\boldsymbol{e}_j\dfrac{\partial x_j}{\partial\bar{x}_i}$ or $\left[\bar{J}^{(0)}\right]=\left[\dfrac{\partial\{x\}}{\partial\{\bar{x}\}}\right]$ is Eulerian description of Jacobian of deformation. Rows of $\left[\bar{J}^{(0)}\right]$ are contravariant base vectors. Hence, quantities derived using $\bar{J}^{(0)}$ are in contravariant basis and are Eulerian descriptions. The convected time derivatives of the rotation gradient tensors in covariant and contravariant bases must be derived using rotation gradient tensor obtained using $\boldsymbol{J}_{(0)}$ and $\bar{\boldsymbol{J}}^{(0)}$. Details are presented in the following.

In finite deformation, a tetrahedron in the undeformed configuration with its orthogonal edges deforms into one in which the edges are non-orthogonal covariant base vectors and the vectors normal to the faces of the deformed tetrahedron are contravariant non-orthogonal base vectors that are reciprocal to the covariant base vectors. The covariant and contravariant bases are fundamental in the measures of finite deformation, rotations, etc. Consider deformed coordinates $\bar{\boldsymbol{x}}$ of a material point in the current configuration with undeformed coordinates $\boldsymbol{x}$ in the reference configuration. Then

$$\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{x} = \boldsymbol{x}(\bar{\boldsymbol{x}}) \tag{10}$$

2.1. Covariant Basis: Internal Rotations, Rotation Matrix, Rotation Gradient Tensor and Their Convected Time Derivatives

(a) Internal rotations and rotation matrix

Consider decomposition of the Jacobian of deformation $\boldsymbol{J}_{(0)}$ into symmetric and skew-symmetric tensors.

$$\left[J_{(0)}\right] = \left[\frac{\partial\{\bar{x}\}}{\partial\{x\}}\right] = \left[{}_sJ_{(0)}\right] + \left[{}_aJ_{(0)}\right] \tag{11}$$

$$\left[{}_sJ_{(0)}\right] = \frac{1}{2}\left(\left[J_{(0)}\right] + \left[J_{(0)}\right]^{\mathrm{T}}\right) \tag{12}$$

$$\left[{}_aJ_{(0)}\right] = \frac{1}{2}\left(\left[J_{(0)}\right] - \left[J_{(0)}\right]^{\mathrm{T}}\right) \tag{13}$$

Let $\left\{\Theta_{(0)}\right\} = \left[\left(\Theta_{x_1}\right)_{(0)}, \left(\Theta_{x_2}\right)_{(0)}, \left(\Theta_{x_3}\right)_{(0)}\right]^{\mathrm{T}}$ be the components of the rotations expressed as rotations about ox_1, ox_2, and ox_3 axes of the *x*-frame, then one can write

$$\left[{}_aJ_{(0)}\right] = \begin{bmatrix} 0 & \left(\Theta_{x_3}\right)_{(0)} & -\left(\Theta_{x_2}\right)_{(0)} \\ -\left(\Theta_{x_3}\right)_{(0)} & 0 & \left(\Theta_{x_1}\right)_{(0)} \\ \left(\Theta_{x_2}\right)_{(0)} & -\left(\Theta_{x_1}\right)_{(0)} & 0 \end{bmatrix} \tag{14}$$

in which

$$\left(\Theta_{x_1}\right)_{(0)} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right); \quad \left(\Theta_{x_2}\right)_{(0)} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right); \quad \left(\Theta_{x_3}\right)_{(0)} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tag{15}$$

Alternatively one can also derive (15) as follows.

$$\nabla \times \boldsymbol{u} = \boldsymbol{e}_i \times \boldsymbol{e}_j \frac{\partial u_j}{\partial x_i} = \epsilon_{ijk} \boldsymbol{e}_k \frac{\partial u_j}{\partial x_i} \tag{16}$$

$$\nabla \times \boldsymbol{u} = \boldsymbol{e}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \boldsymbol{e}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \boldsymbol{e}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{17}$$

$$\nabla \times \boldsymbol{u} = \boldsymbol{e}_1\left(-2\left(\Theta_{x_1}\right)_{(0)}\right) + \boldsymbol{e}_2\left(-2\left(\Theta_{x_2}\right)_{(0)}\right) + \boldsymbol{e}_3\left(-2\left(\Theta_{x_3}\right)_{(0)}\right) \tag{18}$$

ϵ_{ijk} is the permutation tensor.

The sign differences between (15) and (18) are due to clockwise and counterclockwise internal rotations and will only affect sign of $\bar{\boldsymbol{M}}$ term in the balance of angular momenta. If one uses (15) as the definition of rotations then the term containing $\bar{\boldsymbol{M}}$ in the balance of angular momenta must have negative sign. If the rotations in (18) are defined as $\left(\Theta_{x_1}\right)_{(0)}$, $\left(\Theta_{x_2}\right)_{(0)}$, and $\left(\Theta_{x_3}\right)_{(0)}$ then the term containing $\bar{\boldsymbol{M}}$ in the balance of angular momenta must have positive sign. Regardless, the resulting equations and the following derivations are not affected. One notes that decomposition in (11) enables explicit description of stretches and rotations contained in $\boldsymbol{J}_{(0)}$ due to deformation of solid matter. The stretch tensor and the rotation tensor can also be obtained using polar decomposition of $\boldsymbol{J}_{(0)}$ into right stretch tensor $\left(\boldsymbol{S}_r\right)_{(0)}$ or left stretch tensor $\left(\boldsymbol{S}_l\right)_{(0)}$ and pure rotation tensor $\boldsymbol{R}_{(0)}$ [1] [2] [3] [4].

$$\left[J_{(0)}\right]=\left[R_{(0)}\right]\left[\left(S_r\right)_{(0)}\right]=\left[\left(S_l\right)_{(0)}\right]\left[R_{(0)}\right] \tag{19}$$

The stretch tensors $\left(\boldsymbol{S}_r\right)_{(0)}$ and $\left(\boldsymbol{S}_l\right)_{(0)}$ are symmetric and positive-definite and the rotation tensor $\boldsymbol{R}_{(0)}$ is orthogonal. Since $\boldsymbol{R}_{(0)}$ in (19) and $\boldsymbol{\Theta}_{(0)}$ in (15) are both obtained from the same deformation in $\boldsymbol{J}_{(0)}$, these contain details of the same internal rotation physics but in different forms. One may make the following remarks.

1) $\boldsymbol{R}_{(0)}$ is rotation matrix, hence relates undeformed orthogonal frame to a new orthogonal rotated frame (due to deformation).

2) $\boldsymbol{\Theta}_{(0)}$ on the other hand contains rotation angles due to deformation about the axes of the *x*-frame.

3) Determination of $\boldsymbol{\Theta}_{(0)}$ from $\boldsymbol{R}_{(0)}$ or determination of $\boldsymbol{R}_{(0)}$ from $\boldsymbol{\Theta}_{(0)}$ is not necessary. Two different mathematical forms of rotation physics in $\boldsymbol{R}_{(0)}$ and $\boldsymbol{\Theta}_{(0)}$ is sufficient. However, this process of obtaining $\boldsymbol{\Theta}_{(0)}$ from $\boldsymbol{R}_{(0)}$ or $\boldsymbol{R}_{(0)}$ from $\boldsymbol{\Theta}_{(0)}$ in general is not unique and may not even be possible without some approximation [5] [6] [7] [8].

4) It suffices to note that internal rotations at a material point present in $\boldsymbol{J}_{(0)}$ can be expressed either in $\boldsymbol{R}_{(0)}$ or in $\boldsymbol{\Theta}_{(0)}$. Both forms contain mathematical description of same physics, hence either can be used as deemed suitable, but determination of $\boldsymbol{\Theta}_{(0)}$ from $\boldsymbol{R}_{(0)}$ or $\boldsymbol{R}_{(0)}$ from $\boldsymbol{\Theta}_{(0)}$ is not necessary.

5) The internal rotation angles $\boldsymbol{\Theta}_{(0)}$ are present at every material point and are a result of deformation. Between two neighboring material points the variation of $\boldsymbol{\Theta}_{(0)}$ is perhaps small otherwise there may be permanent damage or separation between them. Regardless of the magnitude of $\boldsymbol{\Theta}_{(0)}$, these are strictly deterministic from ${}_a\boldsymbol{J}_{(0)}$, $\nabla\times\boldsymbol{u}$, or the polar decomposition.

(b) Internal rotation gradient tensor and its rates using $\boldsymbol{J}_{(0)}$

The covariant internal rotation tensor ${}_a\boldsymbol{J}_{(0)}$ is a tensor of rank two, hence alternatively one can write

$${}_a\boldsymbol{J}_{(0)}=\boldsymbol{e}_i\otimes\boldsymbol{e}_j\,\frac{1}{2}\left(\frac{\partial\bar{x}_j}{\partial x_i}-\frac{\partial\bar{x}_i}{\partial x_j}\right) \tag{20}$$

Let ${}^{\Theta}\boldsymbol{J}_{(0)}$ be the internal rotation gradient tensor, a tensor of rank three. Using (20) one can define

$$ {}^{\Theta}\boldsymbol{J}_{(0)} = \boldsymbol{e}_k \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{1}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{x}_j}{\partial x_i} - \frac{\partial \overline{x}_i}{\partial x_j}\right) \tag{21} $$

Alternatively (16) can be written as

$$ {}_a\boldsymbol{J}_{(0)} = \epsilon_{ijl}\boldsymbol{e}_l \frac{1}{2}\left(\frac{\partial \overline{x}_j}{\partial x_i} - \frac{\partial \overline{x}_i}{\partial x_j}\right) \tag{22} $$

and then

$$ {}^{\Theta}\boldsymbol{J}_{(0)} = \boldsymbol{e}_k \otimes \boldsymbol{e}_l \epsilon_{ijl} \frac{1}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{x}_j}{\partial x_i} - \frac{\partial \overline{x}_i}{\partial x_j}\right) \tag{23} $$

In (22) the internal rotations ${}_a\boldsymbol{J}_{(0)}$ are expressed as a tensor of rank one (*i.e.* $\left(\Theta_{x_1}\right)_{(0)}$, $\left(\Theta_{x_2}\right)_{(0)}$, $\left(\Theta_{x_3}\right)_{(0)}$ as a vector), hence its gradient ${}^{\Theta}\boldsymbol{J}_{(0)}$ in (23) appears as a tensor of rank two. The representation (22) is more appealing for matrix and vector forms given in the following. Let

$$ \left\{\Theta_{(0)}\right\} = \left[\left(\Theta_{x_1}\right)_{(0)}, \left(\Theta_{x_2}\right)_{(0)}, \left(\Theta_{x_3}\right)_{(0)}\right]^{\mathrm{T}} \tag{24} $$

Then, one defines rotation gradient tensor ${}^{\Theta}\boldsymbol{J}_{(0)}$ and its decomposition into symmetric and skew-symmetric tensors ${}^{\Theta}_{s}\boldsymbol{J}_{(0)}$ and ${}^{\Theta}_{a}\boldsymbol{J}_{(0)}$.

$$ \left[{}^{\Theta}J_{(0)}\right] = \left[\frac{\partial\left\{\Theta_{(0)}\right\}}{\partial\{x\}}\right] = \left[{}^{\Theta}_{s}J_{(0)}\right] + \left[{}^{\Theta}_{a}J_{(0)}\right] \tag{25} $$

$$ \left[{}^{\Theta}_{s}J_{(0)}\right] = \frac{1}{2}\left(\left[{}^{\Theta}J_{(0)}\right] + \left[{}^{\Theta}J_{(0)}\right]^{\mathrm{T}}\right) \tag{26} $$

$$ \left[{}^{\Theta}_{a}J_{(0)}\right] = \frac{1}{2}\left(\left[{}^{\Theta}J_{(0)}\right] - \left[{}^{\Theta}J_{(0)}\right]^{\mathrm{T}}\right) \tag{27} $$

One can also define the velocity gradients as

$$ \frac{\partial\{v\}}{\partial\{x\}} = \left[L_{(0)}\right] = \left[D_{(0)}\right] + \left[W_{(0)}\right] \tag{28} $$

in which

$$ \left[D_{(0)}\right] = \frac{1}{2}\left(\left[L_{(0)}\right] + \left[L_{(0)}\right]^{\mathrm{T}}\right) \tag{29} $$

$$ \left[W_{(0)}\right] = \frac{1}{2}\left(\left[L_{(0)}\right] - \left[L_{(0)}\right]^{\mathrm{T}}\right) \tag{30} $$

Likewise if ${}_t\boldsymbol{\Theta}_{(0)}$ or $\dot{\boldsymbol{\Theta}}_{(0)}$ is the rotation rate then its gradients are given by

$$ \frac{\partial\left\{{}_t\Theta_{(0)}\right\}}{\partial\{x\}} = \left[{}^{\Theta}L_{(0)}\right] = \left[{}^{\Theta}D_{(0)}\right] + \left[{}^{\Theta}W_{(0)}\right] \tag{31} $$

$$ \left[{}^{\Theta}D_{(0)}\right] = \frac{1}{2}\left(\left[{}^{\Theta}L_{(0)}\right] + \left[{}^{\Theta}L_{(0)}\right]^{\mathrm{T}}\right) \tag{32} $$

$$\left[{}^{\Theta}W_{(0)}\right] = \frac{1}{2}\left(\left[{}^{\Theta}L_{(0)}\right] - \left[{}^{\Theta}L_{(0)}\right]^{\mathrm{T}}\right) \tag{33}$$

Remarks

1) Symmetric rotation gradient tensor in (26) is a covariant measure in Lagrangian description. It describes symmetric part of the gradients in x-frame of rotations about covariant axes expressed about the axes of the x-frame.

2) Since this measure is covariant rotation rate its work conjugate measure will be contravariant.

3) The covariant nature of this measure is intrinsic in its derivation due to $\boldsymbol{J}_{(0)}$, hence can not be changed. However, by replacing $\boldsymbol{J}_{(0)}$ with $\bar{\boldsymbol{J}}_{(0)}^{-1}$ these measures can be converted to Eulerian description.

4) Convected time derivatives of ${}^{\Theta}_{s}\boldsymbol{J}_{(0)}$ of orders up to ${}^{1}n$ are defined as $\left[{}_{\Theta}\gamma_{(k)}\right]; k = 1, 2, \cdots, {}^{1}n$.

(c) Second Piola-Kirchhoff covariant rotation gradient tensor

Consider isotropic, homogeneous, compressible matter. Let $d\bar{A}_n$ and dA_n be the scalar and $\{d\bar{A}\}$ and $\{dA\}$ be the vector areas of the oblique planes of the deformed and the undeformed tetrahedra. Let the resultant rotation gradient vector acting on these areas be $\{dF_{\Theta_g}\}$ and $\{d\bar{F}_{\Theta_g}\}$ and the average rotation gradient vectors on these areas be $\{P_{\Theta_g}\}$ and $\{\bar{P}_{\Theta_g}\}$. Let the covariant Cauchy rotation gradient tensor be $\left[{}^{\Theta}J_{(0)}\right]$ acting on the deformed tetrahedron and let $\left[{}^{\Theta}J_{[0]}\right]$ be the corresponding second Piola-Kirchhoff covariant rotation gradient tensor acting on the undeformed tetrahedron. Consider the following correspondence rule [4].

$$\{d\bar{F}_{\Theta_g}\} = \left[\bar{J}\right]^{\mathrm{T}}\{dF_{\Theta_g}\} \tag{34}$$

$$\{d\bar{A}\} = |J|[J]\{dA\} \tag{35}$$

Thus, one obtains

$$\{dF_{\Theta_g}\} = dA_n\{P_{\Theta_g}\} = dA_n\left[{}^{\Theta}J_{[0]}\right]^{\mathrm{T}}\{n\} = \left[{}^{\Theta}J_{[0]}\right]^{\mathrm{T}}\{dA\} \tag{36}$$

$$\{d\bar{F}_{\Theta_g}\} = d\bar{A}_n\{\bar{P}_{\Theta_g}\} = d\bar{A}_n\left[{}^{\Theta}\bar{J}_{(0)}\right]^{\mathrm{T}}\{\bar{n}\} = \left[{}^{\Theta}\bar{J}_{(0)}\right]^{\mathrm{T}}\{d\bar{A}\} \tag{37}$$

using (36) and (37) in (34) one obtains

$$\left[{}^{\Theta}\bar{J}_{(0)}\right]^{\mathrm{T}}\{d\bar{A}\} = \left[\bar{J}\right]^{\mathrm{T}}\left[{}^{\Theta}J_{[0]}\right]^{\mathrm{T}}\{dA\} \tag{38}$$

using (35) in (38)

$$\left[{}^{\Theta}J_{(0)}\right]^{\mathrm{T}}|J|[J]\{dA\} = \left[\bar{J}\right]^{\mathrm{T}}\left[{}^{\Theta}J_{[0]}\right]^{\mathrm{T}}\{dA\} \tag{39}$$

hence, one obtains

$$\left.\begin{aligned} \left[{}^{\Theta}J_{[0]}\right]^{\mathrm{T}} &= |J|[J]^{\mathrm{T}}\left[{}^{\Theta}J_{(0)}\right]^{\mathrm{T}}[J] \\ \text{or}\quad \left[{}^{\Theta}J_{[0]}\right] &= |J|[J]^{\mathrm{T}}\left[{}^{\Theta}J_{(0)}\right][J] \end{aligned}\right\} \tag{40}$$

and

$$\left.\begin{aligned} \left[{}^{\Theta}\bar{J}_{[0]}\right]^{\mathrm{T}} &= \left|\left[\bar{J}\right]^{-1}\right| \left[\left[\bar{J}\right]^{-1}\right]^{\mathrm{T}} \left[{}^{\Theta}\bar{J}_{(0)}\right]^{\mathrm{T}} \left[\bar{J}\right]^{-1} \\ \text{or} \quad \left[{}^{\Theta}\bar{J}_{[0]}\right] &= \left|\left[\bar{J}\right]^{-1}\right| \left[\left[\bar{J}\right]^{-1}\right]^{\mathrm{T}} \left[{}^{\Theta}\bar{J}_{(0)}\right] \left[\bar{J}\right]^{-1} \end{aligned}\right\} \tag{41}$$

Equations (40) and (41) are Lagrangian and Eulerian descriptions for second Piola-Kirchhoff covariant rotation gradient tensor. These are useful in deriving covariant convected time derivatives of the rotation gradient tensor $\left[{}^{\Theta}\bar{J}_{(0)}\right]$. For incompressible fluent continua $|J| = \left|\left[\bar{J}^{-1}\right]\right| = 1$, hence for this case (40) and (41) can be modified.

(d) Convected time derivatives of the covariant rotation gradient tensor: compressible matter

In this section derivation of convected time derivative of the covariant rotation gradient tensor $\left[{}^{\Theta}\bar{J}_{(0)}\right]$ for compressible matter is presented. Consider

$$\left[{}^{\Theta}\bar{J}_{[0]}\right]^{\mathrm{T}} = |J| [J]^{\mathrm{T}} \left[{}^{\Theta}\bar{J}_{(0)}\right]^{\mathrm{T}} [J] \tag{42}$$

One intentionally chooses Eulerian description for Cauchy and second Piola-Kirchhoff tensor as this is what is needed in the case of mathematical model for fluent continua. Consider material derivative of $\left[{}^{\Theta}\bar{J}_{[0]}\right]$

$$\begin{aligned} \frac{D}{Dt}\left[{}^{\Theta}\bar{J}_{[0]}\right] &= \frac{D}{Dt}\left(|J| [J]^{\mathrm{T}} \left[{}^{\Theta}\bar{J}_{(0)}\right][J]\right) \\ &= \frac{D}{Dt}(|J|)[J]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right][J] + |J|\frac{D}{Dt}\left([J]^{\mathrm{T}}\right)\left[{}^{\Theta}\bar{J}_{(0)}\right][J] \\ &\quad + |J|[J]^{\mathrm{T}}\frac{D}{Dt}\left(\left[{}^{\Theta}\bar{J}_{(0)}\right]\right)[J] + |J|[J]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right]\frac{D}{Dt}([J]) \end{aligned} \tag{43}$$

using

$$\frac{D}{Dt}([J]) = \left[\bar{L}\right][J] \tag{44}$$

$$\frac{D}{Dt}(|J|) = |J| \operatorname{tr}\left(\left[\bar{L}\right]\right) \tag{45}$$

in (43), factoring and regrouping, one can write

$$\begin{aligned} &\frac{D}{Dt}\left[{}^{\Theta}\bar{J}_{[0]}\right] \\ &= |J|[J]^{T}\left(\frac{D}{Dt}\left[{}^{\Theta}\bar{J}_{(0)}\right] + \left[\bar{L}\right]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right] + \left[{}^{\Theta}\bar{J}_{(0)}\right]\left[\bar{L}\right] + \left[{}^{\Theta}\bar{J}_{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right)\right)[J] \end{aligned} \tag{46}$$

If one defines

$$\frac{D}{Dt}\left[{}^{\Theta}\bar{J}_{[0]}\right] = \left[{}^{\Theta}\bar{J}_{[1]}\right] = \left[{}_{\Theta}\gamma_{[1]}\right] \quad (\text{Def.}) \tag{47}$$

$$\left[{}_{\Theta}\gamma_{(1)}\right] = \frac{D}{Dt}\left[{}^{\Theta}\bar{J}_{(0)}\right] + \left[\bar{L}\right]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right] + \left[{}^{\Theta}\bar{J}_{(0)}\right]\left[\bar{L}\right] + \left[{}^{\Theta}\bar{J}_{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right) \quad (\text{Def.}) \tag{48}$$

then one obtains the following from (46)

$$\left[{}_{\Theta}\gamma_{[1]}\right] = |J|[J]^{\mathrm{T}}\left[{}_{\Theta}\gamma_{(1)}\right][J] \tag{49}$$

$\left[{}_{\Theta}\gamma_{(1)}\right]$ is the first convected time derivative of the covariant rotation gradient tensor $\left[{}^{\Theta}\bar{J}_{(0)}\right]$ for compressible matter. To obtain the second convected time derivative of the covariant rotation gradient tensor one can take material derivative of (49), and following the same steps as in case of $\left[{}^{\Theta}\bar{J}_{[1]}\right]$ one obtains:

$$\left[{}_{\Theta}\gamma_{[2]}\right] = \frac{D}{Dt}\left[{}_{\Theta}\gamma_{[1]}\right] = |J|\left[J\right]^{\mathrm{T}}\left[{}_{\Theta}\gamma_{(2)}\right]\left[J\right] \tag{50}$$

where

$$\left[{}_{\Theta}\gamma_{(2)}\right] = \frac{D}{Dt}\left[{}_{\Theta}\gamma_{(1)}\right] + \left[\bar{L}\right]^{\mathrm{T}}\left[{}_{\Theta}\gamma_{(1)}\right] + \left[{}_{\Theta}\gamma_{(1)}\right]\left[\bar{L}\right] + \left[{}_{\Theta}\gamma_{(1)}\right]\mathrm{tr}\left(\left[\bar{L}\right]\right) \tag{51}$$

In general one can write the following recursive relations that can be used to obtain the convected time derivative of any desired order k of the covariant rotation gradient tensor $\left[{}^{\Theta}\bar{J}_{(0)}\right]$ for compressible continua.

$$\left.\begin{aligned} &\frac{D}{Dt}\left[{}_{\Theta}\gamma_{[k-1]}\right] = \left[{}_{\Theta}\gamma_{[k]}\right] \\ &\left[{}_{\Theta}\gamma_{(k)}\right] = |J|\left[J\right]^{\mathrm{T}}\left[{}_{\Theta}\gamma_{(k)}\right]\left[J\right] \\ &\left[{}_{\Theta}\gamma_{(k)}\right] = \frac{D}{Dt}\left[{}_{\Theta}\gamma_{(k-1)}\right] + \left[\bar{L}\right]^{\mathrm{T}}\left[{}_{\Theta}\gamma_{(k-1)}\right] + \left[{}_{\Theta}\gamma_{(k-1)}\right]\left[\bar{L}\right] + \left[{}_{\Theta}\gamma_{(k-1)}\right]\mathrm{tr}\left(\left[\bar{L}\right]\right) \end{aligned}\right\} k = 1, 2, \cdots, {}^{1}n \tag{52}$$

For incompressible case $|J| = 1$ and $\mathrm{tr}\left(\left[\bar{L}\right]\right) = 0$ in (52).

2.2. Contravariant Basis: Internal Rotations, Rotation Matrix, Rotation Gradient Tensor and Their Convected Time Derivatives

(a) Internal rotations and rotation matrix

Following the derivations for covariant measures, one can derive the following if one considers Jacobian of deformation $\bar{\boldsymbol{J}}^{(0)}$ in contravariant basis. Rows of $\bar{\boldsymbol{J}}^{(0)}$ are contravariant base vectors. Consider decomposition of $\bar{\boldsymbol{J}}^{(0)}$ into symmetric and skew-symmetric tensors.

$$\left[\bar{J}^{(0)}\right] = \left[\frac{\partial\{x\}}{\partial\{\bar{x}\}}\right] = \left[{}_{s}\bar{J}^{(0)}\right] + \left[{}_{a}\bar{J}^{(0)}\right] \tag{53}$$

$$\left[{}_{s}\bar{J}^{(0)}\right] = \frac{1}{2}\left(\left[\bar{J}^{(0)}\right] + \left[\bar{J}^{(0)}\right]^{\mathrm{T}}\right) \tag{54}$$

$$\left[{}_{a}\bar{J}^{(0)}\right] = \frac{1}{2}\left(\left[\bar{J}^{(0)}\right] - \left[\bar{J}^{(0)}\right]^{\mathrm{T}}\right) \tag{55}$$

Let $\left\{\bar{\Theta}^{(0)}\right\} = \left[\bar{\Theta}^{(0)}_{x_1}, \bar{\Theta}^{(0)}_{x_2}, \bar{\Theta}^{(0)}_{x_3}\right]^{\mathrm{T}}$ be the components of the rotations about ox_1, ox_2, and ox_3 axes of the x-frame, then one can write

$$\left[{}_{a}\bar{J}^{(0)}\right] = \begin{bmatrix} 0 & \bar{\Theta}^{(0)}_{x_3} & -\bar{\Theta}^{(0)}_{x_2} \\ -\bar{\Theta}^{(0)}_{x_3} & 0 & \bar{\Theta}^{(0)}_{x_1} \\ \bar{\Theta}^{(0)}_{x_2} & -\bar{\Theta}^{(0)}_{x_1} & 0 \end{bmatrix} \tag{56}$$

in which

$$\bar{\Theta}^{(0)}_{x_1} = \frac{1}{2}\left(\frac{\partial\bar{u}_2}{\partial\bar{x}_3} - \frac{\partial\bar{u}_3}{\partial\bar{x}_2}\right);\quad \bar{\Theta}^{(0)}_{x_2} = \frac{1}{2}\left(\frac{\partial\bar{u}_3}{\partial\bar{x}_1} - \frac{\partial\bar{u}_1}{\partial\bar{x}_3}\right);\quad \bar{\Theta}^{(0)}_{x_3} = \frac{1}{2}\left(\frac{\partial\bar{u}_1}{\partial\bar{x}_2} - \frac{\partial\bar{u}_2}{\partial\bar{x}_1}\right) \tag{57}$$

Alternatively one can also derive (57) as follows.

$$\bar{\nabla}\times\bar{\boldsymbol{u}}=\boldsymbol{e}_i\times\boldsymbol{e}_j\frac{\partial\bar{u}_j}{\partial\bar{x}_i}=\epsilon_{ijk}\boldsymbol{e}_k\frac{\partial\bar{u}_j}{\partial\bar{x}_i} \tag{58}$$

$$\bar{\nabla}\times\bar{\boldsymbol{u}}=\boldsymbol{e}_1\left(\frac{\partial\bar{u}_3}{\partial\bar{x}_2}-\frac{\partial\bar{u}_2}{\partial\bar{x}_3}\right)+\boldsymbol{e}_2\left(\frac{\partial\bar{u}_1}{\partial\bar{x}_3}-\frac{\partial\bar{u}_3}{\partial\bar{x}_1}\right)+\boldsymbol{e}_3\left(\frac{\partial\bar{u}_2}{\partial\bar{x}_1}-\frac{\partial\bar{u}_1}{\partial\bar{x}_2}\right) \tag{59}$$

$$\bar{\nabla}\times\bar{\boldsymbol{u}}=\boldsymbol{e}_1\left(-2\bar{\Theta}_{x_1}^{(0)}\right)+\boldsymbol{e}_2\left(-2\bar{\Theta}_{x_2}^{(0)}\right)+\boldsymbol{e}_3\left(-2\bar{\Theta}_{x_3}^{(0)}\right) \tag{60}$$

The reason for the sign difference in (57) and (60) is exactly same as for covariant measures. One notes that decomposition (53) enables explicit description of stretches (elongation per unit length and change in angles between the pair of orthogonal material lines in the undeformed configuration) and rotation tensor contained in $\bar{\boldsymbol{J}}^{(0)}$. The stretch tensors and the rotation tensor can also be obtained using polar decomposition of $\bar{\boldsymbol{J}}^{(0)}$ into right stretch tensor $\bar{\boldsymbol{S}}_r^{(0)}$ or left stretch tensor $\bar{\boldsymbol{S}}_l^{(0)}$ and rotation tensor $\bar{\boldsymbol{R}}^{(0)}$ [1] [2] [3] [4].

$$\left[\bar{J}^{(0)}\right]=\left[\bar{R}^{(0)}\right]\left[\bar{S}_r^{(0)}\right]=\left[\bar{S}_l^{(0)}\right]\left[\bar{R}^{(0)}\right] \tag{61}$$

The stretch tensors $\bar{\boldsymbol{S}}_r^{(0)}$ and $\bar{\boldsymbol{S}}_l^{(0)}$ are symmetric and positive-definite and the rotation tensor $\bar{\boldsymbol{R}}^{(0)}$ is orthogonal. Since $\bar{\boldsymbol{R}}^{(0)}$ in (61) and $\bar{\boldsymbol{\Theta}}^{(0)}$ in (57) are both obtained from the same deformation in $\bar{\boldsymbol{J}}^{(0)}$, these contain details of the same internal rotation physics but in different forms. The following remarks parallel to those for covariant measures can be made.

1) $\bar{\boldsymbol{R}}^{(0)}$ is rotation matrix due to deformation, hence relates two orthogonal frames.

2) $\bar{\boldsymbol{\Theta}}^{(0)}$ on the other hand contains rotation angles due to deformation about the axes of the x-frame due to rotations about contravariant axes.

3) One notes that determination of $\bar{\boldsymbol{\Theta}}^{(0)}$ from $\bar{\boldsymbol{R}}^{(0)}$ or determination of $\bar{\boldsymbol{R}}^{(0)}$ from $\bar{\boldsymbol{\Theta}}^{(0)}$ is not necessary. Two different mathematical forms of rotation physics is sufficient in derivation of the conservation and balance laws. However, this process of obtaining $\bar{\boldsymbol{\Theta}}^{(0)}$ from $\bar{\boldsymbol{R}}^{(0)}$ or $\bar{\boldsymbol{R}}^{(0)}$ from $\bar{\boldsymbol{\Theta}}^{(0)}$ in general is not unique and may not even be possible without some approximation [5] [6] [7] [8].

4) It suffices to note that internal rotations at a material point present in $\bar{\boldsymbol{J}}^{(0)}$ can be expressed either in $\bar{\boldsymbol{R}}^{(0)}$ or in $\bar{\boldsymbol{\Theta}}^{(0)}$. Both forms contain mathematical description of same physics, hence either can be used as deemed suitable, but determination of $\bar{\boldsymbol{\Theta}}^{(0)}$ from $\bar{\boldsymbol{R}}^{(0)}$ or $\bar{\boldsymbol{R}}^{(0)}$ from $\bar{\boldsymbol{\Theta}}^{(0)}$ is not necessary.

5) The internal rotation angles $\bar{\boldsymbol{\Theta}}^{(0)}$ are present at every material point and are a result of deformation. Between two neighboring material points the variation of $\bar{\boldsymbol{\Theta}}^{(0)}$ is perhaps small otherwise there may be permanent damage or separation between them. Regardless of the magnitude of $\bar{\boldsymbol{\Theta}}^{(0)}$, these are strictly deterministic from ${}_a\bar{\boldsymbol{J}}^{(0)}$, $\bar{\nabla}\times\bar{\boldsymbol{u}}$, or the polar decomposition.

(b) Internal rotation gradient tensor using $\bar{\boldsymbol{J}}^{(0)}$

The contravariant internal rotation tensor ${}_a\bar{\boldsymbol{J}}^{(0)}$ is a tensor of rank two, hence alternatively one can define

$$ {}_{a}\bar{\boldsymbol{J}}^{(0)} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{1}{2}\left(\frac{\partial x_j}{\partial \bar{x}_i} - \frac{\partial x_i}{\partial \bar{x}_j} \right) \tag{62} $$

Let ${}^{\Theta}\bar{\boldsymbol{J}}^{(0)}$ be the internal rotation gradient tensor, a tensor of rank three. Using (62) one can define

$$ {}^{\Theta}\bar{\boldsymbol{J}}^{(0)} = \boldsymbol{e}_k \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \frac{1}{2}\frac{\partial}{\partial \bar{x}_k}\left(\frac{\partial x_j}{\partial \bar{x}_i} - \frac{\partial x_i}{\partial \bar{x}_j} \right) \tag{63} $$

Alternatively (62) can be written as

$$ {}_{a}\bar{\boldsymbol{J}}^{(0)} = \epsilon_{ijl}\boldsymbol{e}_l \frac{1}{2}\left(\frac{\partial x_j}{\partial \bar{x}_i} - \frac{\partial x_i}{\partial \bar{x}_j} \right) \tag{64} $$

and then

$$ {}^{\Theta}\bar{\boldsymbol{J}}^{(0)} = \boldsymbol{e}_k \otimes \boldsymbol{e}_l \epsilon_{ijl} \frac{1}{2}\frac{\partial}{\partial \bar{x}_k}\left(\frac{\partial x_j}{\partial \bar{x}_i} - \frac{\partial x_i}{\partial \bar{x}_j} \right) \tag{65} $$

In (64) the internal rotations ${}_{a}\bar{\boldsymbol{J}}^{(0)}$ are expressed as a tensor of rank one (*i.e.* $\bar{\Theta}^{(0)}_{x_1}$, $\bar{\Theta}^{(0)}_{x_2}$, $\bar{\Theta}^{(0)}_{x_3}$ as a vector), hence its gradient ${}^{\Theta}\bar{\boldsymbol{J}}^{(0)}$ in (65) appears as a tensor of rank two. The representation (64) is more appealing for matrix and vector representations given in the following. Let

$$ \{\bar{\Theta}\} = \left[\bar{\Theta}^{(0)}_{x_1}, \bar{\Theta}^{(0)}_{x_2}, \bar{\Theta}^{(0)}_{x_3} \right]^{\mathrm{T}} \tag{66} $$

Then the rotation gradient tensor ${}^{\Theta}\bar{\boldsymbol{J}}^{(0)}$ and its decomposition into symmetric and skew-symmetric tensors ${}^{\Theta}_{s}\bar{\boldsymbol{J}}^{(0)}$ and ${}^{\Theta}_{a}\bar{\boldsymbol{J}}^{(0)}$ are defined as:

$$ \left[{}^{\Theta}\bar{J}^{(0)} \right] = \left[\frac{\partial \{\bar{\Theta}^{(0)}\}}{\partial \{\bar{x}\}} \right] = \left[{}^{\Theta}_{s}\bar{J}^{(0)} \right] + \left[{}^{\Theta}_{a}\bar{J}^{(0)} \right] \tag{67} $$

$$ \left[{}^{\Theta}_{s}\bar{J}^{(0)} \right] = \frac{1}{2}\left(\left[{}^{\Theta}\bar{J}^{(0)} \right] + \left[{}^{\Theta}\bar{J}^{(0)} \right]^{\mathrm{T}} \right) \tag{68} $$

$$ \left[{}^{\Theta}_{a}\bar{J} \right] = \frac{1}{2}\left(\left[{}^{\Theta}\bar{J}^{(0)} \right] - \left[{}^{\Theta}\bar{J}^{(0)} \right]^{\mathrm{T}} \right) \tag{69} $$

Remarks

1) Symmetric rotation gradient tensor in (67) is a contravariant measure in Eulerian description. It describes symmetric part of the gradients of rotations about contravariant axes expressed about the axes of the *x*-frame.

2) Since this measure is contravariant its work conjugate moment measure is expected to be covariant (see derivation of first law of thermodynamics).

3) Contravariant nature of this measure is intrinsic in its derivation, hence can not be changed. However by replacing $\bar{\boldsymbol{J}}^{(0)}$ with $\boldsymbol{J}^{-1}_{(0)}$, these measures will become Lagrangian descriptions.

4) Convected time derivatives of ${}^{\Theta}_{s}\bar{\boldsymbol{J}}^{(0)}$ of orders up to ${}^{1}n$ in contravariant basis are defined as $\left[{}_{\Theta}\gamma^{(k)} \right]; k = 1, 2, \cdots, {}^{1}n$.

(c) Second Piola-Kirchhoff contravariant rotation gradient tensor

Consider isotropic, homogeneous compressible matter. Consider oblique planes of the deformed and the undeformed tetrahedra with scalar areas $d\bar{A}_n$ and dA_n and vector areas $\{d\bar{A}\}$ and $\{dA\}$. Let the resultant rotation gradient vector acting on these areas be $\{dF_{\Theta_g}\}$ and $\{d\bar{F}_{\Theta_g}\}$ and the average rotation gradient vectors on these areas be $\{P_{\Theta_g}\}$ and $\{\bar{P}_{\Theta_g}\}$.

$$\{d\bar{F}_{\Theta_g}\} = [J]\{dF_{\Theta_g}\} \tag{70}$$

Let the contravariant Cauchy rotation gradient tensor be $\left[{}^{\Theta}\bar{J}^{(0)}\right]$ acting on the deformed tetrahedron and let $\left[{}^{\Theta}J^{[0]}\right]$ be the corresponding second Piola-Kirchhoff covariant rotation gradient tensor acting on the faces of the undeformed tetrahedron derived from $\left[{}^{\Theta}\bar{J}^{(0)}\right]$ using correspondence rule (70). Then one can write

$$\{dF_{\Theta_g}\} = dA_n\{P_{\Theta_g}\} = dA_n\left[{}^{\Theta}J^{[0]}\right]^{\mathrm{T}}\{n\} = \left[{}^{\Theta}J^{[0]}\right]^{\mathrm{T}}\{dA\} \tag{71}$$

$$\{d\bar{F}_{\Theta_g}\} = d\bar{A}_n\{\bar{P}_{\Theta_g}\} = d\bar{A}_n\left[{}^{\Theta}\bar{J}^{(0)}\right]^{\mathrm{T}}\{\bar{n}\} = \left[{}^{\Theta}\bar{J}^{(0)}\right]^{\mathrm{T}}\{d\bar{A}\} \tag{72}$$

Substituting (71) and (72) in (70)

$$\left[{}^{\Theta}\bar{J}^{(0)}\right]^{\mathrm{T}}\{d\bar{A}\} = [J]\left[{}^{\Theta}J^{[0]}\right]^{\mathrm{T}}\{dA\} \tag{73}$$

using

$$\{d\bar{A}\} = |J|\left[[J]^{\mathrm{T}}\right]^{-1}\{dA\} \tag{74}$$

$$\left[{}^{\Theta}J^{(0)}\right]^{\mathrm{T}}|J|\left[[J]^{\mathrm{T}}\right]^{-1}\{dA\} = [J]\left[{}^{\Theta}J^{[0]}\right]^{\mathrm{T}}\{dA\} \tag{75}$$

Hence, one obtains

$$\left.\begin{aligned} \left[{}^{\Theta}J^{[0]}\right]^{\mathrm{T}} &= |J|[J]^{-1}\left[{}^{\Theta}J^{(0)}\right]^{\mathrm{T}}\left[[J]^{\mathrm{T}}\right]^{-1} \\ \text{or}\quad \left[{}^{\Theta}J^{[0]}\right] &= |J|[J]^{-1}\left[{}^{\Theta}J^{(0)}\right]\left[[J]^{\mathrm{T}}\right]^{-1} \end{aligned}\right\} \tag{76}$$

and

$$\left.\begin{aligned} \left[{}^{\Theta}\bar{J}^{[0]}\right]^{\mathrm{T}} &= |J|[\bar{J}]\left[{}^{\Theta}\bar{J}^{(0)}\right]^{\mathrm{T}}[\bar{J}]^{\mathrm{T}} \\ \text{or}\quad \left[{}^{\Theta}\bar{J}^{[0]}\right] &= |J|[\bar{J}]\left[{}^{\Theta}\bar{J}^{(0)}\right][\bar{J}]^{\mathrm{T}} \end{aligned}\right\} \tag{77}$$

Also $|J|$ in (77) can be replaced by $\left|[\bar{J}]^{-1}\right|$ if so desired. Equations (76) and (77) define contravariant second Piola-Kirchhoff rotation gradient tensor in Lagrangian and Eulerian descriptions. For incompressible fluent continua $|J| = \left|[\bar{J}]^{-1}\right| = 1$, hence (76) and (77) can be modified for this case.

(d) Convected time derivatives of the contravariant rotation gradient tensor: compressible matter

Consider the material derivative of $\left[{}^{\Theta}\bar{J}^{[0]}\right]$, the second Piola-Kirchhoff rotation gradient tensor (Equation (77)) derived using contravariant Cauchy rotation gradient tensor.

$$\frac{D}{Dt}\left[{}^{\Theta}\bar{J}^{[0]}\right]=\frac{D}{Dt}\left(|J|\left[\bar{J}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{J}\right]^{\mathrm{T}}\right)$$
$$=\frac{D}{Dt}(|J|)\left[\bar{J}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{J}\right]^{\mathrm{T}}+|J|\frac{D}{Dt}\left(\left[\bar{J}\right]\right)\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{J}\right]^{\mathrm{T}} \tag{78}$$
$$+|J|\left[\bar{J}\right]\frac{D}{Dt}\left(\left[{}^{\Theta}\bar{J}^{(0)}\right]\right)\left[\bar{J}\right]^{\mathrm{T}}+|J|\left[\bar{J}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]\frac{D}{Dt}\left(\left[\bar{J}\right]^{\mathrm{T}}\right)$$

using

$$\frac{D}{Dt}\left(\left[\bar{J}\right]\right)=-\left[\bar{J}\right]\left[\bar{L}\right] \tag{79}$$

$$\frac{D}{Dt}(|J|)=|J|\,\mathrm{tr}\left(\left[\bar{L}\right]\right) \tag{80}$$

and regrouping the terms one obtains

$$\frac{D}{Dt}\left[{}^{\Theta}J^{[0]}\right]$$
$$=|J|\left[\bar{J}\right]\left(\frac{D}{Dt}\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[\bar{L}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}^{\Theta}\bar{J}^{(0)}\right]\mathrm{tr}\left(\left[\bar{L}\right]\right)\right)\left[\bar{J}\right]^{\mathrm{T}} \tag{81}$$

If one defines

$$\frac{D}{Dt}\left[{}^{\Theta}\bar{J}^{[0]}\right]=\left[{}^{\Theta}\bar{J}^{[1]}\right]=\left[{}_{\Theta}\gamma^{[1]}\right]\quad(\text{Def.}) \tag{82}$$

$$\left[{}_{\Theta}\gamma^{(1)}\right]=\frac{D}{Dt}\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[\bar{L}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}^{\Theta}\bar{J}^{(0)}\right]\mathrm{tr}\left(\left[\bar{L}\right]\right)\quad(\text{Def.}) \tag{83}$$

then one can write

$$\left[{}_{\Theta}\gamma^{[1]}\right]=|J|\left[\bar{J}\right]\left[{}_{\Theta}\gamma^{(1)}\right]\left[\bar{J}\right]^{\mathrm{T}} \tag{84}$$

Here $\left[{}_{\Theta}\gamma^{(1)}\right]$ is the first convected time derivative of the contravariant Cauchy rotation gradient tensor $\left[{}^{\Theta}\bar{J}^{(0)}\right]$ for compressible matter. To obtain the second convected time derivative of the contravariant Cauchy rotation gradient tensor $\left[{}^{\Theta}\bar{J}^{(0)}\right]$, one takes material derivative of (84) and follows the same steps as in case of $\left[{}_{\Theta}\gamma^{(1)}\right]$, then one obtains the following:

$$\left[{}_{\Theta}\gamma^{[2]}\right]=\frac{D}{Dt}\left[{}_{\Theta}\gamma^{[1]}\right]=|J|\left[\bar{J}\right]\left[{}_{\Theta}\gamma^{(2)}\right]\left[\bar{J}\right]^{\mathrm{T}} \tag{85}$$

where

$$\left[{}_{\Theta}\gamma^{(2)}\right]=\frac{D}{Dt}\left[{}_{\Theta}\gamma^{(1)}\right]-\left[\bar{L}\right]\left[{}_{\Theta}\gamma^{(1)}\right]-\left[{}_{\Theta}\gamma^{(1)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}_{\Theta}\gamma^{(1)}\right]\mathrm{tr}\left(\left[\bar{L}\right]\right) \tag{86}$$

In general one can write the following recursive relation that can be used to obtain the convected time derivative up to any desired order k of the tensor $\left[{}^{\Theta}\bar{J}^{(0)}\right]$ for compressible matter.

$$\left.\begin{aligned}&\frac{D}{Dt}\left[{}_{\Theta}\gamma^{[k-1]}\right]=\left[{}_{\Theta}\gamma^{[k]}\right]\\&\left[{}_{\Theta}\gamma^{[k]}\right]=|J|\left[\bar{J}\right]\left[{}_{\Theta}\gamma^{(k)}\right]\left[\bar{J}\right]^{\mathrm{T}}\\&\left[{}_{\Theta}\gamma^{(k)}\right]=\frac{D}{Dt}\left[{}_{\Theta}\gamma^{(k-1)}\right]-\left[\bar{L}\right]\left[{}_{\Theta}\gamma^{(k-1)}\right]-\left[{}_{\Theta}\gamma^{(k-1)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}_{\Theta}\gamma^{(k-1)}\right]\mathrm{tr}\left(\left[\bar{L}\right]\right)\end{aligned}\right\}k=1,2,\cdots,{}^{1}n \tag{87}$$

For incompressible case $|J|=1$ and $\operatorname{tr}([\bar{L}])=0$, hence the expressions for the convected time derivatives can be modified for this case.

It is advantageous to introduce basis independent notations so that the derivations of conservation and balance laws could be carried out independent of the basis. These can then be made basis dependent by simply replacing the basis independent quantities. Similar to Cauchy stress tensor and Cauchy moment tensor, introduce ${}^{(k)}_{\Theta}\gamma; k=1,2,\cdots,{}^1n$ as basis independent convected time derivatives of the rotation gradient tensor. By choosing ${}^{(k)}_{\Theta}\gamma; k=1,2,\cdots,{}^1n$ to be ${}_{\Theta}\gamma^{(k)}$ or ${}_{\Theta}\gamma_{(k)}$ or ${}^{(k)}_{\Theta}\gamma^{J}; k=1,2,\cdots,{}^1n$ one can obtain convected time derivatives of the rotation gradient tensor in contravariant basis, covariant basis and in Jaumann rates.

2.3. Polar Decomposition of Velocity Gradient Tensor and Consideration of Local Rotation Rates

Polar decomposition of the velocity gradient tensor is helpful in decomposing deformation into stretch rate tensor and rotation rate tensor. Whether one uses left stretch rate tensor or right stretch rate tensor, the rotation rate tensor is unique. Thus, at each location with infinitesimal volume surrounding it, the velocity gradient tensor $[\bar{L}]$ can be decomposed into pure rates of rotation $[{}^t\bar{R}]$ and right or left stretch rate tensors $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$. $[{}^t\bar{R}]$ is orthogonal and $[{}^t\bar{S}_r]$ and $[{}^t\bar{S}_l]$ are symmetric and positive definite. The rotation rate tensor can equivalently be obtained due to rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ at each location in the flow domain. Thus, at each location in the flow domain the rotation rate $[{}^t\bar{R}]$ matrix can be viewed as being due to ${}^t\bar{\boldsymbol{\Theta}}$. If varying rotation rates at varying locations in the flow domain are resisted by the constitution of the fluent continua then this must result in additional dissipation that requires existence of energy conjugate moments $\bar{\boldsymbol{M}}$ in the deforming matter. Thus, at the onset ${}^t\bar{\boldsymbol{\Theta}}$ and its conjugate $\bar{\boldsymbol{M}}$ are considered in the derivation of the polar continuum theory for the fluent continua. Details of polar decomposition of $[\bar{L}]$ and rotation rates ${}^t\bar{\boldsymbol{\Theta}}$ are given in the following. Let

$$[\bar{L}]=[{}^t\bar{R}][{}^t\bar{S}_r]=[{}^t\bar{S}_l][{}^t\bar{R}] \tag{88}$$

Let $({}^t\lambda_i,\{\phi\}_i); i=1,2,3$ be the eigenvalues of $[\bar{L}]^{\mathrm{T}}[\bar{L}]$ in which $\{\phi\}_i^{\mathrm{T}}\{\phi\}_j=\delta_{ij}$, then

$$[\bar{L}]^{\mathrm{T}}[\bar{L}]=[\bar{\Phi}][{}^t\bar{\lambda}][\bar{\Phi}]^{\mathrm{T}}=[{}^t\bar{S}_r]^2 \tag{89}$$

The columns of $[\bar{\Phi}]$ are eigenvectors $\{\phi\}_i$ and $[{}^t\bar{\lambda}]$ is a diagonal matrix of ${}^t\lambda_i, i=1,2,3$. If one chooses

$$[{}^t\bar{S}_r]=[\bar{\Phi}]\left[\sqrt{{}^t\bar{\lambda}}\right][\bar{\Phi}]^{\mathrm{T}} \tag{90}$$

Then (89) holds, hence $[{}^t\bar{S}_r]$ can be defined using (90). $[{}^t\bar{R}]$ can now be determined using (88)

$$[{}^t\bar{R}]=[\bar{L}][{}^t\bar{S}_r]^{-1} \tag{91}$$

Thus, $\left[{}^{t}\bar{R}\right]$ and $\left[{}^{t}\bar{S}_{r}\right]$ are established in polar decomposition (88). Using

$$\left[\bar{L}\right]\left[\bar{L}\right]^{\mathrm{T}}=\left[{}^{t}\bar{S}_{l}\right]^{2} \tag{92}$$

and following a similar procedure one can establish the following

$$\left[{}^{t}\bar{S}_{l}\right]=\left[\Phi\right]\left[\sqrt{{}^{t}\bar{\lambda}}\right]\left[\Phi\right]^{\mathrm{T}} \tag{93}$$

$$\left[{}^{t}\bar{R}\right]=\left[{}^{t}\bar{S}_{l}\right]^{-1}\left[\bar{L}\right] \tag{94}$$

in which $\left({}^{t}\lambda_{i},\{\phi\}_{i}\right); i=1,2,3$ are eigenpairs of $\left[\bar{L}\right]\left[\bar{L}\right]^{\mathrm{T}}$. $\left[{}^{t}\bar{R}\right]$ defined by (91) or (94) is unique. The rate of rotation matrix $\left[{}^{t}\bar{R}\right]$ can equivalently be obtained due to rotation rates ${}^{t}\bar{\boldsymbol{\Theta}}$ at each location. Thus, at each location $\left[{}^{t}\bar{R}\right]$ can be viewed as being due to rates of rotations ${}^{t}\bar{\boldsymbol{\Theta}}$. Rate of energy dissipation due to ${}^{t}\bar{\boldsymbol{\Theta}}$ requires coexistence of moments $\bar{\boldsymbol{M}}$ (per unit area) on the oblique surface of the tetrahedron in the deforming matter. Thus

$$\left[\bar{L}\right]=\frac{\partial\{\bar{v}\}}{\partial\{\bar{x}\}}=\left[{}^{t}\bar{R}\right]\left[{}^{t}\bar{S}_{r}\right]=\left[{}^{t}\bar{S}_{l}\right]\left[{}^{t}\bar{R}\right] \tag{95}$$

where

$$\left[{}^{t}\bar{R}\right]=\left[{}^{t}\bar{R}\left({}^{t}\bar{\boldsymbol{\Theta}}\right)\right] \tag{96}$$

Explicit forms of ${}^{t}\bar{\boldsymbol{\Theta}}$ *i.e.* ${}^{t}\bar{\Theta}_{x_1}$, ${}^{t}\bar{\Theta}_{x_2}$, and ${}^{t}\bar{\Theta}_{x_3}$ or ${}^{t}\bar{\Theta}_{1}$, ${}^{t}\bar{\Theta}_{2}$, and ${}^{t}\bar{\Theta}_{3}$ are defined in terms of velocity gradients. These rotation rates on the oblique plane of the tetrahedron are conjugate with the resultant moment tensor $\bar{\boldsymbol{M}}$, hence result in rate of work. At this stage these are not basis dependent. When $\bar{\boldsymbol{M}}$ is converted to moment tensor $\bar{\boldsymbol{m}}^{(0)}$ or $\bar{\boldsymbol{m}}_{(0)}$ or ${}^{(0)}\bar{\boldsymbol{m}}^{J}$, then there is basis dependency in ${}^{t}\bar{\boldsymbol{\Theta}}$. In the energy equation and likewise in entropy inequality their derivations are continued with ${}^{t}\bar{\boldsymbol{\Theta}}$ until at a later stage when gradients of ${}^{t}\bar{\boldsymbol{\Theta}}$ are needed, convected time derivatives of the rotation gradient tensor in the appropriate basis are introduced.

2.4. Conservation and Balance Laws

In reference [1] [2] conservation and balance laws were derived for internal polar fluent continua. These derivations were presented using ${}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{q}}$ as basis independent constitutive tensors which were then given appropriate definitions of $\bar{\boldsymbol{\sigma}}^{(0)}$, $\bar{\boldsymbol{m}}^{(0)}$, $\bar{\boldsymbol{q}}^{(0)}$; $\bar{\boldsymbol{\sigma}}_{(0)}$, $\bar{\boldsymbol{m}}_{(0)}$, $\bar{\boldsymbol{q}}_{(0)}$; and ${}^{(0)}\bar{\boldsymbol{\sigma}}^{J}$, ${}^{(0)}\bar{\boldsymbol{m}}^{J}$, ${}^{(0)}\bar{\boldsymbol{q}}^{J}$ depending on whether the basis of choice is contravariant, covariant or Jaumann rates. Additionally ${}^{(k)}\gamma; k=0,1,\cdots,n$ were used as argument tensors that are basis independent convected time derivatives of strain tensor that could be $\gamma^{(k)}$ or $\gamma_{(k)}$ or ${}^{(k)}\gamma^{J}; k=1,2,\cdots,n$ depending upon the contravariant or covariant basis or Jaumann rates. The constitutive theory for thermofluids without memory in reference [3] utilizes $\left({}^{(0)}\bar{\boldsymbol{\sigma}},{}^{(k)}\gamma; k=0,1,\cdots,n\right)$ and $\left({}^{(0)}\bar{\boldsymbol{m}},\left[{}^{\Theta}\bar{D}\right]\right)$ as conjugate pairs in the derivation. Since ${}^{(0)}\bar{\boldsymbol{m}}$ is basis dependent, it's conjugate pair(s) must also be basis dependent. Choice of the rotation rate gradient resulting from the velocity gradient tensor as conjugate to ${}^{(0)}\bar{\boldsymbol{m}}$ limits the applicability of the resulting theory to small strain rates and small rotation rates and its gradients.

Furthermore, this choice can not be extended to higher order time derivatives of the rotation gradient tensor as it is not the convected time derivative of rotation gradient tensor in Eulerian description. The work presented in this paper proposes and replaces $\left[{}^{\theta}\bar{D}\right]$ with the true convected time derivatives ${}^{(k)}_{\Theta}\boldsymbol{\gamma}; k=1,2,\cdots,{}^{1}n$ of the rotation gradient tensor that can be considered in a chosen basis. Use of correct convected time derivatives of the rotation gradient tensor requires the rederivation of energy equation and entropy inequality using correct measures. Using ${}^{(0)}\bar{\boldsymbol{\sigma}}, {}^{(0)}\bar{\boldsymbol{m}}, {}^{(0)}\bar{\boldsymbol{q}}$ as measures of Cauchy stress tensor, Cauchy moment tensor, and heat vector, ${}^{(k)}\boldsymbol{\gamma}; k=0,1,\cdots,n$ as convected time derivatives of the appropriate strain tensors and ${}^{(k)}_{\Theta}\boldsymbol{\gamma}; k=1,2,\cdots,{}^{1}n$ as measures of the convected time derivatives of the rotation gradient tensor (all will eventually be basis dependent) one can write the following for conservation of mass, balance of linear momenta, balance of angular momenta and balance of moments of moments (or couples).

$$\frac{\partial\bar{\rho}}{\partial t}+\bar{\nabla}\cdot\left(\bar{\rho}\bar{\boldsymbol{v}}\right)=0 \tag{97}$$

$$\bar{\rho}\frac{\partial\bar{v}_i}{\partial t}+\bar{\rho}\bar{v}_j\frac{\partial\bar{v}_i}{\partial\bar{x}_j}-\bar{\rho}\bar{F}_i^b-\frac{\partial^{(0)}\bar{\sigma}_{ji}}{\partial\bar{x}_j}=0 \tag{98}$$

$${}^{(0)}\bar{m}_{pk,p}-\epsilon_{ijk}{}^{(0)}\bar{\sigma}_{ij}=0 \tag{99}$$

$$\epsilon_{ijk}{}^{(0)}\bar{m}_{ij}=0 \tag{100}$$

At this stage the Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is nonsymmetric but the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ is symmetric (due to moment of moments Equation (100)). In (98) - (100) basis independent ${}^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ have been used as opposed to their contravariant measures, but the details of the derivations remain similar. Since the energy equation and the entropy inequality require rate of work due to ${}^{(0)}\bar{\boldsymbol{m}}$ in addition to rate of work due to ${}^{(0)}\bar{\boldsymbol{\sigma}}$, their derivations in reference [3] does not hold here as the conjugate pair to ${}^{(0)}\bar{\boldsymbol{m}}$ is no longer gradient of rotation rate tensor resulting from the velocity gradient tensor, instead it is convected time derivative of the rotation gradient tensor in appropriate basis depending upon the choice of basis for the Cauchy moment tensor.

2.4.1. First Law of Thermodynamics: Energy Equation

The sum of work and heat added to a deforming volume of matter must result in the increase in energy of the system. Expressing this as a rate statement one can write [4] [9] [10] [11]

$$\frac{D\bar{E}_t}{Dt}=\frac{D\bar{Q}}{Dt}+\frac{D\bar{W}}{Dt} \tag{101}$$

$\bar{E}_t$, $\bar{Q}$, and $\bar{W}$ are total energy, heat added, and work done. These can be written as

$$\frac{D\bar{E}_t}{Dt}=\frac{D}{Dt}\int_{\bar{V}(t)}\bar{\rho}\left(\bar{e}+\frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}}-\bar{\boldsymbol{F}}^b\cdot\bar{\boldsymbol{u}}\right)\mathrm{d}\bar{V} \tag{102}$$

$$\frac{D\bar{Q}}{Dt} = -\int_{\partial\bar{V}(t)} {}^{(0)}\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}} \mathrm{d}\bar{A} \tag{103}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} \left(\bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} + \bar{\boldsymbol{M}} \cdot {}^{t}\bar{\boldsymbol{\Theta}}\right) \mathrm{d}\bar{A} \tag{104}$$

where $\bar{e}$ is specific internal energy, $\bar{\boldsymbol{F}}^b$ is body force per unit mass, $\bar{\boldsymbol{u}}$ are displacement, and ${}^{(0)}\bar{\boldsymbol{q}}$ is rate of heat. Note the additional term $\bar{\boldsymbol{M}} \cdot {}^{t}\bar{\boldsymbol{\Theta}}$ in $\frac{D\bar{W}}{Dt}$ contributes additional rate of work due to rates of rotation in (104). Expand each of the integrals in (102)-(104). Following reference [4], it is straight forward to show that:

$$\frac{D}{Dt}\int_{\bar{V}(t)} \bar{\rho}\left(\bar{e} + \frac{1}{2}\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{u}}\right) \mathrm{d}\bar{V} = \int_{\bar{V}(t)} \bar{\rho}\left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{v}} \cdot \frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{v}}\right) \mathrm{d}\bar{V} \tag{105}$$

$$-\int_{\partial\bar{V}(t)} {}^{(0)}\bar{\boldsymbol{q}} \cdot \bar{\boldsymbol{n}} d\bar{A} = -\int_{\bar{V}(t)} \bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{q}} \mathrm{d}\bar{V}; \quad \text{divergence theorem} \tag{106}$$

$$\frac{D\bar{W}}{Dt} = \int_{\partial\bar{V}(t)} \bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} \mathrm{d}\bar{A} + \int_{\partial\bar{V}(t)} \bar{\boldsymbol{M}} \cdot {}^{t}\bar{\boldsymbol{\Theta}} \mathrm{d}\bar{A} \tag{107}$$

Using basis independent Cauchy stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$, Cauchy principle, and following the details in reference [4] one can write

$$\int_{\partial\bar{V}(t)} \bar{\boldsymbol{P}} \cdot \bar{\boldsymbol{v}} \mathrm{d}\bar{A} = \int_{\bar{V}(t)} \left(\bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}^{(0)}\bar{\sigma}_{ji} \frac{\partial \bar{v}_i}{\partial \bar{x}_j}\right) \mathrm{d}\bar{V} \tag{108}$$

Likewise using basis independent moment tensor (per unit area) ${}^{(0)}\bar{\boldsymbol{m}}$, Cauchy principle, and following the details similar to these used in deriving (108), one can write

$$\int_{\partial\bar{V}(t)} \bar{\boldsymbol{M}} \cdot {}^{t}\bar{\boldsymbol{\Theta}} \mathrm{d}\bar{A} = \int_{\bar{V}(t)} \left({}^{t}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + {}^{(0)}\bar{m}_{ji} \left({}_{\Theta}^{(1)}\gamma_{ij}\right)\right) \mathrm{d}\bar{V} \tag{109}$$

The first convected time derivative of the rotation gradient tensor, $\frac{\partial\left({}^{t}\bar{\Theta}_i\right)}{\partial \bar{x}_j} = {}_{\Theta}^{(1)}\gamma_{ij}$ that is conjugate to the Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ has been used in (109). Using (105)-(109) in (101)

$$\begin{aligned}
&\int_{\bar{V}(t)} \bar{\rho}\left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{v}} \cdot \frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b \cdot \bar{\boldsymbol{v}}\right) \mathrm{d}\bar{V} \\
&= -\int_{\bar{V}(t)} \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{q}} \mathrm{d}\bar{V} + \int_{\bar{V}(t)} \left(\bar{\boldsymbol{v}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right) + {}^{(0)}\bar{\sigma}_{ji} \frac{\partial \bar{v}_i}{\partial \bar{x}_j}\right) \mathrm{d}\bar{V} \\
&\quad + \int_{\bar{V}(t)} \left({}^{t}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right) + {}^{(0)}\bar{m}_{ji} \left({}_{\Theta}^{(1)}\gamma_{ij}\right)\right) \mathrm{d}\bar{V}
\end{aligned} \tag{110}$$

Transferring all terms to left of equality and regrouping

$$\begin{aligned}
&\int_{\bar{V}(t)} \bar{\rho}\left(\bar{\boldsymbol{v}} \cdot \left(\frac{D\bar{\boldsymbol{v}}}{Dt} - \bar{\boldsymbol{F}}^b - \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{\sigma}}\right)\right) \mathrm{d}\bar{V} \\
&+ \int_{\bar{V}(t)} \left(\frac{D\bar{e}}{Dt} + \bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{q}} - {}^{(0)}\bar{\sigma}_{ji} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji} \left({}_{\Theta}^{(1)}\gamma_{ij}\right) - {}^{t}\bar{\boldsymbol{\Theta}} \cdot \left(\bar{\boldsymbol{\nabla}} \cdot {}^{(0)}\bar{\boldsymbol{m}}\right)\right) \mathrm{d}\bar{V} = 0
\end{aligned} \tag{111}$$

Using (98) (balance of linear momenta) and (99) balance of angular momenta, (110) reduces to

$$\int_{\bar{V}(t)} \left(\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{q}} - {}^{(0)}\bar{\sigma}_{ji} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji} \left({}^{(1)}_{\Theta}\gamma_{ij} \right) - {}^{t}\boldsymbol{\Theta} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) \right) \mathrm{d}\bar{V} = 0 \tag{112}$$

Since $\bar{V}(t)$ is arbitrary, (112) implies that

$$\bar{\rho} \frac{D\bar{e}}{Dt} + \bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{q}} - {}^{(0)}\bar{\sigma}_{ji} \frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji} \left({}^{(1)}_{\Theta}\gamma_{ij} \right) - {}^{t}\boldsymbol{\Theta} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) = 0 \tag{113}$$

Equation (113) is the final form of the energy equation in which ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is a nonsymmetric Cauchy stress tensor and ${}^{(0)}\bar{\boldsymbol{m}}$ is a symmetric Cauchy moment tensor. Thus in (113) one can use

$$\begin{aligned} &{}^{(0)}\bar{m}_{mj} \left({}^{(1)}_{\Theta}\gamma_{ij} \right) = {}^{(0)}\bar{m}_{mj} \left({}_{s}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) + {}_{a}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) \right) = {}^{(0)}\bar{m}_{mj} \left({}_{s}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) \right); \\ &\text{as } {}^{(0)}\bar{m}_{mj} \left({}_{a}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) \right) = 0 \end{aligned} \tag{114}$$

In (114) the following decomposition of $\left({}^{(1)}_{\Theta}\boldsymbol{\gamma} \right)$ into symmetric and antisymmetric tensors has been used.

$${}^{(1)}_{\Theta}\boldsymbol{\gamma} = \left({}_{s}\left({}^{(1)}_{\Theta}\boldsymbol{\gamma} \right) + {}_{a}\left({}^{(1)}_{\Theta}\boldsymbol{\gamma} \right) \right) \tag{115}$$

By appropriate choices of ${}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}_{s}\left({}^{(1)}_{\Theta}\boldsymbol{\gamma} \right)$, the explicit form of the energy equation in any desired basis can be obtained.

2.4.2. Second Law of Thermodynamics: Entropy Inequality

If $\bar{\eta}$ is the entropy density in volume $\bar{V}(t)$, $\bar{h}$ is the entropy flux between $\bar{V}(t)$ and the volume of matter surrounding it and $\bar{s}$ is the source of entropy in $\bar{V}$ due to non-contacting bodies, then the rate of increase in entropy in volume $\bar{V}(t)$ is at least equal to that supplied to $\bar{V}(t)$ from all contacting and non-contacting sources [4]. Thus

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} \mathrm{d}\bar{V} \geq \int_{\partial\bar{V}(t)} \bar{h} \mathrm{d}\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} \mathrm{d}\bar{V} \tag{116}$$

Using Cauchy's postulate for $\bar{h}$ *i.e.*

$$\bar{h} = -\bar{\boldsymbol{\Psi}} \cdot \bar{\boldsymbol{n}} \tag{117}$$

Using (117) in (116)

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} \mathrm{d}\bar{V} \geq -\int_{\partial\bar{V}(t)} \bar{\boldsymbol{\Psi}} \cdot \bar{\boldsymbol{n}} \mathrm{d}\bar{A} + \int_{\bar{V}(t)} \bar{s} \bar{\rho} \mathrm{d}\bar{V} \tag{118}$$

One recalls that [4]

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\eta} \bar{\rho} \mathrm{d}\bar{V} = \int_{\bar{V}(t)} \bar{\rho} \frac{D\bar{\eta}}{Dt} \mathrm{d}\bar{V} \tag{119}$$

and

$$-\int_{\partial\bar{V}(t)} \bar{\boldsymbol{\Psi}} \cdot \bar{\boldsymbol{n}} \mathrm{d}\bar{A} = -\int_{\bar{V}(t)} \bar{\nabla} \cdot \bar{\boldsymbol{\Psi}} \mathrm{d}\bar{V} = -\int_{\bar{V}(t)} \bar{\Psi}_{i,i} \mathrm{d}\bar{V}; \quad \text{divergence theorem} \tag{120}$$

Substituting from (119) and (120) in (118) and transferring all terms to the left of inequality

$$\int_{\bar{V}(t)} \left(\bar{\rho} \frac{D\bar{\eta}}{Dt} + \bar{\Psi}_{i,i} - \bar{s}\bar{\rho} \right) \mathrm{d}\bar{V} \geq 0 \tag{121}$$

Since volume $\bar{V}(t)$ is arbitrary, (121) implies

$$\bar{\rho} \frac{D\bar{\eta}}{Dt} + \bar{\Psi}_{i,i} - \bar{s}\bar{\rho} \geq 0 \tag{122}$$

Equation (122) is called the Clausius-Duhem inequality and is the most fundamental form resulting from the second law of thermodynamics. A different form of (122) can be derived if one assumes

$$\bar{\boldsymbol{\Psi}} = \frac{{}^{(0)}\bar{\boldsymbol{q}}}{\bar{\theta}}; \quad \bar{s} = \frac{\bar{r}}{\bar{\theta}} \tag{123}$$

where $\bar{\theta}$ is absolute temperature, ${}^{(0)}\bar{\boldsymbol{q}}$ is heat vector, and $\bar{r}$ is a suitable potential. Using (123)

$$\bar{\Psi}_{i,i} = \frac{{}^{(0)}\bar{q}_{i,i}}{\bar{\theta}} - \frac{{}^{(0)}\bar{q}_i}{\left(\bar{\theta}\right)^2}\bar{\theta}_{,i} = \frac{{}^{(0)}\bar{q}_{i,i}}{\bar{\theta}} - \frac{{}^{(0)}\bar{q}_i}{\left(\bar{\theta}\right)^2}\bar{g}_i; \quad \bar{g}_i = \bar{\theta}_{,i} \tag{124}$$

Substituting for $\bar{s}$ from (123) and for $\bar{\Psi}_{i,i}$ from (124) into (122) and multiplying by $\bar{\theta}$.

$$\bar{\rho}\bar{\theta} \frac{D\bar{\eta}}{Dt} + \left({}^{(0)}\bar{q}_{i,i} - \bar{\rho}\bar{r} \right) - \frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} \geq 0 \tag{125}$$

From energy Equation (113) (after inserting $\bar{\rho}\bar{r}$ term) in basis independent form one can write the following.

$$\bar{\nabla} \cdot {}^{(0)}\bar{\boldsymbol{q}} - \bar{\rho}\bar{r} = {}^{(0)}\bar{q}_{i,i} - \bar{\rho}\bar{r} = -\bar{\rho}\frac{D\bar{e}}{Dt} + {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + {}^{(0)}\bar{m}_{ji}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) + {}^{t}\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) \tag{126}$$

Substituting from (126) into (125)

$$\bar{\rho}\bar{\theta}\frac{D\bar{\eta}}{Dt} - \bar{\rho}\frac{D\bar{e}}{Dt} + {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} + {}^{(0)}\bar{m}_{ji}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) + {}^{t}\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) - \frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} \geq 0 \tag{127}$$

or

$$\bar{\rho}\left(\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt} \right) - {}^{(0)}\bar{\sigma}_{ji}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} - {}^{(0)}\bar{m}_{ji}\left({}^{(1)}_{\Theta}\gamma_{ij} \right) - {}^{t}\bar{\boldsymbol{\Theta}} \cdot \left(\boldsymbol{\epsilon} : {}^{(0)}\bar{\boldsymbol{\sigma}} \right) + \frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} \leq 0 \tag{128}$$

Let $\bar{\Phi}$ be Helmholtz free energy density (specific Helmholtz free energy) defined by

$$\bar{\Phi} = \bar{e} - \bar{\eta}\bar{\theta} \tag{129}$$

Hence

$$\frac{D\bar{e}}{Dt} - \bar{\theta}\frac{D\bar{\eta}}{Dt} = \left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt} \right) \tag{130}$$

Substituting from (130) into (128)

$$\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-{}^{(0)}\bar{\sigma}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j}-{}^{(0)}\bar{m}_{ji}\left(\underset{\Theta}{\overset{(1)}{}}\gamma_{ij}\right)-{}^{t}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)\le 0 \tag{131}$$

or

$$\begin{aligned}&\bar{\rho}\left(\frac{D\bar{\Phi}}{Dt}+\bar{\eta}\frac{D\bar{\theta}}{Dt}\right)+\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-\mathrm{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]^{\mathrm{T}}\left[\bar{L}\right]^{\mathrm{T}}\right)\\&-\mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[\underset{\Theta}{\overset{(1)}{}}\gamma\right]\right)-{}^{t}\bar{\boldsymbol{\Theta}}\cdot\left(\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}\right)\le 0\end{aligned} \tag{132}$$

${}^{(0)}\bar{\boldsymbol{m}}$ is symmetric but ${}^{(0)}\bar{\boldsymbol{\sigma}}$ is not symmetric. Since ${}^{(0)}\bar{\boldsymbol{m}}$ is symmetric, one can use the following in (132).

$$\mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[\underset{\Theta}{\overset{(1)}{}}\gamma\right]\right)=\mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_s\left(\underset{\Theta}{\overset{(1)}{}}\gamma\right)\right]\right) \tag{133}$$

The entropy inequality (132) in contravariant basis, covariant bases and in Jaumann rates can be obtained by replacing ${}^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$ with corresponding quantities in appropriate basis.

3. Stress Decomposition and Balance Laws

It is instructive to decompose stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}$ into symmetric ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ and antisymmetric ${}^{(0)}_{a}\bar{\boldsymbol{\sigma}}$ tensors

$${}^{(0)}\bar{\boldsymbol{\sigma}}={}^{(0)}_{s}\bar{\boldsymbol{\sigma}}+{}^{(0)}_{a}\bar{\boldsymbol{\sigma}} \tag{134}$$

where

$$\begin{aligned}{}^{(0)}_{s}\bar{\boldsymbol{\sigma}}&=\frac{1}{2}\left({}^{(0)}\bar{\boldsymbol{\sigma}}+\left({}^{(0)}\bar{\boldsymbol{\sigma}}\right)^{\mathrm{T}}\right)\\{}^{(0)}_{a}\bar{\boldsymbol{\sigma}}&=\frac{1}{2}\left({}^{(0)}\bar{\boldsymbol{\sigma}}-\left({}^{(0)}\bar{\boldsymbol{\sigma}}\right)^{\mathrm{T}}\right)\end{aligned} \tag{135}$$

Substituting these in the balance of linear momenta (98), balance of angular momenta (99), energy Equation (113), and entropy inequality (132) and noting that

$$\boldsymbol{\epsilon}:{}^{(0)}\bar{\boldsymbol{\sigma}}=\boldsymbol{\epsilon}:\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}+{}^{(0)}_{a}\bar{\boldsymbol{\sigma}}\right)=\boldsymbol{\epsilon}:\left({}^{(0)}_{a}\bar{\boldsymbol{\sigma}}\right);\quad \boldsymbol{\epsilon}:\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)=0 \tag{136}$$

$${}^{(0)}\bar{\sigma}_{ji}\frac{\partial\bar{v}_i}{\partial\bar{x}_j}=\left({}^{(0)}_{s}\bar{\sigma}_{ji}+{}^{(0)}_{a}\bar{\sigma}_{ji}\right)\left(\bar{D}_{ij}+\bar{W}_{ij}\right)=\left({}^{(0)}_{s}\bar{\sigma}_{ji}\right)\bar{D}_{ij}+\left({}^{(0)}_{a}\bar{\sigma}_{ji}\right)\bar{W}_{ij} \tag{137}$$

as

$$\left({}^{(0)}_{s}\bar{\sigma}_{ji}\right)\bar{W}_{ij}=\left({}^{(0)}_{a}\bar{\sigma}_{ji}\right)\bar{D}_{ij}=0 \tag{138}$$

one can write (137) as

$$\mathrm{tr}\left(\left[{}^{(0)}\bar{\sigma}\right]\left[\bar{L}\right]\right)=\mathrm{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right)+\mathrm{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[\bar{W}\right]\right) \tag{139}$$

Using (136)-(139) in (98), (99), (113), and (132) one can obtain

$$\bar{\rho}\frac{\partial\bar{v}_i}{Dt}+\bar{\rho}\bar{v}_j\frac{\partial\bar{v}_i}{\partial\bar{x}_j}-\bar{\rho}\bar{F}_i^b-\frac{\partial\,{}^{(0)}_{s}\bar{\sigma}_{ji}}{\partial\bar{x}_j}-\frac{\partial\,{}^{(0)}_{a}\bar{\sigma}_{ji}}{\partial\bar{x}_j}=0 \tag{140}$$

$$ {}^{(0)}\bar{m}_{mk,m} - \epsilon_{ijk}\left({}^{(0)}_{a}\bar{\sigma}_{ij}\right) = 0 \tag{141}$$

$$ \bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla}\cdot{}^{(0)}\bar{\boldsymbol{q}} - \mathrm{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right) - \mathrm{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[\bar{W}\right]\right) - \mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_{s}\left({}^{(1)}_{\Theta}\gamma\right)\right]\right) - {}^{t}\bar{\boldsymbol{\Theta}}\cdot\left(\epsilon : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) = 0 \tag{142}$$

$$ \bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - \mathrm{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right) - \mathrm{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[\bar{W}\right]\right) - \mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_{s}\left({}^{(1)}_{\Theta}\gamma\right)\right]\right) - {}^{t}\bar{\boldsymbol{\Theta}}\cdot\left(\epsilon : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \le 0 \tag{143}$$

A simple calculation by expanding the terms shows that

$$ \mathrm{tr}\left(\left[{}^{(0)}_{a}\bar{\sigma}\right]\left[\bar{W}\right]\right) = -{}^{t}\bar{\boldsymbol{\Theta}}\cdot\left(\epsilon : {}^{(0)}\bar{\boldsymbol{\sigma}}\right) \tag{144}$$

By substituting (144) in (142) and (143) the energy equation and entropy inequality simplify.

$$ \bar{\rho}\frac{D\bar{e}}{Dt} + \bar{\nabla}\cdot{}^{(0)}\bar{\boldsymbol{q}} - \mathrm{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right) - \mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_{s}\left({}^{(1)}_{\Theta}\gamma\right)\right]\right) = 0 \tag{145}$$

$$ \bar{\rho}\left(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\right) + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - \mathrm{tr}\left(\left[{}^{(0)}_{s}\bar{\sigma}\right]\left[\bar{D}\right]\right) - \mathrm{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}_{s}\left({}^{(1)}_{\Theta}\gamma\right)\right]\right) \le 0 \tag{146}$$

Remarks

1) Equations (140), (141), (145), and (146) can also be expressed in contravariant basis, covariant basis and using Jaumann rates.

2) Equations (97), (140), (141), (145), and (146) constitute a complete mathematical model for internal polar fluent media in Eulerian description.

3) From (145) and (146) one can conclude that ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$, ${}^{(1)}\boldsymbol{\gamma}$ and ${}^{(0)}\bar{\boldsymbol{m}}$, ${}_{s}\left({}^{(1)}_{\Theta}\boldsymbol{\gamma}\right)$ are conjugate pairs, hence are responsible for conversion of mechanical energy into heat or entropy. The conjugate pairs are instrumental in deciding the dependent variables in the constitutive theories and some of their argument tensors. These conjugate pairs suggest that ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ can be expressed as a function of ${}^{(1)}\boldsymbol{\gamma}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ as a function of ${}_{s}\left({}^{(1)}_{\Theta}\boldsymbol{\gamma}\right)$. One notes that ${}^{(0)}\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ are also conjugate, thus ${}^{(0)}\bar{\boldsymbol{q}}$ can be expressed as a function of $\bar{\boldsymbol{g}}$. These details of the constitutive theories are presented in the following sections.

4) This mathematical model has closure once the constitutive theories for ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{q}}$ (total of 15 equations) are added to the already existing eight equations for the conservation and balance laws (conservation of mass, balance of linear momenta, balance of angular momenta and energy equation) giving rise to a total of 23 equations in 23 variables, $\bar{\rho}, \bar{v}_i, {}^{(0)}_{s}\bar{\sigma}_{ij}, {}^{(0)}_{a}\bar{\sigma}_{ij}, {}^{(0)}\bar{m}_{ij}, {}^{(0)}\bar{q}_i$ and $\bar{\theta}$.

4. Dependent Variables in the Constitutive Theories

The choice of dependent variables in the constitutive theories must be consistent with the axiom of casualty [4] [9] [10]. The self observable quantities and those that can be derived from them by simple differentiation and/or integration can not be considered as dependent variables in the constitutive theories. Thus velocities, temperatures, temperature gradients, etc. are ruled out as choices of dependent variables in the constitutive theories. From the entropy inequality one notes that ${}_s^{(0)}\bar{\boldsymbol{\sigma}}, {}^{(0)}\bar{\boldsymbol{m}}, \bar{\Phi}, \bar{\eta}, \bar{\boldsymbol{q}}$ are possible choices of dependent variables in the constitutive theories. The choice of ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$, ${}^{(0)}\bar{\boldsymbol{m}}$, and $\bar{\boldsymbol{q}}$ as dependent variables in the constitutive theories is also supported by balance of linear momenta, balance of angular momenta, and the energy equation. ${}_a^{(0)}\bar{\boldsymbol{\sigma}}$ can not be chosen as dependent variables in the constitutive theories as these are deterministic from the balance of angular momenta. Choice of $\bar{e}, \bar{\eta}$ or $\bar{\Phi}, \bar{\eta}$ is a matter of preference as these are related through $\bar{\Phi}$. In the present work $\bar{\Phi}, \bar{\eta}$ are chosen, hence $\bar{e}$ need not be considered as a dependent variable in the constitutive theories. Thus, ${}_s^{(0)}\bar{\boldsymbol{\sigma}}, {}^{(0)}\bar{\boldsymbol{m}}, \bar{\Phi}, \bar{\eta}$, and $\bar{\boldsymbol{q}}$ are the possible dependent variables in the constitutive theories. At a later stage of the derivation, some of these may be ruled out as dependent variables in the constitutive theories if so warranted by some other considerations.

Possible choices of argument tensors of dependent variables are considered, keeping in mind the principle of equipresence [4] [9] [10], *i.e.* at the onset all dependent variables in the constitutive theories possibly must contain the same argument tensors. For compressible fluent media, density $\bar{\rho}$ is certainly an argument tensor. $\bar{\theta}$ is a natural choice as an argument tensor. The choice of $\bar{\boldsymbol{g}}$ as an argument tensor is necessitated due to the dependent variable $\bar{\boldsymbol{q}}$ in the constitutive theory and the physics of heat conduction. The choice of $\left[{}^{(1)}\gamma\right]$ and $\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]$ as argument tensors is also clear as these are conjugate to ${}_s^{(0)}\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$. From conservation of mass in Lagrangian description $\rho_0 = |J|\rho$ *i.e.* compressibility is due to $|J| = \rho_0/\rho$, hence it is fitting to consider $1/\bar{\rho}$ as an argument tensor in Eulerian description as opposed to $\bar{\rho}$ for the dependent variables in the constitutive theories. At a later stage dependence on $1/\bar{\rho}$ can be replaced by dependence on $\bar{\rho}$ by using calculus. Thus, based on the principle of equipresence [4] [9] [10] all dependent variables have the same argument tensors.

$$
\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\frac{1}{\bar{\rho}}, \left[{}^{(1)}\gamma\right], \left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\eta} &= \bar{\eta}\left(\frac{1}{\bar{\rho}}, \left[{}^{(1)}\gamma\right], \left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}_s^{(0)}\bar{\boldsymbol{\sigma}} &= {}_s^{(0)}\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(1)}\gamma\right], \left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(1)}\gamma\right], \left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right], \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\boldsymbol{q}} &= \bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(1)}\gamma\right], \left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right], \bar{\theta}, \bar{\boldsymbol{g}}\right)
\end{aligned}
\tag{147}
$$

One notes that $\left[{}^{(1)}\gamma\right]$ is the first convected time derivative of the strain tensor (Almansi tensor or Green's tensor or Jaumann rates) and is a fundamental kinematic tensor. In addition to $\left[{}^{(1)}\gamma\right]$, $\left[{}^{(k)}\gamma\right]; k = 2, 3, \cdots, n$, are also fundamental kinematic tensors up to order n that are convected time derivatives of orders $2, 3, \cdots, n$ of strain tensor in a chosen basis. With the choice of $\left[{}^{(1)}\gamma\right]$, the first convected time derivative of the strain tensor only in (147) the resulting constitutive theories would be rate constitutive theories of order one. $\left[{}^{(1)}\gamma\right]$ can be replaced with $\left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n$ as these all are fundamental kinematic tensors to generalize the derivation for the rate constitutive theories to up to order n.

Similarly ${}_s\left({}^{(1)}_{\Theta}\boldsymbol{\gamma}\right)$ is the symmetric part of the first convected derivative of the rotation gradient tensor and is a fundamental kinematic tensor. In addition to ${}_s\left({}^{(1)}_{\Theta}\boldsymbol{\gamma}\right)$, one also has ${}_s\left({}^{(k)}_{\Theta}\boldsymbol{\gamma}\right); k=2,3,\cdots,{}^1n$ as fundamental kinematic tensors that are symmetric part of the convected time derivatives of the rotation gradient tensor. With the use of ${}_s\left({}^{(1)}_{\Theta}\boldsymbol{\gamma}\right)$ only in the constitutive theory, the resulting constitutive theory will be of order one in rotation gradient rates. One replaces ${}_s\left({}^{(1)}_{\Theta}\boldsymbol{\gamma}\right)$ by ${}_s\left({}^{(k)}_{\Theta}\boldsymbol{\gamma}\right); k=1,2,\cdots,{}^1n$ to generalize the derivation of the rate constitutive theories up to order ${}^1n$.

Secondly, since the arguments in (147) are basis dependent the heat vector is no longer $\bar{\boldsymbol{q}}$, but instead is ${}^{(0)}\bar{\boldsymbol{q}}$ indicating that it could be $\bar{\boldsymbol{q}}^{(0)}, \bar{\boldsymbol{q}}_{(0)}$, or ${}^{(0)}\bar{\boldsymbol{q}}^J$ depending upon the choice of the basis. One can write the following for the dependent variables in the constitutive theories and their arguments tensors.

$$\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\eta} &= \bar{\eta}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}_s\bar{\boldsymbol{\sigma}} &= {}^{(0)}_s\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right)
\end{aligned} \tag{148}$$

From the entropy inequality one notes that $\left(\left[{}^{(0)}_s\bar{\sigma}\right], \left[{}^{(1)}\gamma\right]\right)$ and $\left(\left[{}^{(0)}\bar{m}\right], \left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]\right)$ are conjugate pairs *i.e.* $\left[{}^{(0)}_s\bar{\sigma}\right]$ has no dependence on $\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]$ and likewise $\left[{}^{(0)}\bar{m}\right]$ has no dependence on $\left[{}^{(1)}\gamma\right]$. Thus one can modify the argument tensors of ${}^{(0)}_s\bar{\boldsymbol{\sigma}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ in (148).

$$\begin{aligned}
\bar{\Phi} &= \bar{\Phi}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
\bar{\eta} &= \bar{\eta}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}_s\bar{\boldsymbol{\sigma}} &= {}^{(0)}_s\bar{\boldsymbol{\sigma}}\left(\frac{1}{\bar{\rho}}, \left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
{}^{(0)}\bar{\boldsymbol{m}} &= {}^{(0)}\bar{\boldsymbol{m}}\left(\frac{1}{\bar{\rho}}, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n, \bar{\theta}, \bar{\boldsymbol{g}}\right)
\end{aligned} \tag{149}$$

5. Entropy Inequality and Constitutive Theories

Consider the entropy inequality (146) with the arguments of $\bar{\Phi}$ defined in (149). Now $\frac{D\bar{\Phi}}{Dt} = \dot{\bar{\Phi}}$ can be obtained which is needed in the entropy inequality.

$$\dot{\bar{\Phi}} = \frac{\partial \bar{\Phi}}{\partial \left(\frac{1}{\bar{\rho}}\right)} \left(-\frac{1}{\bar{\rho}^2}\right) \dot{\bar{\rho}} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik} \right)} \left({}^{(j)}\dot{\gamma}_{ik} \right) + \sum_{m=1}^{{}^1 n} \frac{\partial \bar{\Phi}}{\partial_s \left({}^{(m)}_{\Theta}\gamma_{ik} \right)} \left({}_s\left({}^{(m)}_{\Theta}\dot{\gamma}_{ik} \right) \right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i} \dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} \tag{150}$$

From the continuity Equation (97) (its alternate from in $\frac{D\bar{\rho}}{Dt} = \dot{\bar{\rho}}$)

$$\dot{\bar{\rho}} = -\bar{\rho}\bar{\boldsymbol{\nabla}} \cdot \bar{\boldsymbol{v}} = -\bar{\rho}\bar{D}_{kk} = -\bar{\rho}\,{}^{(1)}\gamma_{kk} = -\bar{\rho}\,{}^{(1)}\gamma_{ik}\delta_{ik} \tag{151}$$

Using (151) in (150)

$$\dot{\bar{\Phi}} = \frac{1}{\bar{\rho}} \frac{\partial \bar{\Phi}}{\partial \left(\frac{1}{\bar{\rho}}\right)} {}^{(1)}\gamma_{ik}\delta_{ik} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik} \right)} \left({}^{(j)}\dot{\gamma}_{ik} \right) + \sum_{m=1}^{{}^1 n} \frac{\partial \bar{\Phi}}{\partial_s \left({}^{(m)}_{\Theta}\gamma_{ik} \right)} \left({}_s\left({}^{(m)}_{\Theta}\dot{\gamma}_{ik} \right) \right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i} \dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} \tag{152}$$

One notes that

$$-\frac{\partial \bar{\Phi}}{\partial \left(\frac{1}{\bar{\rho}}\right)} = \bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \tag{153}$$

Using (153) in (152)

$$\dot{\bar{\Phi}} = -\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} {}^{(1)}\gamma_{ik}\delta_{ik} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik} \right)} \left({}^{(j)}\dot{\gamma}_{ik} \right) + \sum_{m=1}^{{}^1 n} \frac{\partial \bar{\Phi}}{\partial_s \left({}^{(m)}_{\Theta}\gamma_{ik} \right)} \left({}_s\left({}^{(m)}_{\Theta}\dot{\gamma}_{ik} \right) \right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i} \dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} \tag{154}$$

Substituting $\dot{\bar{\Phi}}$ from (154) in the entropy inequality (146)

$$\bar{\rho} \left(-\bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} {}^{(1)}\gamma_{ik}\delta_{ik} + \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik} \right)} \left({}^{(j)}\dot{\gamma}_{ik} \right) + \sum_{m=1}^{{}^1 n} \frac{\partial \bar{\Phi}}{\partial_s \left({}^{(m)}_{\Theta}\gamma_{ik} \right)} \left({}_s\left({}^{(m)}_{\Theta}\dot{\gamma}_{ik} \right) \right) + \frac{\partial \bar{\Phi}}{\partial \bar{g}_i} \dot{\bar{g}}_i + \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \dot{\bar{\theta}} + \bar{\eta}\dot{\bar{\theta}} \right) + \frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} - {}^{(0)}_{s}\bar{\sigma}_{ik}\,{}^{(1)}\gamma_{ik} - {}^{(0)}\bar{m}_{ik} \left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik} \right) \right) \le 0 \tag{155}$$

Regrouping terms in (155)

$$\left(-\bar{\rho}^2 \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} \delta_{ik} - {}^{(0)}_{s}\bar{\sigma}_{ik} \right) {}^{(1)}\gamma_{ik} + \bar{\rho} \sum_{j=1}^{n} \frac{\partial \bar{\Phi}}{\partial \left({}^{(j)}\gamma_{ik} \right)} \left({}^{(j)}\dot{\gamma}_{ik} \right) + \bar{\rho} \sum_{m=1}^{{}^1 n} \frac{\partial \bar{\Phi}}{\partial_s \left({}^{(m)}_{\Theta}\gamma_{ik} \right)} \left({}_s\left({}^{(m)}_{\Theta}\dot{\gamma}_{ik} \right) \right) + \bar{\rho} \left(\frac{\partial \bar{\Phi}}{\partial \bar{\theta}} + \bar{\eta} \right) \dot{\bar{\theta}} - {}^{(0)}\bar{m}_{ik} \left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik} \right) \right) + \bar{\rho} \frac{\partial \bar{\Phi}}{\partial \bar{g}_i} \dot{\bar{g}}_i + \frac{{}^{(0)}\bar{q}_i \bar{g}_i}{\bar{\theta}} \le 0 \tag{156}$$

For (156) to hold for arbitrary but admissible $\left[{}^{(j)}\dot{\gamma}\right];\ j=1,2,\cdots,n$, ${}_s\left({}^{(m)}_{\Theta}\dot{\gamma}\right); m=1,2,\cdots,{}^1n$, $\dot{\bar{\boldsymbol{g}}}$ and $\dot{\bar{\theta}}$, the following must hold

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\bar{g}_i}=0 \Rightarrow \frac{\partial\bar{\Phi}}{\partial\bar{g}_i}=0 \tag{157}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\gamma_{ik}\right)}=0 \Rightarrow \frac{\partial\bar{\Phi}}{\partial\left({}^{(j)}\gamma_{ik}\right)}=0;\quad j=1,2,\cdots,n \tag{158}$$

$$\bar{\rho}\frac{\partial\bar{\Phi}}{\partial_s\left({}^{(m)}_{\Theta}\gamma_{ik}\right)}=0 \Rightarrow \frac{\partial\bar{\Phi}}{\partial_s\left({}^{(m)}_{\Theta}\gamma_{ik}\right)}=0;\quad m=1,2,\cdots,{}^1n \tag{159}$$

$$\bar{\rho}\left(\frac{\partial\bar{\Phi}}{\partial\bar{\theta}}+\bar{\eta}\right)=0 \Rightarrow \frac{\partial\bar{\Phi}}{\partial\bar{\theta}}+\bar{\eta}=0 \tag{160}$$

$$\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik}-{}^{(0)}_{s}\bar{\sigma}_{ik}\right){}^{(1)}\gamma_{ik}+\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-{}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right)\le 0 \tag{161}$$

Equations (157)-(161) are fundamental relations from the entropy inequality

Remarks

1) Equation (157) implies that $\bar{\Phi}$ is not a function of $\bar{\boldsymbol{g}}$.

2) Equation (158) implies that $\bar{\Phi}$ is not a function of $\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n$.

3) Equation (159) implies that $\bar{\Phi}$ is not a function of $\left[{}_s\left({}^{(m)}_{\Theta}\gamma\right)\right]; m=1,2,\cdots,{}^1n$.

4) Based on (160), $\bar{\eta}$ is not a dependent variable in the constitutive theories as $\bar{\eta}=-\frac{\partial\bar{\Phi}}{\partial\bar{\theta}}$, hence $\bar{\eta}$ is deterministic from $\bar{\Phi}$.

5) The last inequality is essential in the form it is stated. For example the following (or any other separation of terms)

$$-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik}-{}^{(0)}_{s}\bar{\sigma}_{ik}=0 \quad\text{and}\quad \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-{}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right)\le 0 \tag{162}$$

are inappropriate due to the fact that these imply that ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is not a function of $\bar{\boldsymbol{g}}$, $\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n$ as $\bar{\Phi}$ is not a function of these. This is contrary to (149). Inequality (161) in this form is unable to provide one with further details regarding the derivation of the constitutive theories.

In view of these remarks the arguments of the dependent variables in the constitutive theories in (149) can be modified. One can use $\bar{\rho}$ instead of $\frac{1}{\bar{\rho}}$.

$$\begin{aligned}
&\bar{\Phi}=\bar{\Phi}\left(\bar{\rho},0,0,\bar{\theta},0\right)\\
&{}^{(0)}_{s}\bar{\boldsymbol{\sigma}}={}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\left(\bar{\rho},\left[{}^{(k)}\gamma\right];k=1,2,\cdots,n,\bar{\theta},\bar{\boldsymbol{g}}\right)\\
&{}^{(0)}\bar{\boldsymbol{m}}={}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho},\left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right];\ j=1,2,\cdots,{}^1n,\bar{\theta},\bar{\boldsymbol{g}}\right)\\
&{}^{(0)}\bar{\boldsymbol{q}}={}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho},\left[{}^{(k)}\gamma\right];k=1,2,\cdots,n,\left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right];\ j=1,2,\cdots,{}^1n,\bar{\theta},\bar{\boldsymbol{g}}\right)
\end{aligned} \tag{163}$$

One notes that there are no mechanisms or conditions that permit eliminating

$\left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n$ and $\left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^1n$ from the argument list of ${}^{(0)}\bar{\boldsymbol{q}}$, hence one must keep them as in (163). Based on (160) and remark (4), $\bar{\eta}$ is no longer a dependent variable in the constitutive theories. With (161) and (163) one has no further mechanisms to proceed with the derivation of the constitutive theories.

5.1. Decomposition of Stress Tensor ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$

In order to remedy the situation discussed in remark (5), one considers decomposition of symmetric Cauchy stress tensor into equilibrium Cauchy stress tensor ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ and deviatoric Cauchy stress tensor ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ *i.e.*

$$ {}^{(0)}_{s}\bar{\boldsymbol{\sigma}} = {}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right) + {}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right) \tag{164} $$

in which one considers the following

$$ \begin{aligned} &{}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right) = {}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)\left(\bar{\rho},0,0,\bar{\theta},0\right) \\ &{}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right) = {}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)\left(\bar{\rho},\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n,\bar{\theta},\bar{\boldsymbol{g}}\right) \\ &\text{and}\quad {}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right) = {}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)\left(\bar{\rho},0,\bar{\theta},0\right) = 0 \end{aligned} \tag{165} $$

That is ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ is not a function of $\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n$ and $\bar{\boldsymbol{g}}$ and ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ vanishes when $\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n$ and $\bar{\boldsymbol{g}}$ are zero. Substituting (164) into (161)

$$ \left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_e\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right) - {}_d\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right) \le 0 \tag{166} $$

or

$$ \left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} - {}_e\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_d\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\left({}^{(1)}\gamma_{ik}\right) - {}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right) \le 0 \tag{167} $$

5.1.1. Constitutive Theory for Equilibrium Stress ${}_e\left({}^{(0)}_{s}\bar{\sigma}\right)$: Compressible Internal Polar Thermofluids

Since $\bar{\Phi}$ is not a function of ${}^{(1)}\gamma_{ik}$ and $\bar{\boldsymbol{g}}$ and neither is ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ (due to (165)), then the constitutive theory for ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ must be derivable from

$$ \begin{aligned} &{}_e\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right) = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik} = \bar{p}\left(\bar{\rho},\bar{\theta}\right)\delta_{ik} \\ &\left[{}_e\left({}^{(0)}_{s}\bar{\sigma}\right)\right] = \bar{p}\left(\bar{\rho},\bar{\theta}\right)[I] \end{aligned} \tag{168} $$

in which

$$ \bar{p}\left(\bar{\rho},\bar{\theta}\right) = -\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{169} $$

$\bar{p}\left(\bar{\rho},\bar{\theta}\right)$ is called thermodynamic pressure for compressible internal polar thermofluids and is generally referred to as an equation of state [4] [9] in which $\bar{p}$ is expressed as a function of $\bar{\rho}$ and $\bar{\theta}$ or $\bar{v} = \dfrac{1}{\bar{\rho}}$ and $\bar{\theta}$, where $\bar{v}$ is specific volume. If one assumes the compressive pressure to be positive, then $\bar{p}(\bar{\rho},\bar{\theta})$ in (168) can be replaced by $-\bar{p}(\bar{\rho},\bar{\theta})$. Using (168), inequality (167) reduces to

$$-{}_d\left({}^{(0)}_s\bar{\sigma}_{ik}\right)\left({}^{(1)}\gamma_{ik}\right)+\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-{}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right)\le 0 \tag{170}$$

Inequality (170) is satisfied if

$${}^s\psi_d={}^{(0)}_s\bar{\sigma}_{ik}\left({}^{(1)}\gamma_{ik}\right)>0;\quad {}^m\psi_d={}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right)>0 \tag{171}$$

and

$$\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}\le 0 \tag{172}$$

Inequalities (171) imply that the rate of work due to ${}^{(0)}_s\bar{\boldsymbol{\sigma}}$ *i.e.* ${}^s\psi_d$ and due to ${}^{(0)}\bar{\boldsymbol{m}}$ *i.e.* ${}^m\psi_d$ must be positive. In view of (168) one can write the following for compressible internal polar thermofluids.

$$\begin{aligned}
&\left[{}^{(0)}_s\bar{\sigma}\right]=\left[{}_e\left({}^{(0)}_s\bar{\sigma}\right)(\bar{\rho},\bar{\theta})\right]+\left[{}_d\left({}^{(0)}_s\bar{\sigma}\right)\left(\bar{\rho},\left[{}^{(k)}\gamma\right];k=1,2,\cdots,n,\bar{\theta},\bar{\boldsymbol{g}}\right)\right]\\
&{}^{(0)}\bar{\boldsymbol{q}}={}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho},\left[{}^{(k)}\gamma\right];k=1,2,\cdots,n,\left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right];j=1,2,\cdots,{}^1n,\bar{\theta},\bar{\boldsymbol{g}}\right)\\
&{}^{(0)}\bar{\boldsymbol{m}}={}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho},\left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right];j=1,2,\cdots,{}^1n,\bar{\theta},\bar{\boldsymbol{g}}\right)\\
&\bar{\Phi}=\bar{\Phi}(\bar{\rho},\bar{\theta})\\
&\left[{}_e\left({}^{(0)}_s\bar{\sigma}\right)\right]=\bar{p}(\bar{\rho},\bar{\theta})[I];\quad \bar{p}(\bar{\rho},\bar{\theta})=-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}
\end{aligned} \tag{173}$$

Constitutive theories for ${}_d\left({}^{(0)}_s\bar{\boldsymbol{\sigma}}\right)$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (171) and (172).

5.1.2. Constitutive Theory for Equilibrium Stress ${}_e\left({}^{(0)}_s\bar{\sigma}\right)$: Incompressible Matter

For incompressible matter density is constant, hence $\bar{\rho}=\rho_0$, thus for this case $\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}=0$, hence the constitutive theory for the incompressible case must consider $|J|=1$ as $\bar{\rho}=\rho_0$. The incompressibility condition given by (174) must be incorporated in the entropy inequality.

$$\bar{\nabla}\cdot\bar{v}=\text{tr}\left[\bar{D}\right]=\text{tr}\left[{}^{(1)}\gamma\right]={}^{(1)}\gamma_{ik}\delta_{ik}=0 \tag{174}$$

The incompressibility condition must be enforced. Based on (174) one can add

$$\bar{p}(\bar{\theta})\,{}^{(1)}\gamma_{ik}\delta_{ik}=0 \tag{175}$$

to (167). $\bar{p}(\bar{\theta})$ is arbitrary Lagrange multiplier.

$$\begin{aligned}
&\left(-\bar{\rho}^2\frac{\partial\bar{\Phi}}{\partial\bar{\rho}}\delta_{ik}-{}_e\left({}^{(0)}_s\bar{\sigma}_{ik}\right)\right){}^{(1)}\gamma_{ik}+\bar{p}(\bar{\theta})\,{}^{(1)}\gamma_{ik}\delta_{ik}\\
&+\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}}-{}_d\left({}^{(0)}_s\bar{\sigma}_{ik}\right)\left({}^{(1)}\gamma_{ik}\right)-{}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right)\le 0
\end{aligned} \tag{176}$$

Using $\frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0$ in (176) and regrouping terms.

$$\left(\bar{p}\left(\bar{\theta}\right)\delta_{ik} - {}_e\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\right){}^{(1)}\gamma_{ik} + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} - {}_d\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\left({}^{(1)}\gamma_{ik}\right) - {}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right) \le 0 \quad (177)$$

In the case of incompressible internal polar thermofluids ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ is a function of $\bar{\theta}$ only, hence from (177) one obtains

$$ {}_e\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right) = \bar{p}\left(\bar{\theta}\right)\delta_{ik} \quad \text{or} \quad \left[{}_e\left({}^{(0)}_{s}\bar{\sigma}\right)\right] = \bar{p}\left(\bar{\theta}\right)[I] \quad (178)$$

$\bar{p}\left(\bar{\theta}\right)$ is called mechanical pressure. Since $\bar{p}\left(\bar{\theta}\right)$ is an arbitrary Lagrange multiplier, $\bar{p}\left(\bar{\theta}\right)$ is not deterministic from the deformation field. In view of (178), (177) reduces to

$$-{}_d\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\left({}^{(1)}\gamma_{ik}\right) - {}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right) + \frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} \le 0 \quad (179)$$

Inequality (179) will hold if

$$ {}^{s}\psi_d = {}_d\left({}^{(0)}_{s}\bar{\sigma}_{ik}\right)\left({}^{(1)}\gamma_{ik}\right) > 0; \quad {}^{m}\psi_d = {}^{(0)}\bar{m}_{ik}\left({}_s\left({}^{(1)}_{\Theta}\gamma_{ik}\right)\right) > 0 \quad (180)$$

and

$$\frac{{}^{(0)}\bar{q}_i\bar{g}_i}{\bar{\theta}} \le 0 \quad (181)$$

Conditions (180) and (181) are the same for the compressible case *i.e.* the rate of work due to ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ and ${}^{(0)}\bar{\boldsymbol{m}}$ must be positive and the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (181). In view of (178) one can write the following for incompressible internal polar thermofluids.

$$\begin{aligned}
&\left[{}^{(0)}_{s}\bar{\sigma}\right] = \left[{}_e\left({}^{(0)}_{s}\bar{\sigma}\right)\left(\bar{\theta}\right)\right] + \left[{}_d\left({}^{(0)}_{s}\bar{\sigma}\right)\left(\left[{}^{(k)}\gamma\right]; k = 1,2,\cdots,n, \bar{\theta}, \bar{\boldsymbol{g}}\right)\right] \\
&{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left(\left[{}^{(k)}\gamma\right]; k = 1,2,\cdots,n, \left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j = 1,2,\cdots,{}^{1}n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
&{}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}\left(\left[{}_s\left({}^{(j)}_{\Theta}\gamma\right)\right]; j = 1,2,\cdots,{}^{1}n, \bar{\theta}, \bar{\boldsymbol{g}}\right) \\
&\bar{\Phi} = \bar{\Phi}\left(\bar{\theta}\right) \\
&\left[{}_e\left({}^{(0)}_{s}\bar{\sigma}\right)\right] = \bar{p}\left(\bar{\theta}\right)[I]; \quad \frac{\partial \bar{\Phi}}{\partial \bar{\rho}} = 0
\end{aligned} \quad (182)$$

Constitutive theories for ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$, ${}^{(0)}\bar{\boldsymbol{m}}$, and ${}^{(0)}\bar{\boldsymbol{q}}$ must satisfy (180) and (181).

Remarks

1) Conditions resulting from the entropy inequality require decomposition of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ into equilibrium and deviatoric stresses ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ and ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ (164) to proceed further.

2) Use of stress decomposition (164) in the conditions resulting from the entropy inequality permits determination of the constitutive theory for equilibrium stress tensor for compressible as well as incompressible internal polar thermofluids in terms of thermodynamic pressure and mechanical pressure.

3) The inequalities (170) or (179) require the rate of work due to ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ and ${}^{(0)}\bar{\boldsymbol{m}}$

be positive but provide no mechanisms for deriving constitutive theories for ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ and ${}^{(0)}\bar{\boldsymbol{m}}$.

4) The inequality (172) or (181) can be used (shown later) to derive a simple constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ (Fourier heat conduction law), but better constitutive theories are possible for ${}^{(0)}\bar{\boldsymbol{q}}$ (shown in subsequent sections)

5) The equilibrium stress ${}_e\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ is independent of the basis for compressible as well as incompressible polar thermofluids due to the fact that $[I]$ is basis independent. This implies that

$$
\begin{aligned}
&\left[{}_e\left({}_s\bar{\sigma}^{(0)}\right)\right]=\left[{}_e\left({}_s\bar{\sigma}_{(0)}\right)\right]=\left[{}_e\left({}^{(0)}_{s}\bar{\sigma}^{J}\right)\right]=\bar{p}\left(\bar{\rho},\bar{\theta}\right)[I];\quad \text{Compressible matter}\\
&\text{and}\quad \left[{}_e\left({}_s\bar{\sigma}^{(0)}\right)\right]=\left[{}_e\left({}_s\bar{\sigma}_{(0)}\right)\right]=\left[{}_e\left({}^{(0)}_{s}\bar{\sigma}^{J}\right)\right]=\bar{p}\left(\bar{\theta}\right)[I];\quad \text{Incompressible matter}
\end{aligned}
\tag{183}
$$

6) The rate constitutive theories for deviatoric Cauchy stress tensor, Cauchy moment tensor and heat vector are derived using theories of generators and invariants [4] [9] [12]-[27].

5.2. Rate Constitutive Theories of up to Order *n* for Deviatoric Symmetric Cauchy Stress Tensor ${}_d\left({}^{(0)}_{s}\bar{\sigma}\right)$: Compressible

Consider the following (from (173))

$$
{}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)={}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)\left(\bar{\rho},\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n,\bar{\theta},\bar{\boldsymbol{g}}\right)
\tag{184}
$$

Let $\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]; i=1,2,\cdots,N$ be the combined generators of the argument tensors $\left[{}^{(j)}\gamma\right];\ j=1,2,\cdots,n$, and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two [4] and ${}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\cdots,M$ be the combined invariants of the same argument tensors [4]. Then, one can express $\left[{}_d\left({}^{(0)}_{s}\bar{\sigma}\right)\right]$ as a linear combination of $\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]; i=1,2,\cdots,N$ and identity tensor $[I]$.

$$
\left[{}_d\left({}^{(0)}_{s}\bar{\sigma}\right)\right]={}^{\sigma}\underset{\sim}{\alpha}^{0}[I]+\sum_{i=1}^{N}{}^{\sigma}\underset{\sim}{\alpha}^{i}\left[{}^{\sigma}G^{i}\right]
\tag{185}
$$

The coefficients ${}^{\sigma}\underset{\sim}{\alpha}^{0}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\cdots,M$ in the current configuration

$$
{}^{\sigma}\underset{\sim}{\alpha}^{k}={}^{\sigma}\underset{\sim}{\alpha}^{k}\left(\bar{\rho},{}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\cdots,M,\bar{\theta}\right)
\tag{186}
$$

To determine material coefficients from (186), one considers Taylor series expansion of each ${}^{\sigma}\underset{\sim}{\alpha}^{k}; k=0,1,\cdots,N$ in $\theta,{}^{\sigma}\underset{\sim}{I}^{j};\ j=1,2,\cdots,M$ about a known configuration $\underset{\sim}{\Omega}$ and retains only up to linear terms in $\bar{\theta}$ and the invariants (for simplicity).

$$
{}^{\sigma}\underset{\sim}{\alpha}^{i}={}^{\sigma}\underset{\sim}{\alpha}^{i}\Big|_{\Omega}+\sum_{j=1}^{M}\left.\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\right|_{\Omega}\left({}^{\sigma}\underset{\sim}{I}^{j}-\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\Omega}\right)+\left.\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\bar{\theta}}\right|_{\Omega}\left(\bar{\theta}-\bar{\theta}_{\underset{\sim}{\Omega}}\right);\ i=0,1,\cdots,N
\tag{187}
$$

One notes that ${}^{\sigma}\underset{\sim}{\alpha}^{i}\Big|_{\Omega}$, $\left.\dfrac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\right|_{\Omega}$; $j=1,2,\cdots,M$, $\left.\dfrac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\bar{\theta}}\right|_{\Omega}$; $i=0,1,\cdots,N$ are functions of $\bar{\rho}\big|_{\underline{\Omega}}$, $\bar{\theta}\big|_{\underline{\Omega}}$ and $\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}$ whereas ${}^{\sigma}\underset{\sim}{\alpha}^{i}$ are functions of the same quantities but in the current configuration (186). When (187) is substituted in (185), one obtains the final expression for the most general rate constitutive theory of up to order n for ${}_{d}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ for compressible internal polar thermofluids. The final expression defines the material coefficients in the known configuration $\underline{\Omega}$. Details are given in the following. First, substitute (187) in (185),

$$\left[{}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)\right]=\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\big|_{\underline{\Omega}}+\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\left({}^{\sigma}\underset{\sim}{I}^{j}-\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\right)+\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\right)}{\partial\bar{\theta}}\Bigg|_{\underline{\Omega}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\right)[I]$$
$$+\sum_{i=1}^{N}\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}}+\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\left({}^{\sigma}\underset{\sim}{I}^{j}-\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\right)+\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\bar{\theta}}\Bigg|_{\underline{\Omega}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\right)\left[{}^{\sigma}\underset{\sim}{G}^{i}\right] \tag{188}$$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (188) that are defined in the current configuration and also grouping those terms that are completely defined in the known configuration $\underline{\Omega}$. Let

$$\begin{aligned}
{}^{0}\bar{\sigma}\big|_{\underline{\Omega}}&=\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\big|_{\underline{\Omega}}-\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\right)\\
{}^{\sigma}\underline{a}_{j}&=\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\underline{\Omega}};\quad j=1,2,\cdots,M\\
{}^{\sigma}\underline{b}_{i}&={}^{\sigma}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}}-\sum_{j=1}^{M}\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\underline{\Omega}}\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\\
{}^{\sigma}\underline{c}_{ij}&=\frac{\partial\left({}^{\sigma}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{\sigma}\underset{\sim}{I}^{j}\right)}\Bigg|_{\underline{\Omega}};\quad \begin{matrix}i=1,2,\cdots,N\\ j=1,2,\cdots,M\end{matrix}\\
\underline{\alpha}_{tm}&=-\frac{\partial^{\sigma}\underset{\sim}{\alpha}^{0}}{\partial\theta}\Bigg|_{\underline{\Omega}}\\
{}^{\sigma}\underline{d}_{i}&=-\frac{\partial^{\sigma}\underset{\sim}{\alpha}^{i}}{\partial\bar{\theta}}\Bigg|_{\underline{\Omega}};\ i=1,2,\cdots,N
\end{aligned} \tag{189}$$

Using (189), one can write (188) as follows

$$\left[{}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)\right]={}^{0}\bar{\sigma}\big|_{\underline{\Omega}}[I]+\sum_{j=1}^{M}{}^{\sigma}\underline{a}_{j}\,{}^{\sigma}\underset{\sim}{I}^{j}[I]-\underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I]+\sum_{i=1}^{N}{}^{\sigma}\underline{b}_{i}\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]$$
$$+\sum_{i=1}^{N}\sum_{j=1}^{M}{}^{\sigma}\underline{c}_{ij}\,{}^{\sigma}\underset{\sim}{I}^{j}\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]-\sum_{i=1}^{N}{}^{\sigma}\underline{d}_{i}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{\sigma}\underset{\sim}{G}^{i}\right] \tag{190}$$

${}^{\sigma}\underline{a}_{j},{}^{\sigma}\underline{b}_{i},{}^{\sigma}\underline{c}_{ij},{}^{\sigma}\underline{d}_{i}$ and $\underline{\alpha}_{\text{tm}}$ are material coefficients defined in known configurations $\underline{\Omega}$. This constitutive theory requires $\left(M+N+(M)(N)+N+1\right)$ material coefficients. The material coefficients defined in (190) are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and $\left({}^{\sigma}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}$; $j=1,2,\cdots,M$. This constitutive theory is based on integrity, hence it is complete.

5.3. Rate Constitutive Theories of up to Order *n* and ¹*n* for Heat Vector ${}^{(0)}\bar{q}$: Compressible

Consider (from (173))

$$ {}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho},\left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n,\left[{}_{s}\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^{1}n,\bar{\theta},\bar{\boldsymbol{g}}\right) \tag{191}$$

Let $\left\{{}^{q}\underset{\sim}{G}^{i}\right\}; i=1,2,\cdots,\tilde{N}$ be the combined generators of the argument tensors $\left[{}^{(k)}\gamma\right]; k=1,2,\cdots,n$, $\left[{}_{s}\left({}^{(j)}_{\Theta}\gamma\right)\right]; j=1,2,\cdots,{}^{1}n$, and $\bar{\boldsymbol{g}}$ that are tensors of rank one. Let ${}^{q}\underset{\sim}{I}^{j}; j=1,2,\cdots,\tilde{M}$ be the combined invariants of the same argument tensors. Then, one can express $\left\{{}^{(0)}\bar{q}\right\}$ as a linear combination of $\left\{{}^{q}\underset{\sim}{G}^{i}\right\}; i=1,2,\cdots,\tilde{N}$.

$$\left\{{}^{(0)}\bar{q}\right\} = -\sum_{i=1}^{\tilde{N}} {}^{q}\underset{\sim}{\alpha}^{i}\left\{{}^{q}\underset{\sim}{G}^{i}\right\} \tag{192}$$

The absence of unit vector in (192) is due to the fact that uniform temperature field does not contribute to $\left\{{}^{(0)}\bar{q}\right\}$. The negative sign in (192) is because a positive $\left\{{}^{(0)}\bar{q}\right\}$ in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter. The coefficients ${}^{q}\underset{\sim}{\alpha}^{i}; i=1,2,\cdots,\tilde{N}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}^{j}; j=1,2,\cdots,\tilde{M}$ in the current configuration. To determine the material coefficients from ${}^{q}\underset{\sim}{\alpha}^{i}; i=1,2,\cdots,\tilde{N}$ (that are defined in the current configuration) in (192), one considers Taylor series expansion of each ${}^{q}\underset{\sim}{\alpha}^{i}; i=1,2,\cdots,\tilde{N}$ about a known configuration $\underline{\Omega}$ in $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}^{j}; j=1,2,\cdots,\tilde{M}$ and retains only up to linear terms in $\bar{\theta}$ and the invariants and then substitutes these back in (192). If one defines the following in the final expression for $\left\{{}^{(0)}\bar{q}\right\}$

$$\begin{aligned} {}^{q}\underline{b}_{i} &= {}^{q}\underset{\sim}{\alpha}^{i}\Big|_{\underline{\Omega}} - \sum_{j=1}^{\tilde{M}} \left.\frac{\partial\left({}^{q}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{q}\underset{\sim}{I}^{j}\right)}\right|_{\underline{\Omega}} \left({}^{q}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}} \\ {}^{q}\underline{c}_{ij} &= \left.\frac{\partial\left({}^{q}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{q}\underset{\sim}{I}^{j}\right)}\right|_{\underline{\Omega}} \\ {}^{q}\underline{d}_{i} &= \left.\frac{\partial\left({}^{q}\underset{\sim}{\alpha}^{i}\right)}{\partial\bar{\theta}}\right|_{\underline{\Omega}} \end{aligned} \tag{193}$$

for $i=1,2,\cdots,\tilde{N}$ and $j=1,2,\cdots,\tilde{M}$.

Then, using (193) the resulting form of (192) can be written as

$$\left\{{}^{(0)}\bar{q}\right\} = -\sum_{i=1}^{\tilde{N}} {}^{q}\underline{b}_{i}\left\{{}^{q}\underset{\sim}{G}^{i}\right\} - \sum_{i=1}^{\tilde{N}}\sum_{j=1}^{\tilde{M}} {}^{q}\underline{c}_{ij}\,{}^{q}\underset{\sim}{I}^{j}\left\{{}^{q}\underset{\sim}{G}^{i}\right\} - \sum_{i=1}^{\tilde{N}} {}^{q}\underline{d}_{i}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left\{{}^{q}\underset{\sim}{G}^{i}\right\} \tag{194}$$

${}^{q}\underline{b}_{i}$, ${}^{q}\underline{c}_{ij}$, and ${}^{q}\underline{d}_{i}$ are material coefficients defined in known configuration $\underline{\Omega}$. This constitutive theory defined by (194) requires $\left(\tilde{N}+\tilde{N}\tilde{M}+\tilde{N}\right)$ material coefficients. The material coefficients are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and $\left({}^{q}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}; j=1,2,\cdots,\tilde{M}$. This theory is based on integrity, hence is complete.

5.4. Constitutive Theory for Cauchy Moment Tensor ${}^{(0)}\bar{\boldsymbol{m}}$: Compressible

Consider the following (from (173))

$$\left[{}^{(0)}\bar{m}\right]=\left[{}^{(0)}\bar{m}\left(\bar{\rho},\left[{}_{s}\left({}^{(k)}_{\Theta}\gamma\right)\right];k=1,2,\cdots,{}^{1}n,\bar{\theta},\{\bar{g}\}\right)\right] \tag{195}$$

Let $\left[{}^{m}\underset{\sim}{G}^{i}\right];i=1,2,\cdots,\underset{\sim}{N}$ be the combined generators of the argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$ that are symmetric tensors of rank two and ${}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,\underset{\sim}{M}$ be the combined invariants of the same argument tensors of ${}^{(0)}\bar{\boldsymbol{m}}$. One can express $\left[{}^{(0)}\bar{m}\right]$ as a linear combination of $\left[{}^{m}\underset{\sim}{G}^{i}\right];i=1,2,\cdots,\underset{\sim}{N}$ and identity tensor $[I]$.

$$\left[{}^{(0)}\bar{m}\right]={}^{m}\underset{\sim}{\alpha}^{0}[I]+\sum_{i=1}^{N}{}^{m}\underset{\sim}{\alpha}^{i}\left[{}^{m}\underset{\sim}{G}^{i}\right] \tag{196}$$

The coefficients ${}^{m}\underset{\sim}{\alpha}^{i};i=0,1,\cdots,\underset{\sim}{N}$ are functions of $\bar{\rho},\bar{\theta}$ and ${}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,\underset{\sim}{M}$ in the current configuration *i.e.*

$$ {}^{m}\underset{\sim}{\alpha}^{i}={}^{m}\underset{\sim}{\alpha}^{i}\left(\bar{\rho},\bar{\theta},{}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,\underset{\sim}{M}\right);i=0,1,\cdots,\underset{\sim}{N} \tag{197}$$

To determine the material coefficients from (197), one considers Taylor series expansion of each ${}^{m}\underset{\sim}{\alpha}^{i};i=0,1,\cdots,\underset{\sim}{N}$ in $\bar{\theta}$, ${}^{m}\underset{\sim}{I}^{j};\ j=1,2,\cdots,\underset{\sim}{M}$ about a known configuration $\underline{\Omega}$ and retains only up to linear terms in $\bar{\theta}$ and the invariants (for simplicity) and then substitutes these back in (196). Define the following

$$\begin{aligned}
{}^{0}\bar{m}\Big|_{\underline{\Omega}}&={}^{m}\underset{\sim}{\alpha}^{0}\Big|_{\underline{\Omega}}-\sum_{j=1}^{M}\left.\frac{\partial\left({}^{m}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{m}\underset{\sim}{I}^{j}\right)}\right|_{\underline{\Omega}}\left({}^{m}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\\
{}^{m}\underline{a}_{j}&=\left.\frac{\partial\left({}^{m}\underset{\sim}{\alpha}^{0}\right)}{\partial\left({}^{m}\underset{\sim}{I}^{j}\right)}\right|_{\underline{\Omega}};\quad j=1,2,\cdots,\underset{\sim}{M}\\
{}^{m}\underline{b}_{i}&={}^{m}\underset{\sim}{\alpha}^{i}\Big|_{\underline{\Omega}}-\sum_{j=1}^{M}\left.\frac{\partial\left({}^{m}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{m}\underset{\sim}{I}^{j}\right)}\right|_{\underline{\Omega}}\left({}^{m}\underset{\sim}{I}^{j}\right)_{\underline{\Omega}}\\
{}^{m}\underline{c}_{ij}&=\left.\frac{\partial\left({}^{m}\underset{\sim}{\alpha}^{i}\right)}{\partial\left({}^{m}\underset{\sim}{I}^{j}\right)}\right|_{\underline{\Omega}};\quad \begin{array}{l}i=1,2,\cdots,\underset{\sim}{N}\\ j=1,2,\cdots,\underset{\sim}{M}\end{array}\\
{}^{m}\underline{\alpha}_{tm}&=\left.\frac{\partial\left({}^{m}\underset{\sim}{\alpha}^{0}\right)}{\partial\bar{\theta}}\right|_{\underline{\Omega}}\\
{}^{m}\underline{d}_{i}&=\left.\frac{\partial\left({}^{m}\underset{\sim}{\alpha}^{i}\right)}{\partial\bar{\theta}}\right|_{\underline{\Omega}};\quad i=1,2,\cdots,\underset{\sim}{N}
\end{aligned} \tag{198}$$

Then using (198) in (196) can be written as

$$\begin{aligned}
\left[{}^{(0)}\bar{m}\right]={}&{}^{0}\bar{m}\Big|_{\underline{\Omega}}[I]+\sum_{j=1}^{M}{}^{m}\underline{a}_{j}\,{}^{m}\underset{\sim}{I}^{j}[I]-{}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I]+\sum_{i=1}^{N}{}^{m}\underline{b}_{i}\left[{}^{m}\underset{\sim}{G}^{i}\right]\\
&+\sum_{i=1}^{N}\sum_{j=1}^{M}{}^{m}\underline{c}_{ij}\,{}^{m}\underset{\sim}{I}^{j}\left[{}^{m}\underset{\sim}{G}^{i}\right]-\sum_{i=1}^{N}{}^{m}\underline{d}_{i}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{m}\underset{\sim}{G}^{i}\right]
\end{aligned} \tag{199}$$

${}^m\underline{a}_j, {}^m\underline{b}_i, {}^m\underline{c}_{ij}, {}^m\underline{d}_i$ and ${}^m\underline{\alpha}_{tm}$ are material coefficients defined in known configurations $\underline{\Omega}$. This constitutive theory requires $\left(\underset{\sim}{M}+\underset{\sim}{N}+\left(\underset{\sim}{N}\right)\left(\underset{\sim}{M}\right)+\underset{\sim}{N}+1\right)$ material coefficients. The material coefficients defined in (198) are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and $\left({}^m\underset{\sim}{I}^j\right)_{\underline{\Omega}}$; $j=1,2,\cdots,M$. This constitutive theory is based on integrity, hence is complete.

5.5. Remarks

1) The constitutive theories for ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}, {}^{(0)}\bar{\boldsymbol{q}}$ and ${}^{(0)}\bar{\boldsymbol{m}}$ can be made basis specific by choosing these and ${}^{(k)}\gamma; k=1,2,\cdots,n$ and ${}^{(j)}_{\Theta}\gamma; j=1,2,\cdots,{}^1n$ specific to the basis of choice.

2) The configuration $\underline{\Omega}$ can be chosen to be reference configuration (undeformed configuration before the commencement of the evolution) in which case the material coefficients will be independent of the deformation. If one chooses $\underline{\Omega}$ to be a known deformed configuration, then the deformation dependent material coefficients are possible in the constitutive theories. Dependence of the material coefficients on the invariants of the argument tensor in a known configuration $\underline{\Omega}$ permits complex description of material coefficients.

3) An important point to note is that the material coefficients in the final forms of the constitutive theories are defined in a known configuration $\underline{\Omega}$, whereas the constitutive equations hold in the current configuration for which the deformation field is yet to be determined. This of course is a consequence of the Taylor series expansion of the coefficients in the linear combination (using combined generators) about a known configuration. In the currently used constitutive models in the published works [28] for variable material coefficients, the coefficients are expressed as a function of the unknown deformation field in the current configuration. This is obviously not supported by the derivations presented in Sections 5.2-5.4.

4) Using the derivations presented in Sections 5.2-5.4 rate constitutive theories of various orders in desired basis can be derived by choosing values of n and ${}^1n$, the orders of the rate theory. As the orders of the rate theory increase, the number of material constants increases significantly. Thus, the higher order rate theories necessitate elaborate experiments to calibrate them.

5) In the following rate theories of orders one ($n={}^1n=1$) and their further simplifications are considered to present rather simple theories that could be used in model problem studies.

5.6. Rate Constitutive Theories of Order One ($n=1$) for ${}_d\left({}^{(0)}_{s}\bar{\sigma}\right)$: Compressible

This is the simplest possible constitutive theory for ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ in which there is interaction between $\left[{}^{(1)}\gamma\right]$ and $\{\bar{g}\}$. Consider

$$\left[{}_d\left({}^{(0)}_{s}\bar{\sigma}\right)\right]=\left[{}_d\left({}^{(0)}_{s}\bar{\sigma}\right)\left(\bar{\rho},\left[{}^{(1)}\gamma\right],\bar{\theta},\bar{\boldsymbol{g}}\right)\right] \tag{200}$$

In this case the combined generators of $\left[{}^{(1)}\gamma\right]$ and $\{\bar{g}\}$ that are symmetric tensors of rank two are ($N=5$)

$$\begin{aligned}&\left[{}^{\sigma}\underset{\sim}{G}^{1}\right]=\left[{}^{(1)}\gamma\right],\quad\left[{}^{\sigma}\underset{\sim}{G}^{2}\right]=\left[{}^{(1)}\gamma\right]^{2},\quad\left[{}^{\sigma}\underset{\sim}{G}^{3}\right]=\{\bar{g}\}\{\bar{g}\}^{\mathrm{T}}\\&\left[{}^{\sigma}\underset{\sim}{G}^{4}\right]=\{\bar{g}\}\left\{\left[{}^{(1)}\gamma\right]\{\bar{g}\}\right\}^{\mathrm{T}}+\left\{\left[{}^{(1)}\gamma\right]\{\bar{g}\}\right\}\{\bar{g}\}^{\mathrm{T}}\\&\left[{}^{\sigma}\underset{\sim}{G}^{5}\right]=\{\bar{g}\}\left\{\left[{}^{(1)}\gamma\right]^{2}\{\bar{g}\}\right\}^{\mathrm{T}}+\left\{\left[{}^{(1)}\gamma\right]^{2}\{\bar{g}\}\right\}\{\bar{g}\}^{\mathrm{T}}\end{aligned}\tag{201}$$

and the combined invariants of $\left[{}^{(1)}\gamma\right]$, $\{\bar{g}\}$ are ($M=6$)

$$\begin{aligned}&{}^{\sigma}\underset{\sim}{I}^{1}=\mathrm{tr}\left[{}^{(1)}\gamma\right],\quad{}^{\sigma}\underset{\sim}{I}^{2}=\mathrm{tr}\left(\left[{}^{(1)}\gamma\right]^{2}\right),\quad{}^{\sigma}\underset{\sim}{I}^{3}=\mathrm{tr}\left(\left[{}^{(1)}\gamma\right]^{3}\right)\\&{}^{\sigma}\underset{\sim}{I}^{4}=\{\bar{g}\}^{\mathrm{T}}\{\bar{g}\},\quad{}^{\sigma}\underset{\sim}{I}^{5}=\{\bar{g}\}^{\mathrm{T}}\left\{\left[{}^{(1)}\gamma\right]\{\bar{g}\}\right\},\quad{}^{\sigma}\underset{\sim}{I}^{6}=\{\bar{g}\}^{\mathrm{T}}\left\{\left[{}^{(1)}\gamma\right]^{2}\{\bar{g}\}\right\}\end{aligned}\tag{202}$$

Thus, one can write

$$\left[{}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)\right]={}^{\sigma}\alpha^{0}[I]+\sum_{i=1}^{5}{}^{\sigma}\underset{\sim}{\alpha}^{i}\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]\tag{203}$$

Following the general derivations in Section 5.2 for N generators and M invariants, for this specific case one can write

$$\begin{aligned}\left[{}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)\right]=&{}^{0}\bar{\sigma}\big|_{\underline{\Omega}}[I]+\sum_{j=1}^{6}{}^{\sigma}\underline{a}_{j}\,{}^{\sigma}\underset{\sim}{I}^{j}[I]-\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I]+\sum_{i=1}^{5}{}^{\sigma}\underline{b}_{i}\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]\\&+\sum_{i=1}^{5}\sum_{j=1}^{6}{}^{\sigma}\underline{c}_{ij}\,{}^{\sigma}\underset{\sim}{I}^{j}\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]-\sum_{i=1}^{5}{}^{\sigma}\underline{d}_{i}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{\sigma}\underset{\sim}{G}^{i}\right]\end{aligned}\tag{204}$$

The definitions of material coefficients ${}^{\sigma}\underline{a}_{j},{}^{\sigma}\underline{b}_{i},{}^{\sigma}\underline{c}_{ij},{}^{\sigma}\underline{d}_{i}$ and $\underline{\alpha}_{\mathrm{tm}}$ as well as ${}^{0}\bar{\sigma}\big|_{\underline{\Omega}}$ remain the same as defined in (189). This constitutive theory requires 46 material coefficients, still too many to determine experimentally.

5.6.1. Simplified Rate Constitutive Theory of Order One ($n=1$) for ${}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)$: Compressible

Consider a constitutive theory in which ${}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)$ is not dependent on $\{\bar{g}\}$ *i.e.*

$$\left[{}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)\right]=\left[{}_{d}\left({}^{(0)}_{s}\bar{\sigma}\right)\left(\bar{\rho},\left[{}^{(1)}\gamma\right],\bar{\theta}\right)\right]\tag{205}$$

In this case there are only two generators ($N=2$) and three invariants ($M=3$)

$$\begin{aligned}&\left[{}^{\sigma}\underset{\sim}{G}^{1}\right]=\left[{}^{(1)}\gamma\right],\quad\left[{}^{\sigma}\underset{\sim}{G}^{2}\right]=\left[{}^{(1)}\gamma\right]^{2}\\&{}^{\sigma}\underset{\sim}{I}^{1}=\mathrm{tr}\left[{}^{(1)}\gamma\right],\quad{}^{\sigma}\underset{\sim}{I}^{2}=\mathrm{tr}\left(\left[{}^{(1)}\gamma\right]^{2}\right),\quad{}^{\sigma}\underset{\sim}{I}^{3}=\mathrm{tr}\left(\left[{}^{(1)}\gamma\right]^{3}\right)\end{aligned}\tag{206}$$

and the following constitutive theory (using (190) for $N=2$ and $M=3$) is obtained.

$$\begin{aligned}\left[{}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)\right] &= {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)[I] + {}^\sigma\underline{a}_2\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^2\right)[I] \\ &\quad + {}^\sigma\underline{a}_3\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^3\right)[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] + {}^\sigma\underline{b}_1\left[{}^{(1)}\gamma\right] \\ &\quad + {}^\sigma\underline{b}_2\left[{}^{(1)}\gamma\right]^2 + {}^\sigma\underline{c}_{11}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)\left[{}^{(1)}\gamma\right] + {}^\sigma\underline{c}_{12}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^2\right)\left[{}^{(1)}\gamma\right] \\ &\quad + {}^\sigma\underline{c}_{13}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^3\right)\left[{}^{(1)}\gamma\right] + {}^\sigma\underline{c}_{21}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)\left[{}^{(1)}\gamma\right]^2 \\ &\quad + {}^\sigma\underline{c}_{22}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^2\right)\left[{}^{(1)}\gamma\right]^2 + {}^\sigma\underline{c}_{23}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^3\right)\left[{}^{(1)}\gamma\right]^2 \\ &\quad + {}^\sigma\underline{d}_1\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{(1)}\gamma\right] + {}^\sigma\underline{d}_2\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{(1)}\gamma\right]^2\end{aligned} \tag{207}$$

This constitutive theory requires 14 material coefficients and contains up to fifth degree terms in the components of $\left[{}^{(1)}\gamma\right]$.

5.6.2. Simplified Rate Constitutive Theory of Order One ($n=1$) for ${}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)$ That Is Quadratic in the Components of $\left[{}^{(1)}\gamma\right]$: Compressible

Begin with (204) and neglect those terms on the right side of (204) that are of degree higher than two in the components of $\left[{}^{(1)}\gamma\right]$.

$$\begin{aligned}\left[{}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)\right] &= {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)[I] + {}^\sigma\underline{a}_2\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^2\right)[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] \\ &\quad + {}^\sigma\underline{b}_1\left[{}^{(1)}\gamma\right] + {}^\sigma\underline{b}_2\left[{}^{(1)}\gamma\right]^2 + {}^\sigma\underline{c}_{11}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)\left[{}^{(1)}\gamma\right] \\ &\quad + {}^\sigma\underline{d}_1\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{(1)}\gamma\right] + {}^\sigma\underline{d}_2\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}^{(1)}\gamma\right]^2\end{aligned} \tag{208}$$

This constitutive theory requires 8 material coefficients.

If one further neglects the product terms in $\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)$ (last two terms on the right side of (208)) in (208) one obtains

$$\begin{aligned}\left[{}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)\right] &= {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^\sigma\underline{a}_1\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)[I] + {}^\sigma\underline{a}_2\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]^2\right)[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] \\ &\quad + {}^\sigma\underline{b}_1\left[{}^{(1)}\gamma\right] + {}^\sigma\underline{b}_2\left[{}^{(1)}\gamma\right]^2 + {}^\sigma\underline{c}_{11}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)\left[{}^{(1)}\gamma\right]\end{aligned} \tag{209}$$

This constitutive theory requires only six material coefficients. The dependence of the material coefficients on the invariants in (209) can be modified based on the assumptions used here or can be maintained as originally defined in (189).

5.6.3. Simplified Rate Constitutive Theory of Order One ($n=1$) for ${}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)$ That Is Linear in the Components of $\left[{}^{(1)}\gamma\right]$: Compressible

If one neglects quadratic terms in $\left[{}^{(1)}\gamma\right]$ in (209), then one obtains a constitutive theory for ${}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)$ that is linear in $\left[{}^{(1)}\gamma\right]$.

$$\left[{}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)\right] = {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + {}^{\sigma}\underline{a}_1\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)[I] + {}^{\sigma}\underline{b}_1\left[{}^{(1)}\gamma\right] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \tag{210}$$

If one denotes $\kappa_{\underline{\Omega}} = {}^{\sigma}\underline{a}_1$ and $2\eta_{\underline{\Omega}} = {}^{\sigma}\underline{b}_1$, then one can write (210) as

$$\left[{}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)\right] = {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + 2\eta_{\underline{\Omega}}\left[{}^{(1)}\gamma\right] + \kappa_{\underline{\Omega}}\left(\operatorname{tr}\left[{}^{(1)}\gamma\right]\right)[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \tag{211}$$

Material coefficients $\eta_{\underline{\Omega}}$ and $\kappa_{\underline{\Omega}}$ can be functions of $\bar{\rho}\big|_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and invariants of $\left[{}^{(1)}\gamma\right]$ in the known configuration $\underline{\Omega}$. The constitutive theory (211) is the simplest possible constitutive theory for deviatoric symmetric Cauchy stress tensor.

5.7. Remarks on Constitutive Theories for ${}_d\left({}^{(0)}_{\ s}\bar{\sigma}\right)$: Compressible

1) One notes that the arguments of ${}_d\left({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\right)$ are same as those of ${}^{(0)}_{\ d}\bar{\boldsymbol{\sigma}}$, deviatoric Cauchy stress tensor for classical thermofluids [4]. Thus the constitutive theories for ${}_d\left({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\right)$ for internal polar thermofluids are the same as those for ${}^{(0)}_{\ d}\bar{\boldsymbol{\sigma}}$ for classical thermofluids. The fundamental difference is that even though the constitutive theories are the same, they are for different stress measures. ${}^{(0)}_{\ d}\bar{\boldsymbol{\sigma}}$ is the deviatoric part of the total Cauchy stress tensor, whereas ${}_d\left({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\right)$ is the deviatoric part of the symmetric part of the Cauchy stress tensor.

2) Some specific remarks can be made for the simplified rate theory of order one given by (211). When one compares (211) with the similar theory for ${}^{(0)}_{\ d}\bar{\boldsymbol{\sigma}}$, one notes that η and κ are similar to first and second viscosities and $\underline{\alpha}_{\text{tm}}$ is thermal modulus. Since

$$\left[{}^{(1)}\gamma\right] = \left[\gamma^{(1)}\right] = \left[\gamma_{(1)}\right] = \left[{}^{(1)}\gamma^{J}\right] = \left[\bar{D}\right] \tag{212}$$

Equation (211) implies that

$${}_d\left({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}\right) = {}_d\left({}_s\bar{\boldsymbol{\sigma}}^{(0)}\right) = {}_d\left({}_s\bar{\boldsymbol{\sigma}}_{(0)}\right) = {}_d\left({}^{(0)}_{\ s}\bar{\boldsymbol{\sigma}}^{J}\right) = {}_d\left({}_s\bar{\boldsymbol{\sigma}}\right) \tag{213}$$

Hence, one can write (211) as

$$\left[{}_d\left({}_s\bar{\sigma}\right)\right] = {}^0\bar{\sigma}\big|_{\underline{\Omega}}[I] + 2\eta_{\underline{\Omega}}\left[\bar{D}\right] + \kappa_{\underline{\Omega}}\left(\operatorname{tr}\left[\bar{D}\right]\right)[I] - \underline{\alpha}_{\text{tm}}\left(\bar{\theta} - \bar{\theta}_{\underline{\Omega}}\right)[I] \tag{214}$$

That is, the linear constitutive theory of order one in (214) for deviatoric Cauchy stress tensor is basis independent.

3) Since the material coefficients $\eta_{\underline{\Omega}}$ and $\kappa_{\underline{\Omega}}$ are functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and invariants of $\left[\bar{D}\right]$ in the known configuration $\underline{\Omega}$, they can be defined using power law, Carreau-Yasuda model, Sutherland law, etc. similar to classical thermofluids (see reference [28]). In case of incompressible fluid $\operatorname{tr}\left[\bar{D}\right] = 0$ in (214).

5.8. Simplified Constitutive Theories for ${}^{(0)}\bar{m}$: Compressible

The most general constitutive theory for Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ has been presented in Section 5.4. Unfortunately this constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ requires forty seven material coefficients. In this section simplified constitutive theories are presented that are derived using the general constitutive theory presented in Section 5.4.

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5.8.1. Constitutive Theory for ${}^{(0)}\bar{m}$ without $\{\bar{g}\}$ as Argument Tensor: Compressible Case

In this case

$$\left[{}^{(0)}\bar{m}\right]=\left[{}^{(0)}\bar{m}\left(\bar{\rho},\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right],\bar{\theta}\right)\right] \tag{215}$$

and $\underset{\sim}{N}=2$ and $\underset{\sim}{M}=3$. The generators and invariants are

$$\left[{}^{m}\underset{\sim}{G}^{1}\right]=\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right],\quad \left[{}^{m}\underset{\sim}{G}^{2}\right]=\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2} \tag{216}$$

$${}^{m}\underset{\sim}{I}^{1}=\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right],\quad {}^{m}\underset{\sim}{I}^{2}=\operatorname{tr}\left(\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\right),\quad {}^{m}\underset{\sim}{I}^{3}=\operatorname{tr}\left(\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{3}\right) \tag{217}$$

The constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ based on the theory of generators and invariants is given by (199) with $\underset{\sim}{N}=2$ and $\underset{\sim}{M}=3$. This constitutive theory requires fourteen material coefficients, still too many for practical applications. Explicit form is given by the following after Taylor series expansion of the coefficients in the linear combination about a known configuration $\underline{\Omega}$.

$$\begin{aligned}\left[{}^{(0)}\bar{m}\right]=&\;{}^{0}\bar{m}\big|_{\underline{\Omega}}[I]+{}^{m}\underline{a}_{1}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\right)[I]+{}^{m}\underline{a}_{2}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\right)[I]\\&+{}^{m}\underline{a}_{3}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{3}\right)[I]-{}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I]+{}^{m}\underline{b}_{1}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\\&+{}^{m}\underline{b}_{2}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}+{}^{m}\underline{c}_{11}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\\&+{}^{m}\underline{c}_{12}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]+{}^{m}\underline{c}_{13}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{3}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\\&+{}^{m}\underline{c}_{21}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}+{}^{m}\underline{c}_{22}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\\&+{}^{m}\underline{c}_{23}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{3}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}+{}^{m}\underline{d}_{1}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\\&+{}^{m}\underline{d}_{2}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\end{aligned} \tag{218}$$

The material coefficients in (218) are defined by (198).

5.8.2. Constitutive Theory for ${}^{(0)}\bar{m}$ That Is Quadratic in $\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]$ But Independent of $\{\bar{g}\}$: Compressible

One begins with (218) and neglects those terms on the right side of (218) that are of degree higher than two in the components of $\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]$.

$$\begin{aligned}\left[{}^{(0)}\bar{m}\right]=&\;{}^{0}\bar{m}\big|_{\underline{\Omega}}[I]+{}^{m}\underline{a}_{1}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\right)[I]+{}^{m}\underline{a}_{2}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\right)[I]\\&-{}^{m}\underline{\alpha}_{\mathrm{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I]+{}^{m}\underline{b}_{1}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]+{}^{m}\underline{b}_{2}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\\&+{}^{m}\underline{c}_{11}\left(\operatorname{tr}\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]+{}^{m}\underline{d}_{1}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]\\&+{}^{m}\underline{d}_{2}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)\left[{}_{s}\left({}_{\Theta}^{(1)}\gamma\right)\right]^{2}\end{aligned} \tag{219}$$

This constitutive theory requires eight material coefficients. If one further neglects the product terms in $\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)$ (last two terms on the right side of (219)) in (219), then one obtains

$$\left[{}^{(0)}\bar{m}\right]={}^{0}\bar{m}\big|_{\underline{\Omega}}[I]+{}^{m}\underline{a}_1\left(\operatorname{tr}\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]\right)[I]+{}^{m}\underline{a}_2\left(\operatorname{tr}\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]^2\right)[I] \\ -{}^{m}\underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I]+{}^{m}\underline{b}_1\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]+{}^{m}\underline{b}_2\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]^2 \\ +{}^{m}\underline{c}_{11}\left(\operatorname{tr}\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]\right)\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right] \tag{220}$$

This constitutive theory requires only six material coefficients. The dependence of the material coefficients on the invariants in (220) can be modified based on the assumptions used here or can be maintained as originally defined in (198).

5.8.3. Constitutive Theory for ${}^{(0)}\bar{\boldsymbol{m}}$ That Is Linear in $\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]$ But Independent of $\{\bar{g}\}$: Compressible

$$\left[{}^{(0)}\bar{m}\right]={}^{0}\bar{m}\big|_{\underline{\Omega}}[I]+{}^{m}\underline{a}_1\left(\operatorname{tr}\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]\right)[I]+{}^{m}\underline{b}_1\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]-{}^{m}\underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] \tag{221}$$

If one denotes ${}^{m}\kappa_{\underline{\Omega}}={}^{m}\underline{a}_1$ and $2\left({}^{m}\eta\right)_{\underline{\Omega}}={}^{m}\underline{b}_1$, then one can write (221) as

$$\left[{}^{(0)}\bar{m}\right]={}^{0}\bar{m}\big|_{\underline{\Omega}}[I]+2\left({}^{m}\eta\right)_{\underline{\Omega}}\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]+{}^{m}\kappa_{\underline{\Omega}}\left(\operatorname{tr}\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]\right)[I]-{}^{m}\underline{\alpha}_{\text{tm}}\left(\bar{\theta}-\bar{\theta}_{\underline{\Omega}}\right)[I] \tag{222}$$

The material coefficients ${}^{m}\kappa_{\underline{\Omega}}$ and ${}^{m}\eta_{\underline{\Omega}}$ can be functions of $\bar{\rho}_{\underline{\Omega}}$, $\bar{\theta}_{\underline{\Omega}}$ and invariants of $\left[{}_s\left({}^{(1)}_{\Theta}\gamma\right)\right]$ in the known configuration $\underline{\Omega}$. The constitutive theory (222) is the simplest possible constitutive theory for Cauchy moment tensor ${}^{(0)}\bar{\boldsymbol{m}}$ but permits deformation dependent material coefficients. Thus, here also one can use concepts similar to power law, Carreau-Yasuda model, Sutherland law etc. that are used for the material coefficients in the constitutive theory for ${}_d\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$.

5.9. Remarks

In reference [3] instead of $\left[{}_s\left({}^{(k)}_{\Theta}\gamma\right)\right]; k=1,2,\cdots,{}^{1}n$, the symmetric part of the convected time derivatives of the rotation gradient tensor, only $\left[{}^{\Theta}\bar{D}\right]$ was used as conjugate of $\left[{}^{(0)}\bar{m}\right]$. Details are presented in the following. Recall ${}^{(0)}\bar{m}_{ji}\dfrac{\partial\left({}^{t}\bar{\Theta}_i\right)}{\partial\bar{x}_j}$ from the derivation of the energy equation. Since ${}^{(0)}\bar{m}_{ji}$ is basis dependent $\dfrac{\partial\left({}^{t}\bar{\Theta}_i\right)}{\partial\bar{x}_j}$ must also be basis dependent *i.e.* it must be the convected time derivative of the rotation gradient tensor *i.e.* one must replace it with $\left[{}^{(1)}_{\Theta}\gamma\right]$. This treatment is consistent. Authors in [3] use the following.

$${}^{(0)}\bar{m}_{ji}\frac{\partial\left({}^{t}\bar{\Theta}_i\right)}{\partial\bar{x}_j}={}^{(0)}\bar{m}_{ji}\,{}^{\Theta}L_{ij}=\operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{\Theta}L\right]\right) \tag{223}$$

$\left[{}^{\Theta}\bar{L}\right]$ is the gradients of the rotation rates, not the convected time derivatives of the rotation gradient tensor.

$$\left[{}^{\Theta}L\right]=\left[{}^{\Theta}\bar{D}\right]+\left[{}^{\Theta}\bar{W}\right] \tag{224}$$

where

$$\left[{}^{\Theta}\bar{D}\right]=\frac{1}{2}\left(\left[{}^{\Theta}L\right]+\left[{}^{\Theta}L\right]^{\mathrm{T}}\right);\quad \left[{}^{\Theta}\bar{W}\right]=\frac{1}{2}\left(\left[{}^{\Theta}L\right]-\left[{}^{\Theta}L\right]^{\mathrm{T}}\right) \tag{225}$$

Hence,

$$\operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[\left[{}^{\Theta}\bar{D}\right]+\left[{}^{\Theta}\bar{W}\right]\right]\right)=\operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{\Theta}\bar{D}\right]\right) \tag{226}$$

as

$$\left[{}^{(0)}\bar{m}\right]=\left[{}^{(0)}\bar{m}\right]^{\mathrm{T}} \tag{227}$$

Thus

$${}^{(0)}\bar{m}_{ji}\frac{\partial\left({}^{t}\bar{\Theta}_i\right)}{\partial\bar{x}_j}=\operatorname{tr}\left(\left[{}^{(0)}\bar{m}\right]\left[{}^{\Theta}\bar{D}\right]\right) \tag{228}$$

This is an approximation as $\left[{}^{\Theta}\bar{D}\right]$ is an approximation to the symmetric part of the convected time derivative of the rotation gradient tensor. Use of $\left[{}^{\Theta}\bar{D}\right]$ is justified only when the rates of rotation gradients are very small in which case

$$\left[{}_{s}\left({}^{(1)}_{\Theta}\gamma\right)\right]\approx\left[{}^{\Theta}\bar{D}\right] \tag{229}$$

Recall that

$$\left[{}_{\Theta}\gamma_{(1)}\right]=\frac{D}{Dt}\left[{}^{\Theta}\bar{J}_{(0)}\right]+\left[\bar{L}\right]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right]+\left[{}^{\Theta}\bar{J}_{(0)}\right]\left[\bar{L}\right]+\left[{}^{\Theta}\bar{J}_{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right) \tag{230}$$

or

$$\begin{aligned}\left[{}_{\Theta}\gamma_{(1)}\right]=&\frac{\partial}{\partial t}\left[{}^{\Theta}\bar{J}_{(0)}\right]+\left(\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\nabla}}\right)\left[{}^{\Theta}\bar{J}_{(0)}\right]+\left[\bar{L}\right]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right]\\&+\left[{}^{\Theta}\bar{J}_{(0)}\right]\left[\bar{L}\right]+\left[{}^{\Theta}\bar{J}_{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right)\end{aligned} \tag{231}$$

or

$$\left[{}_{\Theta}\gamma_{(1)}\right]=\left[{}^{\Theta}\bar{L}\right]+\left(\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\nabla}}\right)\left[{}^{\Theta}\bar{J}_{(0)}\right]+\left[\bar{L}\right]^{\mathrm{T}}\left[{}^{\Theta}\bar{J}_{(0)}\right]+\left[{}^{\Theta}\bar{J}_{(0)}\right]\left[\bar{L}\right]+\left[{}^{\Theta}\bar{J}_{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right) \tag{232}$$

Likewise

$$\left[{}_{\Theta}\gamma^{(1)}\right]=\frac{D}{Dt}\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[\bar{L}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}^{\Theta}\bar{J}^{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right) \tag{233}$$

or

$$\begin{aligned}\left[{}_{\Theta}\gamma^{(1)}\right]=&\frac{\partial}{\partial t}\left[{}^{\Theta}\bar{J}^{(0)}\right]+\left(\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\nabla}}\right)\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[\bar{L}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]\\&-\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}^{\Theta}\bar{J}^{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right)\end{aligned} \tag{234}$$

or

$$\left[{}_{\Theta}\gamma^{(1)}\right]=\left[{}^{\Theta}\bar{L}\right]+\left(\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\nabla}}\right)\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[\bar{L}\right]\left[{}^{\Theta}\bar{J}^{(0)}\right]-\left[{}^{\Theta}\bar{J}^{(0)}\right]\left[\bar{L}\right]^{\mathrm{T}}+\left[{}^{\Theta}\bar{J}^{(0)}\right]\operatorname{tr}\left(\left[\bar{L}\right]\right) \tag{235}$$

Thus for small rates of rotation gradients one can write

$$\left[{}_s\left({}_\Theta\gamma_{(1)}\right)\right]\approx\left[{}_s\left({}_\Theta\gamma^{(1)}\right)\right]\approx\left[{}^\Theta\bar{D}\right] \tag{236}$$

Thus, (228) used in reference [3] is an approximation to the symmetric part of the convected time derivative of the rotation gradient tensor. Use of ${}^{(k)}_{\Theta}\boldsymbol{\gamma};\,k=1,2,\cdots,{}^1n$ in the constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ is consistent and removes the restriction of small rates of rotation gradients.

6. Summary and Conclusions

In this paper ordered rate constitutive theories of orders n and ${}^1n$ are presented for internal polar non-classical, isotropic, homogeneous thermofluids in which the varying rates of rotations and conjugate moments in addition to usual thermofluid physics (classical) are considered in the derivations of the conservation and balance laws. The constitutive theories are presented in contravariant basis, covariant basis, and using Jaumann rates, but the derivation of the constitutive theories is carried out using basis independent stress tensor, heat vector, and moment tensor. By choosing these in the desired basis, basis dependent constitutive theories can be obtained. The theory of generators and invariants in conjunction with the conditions resulting from entropy inequality form the basis for the derivation of the constitutive theories.

The dependent variables in the constitutive theories are established by examining conservation and balance laws in conjunction with principle of casualty [4] [9]. By examining the conditions resulting from entropy inequality and after introducing stress decomposition one finally arrives at $\bar{\Phi},\bar{\eta},{}^{(0)}\bar{\boldsymbol{q}},{}^{(0)}_{s}\bar{\boldsymbol{\sigma}},{}^{(0)}\bar{\boldsymbol{m}}$ as dependent variables in the constitutive theory. Back superscript zero implies that these quantities are basis dependent. From entropy inequality the conjugate pairs are established:

$$\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}},{}^{(1)}\boldsymbol{\gamma}\right),\left({}^{(0)}\bar{\boldsymbol{m}},{}^{(1)}_{\Theta}\boldsymbol{\gamma}\right),\left({}^{(0)}\bar{\boldsymbol{q}},\boldsymbol{g}\right).$$

where ${}^{(1)}\boldsymbol{\gamma}$ and ${}^{(1)}_{\Theta}\boldsymbol{\gamma}$ are convected time derivatives of appropriate basis dependent strain tensor and rotation gradient tensor. In reference [3] ${}^\Theta\bar{\boldsymbol{D}}$, symmetric part of the gradients of rotation rates is used in place of ${}^{(1)}_{\Theta}\boldsymbol{\gamma}$. This only holds when the rates of rotation gradients are small. This is a fundamental difference between the rate theories presented here and those in [3]. Following [3] one may generalize the derivation of the constitutive theory by replacing ${}^{(1)}\boldsymbol{\gamma}$ with ${}^{(k)}\boldsymbol{\gamma};k=1,2,\cdots,n$. However with the use of ${}^\Theta\bar{\boldsymbol{D}}$ conjugate to ${}^{(0)}\bar{\boldsymbol{m}}$ such generalization was not possible in [3]. This is the second major difference in the constitutive theories presented here and those in [3]. ${}^\Theta\bar{\boldsymbol{D}}$ is also replaced with ${}^{(1)}_{\Theta}\boldsymbol{\gamma}$ at the onset of the derivation of energy equation and subsequently ${}^{(1)}_{\Theta}\boldsymbol{\gamma}$ is replaced with ${}^{(k)}_{\Theta}\boldsymbol{\gamma};\,k=1,2,\cdots,{}^1n$. This gives

$$\bar{\rho},{}^{(k)}\boldsymbol{\gamma};\,k=1,2,\cdots,n,\,{}^{(j)}_{\Theta}\boldsymbol{\gamma};\;j=1,2,\cdots,{}^1n,\bar{\theta}\text{ and }\bar{\boldsymbol{g}}$$

as argument tensors of all dependent variables (based on principle of equipresence [4] [9]) in the constitutive theories for thermofluids. Using the conjugate pairs in entropy inequality it is straightforward to eliminate ${}^{(j)}_{\Theta}\boldsymbol{\gamma};\;j=1,2,\cdots,{}^1n$ as arguments of ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$

and ${}^{(k)}\gamma; k=1,2,\cdots,n$ as arguments of ${}^{(0)}\bar{\boldsymbol{m}}$. ${}^{(0)}_{s}\bar{\boldsymbol{\sigma}}$ is decomposed into equilibrium and deviatoric tensors $\left({}_{e}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right), {}_{d}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)\right)$. The constitutive theories for ${}_{e}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$ as thermodynamic pressure $\bar{p}\left(\bar{\rho},\bar{\theta}\right)$ for compressible matter and mechanical pressure $\bar{p}\left(\bar{\theta}\right)$ for incompressible matter are established using entropy inequality and incompressibility condition. Constitutive theories for

$$ {}_{d}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right) = {}_{d}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)\left(\bar{\rho}, {}^{(k)}\gamma; k=1,2,\cdots,n,\bar{\theta},\bar{\boldsymbol{g}}\right), $$

$$ {}^{(0)}\bar{\boldsymbol{m}} = {}^{(0)}\bar{\boldsymbol{m}}\left(\bar{\rho}, {}^{(j)}_{\Theta}\gamma;\ j=1,2,\cdots,{}^{1}n,\bar{\theta},\bar{\boldsymbol{g}}\right) $$

and

$$ {}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\left(\bar{\rho}, {}^{(k)}\gamma; k=1,2,\cdots,n, {}^{(j)}_{\Theta}\gamma;\ j=1,2,\cdots,{}^{1}n,\bar{\theta},\bar{\boldsymbol{g}}\right) $$

are established using theory of generators ad invariants. General constitutive theories of up to order n for ${}_{d}\left({}^{(0)}_{s}\bar{\boldsymbol{\sigma}}\right)$, of up to order ${}^{1}n$ for ${}^{(0)}\bar{\boldsymbol{m}}$ and of up to orders n and ${}^{1}n$ for ${}^{(0)}\bar{\boldsymbol{q}}$ are derived. The simplified forms for $n=1$ and ${}^{1}n=1$ are also given. Further simplification of the theories of orders $n=1$ and ${}^{1}n=1$ that are linear in ${}^{(1)}\boldsymbol{\gamma}$ and ${}^{(1)}_{\Theta}\boldsymbol{\gamma}$ are also derived.

It is clearly shown in the paper that choice of ${}^{\Theta}\bar{\boldsymbol{D}}$ conjugate to ${}^{(0)}\bar{\boldsymbol{m}}$ is justified when the rates of rotation gradients are small. For finite rates of rotation gradients convected time derivatives of the rotation the gradient tensor (in the basis of choice) must be considered conjugate to ${}^{(0)}\bar{\boldsymbol{m}}$. The derivations of the convected time derivatives of the rotation gradient tensor in contravariant basis and covariant basis in both Lagrangian and Eulerian descriptions are given in the paper. It is shown that use of ${}^{\Theta}\bar{\boldsymbol{D}}$ conjugate to ${}^{(0)}\bar{\boldsymbol{m}}$ is not only limiting in terms of magnitude of the rates of rotation gradients but it also inhibits extension of the constitutive theories for ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{q}}$ to higher rates of rotation gradient tensor. In the derivation of the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$, ${}^{(k)}\gamma; k=1,2,\cdots,n$ as well as ${}^{(j)}_{\Theta}\gamma; j=1,2,\cdots,{}^{1}n$ are retained as its argument tensors as there are no conditions or mechanism that suggest their removal from the argument list of ${}^{(0)}\bar{\boldsymbol{q}}$. The constitutive theory for heat vector as presented in this paper is naturally basis dependent.

The work presented in this paper removes the restriction of small rotation gradient rates due to use of ${}^{\Theta}\bar{\boldsymbol{D}}$ in [3] and presents a general constitutive theory for ${}^{(0)}\bar{\boldsymbol{m}}$ and ${}^{(0)}\bar{\boldsymbol{q}}$ that are functions of the true convected time derivatives of the rotation gradient tensor up to orders ${}^{1}n$ in the basis of choice.

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