

# Rate of convergence in probability to the Marchenko–Pastur law

FRIEDRICH GÖTZE<sup>1,\*</sup> and ALEXANDER TIKHOMIROV<sup>1,\*\*,2</sup>

<sup>1</sup>Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld 1, Germany.

E-mail: \*goetze@mathematik.uni-bielefeld.de; \*\*tichomir@mathematik.uni-bielefeld.de

<sup>2</sup>Faculty of Mathematics, Syktyvkar University, Oktjabrsky Prospect 55, 167001 Syktyvkar, Russia. E-mail: tichomir@syktsu.ru

It is shown that the Kolmogorov distance between the spectral distribution function of a random covariance  $(1/p)\mathbf{X}\mathbf{X}^T$ , where  $\mathbf{X}$  is an  $n \times p$  matrix with independent entries and the distribution function of the Marchenko–Pastur law is of order  $O(n^{-1/2})$  in probability. The bound is explicit and requires that the twelfth moment of the entries of the matrix is uniformly bounded and that  $p/n$  is separated from 1.

*Keywords:* independent random variables; random matrix; spectral distributions

## 1. Introduction and results

Let  $X_{ij}$ ,  $1 \leq i \leq p$ ,  $1 \leq j \leq n$ , be independent random variables with  $EX_{ij} = 0$  and  $EX_{ij}^2 = 1$ , and  $\mathbf{X}_p = (X_{ij})_{\{1 \leq i \leq p, 1 \leq j \leq n\}}$ . Denote by  $\lambda_1 \leq \dots \leq \lambda_p$  the eigenvalues of the symmetric matrix

$$\mathbf{W} := \mathbf{W}_p := \frac{1}{n} \mathbf{X}_p \mathbf{X}_p^T$$

and defined its empirical distribution by

$$F_p(x) = \frac{1}{p} \sum_{k=1}^p I_{\{\lambda_k \leq x\}},$$

where  $I_{\{B\}}$  denotes the indicator of an event  $B$ . We shall investigate the rate of convergence of the expected spectral distribution  $EF_p(x)$  as well as  $F_p(x)$  to the Marchenko–Pastur distribution function  $F_y(x)$  with density

$$f_y(x) = \frac{1}{2xy\pi} \sqrt{(b-x)(x-a)} I_{\{[a,b]\}}(x) + I_{\{(1,\infty)\}}(y)(1-y^{-1})\delta(x),$$

where  $y \in (0, \infty)$  and  $a = (1 - \sqrt{y})^2$ ,  $b = (1 + \sqrt{y})^2$ . Here we denote by  $\delta(x)$  the Dirac delta function and by  $I_{\{[a,b]\}}(x)$  the indicator function of the interval  $[a, b]$ . As in Marchenko and Pastur (1967) and Pastur (1973), assume that  $X_{ij}$ ,  $i, j \geq 1$ , are independent and identically distributed (i.i.d.) random variables such that

$$EX_{ij} = 0, \quad EX_{ij}^2 = 1, \quad E|X_{ij}|^4 \leq \infty, \quad \text{for all } i, j.$$

Then  $EF_p \rightarrow F_y$  and  $F_p \rightarrow F_y$  in probability, where  $y = \lim_{n \rightarrow \infty} y_p := \lim_{n \rightarrow \infty} (p/n) \in (0, \infty)$ . Yin (1986) has shown that the result holds in the i.i.d. case assuming  $EX_{ij}^2 = \sigma^2$  only. Wachter (1978) proved the result for independent  $X_{ij}$  with  $EX_{ij} = 0$ ,  $EX_{ij}^2 = 1$  and  $E|X_{ij}|^{2+\varepsilon} \leq C < \infty$ , for any  $\varepsilon > 0$ .

Let  $y := y_p := p/n$ . We introduce the following distance between the distributions  $EF_p(x)$  and  $F_y(x)$ ,

$$\Delta_p := \sup_x |EF_p(x) - F_y(x)|,$$

as well as another distance between the distributions  $F_p(x)$  and  $F_y(x)$ ,

$$\Delta_p^* := \sup_x |F_p(x) - F_y(x)|.$$

We shall use the notation  $\xi_n = O_p(a_n)$  if, for any  $\varepsilon > 0$ , there exists an  $L > 0$  such that  $P\{|\xi_n| \geq La_n\} \leq \varepsilon$ . Note that, for any  $L > 0$ ,

$$P\left\{\sup_x |F_p(x) - F_y(x)| \geq L\right\} \leq \frac{\Delta_p^*}{L}.$$

Hence bounds for  $\Delta_p^*$  provide bounds for the rate of convergence in probability of the quantity  $\sup_x |F_p(x) - F_y(x)|$  to zero. Using our techniques it is straightforward, though technical, to prove that the rate of almost sure convergence is at least  $O(n^{-1/2+\epsilon})$ , for any  $\epsilon > 0$ . In view of the length of the proofs for the results stated above we refrain from including the details in this paper as well.

Bai (1993b) proved that  $\Delta_p = O(n^{-1/4})$ , assuming  $EX_{ij} = 0$ ,  $EX_{ij}^2 = 1$ ,  $\sup_n \sup_{i,j} EX_{ij}^4 I_{\{|X_{ij}| > M\}} \rightarrow 0$  as  $M \rightarrow \infty$ , and

$$y \in (\theta, \Theta) \text{ such that } 0 < \theta < \Theta < 1 \text{ or } 1 < \theta < \Delta < \infty. \tag{1.1}$$

If  $y$  is close to 1 the limit density and the Stieltjes transform of the limit density have a singularity. In this case the investigation of the rate of convergence is more difficult. Bai (1993b) showed that, if  $0 < \theta \leq y_p \leq \Theta < \infty$ ,  $\Delta_p = O(n^{-5/48})$ . Recently Bai *et al.* (2003) have shown, for  $y_p$  equal to 1 or asymptotically near 1, that  $\Delta_p = O(n^{-1/8})$ . It is clear that the case  $y_p \approx 1$  requires different techniques. Recent results of the authors show that for Gaussian random variables  $X_{ij}$  the rate  $\Delta_p = O(n^{-1})$  is the correct rate of approximation.

In the present paper we shall consider bounds for  $\Delta_p$  in the case (1.1) only. By  $C$  (with or without an index) we shall denote generic absolute constants, whereas  $C(\cdot, \cdot)$  will denote positive constants depending on arguments. For  $k \geq 1$ , we introduce the notation

$$M_k := M_k^{(n)} := \sup_{1 \leq j, k \leq n} E|X_{jk}|^k.$$

Our main results are the following:

**Theorem 1.1.** *Let  $0 < \Theta_1 \leq y \leq \Theta_2 < \infty$  and  $|y - 1| \geq \theta > 0$ . Assume that  $X_{ij}$  satisfies the conditions above and that*

$$M_8 := \sup_{1 \leq j, k \leq n} \mathbb{E}|X_{jk}|^8 \leq \infty. \quad (1.2)$$

Then there exists an absolute constant  $C(\theta, \Theta_1, \Theta_2) > 0$  such that

$$\Delta_p \leq C(\theta, \Theta_1, \Theta_2) M_8^{1/4} n^{-1/2}.$$

**Theorem 1.2.** Let  $0 < \Theta_1 \leq y \leq \Theta_2 < \infty$  and  $|y - 1| \geq \theta > 0$ . Assume that  $X_{ij}$  satisfies the conditions above and condition (1.2), and that

$$M_{12} := \sup_{1 \leq j, k \leq n} \mathbb{E}|X_{jk}|^{12} < \infty. \quad (1.3)$$

Then there exists an absolute constant  $C(\theta, \Theta_1, \Theta_2) > 0$  such that

$$\mathbb{E} \Delta_p^* = \mathbb{E} \sup_x |F_p(x) - G(x)| \leq C(\theta, \Theta_1, \Theta_2) M_{12}^{1/6} n^{-1/2}.$$

## 2. Inequalities for the distance between distributions via Stieltjes transforms

We define the Stieltjes transform  $s(z)$  of a random variable  $\xi$  with distribution function  $F(x)$  (the Stieltjes transform  $s(z)$  of distribution function  $F(x)$ ) by

$$s(z) := \mathbb{E} \frac{1}{\xi - z} = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x), \quad z = u + iv, v > 0.$$

Given  $\varepsilon > 0$ , we introduce the intervals  $I_\varepsilon = [a + \varepsilon, b - \varepsilon]$  and  $I'_\varepsilon = [a + \frac{1}{2}\varepsilon, b - \frac{1}{2}\varepsilon]$ . Recall that  $a = a(y) = (1 - \sqrt{y})^2$  and  $b = b(y) = (1 + \sqrt{y})^2$ .

**Lemma 2.1.** Let  $F$  be a distribution function and let  $F_y$  denote the Marchenko–Pastur distribution function. Denote their Stieltjes transforms by  $s(z)$  and  $s_y(z)$  respectively. Assume that  $\int_{-\infty}^{\infty} |F(x) - F_y(x)| dx < \infty$ . Let  $v > 0$ , and  $d$  and  $\varepsilon$  be positive numbers such that

$$\gamma = \frac{1}{\pi} \int_{|u| \leq d} \frac{1}{u^2 + 1} du = \frac{3}{4}, \quad (2.1)$$

and

$$\varepsilon > 2vd. \quad (2.2)$$

Assume that  $|y - 1| \geq \theta > 0$ . Then there exist some constants  $C_1(\theta), C_2(\theta), C_3(\theta)$ , depending only on  $\theta$ , such that

$$\begin{aligned} \Delta(F, F_y) &:= \sup_x |F(x) - F_y(x)| \\ &\leq C_1 \sup_{x \in I'_\varepsilon} \left| \operatorname{Im} \left( \int_{-\infty}^x (s(z) - s_y(z)) du \right) \right| + C_2 v + C_3 \varepsilon^{3/2}, \end{aligned}$$

where  $z = u + iv$ .

A proof of Lemma 2.1 is given in Götze and Tikhomirov (2000; 2003).

**Corollary 2.2.** *The following inequality holds:*

$$\begin{aligned} \Delta(F, F_y) \leq & C_1 \int_{-\infty}^{\infty} |(s(u + iV) - s_y(u + iV))| du + C_2 v + C_3 \varepsilon^{3/2} \\ & + C_1 \sup_{x \in I'_k} \left| \operatorname{Im} \left\{ \int_v^V (s(x + iu) - s_y(x + iu)) du \right\} \right|. \end{aligned} \tag{2.3}$$

### 3. The main lemma

We shall follow the notation of Bai (1993b). Let

$$s_y(z) = -\frac{y + z - 1 - \sqrt{(y + z - 1)^2 - 4yz}}{2yz}, \quad s_p(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dEF_p(x). \tag{3.1}$$

Note that, for  $z = u + iv$  such that  $a \leq u \leq b$ ,

$$|s_y(z)| \leq \frac{C|y - 1|}{|z|} + \frac{C}{\sqrt{|z|}} \leq \frac{C}{\sqrt{a}}. \tag{3.2}$$

By definition of  $F_p(x)$ , we can write

$$s_p(z) = E \left( \frac{1}{p} \sum_{j=1}^p \frac{1}{\lambda_j - z} \right) = \frac{1}{p} E \operatorname{tr} \mathbf{R} = \frac{1}{p} \sum_{j=1}^p ER(j, j), \tag{3.3}$$

where  $\mathbf{R} := \mathbf{R}(z) := (\mathbf{W} - z\mathbf{I}_p)^{-1} = (R(j, k))_{j,k=1}^p$ . Here  $\mathbf{I}_p$  denotes the  $p \times p$  identity matrix.

Set  $\mathbf{W}(k) = (1/n)\mathbf{X}(k)\mathbf{X}(k)^T$ , where  $\mathbf{X}(k)$  denotes the matrix obtained from  $\mathbf{X}$  by deleting the  $k$ th row, and let  $\mathbf{x}_k^T = (X_{k1}, \dots, X_{kn})$ . Set  $\mathbf{a}_k = (1/n)\mathbf{X}(k)\mathbf{x}_k$ . Write

$$\varepsilon_k = \frac{1}{n} \sum_{j=1}^n (X_{kj}^2 - 1) + y + yzs_p(z) - \mathbf{a}_k^T (\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1} \mathbf{a}_k. \tag{3.4}$$

We introduce the scalar

$$\delta_p(z) = -\frac{1}{n} \sum_{k=1}^p E \frac{\varepsilon_k}{(y + z - 1 + yzs_p(z))(y + z - 1 + yzs_p(z) - \varepsilon_k)} \tag{3.5}$$

and the matrix

$$\mathbf{R}_k = (\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1}.$$

For readers' convenience we state here two algebraic lemmas which are proved in Bai (1993a) and in Götze and Tikhomirov (2003). Let  $\mathbf{A} = (a_{kj})$  denote a matrix of order  $n$  and

$\mathbf{A}_k$  denote the principal submatrix of order  $n - 1$ , that is,  $\mathbf{A}_k$  is obtained from  $\mathbf{A}$  by deleting the  $k$ th row and the  $k$ th column. Let  $\mathbf{A}^{-1} = (a^{jk})$ . Let  $\mathbf{a}_k^T$  denote the vector obtained from the  $k$ th row of  $\mathbf{A}$  by deleting the  $k$ th entry and  $\mathbf{b}_k$  the vector from the  $k$ th column by deleting the  $k$ th entry. Let  $\mathbf{I}$ , with or without a subscript, denote the identity matrix of corresponding order.

**Lemma 3.1.** *Assume that  $\mathbf{A}$  and  $\mathbf{A}_k$  are non-singular. Then*

$$a^{kk} = \frac{1}{a_{kk} - \mathbf{a}_k^T \mathbf{A}_k^{-1} \mathbf{b}_k}.$$

**Lemma 3.2.** *Let  $z = u + iv$ , and  $\mathbf{A}$  be an  $n \times n$  symmetric matrix. Then*

$$\begin{aligned} \operatorname{tr}(\mathbf{A} - z\mathbf{I}_n)^{-1} - \operatorname{tr}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1} &= \frac{1 + \mathbf{a}_k^T (\mathbf{A}_k - z\mathbf{I}_{n-1})^{-2} \mathbf{a}_k}{a_{kk} - z - \mathbf{a}_k^T (\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1} \mathbf{a}_k} \\ &= (1 + \mathbf{a}_k^T (\mathbf{A}_k - z\mathbf{I}_{n-1})^{-2} \mathbf{a}_k) a^{kk} \end{aligned} \quad (3.6)$$

and

$$|\operatorname{tr}(\mathbf{A} - z\mathbf{I}_n)^{-1} - \operatorname{tr}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1}| \leq v^{-1}. \quad (3.7)$$

Applying Lemma 3.1 with  $\mathbf{A} = \mathbf{W}$  and relation (3.3), we may write

$$\begin{aligned} R(j, j) &= -\frac{1}{y + z - 1 + yzs_p(z) - \varepsilon_j} \\ &= -\frac{1}{y + z - 1 + yzs_p(z)} - \frac{\varepsilon_j}{(y + z - 1 + yzs_p(z))(y + z - 1 + yzs_p(z) - \varepsilon_j)}. \end{aligned} \quad (3.8)$$

This implies that

$$s_p(z) = -\frac{1}{y + z - 1 + yzs_p(z)} + \delta_p(z). \quad (3.9)$$

To prove Theorem 1.1 we shall use the result of Corollary 2.2.

The following inequality (3.10) was proved in Bai (1993b), but for readers' convenience we repeat its proof here. Throughout this paper we shall consider  $z = u + iv$  with  $a \leq u \leq b$  and  $0 < v < C$ .

**Lemma 3.3.** *Under the conditions of Theorem 1.1, for any  $v > 0$  and for any  $k = 1, \dots, n$ , we have*

$$|\mathbb{E}\varepsilon_k| \leq \frac{C}{nv}. \quad (3.10)$$

**Proof.** Let  $\mathbb{E}^{(k)}$  denote the conditional expectation given  $X_{ij}$ ,  $i \neq k$ . Note that the random

vector  $\mathbf{x}_k$  and the random matrices  $\mathbf{W}(k)$ ,  $\mathbf{X}_p(k)$  are independent. Using the definition of a vector  $\mathbf{a}_k$  and taking into account the above-mentioned independence, we obtain

$$\begin{aligned} E^{(k)} \mathbf{a}_k^T (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \mathbf{a}_k &= n^{-2} E^{(k)} \mathbf{x}^T(k) \mathbf{X}_p^T(k) (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \mathbf{X}_p(k) \mathbf{x}(k) \\ &= n^{-2} \text{tr} \mathbf{X}_p^T(k) (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \mathbf{x}_p(k) \\ &= \frac{1}{n} \text{tr} (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \mathbf{W}(k). \end{aligned}$$

Furthermore, since

$$(\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \mathbf{W}(k) = z (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} + \mathbf{I}_{p-1},$$

we obtain

$$\begin{aligned} E^{(k)} \mathbf{a}_k^T (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \mathbf{a}_k &= \frac{p-1}{n} + zyp^{-1} \text{tr} (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \\ &= y - \frac{1}{n} + zyp^{-1} \text{tr} (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1}. \end{aligned} \quad (3.11)$$

Equation (3.11) implies that

$$|E \varepsilon_k| = \frac{|zy|}{p} |E[\text{tr} (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} - \text{tr} (\mathbf{W} - z \mathbf{I}_p)^{-1}]| + \frac{1}{n}.$$

Using Lemma 3.2 with  $\mathbf{A} = \mathbf{W}$  and  $\mathbf{A}_k = \mathbf{W}(k)$ , we obtain inequality (3.10).  $\square$

Without loss of generality, we may assume that  $v \geq \beta \Delta_p$  with some constant  $0 < \beta < 1$  depending on  $\theta$  only. Thus, using inequality (3.2), we immediately obtain, for  $z = u + iv$  such that  $a \leq u \leq b$ ,

$$|s_p(z)| \leq |s_y(z)| + |s_p(z) - s_p(z)| \leq C_1(\theta)(1 + \beta^{-1}) \leq C(\theta)\beta^{-1}.$$

The main result of this section is the following:

**Lemma 3.4.** *Let*

$$\text{Im}\{yz\delta_p(z) + z\} \geq 0.$$

*Then there exists a positive constant  $a_1$  depending on  $\theta$ ,  $\Theta_1$ ,  $\Theta_2$  and  $\beta$  such that*

$$|z + y - 1 + yzs_p(z)| \geq a_1.$$

**Proof.** Assume that  $|z| \leq |y-1|/2(1 + yC(\theta)\beta^{-1})$ . This immediately implies that

$$|z + y - 1 + yzs_p(z)| \geq |y-1| - |z|(1 + y|s_p(z)|) \geq \frac{|y-1|}{2} \geq a_1 > 0.$$

Now let

$$|z| \geq \frac{|y-1|}{2(1+yC(\theta)\beta^{-1})}.$$

Equation (3.9) and the assumption of Lemma 3.4 together imply that

$$\operatorname{Im}\{z + yzs_p(z)\} \geq -\operatorname{Im}\left\{\frac{yz}{z + y - 1 + yzs_p(z)}\right\}.$$

Note that

$$\operatorname{Im}\{z + yzs_p(z)\} = v + y(v\operatorname{Re}\{s_p(z)\} + u\operatorname{Im}\{s_p(z)\}) = v + yvE \operatorname{tr} \mathbf{W} |\mathbf{R}|^2 > 0.$$

Hence,

$$|z + y - 1 + yzs_p(z)|^2 \geq \frac{-\operatorname{Im}\{yz(\bar{z} + y - 1 + y\bar{z}s_p(\bar{z}))\}}{\operatorname{Im}\{z + yzs_p(z)\}}, \quad (3.12)$$

where  $\bar{w}$  denotes the complex conjugate of  $w$ . Furthermore,

$$\begin{aligned} -\operatorname{Im}\{yz(\bar{z} + y - 1 + y\bar{z}s_p(\bar{z}))\} &= -y\operatorname{Im}\{|z|^2 + z(y-1) + y|z|^2\bar{s}_p(z)\} \\ &= y^2|z|^2\operatorname{Im}\{s_p(z)\} + y(1-y)v. \end{aligned} \quad (3.13)$$

If  $y \leq 1$ , we have

$$\frac{-\operatorname{Im}\{yz(\bar{z} + y - 1 + y\bar{z}s_p(\bar{z}))\}}{\operatorname{Im}\{z + yzs_p(z)\}} = \frac{y(1-y)v + y^2|z|^2\operatorname{Im}\{s_p(z)\}}{v + yv\operatorname{Re}\{s_p(z)\} + yu\operatorname{Im}\{s_p(z)\}}. \quad (3.14)$$

Assuming  $\operatorname{Im}\{s_p(z)\} \leq v$ , we obtain

$$\begin{aligned} \frac{-\operatorname{Im}\{yz(\bar{z} + y - 1 + y\bar{z}s_p(\bar{z}))\}}{\operatorname{Im}\{z + yzs_p(z)\}} &\geq \frac{y(1-y)v}{v(1+yC(\theta)\beta^{-1} + yb)} \\ &= y(y-1)(1+yC(\theta)\beta^{-1} + yb)^{-1} \geq a_1 > 0. \end{aligned} \quad (3.15)$$

If  $\operatorname{Im}\{s_p(z)\} \geq v$ , then

$$\begin{aligned} \frac{-\operatorname{Im}\{yz(\bar{z} + y - 1 + y\bar{z}s_p(\bar{z}))\}}{\operatorname{Im}\{z + yzs_p(z)\}} &\geq \frac{y^2|z|^2\operatorname{Im}\{s_p(z)\}}{(1+yC(\theta)\beta^{-1} + yb)\operatorname{Im}\{s_p(z)\}} \\ &\geq y^2a^2(1+yC(\theta)\beta^{-1} + yb)^{-1} \geq a_1 > 0. \end{aligned} \quad (3.16)$$

Inequalities (3.12), (3.15) and (3.16) together complete the proof for  $y \leq 1$ .

Consider the case  $y \geq 1$ . Assuming  $\operatorname{Im}\{s_p(z)\} \geq 2v(y-1)/ya^2 \geq 2v(y-1)/y|z|^2$ , we obtain

$$\begin{aligned}
\frac{-\operatorname{Im}\{yz(\bar{z} + y - 1 + y\bar{z}s_p(z))\}}{\operatorname{Im}\{z + yzs_p(z)\}} &\geq \frac{\frac{1}{2}y^2|z|^2 \operatorname{Im}\{s_p(z)\}}{v + yv \operatorname{Re}\{s_p(z)\} + yu \operatorname{Im}\{s_p(z)\}} \\
&\geq \frac{\frac{1}{2}y^2|z|^2 \operatorname{Im}\{s_p(z)\}}{((1 + yC(\theta)\beta^{-1})(ya^2/2(y - 1)) + yb)\operatorname{Im}\{s_p(z)\}} \\
&\geq \frac{\frac{1}{2}y^2|a|^2}{(1 + yC(\theta)\beta^{-1})(ya^2/2(y - 1)) + yb}.
\end{aligned}$$

Write  $B = 2(y - 1)/ya^2$  and assume that

$$\operatorname{Im} s_p(z) \leq Bv. \quad (3.17)$$

If  $\operatorname{Im}\{\delta_p(z)\} \geq 0$ , then (3.9) implies

$$\operatorname{Im}\{s_p(z)\} \geq \frac{v + \operatorname{Im}\{yzs_p(z)\}}{|z + y - 1 + yzs_p(z)|^2}. \quad (3.18)$$

Since  $\operatorname{Im}(yzs_p(z)) \geq 0$ , inequalities (3.17) and (3.18) together imply

$$|z + y - 1 + yzs_p(z)| \geq B^{-1/2} \geq a_1 > 0. \quad (3.19)$$

If  $\operatorname{Im} \delta_p(z) \leq 0$  the condition  $\operatorname{Im}(z + yz\delta_p(z)) > 0$  implies

$$|\operatorname{Im} \delta_p(z)| \leq v \frac{1 + |\delta_p(z)|}{yu}. \quad (3.20)$$

From (3.9) it follows that

$$|\delta_p(z)| \leq |z + y - 1 + yzs_p(z)|^{-1} + \beta^{-1}. \quad (3.21)$$

Without loss of generality, we may assume that

$$|z + y - 1 + yzs_p(z)| \leq \frac{ya}{2}.$$

Thus inequalities (3.9), (3.20) and (3.21) together imply that

$$\operatorname{Im}\{s_p(z)\} \geq \frac{v}{|z + y - 1 + yzs_p(z)|^2} - \frac{v}{ya|z + y - 1 + yzs_p(z)|} - \frac{(1 + \beta^{-1})v}{ya}.$$

From this inequality and assumption (3.17) it follows that

$$|z + y - 1 + yzs_p(z)| \geq y^{-1}a^{-1} \left( B + \frac{1 + \beta^{-1}}{ya} \right)^{-1}.$$

This completes the proof of Lemma 3.4.  $\square$



#### 4. Bounds for the function $\delta_p(z)$

In this section we shall assume that there exist some positive constants  $a_1$  and  $a_2$  depending only on  $\theta$ ,  $\Theta_1$ ,  $\Theta_2$  and  $\beta$  such that

$$a_1 \leq |z + y - 1 + yzs_p(z)| \leq a_2. \quad (4.1)$$

We introduce in addition the following notation:

$$\begin{aligned} \varepsilon_j^{(1)} &= \frac{1}{n} \sum_{l=1}^n (X_{jl}^2 - 1), & \varepsilon_j^{(2)} &= -\frac{1}{n} (\mathbf{x}_j^T \mathbf{X}(j)^T \mathbf{R}_j \mathbf{X}(j) \mathbf{x}_j - \text{tr} \mathbf{R}_j \mathbf{W}(j)), \\ \varepsilon_j^{(3)} &= -\frac{z}{n} (\text{tr} \mathbf{R} - \text{tr} \mathbf{R}_j), & \varepsilon_j^{(4)} &= \frac{1}{n}, & \varepsilon_j^{(5)} &= \frac{z}{n} (\text{tr} \mathbf{R} - \text{E tr} \mathbf{R}). \end{aligned}$$

Note that

$$\text{tr} \mathbf{R}_j \mathbf{W}(j) = \text{tr} \mathbf{I}_{p-1} + z \text{tr} \mathbf{R}_j. \quad (4.2)$$

Using (4.2), we obtain the representation

$$\varepsilon_j = \sum_{v=1}^5 \varepsilon_j^{(v)}. \quad (4.3)$$

This representation implies, for  $j = 1, \dots, p$ ,

$$\text{E} |\varepsilon_j|^2 \leq 5 \sum_{v=1}^5 \text{E} |\varepsilon_j^{(v)}|^2. \quad (4.4)$$

**Lemma 4.1.** *Under condition (4.1) there exist some constants  $C_1(a_1, a_2)$  and  $C_2(a_1, a_2)$  depending on  $a_1$  and  $a_2$  only, such that, for  $u \in [a, b]$  and  $1 \geq v \geq C_1(a_1, a_2) \sqrt{M_4 n^{-1/2}}$ ,*

$$\text{E} |\varepsilon_j^{(1)}|^2 \leq \frac{CM_4}{n}.$$

**Proof.** The proof of this bound is trivial.  $\square$

**Lemma 4.2.** *Under condition (4.1) there exist some constants  $C_1(a_1, a_2)$  and  $C_2(a_1, a_2)$  depending on  $a_1$  and  $a_2$  only, such that, for  $u \in [a, b]$  and  $1 \geq v \geq C_1(a_1, a_2) \sqrt{M_4 n^{-1/2}}$ ,*

$$\text{E} |\varepsilon_j^{(2)}|^2 \leq \frac{C_2(a_1, a_2) M_4}{nv}. \quad (4.5)$$

**Proof.** Since  $\mathbf{x}_j$  and  $\mathbf{X}(j)$  are independent, we have

$$\text{E} |\varepsilon_j^{(2)}|^2 \leq \frac{CM_4}{n^2} \text{E tr} |\mathbf{R}_j \mathbf{W}(j)|^2. \quad (4.6)$$

Here and in what follows we use the notation  $|\mathbf{A}|^2 := \mathbf{A} \bar{\mathbf{A}}^T$ , for any complex matrix  $\mathbf{A}$ . It is easy to check that

$$\operatorname{tr}|\mathbf{R}_j \mathbf{W}(j)|^2 = v^{-1} \operatorname{Im}\{\operatorname{tr} \mathbf{R}_j \mathbf{W}(j)^2\}. \quad (4.7)$$

Using (4.7), we obtain

$$\mathbb{E} \operatorname{tr}|\mathbf{R}_j \mathbf{W}(j)|^2 \leq (p-1) + v^{-1} |z|^2 |\mathbb{E} \operatorname{tr} \mathbf{R}_j|. \quad (4.8)$$

Since, by (3.6),

$$|\mathbb{E} \operatorname{tr} \mathbf{R}_j| \leq |\mathbb{E} \operatorname{tr} \mathbf{R}| + v^{-1}, \quad (4.9)$$

from (4.4) and (4.6) we obtain

$$\mathbb{E} |\varepsilon_j^{(2)}|^2 \leq \frac{CM_4}{n^2 v} |\mathbb{E} \operatorname{tr} \mathbf{R}| + \frac{CM_4}{n^2 v^2} + \frac{CM_4}{nv}. \quad (4.10)$$

Inequality (4.10) and assumption (4.1) together imply

$$\mathbb{E} |\varepsilon_j^{(2)}|^2 \leq \frac{C(a_1, a_2)M_4}{nv} + \frac{C(a_1, a_2)M_4}{n^2 v^2}. \quad (4.11)$$

This concludes the proof.  $\square$

**Lemma 4.3.** *Under condition (4.1) there exist some constants  $C$  such that, for  $u \in [a, b]$  and  $1 \geq v > 0$ ,*

$$\mathbb{E} |\varepsilon_j^{(3)}|^2 \leq \frac{CM_4}{nv}.$$

**Proof.** The proof follows from inequality (3.7) with  $\mathbf{A} = \mathbf{W}$  and  $\mathbf{A}_k = \mathbf{W}(k)$ .  $\square$

**Lemma 4.4.** *Under condition (4.1) there exist some constants  $C_1(a_1, a_2)$  and  $C_2(a_1, a_2)$  depending on  $a_1$  and  $a_2$  only, such that, for  $u \in [a, b]$  and  $1 \geq v \geq C_1(a_1, a_2)M_8^{1/4} n^{-1/2}$ ,*

$$\mathbb{E} |\varepsilon_j^{(5)}|^2 \leq \frac{C_2(a_1, a_2)M_4}{n^2 v^3}.$$

**Proof.** Note that  $\varepsilon_j^{(5)}$  does not depend on  $j = 1, \dots, n$ . To obtain a bound for  $\mathbb{E} |\varepsilon_j^{(5)}|^2$  we shall use the method of martingale differences which was first used for random matrices in Girko (1989); see also Girko (1990). Let

$$\sigma_k = \operatorname{tr}(\mathbf{W} - z \mathbf{I}_p)^{-1} - \operatorname{tr}(\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1} \quad (4.12)$$

and

$$\gamma_k = \mathbb{E}_{k-1} \sigma_k - \mathbb{E}_k \sigma_k. \quad (4.13)$$

Here and in what follows let  $\mathbb{E}_k$  denote the conditional expectation given  $X_{ji}$  with  $k \leq j \leq p$ ,  $1 \leq l \leq n$ . It is easy to check that

$$\mathbb{E} |\operatorname{tr}(\mathbf{W} - z \mathbf{I}_p)^{-1} - \mathbb{E} \operatorname{tr}(\mathbf{W} - z \mathbf{I}_p)|^2 = \sum_{k=1}^p \mathbb{E} |\gamma_k|^2. \quad (4.14)$$

By (3.6), we have

$$\sigma_k = \{(1 + \mathbf{a}_k^T(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{a}_k)R(k, k)\}. \quad (4.15)$$

Write

$$\sigma_k = \sigma_k^{(1)} + \sigma_k^{(2)} + \sigma_k^{(3)}, \quad (4.16)$$

where

$$\begin{aligned} \sigma_k^{(1)} &= \frac{1 + (1/n)\text{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)}{z + y + yz s_p(z)}, & \sigma_k^{(2)} &= \frac{\varepsilon_k \sigma_k}{z + y + yz s_p(z)}, \\ \sigma_k^{(3)} &= \frac{\mathbf{a}^T(k)(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{a}_k - (1/n)\text{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)}{z + y + yz s_p(z)}. \end{aligned}$$

Since  $E_{k-1}\sigma_k^{(1)} - E_k\sigma_k^{(1)} = 0$ , we obtain

$$\begin{aligned} E|\gamma_k|^2 &\leq E|\sigma_k^{(2)}|^2 + E|\sigma_k^{(3)}|^2 \\ &\leq C(a_1, a_2) \left( v^{-2} E|\varepsilon_k|^2 + E \left| \mathbf{a}^T(k)(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{a}_k - \frac{1}{n}\text{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{W}(k) \right|^2 \right). \end{aligned} \quad (4.17)$$

By the representation (4.3) of  $\varepsilon_k$  and by Lemmas 4.1–4.3, we have

$$E|\varepsilon_k|^2 \leq 5 \sum_{\nu=1}^4 |E\varepsilon_k^{(\nu)}|^2 + 5 E|\varepsilon_k^{(5)}|^2 \leq \frac{C(a_1, a_2)M_4}{nv} + 5 E|\varepsilon_k^{(5)}|^2. \quad (4.18)$$

Similarly to the proof of Lemma 4.3, we obtain

$$E \left| \mathbf{a}^T(k)(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{a}(k) - \frac{1}{n}\text{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{W}(k) \right|^2 \leq \frac{CM_4}{n^2} E \text{tr} \mathbf{G}^{(1)}(k) \overline{\mathbf{G}^{(1)}(k)}, \quad (4.19)$$

where  $\mathbf{G}^{(1)}(k) = (\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{W}(k)$ . It is easy to check that

$$E \text{tr} \mathbf{G}^{(1)}(k) \overline{\mathbf{G}^{(1)}(k)} \leq v^{-3} \text{Im}\{E \text{tr}((\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1}\mathbf{W}^2(k))\}.$$

Using (4.7), we obtain

$$E \text{tr} \mathbf{G}^{(1)}(k) \overline{\mathbf{G}^{(1)}(k)} \leq v^{-3}(v(p-1) + |z|^2 |E \text{tr} \mathbf{R}_k|). \quad (4.20)$$

Inequalities (4.1) and (4.20) together imply

$$E \text{tr} \mathbf{G}^{(1)}(k) \overline{\mathbf{G}^{(1)}(k)} \leq v^{-3}(v(p-1) + nC(a_1, a_2)). \quad (4.21)$$

From (4.19) and (4.21), it follows that

$$E \left| \mathbf{a}^T(k)(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{a}(k) - \frac{1}{n}\text{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-2}\mathbf{W}(k) \right|^2 \leq \frac{C(a_1, a_2)M_4}{nv^2} \left( 1 + \frac{1}{v} \right). \quad (4.22)$$

The relations (4.14), (4.17), (4.18) and (4.22) together imply

$$\mathbb{E}|\varepsilon_k^{(5)}|^2 \leq \frac{C(a_1, a_2)M_4}{n^2v^3} + \frac{C(a_1, a_2)M_4}{nv^2} \mathbb{E}|\varepsilon_k^{(5)}|^2. \quad (4.23)$$

Inequality (4.23) implies that, for some positive constant  $C_1(a_1, a_2)$  and for  $v \geq C_1(a_1, a_2)\sqrt{M_4}n^{-1/2}$ ,

$$\frac{1}{n^2} \mathbb{E}|\operatorname{tr}(\mathbf{W} - z\mathbf{I}_p)^{-1} - \mathbb{E} \operatorname{tr}(\mathbf{W} - z\mathbf{I}_p)^{-1}|^2 \leq C(a_1, a_2)M_4n^{-2}v^{-3}, \quad (4.24)$$

which proves the lemma.  $\square$

Let us introduce the matrices

$$\begin{aligned} \mathbf{G} &= (G(j, k))_{j,k=1}^p = \frac{1}{n} \mathbf{X}_p^T (\mathbf{W}_p - z\mathbf{I}_p)^{-1} \mathbf{X}_p, \\ \mathbf{G}(k) &= (G_k(j, l)) = \frac{1}{n} \mathbf{X}_p^T(k) (\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1} \mathbf{X}_p(k), \\ \mathbf{W}(k, d) &= \frac{1}{n} \mathbf{X}_p(k, d) \mathbf{X}_p^T(k, d), \\ \mathbf{R}_{kd} &= (\mathbf{W}(k, d) - z\mathbf{I}_{p-2})^{-1}, \\ \mathbf{G}(k, d) &= (G_{kd}(j, l)) = \frac{1}{n} \mathbf{X}_p^T(k, d) \mathbf{R}_{kd} \mathbf{X}_p(k, d), \end{aligned}$$

where  $\mathbf{X}_p(k, d)$  is obtained from  $\mathbf{X}_p$  by deleting the  $k$ th and  $d$ th rows. Note that

$$\operatorname{tr} \mathbf{G} = \operatorname{tr}(\mathbf{W} - z\mathbf{I}_p)^{-1} \mathbf{W}, \quad \operatorname{tr} \mathbf{G}(k) = \operatorname{tr}(\mathbf{W}(k) - z\mathbf{I}_{p-1})^{-1} \mathbf{W}(k).$$

The next lemma is similar to Lemma 5.4 in Götze and Tikhomirov (2003).

**Lemma 4.5.** *There exist positive constants  $C_1(a_1, a_2)$  and  $C_2(a_1, a_2)$  such that, for any  $1 \geq v \geq C_1(a_1, a_2)\sqrt{M}n^{-1/2}$ ,*

$$\frac{1}{p} \sum_{k=1}^p \mathbb{E}|R(k, k)|^2 \leq C_2(a_1, a_2).$$

**Proof.** Equation (3.5), condition (4.1) and the representation (4.3) together imply

$$\mathbb{E}|R(k, k)|^2 \leq C(a_1, a_2) \left( 1 + \sum_{v=1}^5 \mathbb{E}|\varepsilon_k^{(v)}|^2 |R(k, k)|^2 \right). \quad (4.25)$$

It is obvious that

$$\mathbb{E}|\varepsilon_k^{(1)}|^2 |R(k, k)|^2 \leq \frac{CM_4}{nv^2}, \quad \mathbb{E}|\varepsilon_k^{(4)}|^2 |R(k, k)|^2 \leq \frac{1}{n^2v^2}. \quad (4.26)$$

By Lemma 3.2, we have

$$\mathbb{E}|\varepsilon_k^{(3)}|^2 |R(k, k)|^2 \leq \frac{1}{n^2 v^4}. \quad (4.27)$$

Using Rosenthal's inequality for quadratic forms or direct calculation, we obtain

$$\mathbb{E}|\varepsilon_k^{(2)}|^4 \leq \frac{CM_8}{n^4} \mathbb{E} \left( \sum_{\substack{l,m=1 \\ l \neq k, m \neq k}}^n |G_k(l, m)|^2 \right)^2.$$

We may write

$$\mathbb{E} \left( \sum_{\substack{l,m=1 \\ l \neq k, m \neq k}}^n |G_k(l, m)|^2 \right)^2 \leq |\mathbb{E} \operatorname{tr} |\mathbf{G}(k)|^2|^2 + \mathbb{E} |\operatorname{tr} |\mathbf{G}(k)|^2 - \mathbb{E} \operatorname{tr} |\mathbf{G}(k)|^2|^2.$$

By relations (4.2), (4.7) and condition (4.1), we have

$$\mathbb{E} \operatorname{tr} |\mathbf{G}(k)|^2 \leq \frac{C(a_1, a_2)n}{v}. \quad (4.28)$$

Similarly as in the bounds for  $\mathbb{E} |\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^2$ , we introduce the random variables

$$\tilde{\gamma}_d(k) = \mathbb{E}_{d-1} \operatorname{tr} |\mathbf{G}(k)|^2 - \mathbb{E}_d \operatorname{tr} |\mathbf{G}(k)|^2 = \mathbb{E}_{d-1} \tilde{\sigma}_k(d) - \mathbb{E}_d \tilde{\sigma}_k(d),$$

with  $\tilde{\sigma}_d(k) = \operatorname{tr} |\mathbf{G}(k)|^2 - \operatorname{tr} |\mathbf{G}(k, d)|^2$ . Since the  $\tilde{\gamma}_d(k)$  are orthogonal, for  $d = 1, \dots, p$ , we obtain

$$\frac{1}{n^4} \mathbb{E} |\operatorname{tr} |\mathbf{G}(k)|^2 - \mathbb{E} \operatorname{tr} |\mathbf{G}(k)|^2|^2 \leq \frac{1}{n^4} \sum_{d=1}^p \mathbb{E} |\tilde{\gamma}_d(k)|^2.$$

Note that, according to (4.7),

$$|\operatorname{tr} |\mathbf{G}(k)|^2 - \operatorname{tr} |\mathbf{G}(k, d)|^2| = \frac{1}{v} |\operatorname{Im}\{z^2(\operatorname{tr} \mathbf{R}_k - \operatorname{tr} \mathbf{R}_{k,d}) + z\}| \leq \frac{C}{v^2}.$$

This implies that  $|\tilde{\gamma}_k(d)| \leq Cv^{-2}$  and

$$\frac{1}{n^4} \mathbb{E} |\operatorname{tr} |\mathbf{G}(k)|^2 - \mathbb{E} \operatorname{tr} |\mathbf{G}(k)|^2|^2 \leq \frac{C}{n^3 v^4}. \quad (4.29)$$

Inequalities (4.27)–(4.29) together imply that, for  $v \geq C(a_1, a_2)M_8^{1/4}n^{-1/2}$ ,

$$\mathbb{E}|\varepsilon_k^{(2)}|^4 \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (4.30)$$

Using Cauchy's inequality, we obtain

$$\begin{aligned} \frac{1}{p} \sum_{k=1}^p \mathbb{E} |\varepsilon_k^{(2)}|^2 |R(k, k)|^2 &\leq v^{-1} \left( \frac{1}{p} \sum_{k=1}^p \mathbb{E} |\varepsilon_k^{(2)}|^4 \right)^{1/2} \left( \frac{1}{p} \sum_{k=1}^p \mathbb{E} |R(k, k)|^2 \right)^{1/2} \\ &\leq \frac{CM_8^{1/4}}{nv^2} \left( \frac{1}{p} \sum_{k=1}^p \mathbb{E} |R(k, k)|^2 \right)^{1/2}. \end{aligned} \quad (4.31)$$

Notice that

$$\begin{aligned} \frac{1}{p} \sum_{k=1}^p \mathbb{E} |\varepsilon_k^{(5)}|^2 |R(k, k)|^2 &= \mathbb{E} |\varepsilon_1^{(5)}|^2 \left( \frac{1}{p} \sum_{k=1}^p |R(k, k)|^2 \right) \\ &\leq \mathbb{E} |\varepsilon_1^{(5)}|^2 \left( \frac{1}{p} \sum_{k,j=1}^p |R(k, j)|^2 \right) = v^{-1} \mathbb{E} |\varepsilon_1^{(5)}|^2 \operatorname{Im} \left\{ \frac{1}{p} \operatorname{tr} \mathbf{R} \right\} \\ &\leq \frac{1}{vp} \mathbb{E} |\varepsilon_1^{(5)}|^2 |\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}| + \frac{|\mathbb{E} \operatorname{tr} \mathbf{R}|}{pv} \mathbb{E} |\varepsilon_1^{(5)}|^2. \end{aligned} \quad (4.32)$$

Furthermore,

$$\frac{1}{vp} \mathbb{E} |\varepsilon_1^{(5)}|^2 |\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}| \leq \frac{C}{vn^3} \mathbb{E} |\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^3.$$

By Burkholder's inequality for martingales (see Hall and Heyde 1980, p. 24), we obtain

$$\mathbb{E} |\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^3 \leq C\sqrt{p} \sum_{k=1}^p \mathbb{E} |\gamma_k|^3, \quad (4.33)$$

with  $\gamma_k$  defined in (4.13). Inequalities (4.17), (4.18) and (4.22) together imply that, for  $1 \geq v \geq C_1(a_1, a_2)M_8^{1/4}n^{-1/2}$ ,

$$\mathbb{E} |\gamma_k|^2 \leq \frac{C(a_1, a_2)M_8^{1/2}}{nv^3}.$$

Since  $|\gamma_k| \leq 2v^{-1}$ , we obtain

$$\mathbb{E} |\gamma_k|^3 \leq \frac{C(a_1, a_2)}{nv^4}. \quad (4.34)$$

From (4.33) and (4.34) we obtain

$$\frac{1}{vn^3} \mathbb{E} |\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^3 \leq \frac{C(a_1, a_2)M_8^{1/2}}{\sqrt{n^5v^{10}}}. \quad (4.36)$$

Inequalities (4.32), (4.36), (4.1) and Lemma 4.4 together imply

$$\frac{1}{n} \sum_{k=1}^n \mathbb{E} |\varepsilon_k^{(5)}|^2 |G_{kk}|^2 \leq \frac{C(a_1, a_2)M_8^{1/2}}{\sqrt{n^5v^{10}}} + \frac{C(a_1, a_2)M_8^{1/2}}{nv^2}. \quad (4.37)$$

Inequalities (4.25)–(4.27), (4.31) and (4.37) finally yield, for  $v \geq C(a_1, a_2)M_8^{1/4}n^{-1/2}$ ,

$$\frac{1}{p} \sum_{j=1}^p \mathbb{E}|R(j, j)|^2 \leq C_1(a_1, a_2) + C_2(a_1, a_2) \left( \frac{1}{p} \sum_{j=1}^p \mathbb{E}|R(j, j)|^2 \right)^{1/2}.$$

From the last inequality it follows that, for  $\nu \geq C(a_1, a_2)M_8^{1/4}n^{-1/2}$ ,

$$\frac{1}{p} \sum_{j=1}^p \mathbb{E}|R(j, j)|^2 \leq C(a_1, a_2),$$

which completes the proof.  $\square$

**Lemma 4.6.** *There exist positive constants  $C_1(a_1, a_2)$  and  $C_2(a_1, a_2)$  such that, for any  $\nu \geq C_1(a_1, a_2)M_8^{1/4}n^{-1/2}$ ,*

$$|\delta_p(z)| \leq \frac{C_2(a_1, a_2)\sqrt{M_8}}{n\nu}.$$

**Proof.** By the definition of  $\delta_p(z)$ , (3.4) implies that

$$|\delta_p(z)| \leq |y + z - 1 + yz s_p(z)|^{-1} \left| \frac{1}{p} \sum_{j=1}^p \mathbb{E} \varepsilon_j R(j, j) \right|. \quad (4.38)$$

Taking into account that  $\mathbb{E} \varepsilon_j^{(\nu)} = 0$ , for  $\nu = 1, 2$ ,  $|\mathbb{E} \varepsilon_j^{(\nu)}| \leq 1/n\nu$ , for  $\nu = 3$ , and  $|\mathbb{E} \varepsilon_j^\nu| \leq 1/n$ , for  $\nu = 4$ , and expanding  $R(j, j)$  into the parts  $\varepsilon_j$  defined in (4.3), we obtain

$$\begin{aligned} |\delta_p(z)| &\leq \sum_{\nu=1}^3 \frac{C}{p} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(\nu)}|^2 |R(j, j)| \\ &\quad + \sum_{\nu=1}^3 \frac{C}{p} \left| \sum_{j=1}^p \mathbb{E} \varepsilon_j^{(\nu)} \varepsilon_j^{(5)} R(j, j) \right| + \frac{C}{n\nu} + \frac{C}{p} \left| \sum_{j=1}^p \mathbb{E} \varepsilon_j^{(5)} R(j, j) \right|. \end{aligned} \quad (4.39)$$

Since  $|R(j, j)| \leq \nu^{-1}$  and  $\mathbb{E} |\varepsilon_j^{(1)}|^2 \leq C\sqrt{M_8}n^{-1}$ , we obtain

$$\frac{1}{p} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(1)}|^2 |R(j, j)| \leq \frac{C\sqrt{M_8}}{n\nu}. \quad (4.40)$$

Lemma 3.2 and the definition of both  $\varepsilon_j^{(3)}$  and  $\varepsilon_j^{(4)}$  together imply that

$$\frac{1}{p} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(3)}|^2 |R(j, j)| \leq \frac{C}{n^2\nu^3}, \quad \frac{1}{p} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(4)}|^2 |R(j, j)| \leq \frac{C}{n^2\nu}. \quad (4.41)$$

Applying Hölder's inequality, inequality (4.30) and Lemma 4.5, we obtain that

$$\frac{C}{p} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(2)}|^2 |R(j, j)| \leq \frac{C}{p} \left( \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(2)}|^4 \right)^{1/2} \left( \sum_{j=1}^p \mathbb{E} |R(j, j)|^2 \right)^{1/2} \leq \frac{C\sqrt{M_8}}{n\nu}. \quad (4.42)$$

Consider now the summand with  $\varepsilon_j^{(5)}$ . Write

$$\varepsilon_j^{(5)} = \varepsilon'_j + \varepsilon''_j,$$

where

$$\varepsilon'_j = \frac{1}{n}(\operatorname{tr} \mathbf{R}_j - \mathbb{E} \operatorname{tr} \mathbf{R}_j), \quad \varepsilon''_j = \frac{1}{n}(\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j) - \frac{1}{n} \mathbb{E}(\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j).$$

By (3.7),  $|\varepsilon''_j| \leq 2(n\nu)^{-1}$ . This inequality and Lemma 4.3 together imply that, for  $\nu = 1, 2, 3, 4$ ,

$$\begin{aligned} \frac{1}{n} \left| \sum_{j=1}^p \mathbb{E} \varepsilon_j^{(\nu)} \varepsilon''_j R(j, j) \right| &\leq \frac{C}{n\nu} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(\nu)}|^2 \right)^{1/2} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |R(j, j)|^2 \right)^{1/2} \\ &\leq \frac{C(a_1, a_2)}{n\nu} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\varepsilon_j^{(\nu)}|^2 \right)^{1/2} \leq \frac{C(a_1, a_2) \sqrt{M_8}}{n\nu}. \end{aligned} \quad (4.43)$$

Since the random variables  $X_{jl}$ ,  $l = 1, \dots, n$ , and the random matrix  $\mathbf{R}_j$  are independent, we obtain, for  $\nu = 1, 2, 3$ ,

$$\mathbb{E} |\varepsilon_j^{(1)} \varepsilon'_j|^2 = \mathbb{E} |\varepsilon_j^{(1)}|^2 \mathbb{E} |\varepsilon'_j|^2 \leq \frac{C\sqrt{M_8}}{n} \mathbb{E} |\varepsilon'_j|^2 \leq \frac{CM_8}{n^3\nu^3}, \quad (4.44)$$

$$\mathbb{E} |\varepsilon_j^{(2)} \varepsilon'_j|^2 \leq \frac{M_8 C}{n^2} \mathbb{E} \operatorname{tr} |\mathbf{G}(j)|^2 |\varepsilon'_j|^2, \quad (4.45)$$

$$\mathbb{E} |\varepsilon_j^{(3)} \varepsilon'_j|^2 \leq \frac{C}{n^2\nu^2} \mathbb{E} |\varepsilon'_j|^2. \quad (4.46)$$

By definition of the matrix  $\mathbf{G}(j)$ , we have

$$\begin{aligned} \operatorname{tr} |\mathbf{G}(j)|^2 &= \frac{1}{\nu} \operatorname{Im} \{ (\operatorname{tr} (\mathbf{W}(j) - z \mathbf{I}_{p-1})^{-1} \mathbf{W}(j))^2 \} \\ &= \frac{1}{\nu} \operatorname{Im} \{ z^2 \operatorname{tr} \mathbf{R}_j + z \operatorname{tr} \mathbf{I}_{p-1} \}. \end{aligned} \quad (4.47)$$

The relations (4.45) and (4.47) together imply that

$$\mathbb{E} |\varepsilon_j^{(2)} \varepsilon'_j|^2 \leq \frac{C\sqrt{M_8}}{n\nu} \mathbb{E} |\varepsilon'_j|^2 + \frac{C\sqrt{M_8}}{n^2\nu} |\mathbb{E} \operatorname{tr} \mathbf{R}_j| \mathbb{E} |\varepsilon'_j|^2. \quad (4.48)$$

Using the definition of  $\varepsilon'_j$ , we obtain

$$\mathbb{E} |\varepsilon_j^{(2)} \varepsilon'_j|^2 \leq \frac{C\sqrt{M_8}}{n\nu} \mathbb{E} |\varepsilon'_j|^2 + \frac{C\sqrt{M_8}}{n^2\nu} |\mathbb{E} \operatorname{tr} \mathbf{R}_j| \mathbb{E} |\varepsilon'_j|^2 + \frac{C\sqrt{M_8}}{n\nu} \mathbb{E} |\varepsilon'_j|^3. \quad (4.49)$$

Using (4.36) and (4.24), simple calculations yield

$$\mathbb{E} |\varepsilon_j^{(2)} \varepsilon'_j|^2 \leq \frac{CM_8}{n^3\nu^4} + \frac{CM_8}{\sqrt{n^7\nu^{10}}}. \quad (4.50)$$

Furthermore, by (4.46) and (4.24), we have, for  $\nu \geq CM_8^{1/4} n^{-1/2}$ ,



$$E|\varepsilon_j^{(3)} \varepsilon_j'|^2 \leq \frac{CM_8}{n^4 v^5}. \tag{4.51}$$

Applying Hölder’s inequality and (4.44), (4.50) and (4.51), we obtain, for  $\nu = 1, 2, 3$ ,

$$\begin{aligned} \frac{1}{n} \left| \sum_{k=1}^p E \varepsilon_k^{(\nu)} \varepsilon_k' R(k, k) \right| &\leq \left( \frac{1}{n} \sum_{k=1}^p E |\varepsilon_k^{(\nu)} \varepsilon_k'|^2 \right)^{1/2} \left( \frac{1}{n} \sum_{k=1}^p E |R_{kk}|^2 \right)^{1/2} \\ &\leq \frac{C(a_1, a_2) \sqrt{M_8}}{n\nu}. \end{aligned} \tag{4.52}$$

Inequalities (4.52) and (4.39)–(4.43) conclude the proof.  $\square$

### 5. Proof of Theorem 1.1

The rest of the proof of Theorem 1.1 is similar to the proof of the results for a Wigner matrix in Götze and Tikhomirov (2003). First we prove that there exists some constant  $C$  such that, for any

$$v \geq v_0 := \max\{\beta\Delta_n, CM_8^{1/4} n^{-1/2}\}, \tag{5.1}$$

the inequality  $\text{Im}\{z + yz\delta_p(z)\} > 0$  holds. Assume that

$$\text{Im}\{z + yz\delta_p(z)\} = 0. \tag{5.2}$$

Then according to Lemma 3.2, there exists some constant  $a_1 > 0$  such that

$$|y + 1 - z + yzs_p(z)| \geq a_1. \tag{5.3}$$

In addition, we have

$$\begin{aligned} |s_p(z) - s_y(z)| &= \left| \int_{-\infty}^{\infty} \frac{1}{x - z} d(EF_p(x) - F_y(x)) \right| \\ &= \left| \int_{-\infty}^{\infty} \frac{EF_p(x) - F_y(x)}{(x - z)^2} dx \right| \leq \frac{\Delta_n}{v} \leq \frac{1}{\beta}. \end{aligned}$$

Since  $|s_y(z)| \leq C(\theta, \Theta, a, b) = C$ , we obtain

$$|y + z - 1 + yzs_p(z)| \leq C(\theta, \Theta, a, b, \beta) = a_2. \tag{5.4}$$

By Lemma 4.6, there exist constants  $C_1(a_1, a_2)$  and  $C_2(a_1, a_2)$  such that, for  $v \geq C_1(a_1, a_2)M_8^{1/4} n^{-1/2}$ ,

$$|\delta_p(z)| \leq \frac{C_2(a_1, a_2) \sqrt{M_8}}{nv}. \tag{5.5}$$

Recall that  $v_0 = \max\{\beta\Delta_n, n^{-1/2}C_1\sqrt{M_8}\}$ , with  $1 > \beta > 0$  to be chosen later. The constant  $C_1$  is chosen such that, for any  $1 \geq v \geq v_0$ , we have

$$|\delta_p(z)| \leq \frac{v}{2\Theta(b+1)}. \quad (5.6)$$

This inequality contradicts condition (5.2), since it implies that  $|\delta_p(z)| \geq v/\Theta(b+1)$ . Now choose  $v = 1$ . It is easy to see that

$$|y+z-1+yzs_p(z)| \geq \operatorname{Im}\{y+z-1+yzs_p(z)\} \geq 1, \quad (5.7)$$

and since  $|s_p(z)| \leq v^{-1} \leq 1$ ,

$$|z+y-1+yzs_p(z)| \leq (b+1)(\Theta+1). \quad (5.8)$$

By Lemma 4.6,

$$|\delta_p(z)| \leq \frac{C}{n}. \quad (5.9)$$

This inequality implies, for  $v = 1$ ,

$$\operatorname{Im}(z+yz\delta_p(z)) > 0, \quad (5.10)$$

and since  $\operatorname{Im}(z+yz\delta_p(z)) \neq 0$ , we obtain that (5.10) holds, for  $v \geq v_0$ .

By Lemma 3.2, for  $z = u + iv$  with  $u \in [a, b]$  and  $1 \geq v \geq v_0$ , we have

$$|z+y-1+yzs_p(z)| \geq C_1(\beta, \theta, \Theta_1, \Theta_2). \quad (5.11)$$

Using inequality (5.11) and (5.4), by Lemma 4.6, we obtain that

$$|\delta_p(z)| \leq \frac{C\sqrt{M_8}}{nv}.$$

It is straightforward to check that

$$|s_p(z) - s_y(z)| \leq |\delta_p(z)| |y+z-1+yzs_p(z) + yzs_y(z)|^{-1}. \quad (5.12)$$

Since  $\operatorname{Im}(yzs_y(z)) > 0$  and  $\operatorname{Im}(yzs_p(z)) > 0$ , we obtain

$$|y+z-1+yzs_p(z) + yzs_y(z)| \geq \operatorname{Im}\{y+z-1+yzs_p(z) + yzs_y(z)\} \geq v.$$

These inequalities together imply that

$$|s_p(z) - s_y(z)| \leq \frac{C\sqrt{M_8}}{nv^2}.$$

Integrating this inequality yields

$$\int_{v_0}^V |s_p(u+iv) - s_y(u+iv)| dv \leq \frac{C}{nv_0}. \quad (5.13)$$

Choose  $V = 1$  and consider the first integral in (2.3). By (5.12), we obtain

$$\int_{-\infty}^{\infty} |s_p(z) - s_y(z)| du \leq \int_{-\infty}^{\infty} |\delta_p(z)| du. \quad (5.14)$$

By definition of  $\delta_p(z)$ , we have

$$|\delta_p(z)| \leq C|z + y - 1 + yzs_p(z)|^{-2} \left( \frac{1}{p} \sum_{k=1}^p |\mathbb{E}\varepsilon_k| + \frac{1}{p} \sum_{k=1}^p \mathbb{E}|\varepsilon_k|^2 |R(k, k)| \right). \quad (5.15)$$

Using Lemma 3.1 and inequality (3.31) in Bai (1993b), we obtain

$$|\delta_p(z)| \leq \frac{C}{n} |z + y - 1 + yzs_p(z)|^{-2}.$$

Finally, applying (3.6) gives

$$|\delta_p(z)| \leq \frac{C}{n} (|s_p(z)|^2 + |\delta_p(z)|^2). \quad (5.16)$$

Without loss of generality we may assume that  $|\delta_p(z)| \leq 1/4$ . Inequalities (5.14)–(5.16) together imply that

$$\int_{-\infty}^{\infty} |\delta_p(z)| du \leq \frac{C}{n} \int_{-\infty}^{\infty} |s_p(z)|^2 du. \quad (5.17)$$

It is easy to check that

$$\int_{-\infty}^{\infty} |s_p(z)|^2 du \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(u-x)^2 + v^2} du d\mathbb{E} F_p(x) \leq \frac{1}{v}.$$

Applying this inequality, for  $v = 1$ , we obtain

$$\int_{-\infty}^{\infty} |s_p(z) - s_y(z)| du \leq \frac{C}{n}. \quad (5.18)$$

Now choose  $\varepsilon = v_0^{2/3}$  and apply Lemma 2.1. We obtain that

$$\Delta_n \leq \frac{C_1 \sqrt{M_8}}{nv_0} + \frac{C_2}{n} + C_3 v_0.$$

Note that the constant  $C_3$  does not depend on  $\beta$ . We choose  $\beta < 1(2C_3)^{-1}$  and  $v_0 = CM_8^{1/4} n^{-1/2}$ . We finally conclude

$$\Delta_n \leq \frac{CM_8^{1/4}}{\sqrt{n}},$$

which proves Theorem 1.1. □

## 6. An improved bound for $\mathbb{E}|\text{tr } \mathbf{R} - \mathbb{E} \text{tr } \mathbf{R}|^2$

Recall that

$$\begin{aligned} \mathbf{W} &= \frac{1}{p} \mathbf{X} \mathbf{X}^\top, & \mathbf{R} &= (\mathbf{W} - z \mathbf{I}_p)^{-1}, & \mathbf{W}(k) &= \frac{1}{p} \mathbf{X}(k) \mathbf{X}(k)^\top, \\ \mathbf{R}_k &= (\mathbf{W}(k) - z \mathbf{I}_{p-1})^{-1}, & s_p(z) &= \frac{1}{p} \mathbb{E} \text{tr } \mathbf{R}, & s_{pk}(z) &= \frac{1}{p} \mathbb{E} \text{tr } \mathbf{R}_k \end{aligned}$$

and

$$R(j, j) = -\frac{1}{y+z-1+yzs_p(z)} + \frac{\varepsilon_j}{y+z-1+yzs_p(z)} R(j, j), \quad (6.1)$$

where

$$\varepsilon_j = \frac{1}{n} \sum_{k=1}^n (X_{kj}^2 - 1) + y + yzs_p(z) - \mathbf{a}^\top(k)(\mathbf{W}_p(k) - z\mathbf{I}_{p-1})^{-1} \mathbf{a}_k.$$

From this representation it follows that

$$\frac{1}{p} \operatorname{tr} \mathbf{R} = -\frac{1}{y+z-1+yzs_p(z)} + \delta_p(z), \quad (6.2)$$

where

$$\delta_p(z) = \frac{1}{p(y+z-1+yzs_p(z))} \sum_{j=1}^p \varepsilon_j R(j, j).$$

Write

$$\varepsilon_j = \varepsilon_j^{(1)} + \varepsilon_j^{(2)} + \varepsilon_j^{(3)} + \varepsilon_j^{(4)}, \quad (6.3)$$

with

$$\begin{aligned} \varepsilon_j^{(1)} &= \frac{1}{n} \sum_{k=1}^n (X_{kj}^2 - 1), & \varepsilon_j^{(2)} &= -\left( \mathbf{a}_j^\top \mathbf{R}_j \mathbf{a}_j - \frac{1}{n} \operatorname{tr} \mathbf{R}_j \mathbf{W}(j) \right), \\ \varepsilon_j^{(3)} &= \frac{1}{n} (\operatorname{tr} \mathbf{R} \mathbf{W} - \operatorname{tr} \mathbf{R}_j \mathbf{W}(j)), & \varepsilon_j^{(4)} &= \frac{1}{n} \operatorname{tr} \mathbf{R} \mathbf{W} - y - yzs_p(z). \end{aligned}$$

Recall that  $y = p/n$ . Note that  $\operatorname{tr} \mathbf{R} \mathbf{W} = \operatorname{tr} \mathbf{I}_p + z \operatorname{tr} \mathbf{R}$ . These relations and the definition of  $s_p(z)$  imply that

$$\frac{1}{n} \operatorname{E} \operatorname{tr} \mathbf{R} \mathbf{W} = y + yzs_p(z).$$

We can now write

$$\varepsilon_j^{(4)} = \frac{z}{n} (\operatorname{tr} \mathbf{R} - \operatorname{E} \operatorname{tr} \mathbf{R}) = \frac{yz}{p} (\operatorname{tr} \mathbf{R} - \operatorname{E} \operatorname{tr} \mathbf{R}). \quad (6.4)$$

Furthermore,

$$\operatorname{tr} \mathbf{R} \mathbf{W} - \operatorname{tr} \mathbf{R}_j \mathbf{W}(j) = 1 + z(\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j). \quad (6.5)$$

Hence, it follows that

$$\varepsilon_j^{(3)} = \varepsilon_j^{(5)} + \varepsilon_j^{(6)}, \quad (6.6)$$

where

$$\varepsilon_j^{(5)} = \frac{1}{n}, \quad \varepsilon_j^{(6)} = \frac{z}{n}(\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j).$$

We introduce the notation

$$M_q = \sup_{1 \leq j, k \leq n} \mathbb{E}|X_{jk}|^q.$$

**Proposition 6.1.** *Assuming the conditions of Theorem 1.1, we have, for any  $v > v_0 := \gamma M_{12}^{1/6} n^{-1/2}$  with sufficiently large  $\gamma > 0$ ,*

$$\frac{1}{n^2} \mathbb{E}|\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^2 \leq \frac{CM_8}{n^2 v^2 |y + z - 1 + 2yzs(z)|^2}. \quad (6.7)$$

**Proof.** To prove Proposition 6.1 we shall use the following facts: for  $v \geq v_0$ , we have

$$\frac{1}{n} \sum_{j=1}^p \mathbb{E}|R(j, j)|^2 \leq C \quad (6.8)$$

and

$$\mathbb{E}|\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^2 \leq CM_4 v^{-3}. \quad (6.9)$$

These inequalities were proved in Lemmas 4.4 and 4.5. We shall use also the following lemma:

**Lemma 6.2.** *Under the conditions of Theorem 1.2 we have, for any  $q \geq 4$  and for  $v \geq v_0$ ,*

$$\mathbb{E} \left| \frac{1}{p} (\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}) \right|^q \leq \frac{C(q)M_{2q}}{(n^2 v^3)^{q/2}}. \quad (6.10)$$

**Proof.** By Burkholder's inequality for martingales, we have

$$\mathbb{E}|\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^q \leq C(q)n^{q/2-1} \sum_{j=1}^n \mathbb{E}|\gamma_j|^q, \quad (6.11)$$

where the martingale difference  $\gamma_j$  is defined in (4.12) and (4.13). Furthermore, using the representation (4.16) which is similar to inequality (4.17), we obtain

$$\begin{aligned} \mathbb{E}|\gamma_j|^q &\leq C(q)\mathbb{E}|\sigma_j^{(2)}|^q + C(q)\mathbb{E}|\sigma_j^{(3)}|^q \\ &\leq C(q)\frac{1}{n^q} \mathbb{E}|\mathbf{a}(j)^\top \mathbf{R}_j^2 \mathbf{a}(j) - \operatorname{tr} \mathbf{R}_j^2 \mathbf{W}(j)|^q + C(q)\mathbb{E}|\varepsilon_j(\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j)|^q. \end{aligned} \quad (6.12)$$

By the definition (6.3) of  $\varepsilon_j$ , using Rosenthal's inequality for quadratic forms (see, for example, Götze and Tikhomirov 2003), we obtain

$$\begin{aligned}
E|\gamma_j|^q &\leq \frac{C(q)M_{2q}}{n^q} E(\operatorname{tr}(|\mathbf{R}_j^2| \mathbf{W}(j))^2)^{q/2} \\
&+ E \left| \frac{1}{n} (\mathbf{X}_j^T \mathbf{G}_j \mathbf{X}_j - \operatorname{tr} \mathbf{G}_j) \right|^q |\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j|^q \\
&+ \frac{C(q)}{n^q} E \left| \sum_{k=1}^n (X_{kj}^2 - 1) \right|^q |\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j|^q \\
&+ \frac{C(q)}{n^q} E |\operatorname{tr} \mathbf{R} - E \operatorname{tr} \mathbf{R}|^q |\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j|^q \\
&+ \frac{1}{n^q} E |\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j|^{2q}. \tag{6.13}
\end{aligned}$$

Inequalities (6.11), (6.13) and the inequality  $|\operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j| \leq v^{-1}$  together imply

$$\begin{aligned}
E |\operatorname{tr} \mathbf{R} - E \operatorname{tr} \mathbf{R}|^q &\leq \frac{C(q)M_{2q}}{n^{q/2}} \max_{1 \leq j \leq n} E(\operatorname{tr}(|\mathbf{R}_j^2| \mathbf{W}(j))^2)^{q/2} + \frac{C(q)M_{2q}}{v^q} \\
&+ \frac{C(q)}{n^{q/2} v^{2q}} + \frac{C(q)M_{2q}}{n^{q/2} v^q} \max_{1 \leq j \leq n} E(\operatorname{tr}(|\mathbf{R}_j|^2 \mathbf{W}(j)^2)^{q/2}) \\
&+ \frac{C(q)}{n^{q/2} v^q} E |\operatorname{tr} \mathbf{R} - E \operatorname{tr} \mathbf{R}|^q. \tag{6.14}
\end{aligned}$$

From (6.14) we obtain for  $v \geq v_0$ ,

$$\begin{aligned}
E |\operatorname{tr} \mathbf{R} - E \operatorname{tr} \mathbf{R}|^q &\leq \frac{C(q)M_{2q}}{n^{q/2}} \max_{1 \leq j \leq n} E(\operatorname{tr}(|\mathbf{R}_j^2| \mathbf{W}(j))^2)^{q/2} \\
&+ \frac{C(q)M_{2q}}{v^q} + \frac{C(q)M_{2q}}{n^{q/2} v^q} \max_{1 \leq j \leq n} E(\operatorname{tr}|\mathbf{R}_j|^2 \mathbf{W}(j)^2)^{q/2}. \tag{6.15}
\end{aligned}$$

Note that

$$\operatorname{tr}|\mathbf{R}_j|^4 \mathbf{W}(j)^2 \leq v^{-2} \operatorname{tr}|\mathbf{R}_j|^2 \mathbf{W}(j)^2 = v^{-3} \operatorname{Im}\{\operatorname{tr} \mathbf{R}_j \mathbf{W}(j)^2\}. \tag{6.16}$$

Furthermore, it is easy to check that

$$\operatorname{tr} \mathbf{R}_j \mathbf{W}(j)^2 = z^2 \operatorname{tr} \mathbf{R}_j + z \operatorname{tr} \mathbf{I}_{p-1} + \operatorname{tr} \mathbf{W}(j). \tag{6.17}$$

Relations (6.16) and (6.17) imply that

$$E(\operatorname{tr}|\mathbf{R}_j|^4 \mathbf{W}^2)^{q/2} \leq C(q)v^{-3q/2} (E|\operatorname{tr} \mathbf{R}_j|^{q/2} + (\operatorname{tr} \mathbf{I}_{p-1})^{q/2}). \tag{6.18}$$

Analogously,

$$\begin{aligned}
E(\operatorname{tr}|\mathbf{R}_j|^2 \mathbf{W}(j)^2)^{q/2} &\leq C(q)v^{-q/2} E|\operatorname{tr} \mathbf{R}_j \mathbf{W}(j)^2|^{q/2} \\
&\leq C(q)v^{-q/2} (E|\operatorname{tr} \mathbf{R}_j|^{q/2} + |\operatorname{tr} \mathbf{I}_{p-1}|^{q/2}). \tag{6.19}
\end{aligned}$$

By Burkholder's inequality for the martingales, we have, for  $v \geq v_0$ ,

$$\mathbb{E}|\operatorname{tr} \mathbf{R}_j - \mathbb{E} \operatorname{tr} \mathbf{R}_j|^{q/2} \leq C(q)n^{q/4-1} \sum_{\substack{l=1 \\ l \neq j}}^n \mathbb{E}|\gamma_{jl}|^{q/2} \leq \frac{C(q)n^{q/2}}{n^{q/4}v^{q/2}} \leq C(q)n^{q/2}. \quad (6.20)$$

Inequality (6.20) implies that for,  $v \geq v_0$ ,

$$\mathbb{E}|\operatorname{tr} \mathbf{R}_j|^{q/2} \leq C(q)|\mathbb{E} \operatorname{tr} \mathbf{R}_j|^{q/2} + C(q)\mathbb{E}|\operatorname{tr} \mathbf{R}_j - \mathbb{E} \operatorname{tr} \mathbf{R}_j|^{q/2} \leq C(q)n^{q/2}. \quad (6.21)$$

From inequalities (6.17)–(6.21) it follows that

$$\mathbb{E}\left(\operatorname{tr}|\mathbf{R}_j|^4 \mathbf{W}(j)^2\right)^{q/2} \leq C(q)v^{-3q/2}n^{q/2}, \quad \mathbb{E}\left(\operatorname{tr}|\mathbf{R}_j|^2 \mathbf{W}(j)^2\right)^{q/2} \leq C(q)v^{-q/2}n^{q/2}. \quad (6.22)$$

Combining the inequalities (6.15) and (6.22), we obtain, for  $v \geq v_0$ ,

$$\mathbb{E}|\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^q \leq \frac{C(q)M_{2q}}{v^{3q/2}} + \frac{C(q)M_{2q}}{v^q} \leq \frac{C(q)M_{2q}}{v^{3q/2}}, \quad (6.23)$$

which concludes the proof of the lemma.  $\square$

We may prove a rougher bound than (6.10), assuming  $M_8 < \infty$  only.

**Remark 6.1.** We have, for  $v \geq v_0$  and for  $q \geq 4$ ,

$$\mathbb{E}\left|\frac{1}{n}(\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R})\right|^q \leq \frac{CM_8}{(\sqrt{nv})^{q-4}} \frac{1}{n^4 v^6}. \quad (6.24)$$

**Proof.** To prove (6.24) we use Burkholder's inequality for martingales. We obtain

$$\mathbb{E}\left|\frac{1}{n}(\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R})\right|^q \leq \frac{C}{n^{q/2+1}} \sum_{j=1}^n \mathbb{E}|\gamma_j|^q \leq \frac{C}{n^{q/2-2}v^{q-4}} \left(\frac{1}{n^3} \sum_{j=1}^n \mathbb{E}|\gamma_j|^4\right).$$

Applying now arguments similar to the relations (6.13)–(6.18) for  $(1/n^3)\sum_{j=1}^n \mathbb{E}|\gamma_j|^4$ , we obtain inequality (6.24).  $\square$

We now continue with our proof of Proposition 6.1. In order to simplify the exposition we introduce the following notation:

$$\begin{aligned} \Delta_E(\mathbf{R}) &:= \overline{\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}}, & \Delta_E^{(j)}(\mathbf{R}) &:= \operatorname{tr} \mathbf{R}_j - \mathbb{E} \operatorname{tr} \mathbf{R}_j, \\ \mathcal{X}_j &= \sum_{l=1}^n (X_{jl}^2 - 1), & \mathcal{D}_j(\mathbf{R}) &:= \operatorname{tr} \mathbf{R} - \operatorname{tr} \mathbf{R}_j, & \mathcal{C}(\mathbf{R}) &:= \left(\frac{1}{n} \sum_{l=1}^n \mathbb{E}|R(j, l)|^2\right)^{1/2}, \\ \mathcal{Q}_j(\mathbf{R}) &:= \mathbf{a}_j^\top \mathbf{R}_j \mathbf{a}_j - \operatorname{tr} \mathbf{R}_j \mathbf{W}(j), & \mathcal{Q}_j(\mathbf{R}^2) &:= \mathbf{a}_j^\top \mathbf{R}_j^2 \mathbf{a}_j - \operatorname{tr} \mathbf{R}_j^2 \mathbf{W}(j), \\ a_n(z) &:= (yzs_p(z) + z + y - 1)^{-1}, & b_n(z) &:= (2yzs_p(z) + y + z - 1)^{-1}. \end{aligned}$$

Using this notation, we have

$$\varepsilon_j = \frac{1}{n}\mathcal{X}_j - \frac{1}{n}\mathcal{Q}_j(\mathbf{R}) + \frac{1}{n} + \frac{z}{n}\mathcal{D}_j(\mathbf{R}) - \frac{z}{n}\Delta_E(\mathbf{R}). \quad (6.25)$$

Consider the representation

$$\begin{aligned} \mathbb{E}|\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^2 &= \mathbb{E}(\overline{\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}})(\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}) \\ &= \mathbb{E}(\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R})\operatorname{tr} \mathbf{R} = a_n(z) \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \varepsilon_j R(j, j). \end{aligned}$$

Using (6.25), we may rewrite this equality as follows

$$\frac{1}{n^2} \mathbb{E}|\operatorname{tr} \mathbf{R} - \mathbb{E} \operatorname{tr} \mathbf{R}|^2 = a_n(z)(A_1 + A_2 + zA_3 + zA_4 + zA_5), \quad (6.26)$$

where

$$\begin{aligned} A_1 &= \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j R(j, j), & A_2 &= -\frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_3 &= \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j, j), & A_4 &= -\frac{1}{n^3} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 R(j, j), \\ A_5 &= \frac{1}{n^3} \sum_{j=1}^p \Delta_E(\mathbf{R}) R(j, j) = \frac{1}{n^3} \mathbb{E} |\Delta_E(\mathbf{R})|^2. \end{aligned}$$

We first consider  $A_4$ .

**Lemma 6.3.** *Assuming the conditions of Theorem 1.2, the following representation holds:*

$$A_4 = y(a_n(z) - \delta_n(z)yzs_p(z)b_n(z)) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 + b_n(z)(A_6 + A_7 + zA_8) + \Gamma_1, \quad (6.27)$$

where

$$\begin{aligned} A_6 &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 \mathcal{X}_j R(j, j), \\ A_7 &= \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_8 &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 \mathcal{D}_j(\mathbf{R}) R(j, j), \end{aligned}$$

and  $\Gamma_1$  satisfies the inequality

$$|\Gamma_1| \leq \frac{CM_8 |b_n(z)|}{n^4 v^6}. \quad (6.28)$$



**Proof.** We have

$$A_4 = -\mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \left( \frac{1}{n} \text{tr} \mathbf{R} \right).$$

Adding and subtracting  $ys_p(z) = \frac{1}{n} \mathbb{E} \text{tr} \mathbf{R}$ , we rewrite this equality as

$$A_4 = A_9 + A_{10}, \quad (6.29)$$

where

$$A_9 = -ys_p(z) \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2, \quad A_{10} = -\mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \left( \frac{1}{n} \Delta_E(\mathbf{R}) \right).$$

To investigate the asymptotics of  $A_{10}$  we derive some recursion relations. Using the representation (6.2), we obtain

$$A_{10} = yA_{11} + a_n(z)(A_6 + A_7 + zA_8 + zA_{12}), \quad (6.30)$$

where

$$A_{11} = \frac{a_n(z) + s_p(z)}{n^2} \mathbb{E} |\Delta_E(\mathbf{R})|^2,$$

$$A_{12} = \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 \bar{\Delta}_E(\mathbf{R}) R(j, j) = \frac{1}{n^4} \mathbb{E} |\Delta_E(\mathbf{R})|^2 \bar{\Delta}_E(\mathbf{R}) \text{tr} \mathbf{R}.$$

Adding and subtracting  $ys_p(z)$  again, we write the term  $A_{12}$  in the form

$$A_{12} = \frac{1}{n^3} \left( ys_p(z) \mathbb{E} |\Delta_E(\mathbf{R})|^2 \bar{\Delta}_E(\mathbf{R}) + \frac{1}{n} \mathbb{E} |\Delta_E(\mathbf{R})|^2 (\bar{\Delta}_E(\mathbf{R}))^2 \right)$$

$$= -ys_p(z) A_{10} + \frac{1}{n} \mathbb{E} |\Delta_E(\mathbf{R})|^2 (\bar{\Delta}_E(\mathbf{R}))^2. \quad (6.31)$$

Comparing (6.30) and (6.31), we obtain

$$A_{10} = yzs_p(z)a_n(z)A_{10} + a_n(z) \left( \frac{z}{n^4} \mathbb{E} |\Delta_E(\mathbf{R})|^2 (\Delta_E(\mathbf{R}))^2 + A_6 + A_7 + zA_8 \right) + yA_{11}.$$

Combining the left-hand side with the first term on the right-hand side, we obtain, after multiplication with  $(a_n(z))^{-1}b_n(z) = (1 - yzs_p(z)a_n(z))^{-1}$ ,

$$A_{10} = b_n(z) \left( y(a_n(z))^{-1} A_{11} + \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \left( \frac{1}{n} \bar{\Delta}_E(\mathbf{R}) \right)^2 + A_6 + A_7 + zA_8 \right). \quad (6.32)$$

From the definition of  $A_{11}$  and (6.2), it follows that

$$A_{11} = \delta_p(z) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2.$$

This relation and (6.2), (6.31), (6.32) together imply that

$$\begin{aligned}
A_4 &= -y s_p(z) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 + A_{10} \\
&= y a_n(z) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 - y \delta_p(z) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 + A_{10} \\
&= y (a_n(z) - \delta_p(z) y z s_p(z) b_n(z)) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 \\
&\quad + b_n(z) (A_6 + A_7 + z A_8) + \Gamma_1,
\end{aligned} \tag{6.33}$$

where

$$\Gamma_1 = \frac{a_n(z) b_n(z)}{n^4} \mathbb{E} |\Delta_E(\mathbf{R})|^2 (\overline{\Delta}_E(\mathbf{R}))^2.$$

According to Remark 6.1 and inequality (5.11), we obtain

$$|\Gamma_1| \leq \frac{CM_8 |b_n(z)|}{n^4 v^6}. \tag{6.34}$$

The relations (6.33) and (6.34) conclude the proof of the lemma.  $\square$

We now turn to  $A_1$ .

**Lemma 6.4.** *Under the conditions of Theorem 1.2 the following inequality holds, for  $v \geq v_0$ :*

$$|A_1| \leq \frac{\sqrt{M_8}}{n^2 v^2}. \tag{6.35}$$

**Proof.** Using (6.1), we obtain

$$A_1 = \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j R(j, j) = A_{13} + A_{14}, \tag{6.36}$$

where

$$A_{13} = -\frac{a_n(z)}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j, \quad A_{14} = \frac{a_n(z)}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j \varepsilon_j R(j, j).$$

Using the equality  $\mathcal{D}_j(\mathbf{R}) = (1 + \mathbf{a}_j^\top R_j^2 \mathbf{a}_j) R(j, j)$ , which follows from (3.6), we obtain

$$A_{13} = -a_n(z) \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \mathcal{D}_j(\mathbf{R}) \mathcal{X}_j = -a_n(z) \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^\top \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j R(j, j). \tag{6.37}$$

Applying (6.1) and (6.25), we can write

$$A_{13} = A_{15} + A_{16} + A_{17} + A_{18} + A_{19} + A_{20}, \tag{6.38}$$

where

$$\begin{aligned}
 A_{15} &= a_n^2(z) \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j, \\
 A_{16} &= a_n^2(z) \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j^2 R(j, j), \\
 A_{17} &= a_n^2(z) \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j \mathcal{D}_j(\mathbf{R}) R(j, j), \\
 A_{18} &= a_n^2(z) \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j \mathcal{Q}_j(\mathbf{R}) R(j, j), \\
 A_{19} &= a_n^2(z) \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j \overline{\Delta}_E(\mathbf{R}) R(j, j), \\
 A_{20} &= a_n^2(z) \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \right) \mathcal{X}_j R(j, j).
 \end{aligned}$$

Note that

$$A_{15} = a_n(z)^2 \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j \mathcal{X}_j = a_n(z)^2 \frac{1}{n^4} \sum_{j=1}^p \sum_{l=1}^n \mathbb{E} (X_{jl}^2 - 1) X_{jl}^2 \mathbb{E} \mathbf{G}_j(l, l), \quad (6.39)$$

where  $\mathbf{G}_j = \mathbf{X}(j)^T \mathbf{R}_j^2 \mathbf{X}(j)$ . Since  $\text{tr}|\mathbf{G}_j| = \text{tr}|\mathbf{R}_j^2 \mathbf{W}(j)| \leq C n \nu^{-1}$ , we obtain, from (6.39),

$$|A_{15}| \leq \frac{C M_4}{n^2 \nu}. \quad (6.40)$$

To bound  $A_{16} - A_{20}$  we use (3.6) again. It is easy to check that

$$|A_{16}| \leq \frac{C M_4}{n^2 \nu} \quad (6.41)$$

Furthermore, the inequality  $|\mathcal{D}_j(\mathbf{R})| \leq \nu^{-1}$ , which follows from (3.7), implies

$$|A_{17}| \leq \frac{C}{n^{5/2} \nu^2} \leq \frac{C}{n^2 \nu^2}. \quad (6.42)$$

Applying Cauchy’s inequality gives

$$|A_{18}| \leq \frac{C}{n^4 \nu} \sum_{j=1}^p \mathbb{E}^{1/2} |\mathcal{Q}_j(\mathbf{R})|^2 \mathbb{E}^{1/2} |\mathcal{X}_j|^2 \leq \frac{C}{n^2 \nu^2}. \quad (6.43)$$

Using Cauchy’s inequality and (6.9), we obtain

$$|A_{19}| \leq \frac{C}{n^4 \nu} \sum_{j=1}^p \mathbb{E}^{1/2} |\Delta_E(\mathbf{R})|^2 \mathbb{E}^{1/2} |\mathcal{X}_j|^2 \leq \frac{C}{n^{5/2} \nu^{5/2}} \leq \frac{C}{n^2 \nu^2}. \quad (6.44)$$

For the term  $A_{20}$  we have a similar bound:

$$|A_{20}| \leq \frac{C}{n^4 v} \sum_{j=1}^p \mathbb{E}^{1/2} |\mathcal{X}_j|^2 \leq \frac{C}{n^2 v^2}. \quad (6.45)$$

Inequalities (6.39)–(6.45) together imply that

$$|A_{13}| \leq \frac{C}{n^2 v^2}. \quad (6.46)$$

We write the term  $A_{14}$  as

$$A_{14} = \frac{a_n(z)}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j \varepsilon_j R(j, j) = A_{21} + A_{22} + A_{23} + A_{24}, \quad (6.47)$$

where

$$\begin{aligned} A_{21} &= \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j^2 R(j, j), & A_{22} &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_{23} &= \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{X}_j \mathcal{D}_j(\mathbf{R}) R(j, j), & A_{24} &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 \mathcal{X}_j R(j, j). \end{aligned}$$

Note that

$$\begin{aligned} |A_{21}| &\leq \frac{C \mathcal{C}(\mathbf{R})}{n^3} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 |\mathcal{X}_j|^4 \right)^{1/2} \\ &\leq \frac{C M_8^{1/2}}{n} \left( \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 + \frac{1}{nv} \right) \leq \frac{C M_8^{1/2}}{n^2 v^2}. \end{aligned} \quad (6.48)$$

Using the inequality  $|\Delta_E(\mathbf{R})| \leq |\Delta_E(\mathbf{R}_j)| + v^{-1}$ , we obtain

$$\begin{aligned} |A_{22}| &\leq \frac{C \mathcal{C}(\mathbf{R})}{n^3} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2} \\ &\leq \frac{C}{n^3} \left\{ \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2} \right. \\ &\quad \left. + v^{-1} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2} \right\}. \end{aligned} \quad (6.49)$$

Applying Cauchy's inequality, we obtain

$$\begin{aligned} \mathbb{E}\{|\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 | \mathbf{R}_j\} &\leq \mathbb{E}^{1/2}\{|\mathcal{Q}_j(\mathbf{R})|^4 | \mathbf{R}_j\} \mathbb{E}^{1/2} \mathcal{X}_j^4 \\ &\leq CM_8 n \operatorname{tr} |\mathbf{R}_j|^2 \leq CM_8 n v^{-1} |\operatorname{tr} \mathbf{R}_j|. \end{aligned} \quad (6.50)$$

Furthermore,

$$\begin{aligned} \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 |\operatorname{tr} \mathbf{R}_j| &\leq Cn |s_{pj}(z)| \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 + C \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^3 \\ &\leq Cn \mathbb{E}|\Delta_E(\mathbf{R})|^2 + C \mathbb{E}|\Delta_E(\mathbf{R})|^3 + C v^{-3} + C n v^{-2}. \end{aligned} \quad (6.51)$$

By Burkholder's inequality for martingales, we have, for  $v \geq v_0$ ,

$$\mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^3 \leq \frac{C}{\sqrt{nv}} \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \leq C \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2. \quad (6.52)$$

Inequalities (6.50)–(6.52) together imply that, for  $v \geq v_0$ ,

$$|A_{22}| \leq \frac{CM_8^{1/2}}{n^2 v^2}. \quad (6.53)$$

Using the inequality  $|\mathcal{D}_j(\mathbf{R})| \leq v^{-1}$  and Cauchy's inequality, we obtain that, for  $v \geq v_0$ ,

$$|A_{23}| \leq \frac{CC(\mathbf{R})}{n^3 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E}|\Delta_E(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2}. \quad (6.54)$$

Since  $\Delta_E^{(j)}$  and  $\mathcal{X}_j$  are independent,

$$\mathbb{E}|\Delta_E(\mathbf{R})|^2 \mathcal{X}_j^2 \leq C \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 \mathbb{E} \mathcal{X}_j^2 + \frac{C}{v^2} \mathbb{E} \mathcal{X}_j^2 \leq Cn M_8 (\mathbb{E}|\Delta_E(\mathbf{R})|^2 + v^{-2}). \quad (6.55)$$

Inequalities (6.8), (6.9), (6.54) and (6.55) together imply

$$|A_{23}| \leq \frac{CM_8^{1/2} M_4^{1/2}}{n^{5/2} v^{5/2}} \leq \frac{CM_8^{1/2}}{n^2 v^2}. \quad (6.56)$$

Using Cauchy's inequality,

$$\begin{aligned} |A_{24}| &\leq \frac{CC(\mathbf{R})}{n^3} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E}|\Delta_E(\mathbf{R})|^4 \mathcal{X}_j^2 \right)^{1/2} \leq \frac{C}{n^3} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^4 \mathbb{E} \mathcal{X}_j^2 + v^{-4} \mathbb{E} \mathcal{X}_j^2 \right)^{1/2} \\ &\leq \frac{CM_4^{1/2}}{n^{5/2}} (\mathbb{E}|\Delta_E(\mathbf{R})|^4 + v^{-4})^{1/2} \leq \frac{CM_8^{1/2} M_4^{1/2}}{n^{5/2} v^3} \leq \frac{CM_8^{1/2}}{n^2 v^2}. \end{aligned} \quad (6.57)$$

The relations (6.36), (6.46), (6.47), (6.48), (6.53), (6.56) and (6.57) together imply that, for  $v \geq v_0$ ,

$$|A_1| \leq \frac{\sqrt{M_8}}{n^2 v^2}. \quad (6.58)$$

This concludes the proof of the lemma.  $\square$

Furthermore, using Cauchy's inequality and Lemma 4.5, we obtain

$$|A_3| \leq \frac{C(\mathbf{R})}{nv} E^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \leq \frac{C}{nv} E^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2. \quad (6.59)$$

Using Cauchy's inequality again,

$$|A_6| \leq \frac{CC(\mathbf{R})}{\sqrt{n}} \left( \frac{1}{n} \sum_{j=1}^p E \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^4 \mathcal{X}_j^2 \right)^{1/2}. \quad (6.60)$$

Applying the inequality  $|\Delta(\mathbf{R})| \leq |\Delta^{(j)}(\mathbf{R})| + 2v^{-1}$ ,

$$\begin{aligned} E \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^4 \mathcal{X}_j^2 &\leq CE \left| \frac{1}{n} (\Delta_E^{(j)}(\mathbf{R})) \right|^4 E \mathcal{X}_j^2 + \frac{C}{n^4 v^4} E \mathcal{X}_j^2 \\ &\leq CE \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^4 E \mathcal{X}_j^2 + \frac{CM_4}{n^3 v^4}. \end{aligned} \quad (6.61)$$

Hence inequalities (6.8), (6.60), (6.61) and Remark 6.1 imply that, for  $v \geq v_0$ ,

$$|A_6| \leq \frac{CM_8^{1/2}}{n^2 v^2}. \quad (6.62)$$

We have the bound

$$\begin{aligned} |A_8| &= \frac{1}{n^4} \sum_{j=1}^p E |\Delta_E^{(j)}(\mathbf{R})|^2 \mathcal{D}_j R(j, j) \\ &\leq \frac{CC(\mathbf{R})}{n^3 v} \left( \frac{1}{n} \sum_{j=1}^p E |\Delta_E^{(j)}(\mathbf{R})|^4 \right)^{1/2} \leq \frac{C\sqrt{M_8}}{n^3 v^4} \leq \frac{C}{n^2 v^2}. \end{aligned} \quad (6.63)$$

Now consider  $A_7$ . We may write

$$A_7 = -\frac{1}{n^4} \sum_{j=1}^p E |\Delta_E(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) R(j, j) = A_{25} + A_{26}, \quad (6.64)$$

where

$$\begin{aligned} A_{25} &= \frac{1}{n^4} \sum_{j=1}^p E \left| \Delta_E^{(j)}(\mathbf{R}) \right|^2 \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_{26} &= \frac{1}{n^4} \sum_{j=1}^p E (|\Delta_E(\mathbf{R})|^2 - |\Delta_E^{(j)}(\mathbf{R})|^2) \mathcal{Q}_j(\mathbf{R}) R(j, j). \end{aligned}$$

**Lemma 6.5.** *Under the conditions of Theorem 1.2, there exists some absolute constant  $C$  such that, for  $v \geq v_0$ ,*

$$|A_{26}| \leq \frac{CM_8^{1/2}}{n^2 v^2}.$$

**Proof.** We decompose  $A_{26}$  into

$$A_{26} = A_{27} + A_{28} + A_{29} + A_{30} + A_{31}, \quad (6.65)$$

where

$$\begin{aligned} A_{27} &= \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} |\mathcal{D}_j(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_{28} &= \frac{1}{n^4} |(\mathbb{E} \mathcal{D}_j(\mathbf{R}))|^2 \mathbb{E} \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_{29} &= -\frac{2}{n^4} \sum_{j=1}^p \mathbb{E} [\operatorname{Re}\{\overline{\mathcal{D}}_j(\mathbf{R})\} \mathbb{E} \mathcal{D}_j(\mathbf{R})] \mathcal{Q}_j(\mathbf{R}) \{R(j, j)\}, \\ A_{30} &= \frac{2}{n^4} \sum_{j=1}^p \mathbb{E} [\operatorname{Re}\{\Delta_E^{(j)}(\mathbf{R}) \overline{\mathcal{D}}_j(\mathbf{R})\}] \mathcal{Q}_j(\mathbf{R}) R(j, j), \\ A_{31} &= -\frac{2}{n^4} \sum_{j=1}^p \mathbb{E} [\operatorname{Re}\{\Delta_E^{(j)}(\mathbf{R}) \mathbb{E} \mathcal{D}_j(\mathbf{R})\}] \mathcal{Q}_j(\mathbf{R}) R(j, j). \end{aligned}$$

Using Cauchy's inequality and Lemma 4.5, we obtain that, for  $v \geq v_0$ ,

$$\max\{|A_{27}|, |A_{28}|, |A_{29}|\} \leq \frac{CC(\mathbf{R})}{n^3 v^2} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} \leq \frac{C\sqrt{M_4}}{(nv)^2}. \quad (6.66)$$

Furthermore, using Cauchy's inequality again,

$$\max\{|A_{30}|, |A_{31}|\} \leq \frac{CC(\mathbf{R})}{n^3 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2}. \quad (6.67)$$

Since  $\mathbf{x}_j$  and  $\mathbf{R}_j \mathbf{W}(j)$  are independent, we have

$$\mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \leq CM_4 \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 (\operatorname{tr} |\mathbf{R}_j|^2 \mathbf{W}(j)^2). \quad (6.68)$$

Note that

$$\operatorname{tr} |\mathbf{R}_j|^2 \mathbf{W}(j)^2 = \frac{1}{v} \operatorname{Im}\{\operatorname{tr} \mathbf{R}_j \mathbf{W}(j)^2\} = \frac{1}{v} \mathbf{I} \{ \operatorname{tr} \mathbf{W}(j) + z \operatorname{tr} \operatorname{Im}_{p-1} + z \operatorname{tr} \mathbf{R}_j \}. \quad (6.69)$$

With a similar argument to (6.50), we obtain, for  $v \geq v_0$ ,

$$\begin{aligned} \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2|\mathcal{Q}_j(\mathbf{R})|^2 &\leq \frac{CM_4}{\nu} \{(n(|s_{pj}(z)| + 1)\mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 + \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^3)\} \\ &\leq \frac{CM_4}{\nu} \left\{ n\mathbb{E}|\Delta_E(\mathbf{R})|^2 + \mathbb{E}|\Delta_E(\mathbf{R})|^3 + \frac{n}{\nu^2} \right\} \leq \frac{CM_4^2 n}{\nu^4}. \end{aligned} \quad (6.70)$$

Inequalities (6.8), (6.67) and (6.70) together imply that, for  $\nu \geq \nu_0$  ( $\sqrt{n\nu} \geq C > 0$ ),

$$\max\{|A_{30}|, |A_{31}|\} \leq \frac{CM_4}{n^{5/2}\nu^3} \leq \frac{C\sqrt{M_8}}{n^2\nu^2} \quad (6.71)$$

From inequalities (6.66) and (6.71) it follows that, for  $\nu \geq \nu_0$ ,

$$|A_{26}| \leq \frac{C\sqrt{M_8}}{n^2\nu^2}. \quad (6.72)$$

The last inequality concludes the proof.  $\square$

We continue with  $A_{25}$ .

**Lemma 6.6.** *Under conditions of Theorem 1.2, there exists some constant  $C_1$  such that, for  $\nu \geq \nu_0$ ,*

$$|A_{25}| \leq \frac{C_1 M_8^{1/2}}{n^2 \nu^2}.$$

**Proof.** We express  $A_{25}$  in the form

$$A_{25} = \frac{1}{n^4} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) R(j, j) = A_{32} + A_{33} + A_{34} + A_{35}, \quad (6.73)$$

where

$$A_{32} = -\frac{1}{n^5} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) \mathcal{X}_j(R(j, j)),$$

$$A_{33} = \frac{1}{n^5} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 (\mathcal{Q}_j(\mathbf{R}))^2 R(j, j),$$

$$A_{34} = -\frac{1}{n^5} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j, j),$$

$$A_{35} = \frac{1}{n^5} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2 \Delta_E(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) R(j, j).$$

Using Cauchy's inequality,



$$|A_{32}| \leq \frac{CC(\mathbf{R})}{n^4} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2}. \quad (6.74)$$

Applying inequality (6.50), we obtain

$$\frac{1}{n^8} \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \leq \frac{CM_8}{n^2 v} \left( \mathbb{E} \left| \frac{1}{n} \Delta_E^{(j)}(\mathbf{R}) \right|^4 + \mathbb{E} \left| \frac{1}{n} \Delta_E^{(j)}(\mathbf{R}) \right|^5 \right).$$

This inequality and (6.74), (6.24) and together imply

$$|A_{32}| \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.75)$$

Similarly,

$$|A_{33}| \leq \frac{CC(\mathbf{R})}{n^4} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 |\mathcal{Q}_j(\mathbf{R})|^4 \right)^{1/2}. \quad (6.76)$$

According to Rosenthal's inequality for quadratic forms, we have

$$\mathbb{E} \{ |\mathcal{Q}_j(\mathbf{R})|^4 |X(j)\} \leq CM_8 (\text{tr} |\mathbf{R}_j|^2 \mathbf{W}(j))^2.$$

Similar to inequality (6.70), we obtain that, for  $v \geq v_0$ ,

$$\begin{aligned} \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 |\mathcal{Q}_j(\mathbf{R})|^4 &\leq \frac{CM_8}{v^2} \left( |ns_{nj}(z)|^2 \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 + \mathbb{E} \left| \Delta_E^{(j)}(\mathbf{R}) \right|^6 \right) \\ &\leq \frac{CM_8 n^2}{v^2} \left( \mathbb{E} |\Delta_E(\mathbf{R})|^4 + \frac{1}{n^2} \mathbb{E} |\Delta_E(\mathbf{R})|^6 + \frac{1}{v^4} \right). \end{aligned} \quad (6.77)$$

Using the last inequality, the relations (6.69), (6.24) and the inequalities (6.74)–(6.77), we obtain, for  $v \geq v_0$ ,

$$|A_{33}| \leq \frac{CM_8}{n^3 v^4} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.78)$$

For  $A_{34}$  the following bound holds:

$$|A_{34}| \leq \frac{CC(\mathbf{R})}{n^4 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2}.$$

Analogously to (6.77), (6.78), we obtain that, for  $v \geq v_0$ ,

$$|A_{34}| \leq \frac{CM_4 \sqrt{M_8}}{n^{7/2} v^{9/2}} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.79)$$

Applying Cauchy's inequality, Rosenthal's inequality for quadratic forms and inequalities (6.24) and (6.68), we obtain that, for  $v \geq v_0$

$$\begin{aligned}
 |A_{35}| &= \frac{1}{n^5} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 \Delta_E(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) R(j, j) \\
 &\leq \frac{1}{n^5} \left( \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^3 |\mathcal{Q}_j(\mathbf{R})| R(j, j) + v^{-1} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})| R(j, j) \right) \\
 &\leq \frac{C(\mathbf{R})}{n^4} \left( \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^6 |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} + v^{-1} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} \right) \\
 &\leq \frac{C(\mathbf{R})}{n^4} \left( \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^6 \text{tr} |\mathbf{R}_j|^2 \mathbf{W}(j)^2 \right)^{1/2} + v^{-1} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4 \text{tr} |\mathbf{R}_j|^2 \mathbf{W}(j)^2 \right)^{1/2} \right) \\
 &\leq \frac{C\sqrt{M_8}}{n^4 v^{1/2}} \left( \left( n \mathbb{E} |\Delta_E(\mathbf{R})|^6 + \mathbb{E} |\Delta_E(\mathbf{R})|^7 + \frac{n}{v^6} + \frac{1}{v^7} \right)^{1/2} \right. \\
 &\quad \left. + v^{-1} \left( n \mathbb{E} |\Delta_E(\mathbf{R})|^4 + \mathbb{E} |\Delta_E(\mathbf{R})|^5 + \frac{n}{v^4} + \frac{1}{v^5} \right)^{1/2} \right) \\
 &\leq \frac{C\sqrt{M_8 M_{12}}}{n^{7/2} v^5} = \frac{C\sqrt{M_8}}{n^2 v^2} \frac{\sqrt{M_{12}}}{n^{3/2} v^3} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.80}
 \end{aligned}$$

The relations (6.75), (6.78), (6.79), and (6.80) together imply

$$|A_{25}| \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \tag{6.81}$$

This concludes the proof. □

Lemmas 6.5 and 6.6 imply that, for  $v \geq v_0$ ,

$$|A_7| \leq \frac{CM_8^{1/2}}{n^2 v^2} \tag{6.82}$$

We now continue with  $A_2$  as follows:

$$A_2 = -\frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \Delta_E(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) R(j, j) = A_{36} + a_n(z)(A_{37} + A_{38} + zA_{39} + zA_{40}), \tag{6.83}$$

where

$$A_{36} = -\frac{1}{n^3} \sum_{j=1}^p \mathbb{E}(\overline{\mathcal{D}}_j(\mathbf{R}) - \mathbb{E}\overline{\mathcal{D}}_j(\mathbf{R}))\mathcal{Q}_j(\mathbf{R})R(j, j),$$

$$A_{37} = -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})\mathcal{X}_jR(j, j),$$

$$A_{38} = \frac{1}{n^4} \sum_{j=1}^p \mathbb{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})^2R(j, j),$$

$$A_{39} = -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E}\Delta_E^{(j)}(\mathbf{R})\mathcal{Q}_j(\mathbf{R})\mathcal{D}_j(\mathbf{R})R(j, j),$$

$$A_{40} = \frac{1}{n^4} \sum_{j=1}^p \mathbb{E}|\Delta_E^{(j)}(\mathbf{R})|^2\mathcal{Q}_j(\mathbf{R})R(j, j).$$

**Lemma 6.7.** *Under the conditions of Theorem 1.2, for  $v \geq v_0$ ,*

$$|A_{36}| \leq \frac{CM_8^{1/2}}{n^2v^2}.$$

*Proof.* Write

$$A_{36} = A_{41} + A_{42}, \tag{6.84}$$

where

$$A_{41} = \frac{1}{n^3} \sum_{j=1}^p \mathbb{E}\overline{\mathcal{D}}_j(\mathbf{R})\mathcal{Q}_j(\mathbf{R})R(j, j), \quad A_{42} = -\frac{1}{n^3} \sum_{j=1}^p \mathbb{E}\overline{\mathcal{D}}_j(\mathbf{R})\mathbb{E}\mathcal{Q}_j(\mathbf{R})R(j, j).$$

Using (6.1) and (6.25), we obtain

$$A_{42} = a_n(z)(A_{43} + A_{44} + zA_{45} + zA_{46}), \tag{6.85}$$

where

$$\begin{aligned}
A_{43} &= \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \overline{\mathcal{D}}_j(\mathbf{R}) \mathbb{E} \mathcal{Q}_j(\mathbf{R}) \mathcal{X}_j R(j, j), \\
A_{44} &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \overline{\mathcal{D}}_j(\mathbf{R}) \mathbb{E} (\mathcal{Q}_j(\mathbf{R}))^2 R(j, j), \\
A_{45} &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \overline{\mathcal{D}}_j(\mathbf{R}) \mathbb{E} \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j, j), \\
A_{46} &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \overline{\mathcal{D}}_j(\mathbf{R}) \mathbb{E} \mathcal{Q}_j(\mathbf{R}) \Delta_E(\mathbf{R}) R(j, j).
\end{aligned}$$

Applying Cauchy's inequality and (6.49),

$$|A_{43}| \leq \frac{CC(\mathbf{R})}{n^3 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2} \leq \frac{C\sqrt{M_8}}{n^2 v^{3/2}} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.86)$$

Furthermore,

$$|A_{44}| \leq \frac{CC(\mathbf{R})}{n^3 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^4 \right)^{1/2} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.87)$$

Analogously we obtain, for  $v \geq v_0$ ,

$$|A_{45}| \leq \frac{CC(\mathbf{R})}{n^3 v^2} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} \leq \frac{C\sqrt{M_4}}{n^2 v^2}. \quad (6.88)$$

Finally, for  $A_{46}$  we have the following bounds:

$$\begin{aligned}
 |A_{46}| &\leq \frac{CC(\mathbf{R})}{n^3v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 |\Delta_E(\mathbf{R})|^2 \right)^{1/2} \\
 &\leq \frac{C}{n^3v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 |\Delta_E^{(j)}(\mathbf{R})|^2 \right)^{1/2} + \frac{C\sqrt{M_4}}{n^2v^2} \\
 &\leq \frac{C\sqrt{M_4}}{n^3v^{3/2}} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 \operatorname{Im}\{\operatorname{tr} \mathbf{R}_j \mathbf{W}(j)^2\} \right)^{1/2} + \frac{C\sqrt{M_4}}{n^2v^2} \\
 &\leq \frac{C\sqrt{M_4}}{n^3v^{3/2}} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 (v(p-1) + |z^2| \|\operatorname{tr} \mathbf{R}_j\|) \right)^{1/2} + \frac{C\sqrt{M_4}}{n^2v^2} \\
 &\leq \frac{C\sqrt{M_4}}{(nv)^{3/2}} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 + \frac{C\sqrt{M_8}}{n^2v^2} \leq \frac{C\sqrt{M_8}}{n^2v^2}.
 \end{aligned} \tag{6.89}$$

We now turn to the estimation of  $A_{41}$ . Using Lemma 3.3, we may write

$$A_{41} = \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^\top \overline{\mathbf{R}}_j^2 \mathbf{a}_j \right) \mathcal{Q}_j(\mathbf{R}) |R(j, j)|^2. \tag{6.90}$$

By (4.6), we obtain

$$A_{41} = A_{47} + A_{48} + A_{49} + A_{50}, \tag{6.91}$$

where

$$\begin{aligned}
 A_{47} &= \frac{|a_n(z)|^2}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^\top j^\top \overline{\mathbf{R}}_j^2 \mathbf{a}_j \right) \mathcal{Q}_j(\mathbf{R}), \\
 A_{48} &= \frac{\overline{a_n(z)}}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^\top j^\top \overline{\mathbf{R}}_j^2 \mathbf{a}_j \right) \mathcal{Q}_j(\mathbf{R}) \varepsilon_j R(j, j), \\
 A_{49} &= \frac{a_n(z)}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^\top j^\top \overline{\mathbf{R}}_j^2 \mathbf{a}_j \right) \mathcal{Q}_j(\mathbf{R}) \overline{\varepsilon_j R(j, j)}, \\
 A_{50} &= \frac{1}{n^3} \sum_{j=1}^p \mathbb{E} \left( 1 + \frac{1}{n} \mathbf{a}_j^\top j^\top \overline{\mathbf{R}}_j^2 \mathbf{a}_j \right) \mathcal{Q}_j(\mathbf{R}) |\varepsilon_j|^2 |R(j, j)|^2.
 \end{aligned}$$

Using Cauchy’s inequality,

$$|A_{47}| \leq \frac{C}{n^4} \sum_{j=1}^p \mathbb{E}^{1/2} |\mathcal{Q}_j(\mathbf{R}^2)|^2 \mathbb{E}^{1/2} |\mathcal{Q}_j(\mathbf{R})|^2 \leq \frac{CM_4}{n^2v^2}. \tag{6.92}$$

Since  $|\text{tr } \mathbf{R} - \text{tr } \mathbf{R}_j| = |1 + (1/n)\mathbf{a}_j^T \mathbf{R}_j^2 \mathbf{a}_j| R(j, j) \leq v^{-1}$ , we have

$$\max\{|A_{48}|, |A_{49}|\} \leq \frac{C}{n^2 v} \max_{1 \leq j \leq n} \{E^{1/2} |\mathcal{Q}_j(\mathbf{R})|^2 E^{1/2} |\varepsilon_j|^2\} \leq \frac{CM_4}{n^2 v^2}. \quad (6.93)$$

Furthermore, using (6.25),

$$|A_{50}| \leq A_{51} + A_{52} + A_{53} + A_{54}, \quad (6.94)$$

where

$$\begin{aligned} A_{51} &= \frac{C}{n^5 v} \sum_{j=1}^p E |\mathcal{Q}_j(\mathbf{R})| \mathcal{X}_j^2 |R(j, j)|, \\ A_{52} &= \frac{C}{n^5 v} \sum_{j=1}^p E |\mathcal{Q}_j(\mathbf{R})|^3 |R(j, j)|, \\ A_{53} &= \frac{C}{n^5 v} \sum_{j=1}^p E |\mathcal{Q}_j(\mathbf{R})| |\mathcal{D}_j(\mathbf{R})|^2 |R(j, j)|, \\ A_{54} &= \frac{C}{n^5 v} \sum_{j=1}^p E |\mathcal{Q}_j(\mathbf{R})| |\Delta_E(\mathbf{R})|^2 |R(j, j)|. \end{aligned}$$

For  $A_{51}$  we have the obvious bound, for  $v \geq v_0$ ,

$$\begin{aligned} A_{51} &\leq \frac{C}{n^4 v^2} \max_{1 \leq j \leq n} E |\mathcal{Q}_j(\mathbf{R})| \mathcal{X}_j^2 \leq \frac{C}{n^4 v^2} \max_{1 \leq j \leq n} E^{1/2} |\mathcal{Q}_j(\mathbf{R})|^2 E^{1/2} \mathcal{X}_j^4 \\ &\leq \frac{C\sqrt{M_8}}{n^3 v^2} E^{1/2} \text{tr} |\mathbf{R}_j|^2 \mathbf{W}(j)^2 \leq \frac{C\sqrt{M_8}}{n^3 v^{5/2}} |E \text{tr } \mathbf{R}_j \mathbf{W}(j)^2|^{1/2} \\ &\leq \frac{C\sqrt{M_8}}{n^{5/2} v^{5/2}} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \end{aligned} \quad (6.95)$$

Applying Cauchy's inequality, we obtain, for  $v \geq v_0$ ,

$$A_{52} \leq \frac{CC(\mathbf{R})^{1/2}}{n^4 v \sqrt{v}} \left( \frac{1}{n} \sum_{j=1}^p E |\mathcal{Q}_j(\mathbf{R})|^4 \right)^{3/4} \leq \frac{CM_8^{3/4}}{(nv)^{5/2} \sqrt{v}} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.96)$$

Analogously we obtain, for  $v \geq v_0$ ,

$$A_{53} \leq \frac{CC(\mathbf{R})}{n^4 v^3} \left( \frac{1}{n} \sum_{j=1}^p E |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} \leq \frac{C\sqrt{M_4}}{n^2 v^2}, \quad (6.97)$$

and

$$A_{54} \leq \frac{CC(\mathbf{R})}{n^4 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\mathcal{Q}_j(\mathbf{R})|^2 |\Delta_E(\mathbf{R})|^4 \right)^{1/2} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.98)$$

Inequalities (6.71)–(6.85) together imply that, for  $v \geq v_0$ ,

$$|A_{36}| \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.99)$$

This completes the proof.  $\square$

**Lemma 6.8.** *Under the conditions of Theorem 1.2, there exists constants  $C$  such that*

$$\begin{aligned} \max\{|A_{37}|, |A_{40}|\} &\leq \frac{C\sqrt{M_8}}{n^2 v^2}, \\ \max\{|A_{38}|, |A_{39}|\} &\leq \frac{C\sqrt{M_8}}{n^2 v^2} + \frac{C\sqrt{M_8}}{nv} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2. \end{aligned}$$

**Proof.** Using Cauchy's inequality, we obtain

$$\begin{aligned} |A_{37}| &= \frac{1}{n^4} \left| \sum_{j=1}^p \mathbb{E} \Delta_E^{(j)}(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) \mathcal{X}_j R(j, j) \right| \\ &\leq \frac{CC(\mathbf{R})}{n^3} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \mathcal{X}_j^2 \right)^{1/2}. \end{aligned} \quad (6.100)$$

Inequalities (6.47), (6.8), and (6.85) together imply that, for  $v \geq v_0$ ,

$$|A_{37}| \leq \frac{C\sqrt{M_8}}{n^{3/2} \sqrt{v}} \left( n \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 + \frac{M_4}{n^2 v^3} \right)^{1/2} \leq \frac{C\sqrt{M_8}}{n^2 v^2}. \quad (6.101)$$

Analogously to this inequality, we obtain

$$\begin{aligned} |A_{38}| &= \frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \Delta_E^{(j)}(\mathbf{R}) \mathcal{Q}_j(\mathbf{R})^2 R(j, j) \\ &\leq \frac{C(\mathbf{R})}{n^3} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^4 \right)^{1/2}. \end{aligned}$$

Since  $\mathbf{R}_j$  and  $\mathbf{x}_j$  are independent, we have

$$\frac{1}{n^6} \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^4 \leq \frac{CM_8}{n^6} \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 (\text{tr} |\mathbf{R}_j|^2 \mathbf{W}(j)^2).$$

Using (6.68),

$$\frac{1}{n^6} \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^4 \leq \frac{CM_8}{n^6} (\mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 (v^{-2} |\mathbb{E} \operatorname{tr} R_j|^2 + n^2) + v^{-2} \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^4).$$

The last three inequalities and (6.19) together imply that, for  $v \geq v_0$ ,

$$|A_{38}| \leq \frac{C\sqrt{M_8}}{n^2 v^2} + \frac{C\sqrt{M_8}}{nv} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 + \frac{CM_8}{n^3 v^4}. \quad (6.102)$$

For  $A_{39}$  the following inequality holds:

$$\begin{aligned} |A_{39}| &= -\frac{1}{n^4} \sum_{j=1}^p \mathbb{E} \Delta_E^{(j)}(\mathbf{R}) \mathcal{Q}_j(\mathbf{R}) \mathcal{D}_j(\mathbf{R}) R(j, j) \\ &\leq \frac{C(\mathbf{R})}{n^3 v} \left( \frac{1}{n} \sum_{j=1}^p \mathbb{E} |\Delta_E^{(j)}(\mathbf{R})|^2 |\mathcal{Q}_j(\mathbf{R})|^2 \right)^{1/2} \\ &\leq \frac{C\sqrt{M_4}}{n^2 v^2} + \frac{C\sqrt{M_4}}{nv} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2. \end{aligned} \quad (6.103)$$

For  $A_{40}$  the same bound holds as for  $A_{25}$  (see (6.61) and (6.70)).  $\square$

Equation (6.82) and Lemmas 6.7 and 6.8 together imply that, for  $v \geq v_0$ ,

$$|A_2| \leq \frac{C\sqrt{M_8}}{n^2 v^2} + \frac{C\sqrt{M_8}}{nv} \mathbb{E}^{1/2} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2. \quad (6.104)$$

From Lemma 6.3 and from relations (6.62), (6.63), (6.82) we conclude that

$$A_4 = y(a_n(z) - yz\delta_n(z)b_n(z)) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 + \theta \frac{C|b_n(z)|\sqrt{M_8}}{n^2 v^2}, \quad (6.105)$$

with some  $\theta$  such that  $|\theta| \leq 1$ . Lemma 6.4 and the relations (6.26), (6.59), (6.104), (6.105) together imply that, for  $v \geq v_0$ ,

$$\begin{aligned} \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 &= yza_n(z)(a_n(z) - yz\delta_n(z)b_n(z)) \mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 \\ &\quad + C\theta \left( \frac{|b_n(z)|\sqrt{M_8}}{n^2 v^2} + \frac{1}{nv} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \right), \end{aligned} \quad (6.106)$$

with some  $\theta$  such that  $|\theta| \leq 1$ . Finally, we investigate the quantity  $\kappa_n(z) = 1 - yza_n^2(z)$ .

**Lemma 6.9.** *Under the conditions of Theorem 1.2, there exists a positive constant  $C$  such that, for  $v \geq v_0$ ,*

$$|\kappa_n(z)|^{-1} \leq C|a_n(z)| |y + z - 1 + 2yzy(z)|. \quad (6.107)$$

**Proof.** We may write



$$\begin{aligned} \kappa_n(z) &= 1 + \frac{yzs_p(z)}{z + y - 1 + yzs_p(z)} + \theta C y z |a_n(z)\delta_p(z)| \\ &= a_n(z)(b_n(z))^{-1} + \theta C |yza_n(z)\delta_p(z)|. \end{aligned} \tag{6.108}$$

Note that, for  $z = u + iv$  such that  $\sqrt{(u - a)(b - u)} \geq C\sqrt{v_0}$  and  $v \geq v_0$  according to (5.1)–(5.6) we have  $\text{Im}(z \pm \delta_n(z)) < 0$ . We can write that

$$\begin{aligned} s_p(z) &= \frac{y + z - 1 - yz\delta_p(z)}{2yz} + \frac{\sqrt{(y + z - 1 - yz\delta_n(z))^2 + 4yz\delta_n(z) - 4yz}}{2yz} \\ &= s_y(z + yz\delta_n(z)) - \frac{\delta_n(z)}{2}. \end{aligned}$$

This implies that

$$\begin{aligned} |s_p(z) - s(z)| &\leq |\delta_n(z)| + \frac{|\sqrt{(y + z - 1 + yz\delta_n(z))^2 - 4yz} - \sqrt{(y + z - 1)^2 - 4yz}|}{2|yz|} \\ &\leq c|\delta_n(z)| \left( 1 + \frac{|y + z - 1| + |\delta_n(z)|}{|\sqrt{(y + z - 1 + yz\delta_n(z))^2 - 4yz} + \sqrt{(y + z - 1)^2 - 4yz}|} \right). \end{aligned}$$

It is not difficult to check that for  $z = u + iv$  such that  $|y + u - 1| \geq 3v$  and  $v \geq v_0$ ,

$$\text{sgn}\{\text{Re}\{\sqrt{(z + y - 1)^2 - 4yz}\}\} = \text{sgn}\{\sqrt{(zy - 1 + yz\delta_n(z))^2 - 4yz}\}.$$

This implies that, for such  $z$ ,

$$\begin{aligned} &|\sqrt{(y + z - 1 + yz\delta_n(z))^2 - 4yz} + \sqrt{(y + z - 1)^2 - 4yz}| \\ &\geq |\sqrt{(y + z - 1)^2 - 4yz}| \geq \sqrt{v}. \end{aligned}$$

In this case we have

$$|s_p(z) - s_y(z)| \leq \frac{C|\delta_n(z)|}{\sqrt{v}} \leq \frac{C}{nv^{3/2}}.$$

On the other hand, if  $|y + u - 1| \leq 3v$  then  $|y + z - 1| \leq 4v$  and  $v \geq v_0$ , and we have

$$|s_p(z) - s_y(z)| \leq C|\delta_n(z)| \left( 1 + \frac{|\delta_n(z)|}{v} \right) \leq C|\delta_n(z)| \left( 1 + \frac{1}{nv^2} \right) \leq C|\delta_n(z)|.$$

The last two inequalities imply that, for  $v \geq v_0$ ,

$$|s_p(z) - s_y(z)| \leq \frac{C}{nv^{3/2}} \leq \gamma\sqrt{v_0}, \tag{6.109}$$

with sufficiently small  $\gamma$ . Furthermore, note that

$$z + y - 1 + 2yzs_y(z) = \sqrt{(y + z - 1)^2 - 4yz} = \sqrt{(a - z)(b - z)},$$

where  $a = (1 - \sqrt{y})^2$ ,  $b = (1 + \sqrt{y})^2$ . This implies that there exists some positive constant  $C_1$  such that, for  $v \geq v_0$ ,

$$|z + y - 1 + 2yzs_y(z)| \geq C_1 \sqrt{v_0}. \quad (6.110)$$

These relations imply that

$$\begin{aligned} |b_n(z)|^{-1} &= |z + y - 1 + 2yzs_p(z)| \geq |z + y - 1 + 2yzs_y(z)| - 2|yz||s_p(z) - s_y(z)| \\ &\geq \frac{1}{2}|z + y - 1 + 2yzs_y(z)|. \end{aligned} \quad (6.111)$$

According to Lemma 4.6 and inequality (6.110), we have, for  $v \geq v_0$ ,

$$|\delta_p(z)| \leq C_2 v_0 \leq C_2 |z + y - 1 + 2yzs_y(z)| v_0^{1/2}. \quad (6.112)$$

We may choose the constant in the definition  $v_0$  such that

$$C_1 - C_2 v_0^{1/2} \geq C_3 > 0. \quad (6.113)$$

The relations (6.107), (6.110)–(6.113) together imply that, for  $v \geq v_0$ ,

$$|\varkappa_n(z)| \geq \gamma |a_n(z)| |y + z - 1 + 2yzs(z)|. \quad (6.114)$$

This concludes the proof.  $\square$

Put  $b(z) = (z + y - 1 + 2yzs_y(z))^{-1}$ . Equation (6.106) and Lemma 6.9 together imply that

$$\begin{aligned} \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 &= \theta_1(z) C |b(z) b_n(z)| |\delta_p(z)| \mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \\ &\quad + C \theta_2(z) \left( \frac{|b(z)|^2 \sqrt{M_8}}{|a_n(z)| n^2 v^2} + \frac{|b(z)|}{|a_n(z)| n v} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \right), \end{aligned} \quad (6.115)$$

with some functions  $\theta_1(z)$  and  $\theta_2(z)$  such that  $|\theta_i(z)| \leq 1$ , for  $i = 1, 2$ . Inequalities (6.111) and (6.112) together imply that, for  $v \geq v_0$ ,

$$|C \theta_1(z) C b_n(z) b(z)| |\delta_n(z)| \leq \frac{1}{2}. \quad (6.116)$$

From (6.115), (6.116) and (5.11) we obtain the recursive inequality

$$\mathbb{E} \left| \frac{1}{n} \Delta_E(\mathbf{R}) \right|^2 \leq \frac{C \sqrt{M_8} \mathbb{E}^{1/2} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2}{n v |y + z - 1 + 2yzs_y(z)|} + \frac{\sqrt{M_8}}{n^2 v^2 |y + z - 1 + 2yzs_y(z)|^2}. \quad (6.117)$$

For  $n$  sufficiently large the recursion (6.93) implies that, for  $v \geq v_0$ ,

$$\mathbb{E} \left| \frac{1}{n} (\Delta_E(\mathbf{R})) \right|^2 \leq \frac{C M_8}{n^2 v^2 |y + z - 1 + 2yzs(z)|^2}.$$

The last bound concludes the proof of Proposition 6.1.  $\square$

## 7. Proof of Theorem 1.2

We now consider a modification of the smoothing inequality in Corollary 2.2.

**Lemma 7.1.** *Let  $F_p(x)$  be the empirical spectral distribution function of the matrix  $\mathbf{W}$  and let  $F_y(x)$  denote the Marchenko–Pastur distribution function. Denote their Stieltjes transforms by  $m_p(z)$  and  $s_y(z)$ , respectively. Let  $v_0$ ,  $d$  and  $\varepsilon$  be positive numbers such that*

$$\frac{1}{\pi} \int_{|y| \leq d} \frac{1}{u^2 + 1} du = \frac{3}{4},$$

and

$$\varepsilon < 2v_0d.$$

Then there exist constants  $C_1, \dots, C_4$  such that

$$\begin{aligned} & \mathbb{E} \sup_x |F_p(x) - EF_p(x)| \\ & \leq C_1 \int_{-\infty}^{\infty} |(\mathbb{E} m_n(u + iV) - s_y(u + iV))| du + C_2 v_0 + C_3 \varepsilon^{3/2} \\ & \quad + C_4 \sup_{x \in I'_\varepsilon} \left| \operatorname{Im} \left\{ \int_{v_0}^V (\mathbb{E} m_n(x + iu) - s_y(x + iu)) du \right\} \right| \\ & \quad + C_1 \int_{-\infty}^{\infty} \mathbb{E} \left| \frac{1}{n} (\operatorname{tr} R(u + iV) - \mathbb{E} \operatorname{tr} R(u + iV)) \right| du \\ & \quad + C_1 \int_{v_0}^V \mathbb{E} \left| \frac{1}{n} \operatorname{tr} \mathbf{R}(x_0 + iv) - \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{R}(x_0 + iv) \right| dv \\ & \quad + C_2 \int_{v_0}^V \int_{x \in I_\varepsilon} \mathbb{E} \left| \left( \frac{1}{n} \operatorname{tr} \mathbf{R}^2(x + iu) - \mathbb{E} \frac{1}{n} \operatorname{tr} \mathbf{R}^2(x + iu) \right) \right| dx du. \end{aligned} \tag{7.1}$$

**Proof.** Note that the Stieltjes transform  $m_n(z)$  of distribution function  $F_p(x)$  is equal to  $(1/n)\operatorname{tr} \mathbf{R}$ , and

$$m'_n(z) = \frac{1}{n} \operatorname{tr} \mathbf{R}^2(z). \tag{7.2}$$

Applying Corollary 2.2 to the distribution functions  $F_p(x)$  and  $F_y(x)$ , we obtain

$$\begin{aligned} \Delta_p^* &:= \sup_x |F_p(x) - F_y(x)| \\ &\leq C_1 \int_{-\infty}^{\infty} |(m_n(u + iV) - S_y(u + iV))| du + C_2 v_0 + C_3 \varepsilon^{3/2} \\ &\quad + C_1 \sup_{x \in I'_\varepsilon} \left| \operatorname{Im} \left\{ \int_{v_0}^V (m_n(x + iv) - S_y(x + iv)) dv \right\} \right|. \end{aligned} \tag{7.3}$$

Furthermore, using the obvious inequality

$$|m_n(z) - s_y(z)| \leq |m_n(z) - \mathbb{E}m_n(z)| + |\mathbb{E}m_n(z) - s_y(z)|,$$

we obtain

$$\begin{aligned} \sup_x |F_p(x) - F_y(x)| &\leq C_1 \int_{-\infty}^{\infty} |(\mathbb{E} m_n(u + iV) - S_y(u + iV))| du + C_2 v + C_3 \varepsilon^{3/2} \\ &\quad + C_4 \sup_{x \in I'_\varepsilon} \left| \operatorname{Im} \left\{ \int_{\Rightarrow 0}^V (\mathbb{E} m_n(x + iv) - S_y(x + iv)) dv \right\} \right| \\ &\quad + C_1 \int_{-\infty}^{\infty} |(m_n(u + iV) - \mathbb{E}m_n(u + iV))| du \\ &\quad + C_1 \sup_{x \in I'_\varepsilon} \left| \operatorname{Im} \left\{ \int_{v_0}^V (m_n(x + iu) - \mathbb{E}m_n(x + iu)) du \right\} \right|. \end{aligned} \tag{7.4}$$

By Taylor's formula,

$$\begin{aligned} \sup_{x \in I'_\varepsilon} |m_n(x + iv) - \mathbb{E}m_n(x + iv)| \\ \leq |m_n(x_0 + iv) - \mathbb{E}m_n(x_0 + iv)| + \int_{x \in I'_\varepsilon} |m'_n(u + iv) - \mathbb{E} m'_n(u + iv)| du. \end{aligned} \tag{7.5}$$

Inequalities (7.2)–(7.5) together imply (7.1), thus proving the lemma. □

Note that, for  $v \geq v_0 = \gamma M_{12}^{1/6} n^{-1/2}$  and for  $\varepsilon \geq Cv_0$ , we have

$$\begin{aligned} C_1 \int_{-\infty}^{\infty} |(\mathbb{E}m_n(u + iV) - s_y(u + iV))| du + C_2 v + C_3 \varepsilon^{3/2} \\ + C_4 \sup_{x \in I'_\varepsilon} \left| \operatorname{Im} \left\{ \int_v^V (\mathbb{E}m_n(x + iu) - s_y(x + iu)) du \right\} \right| \leq CM_{12}^{1/6} n^{-1/2}. \end{aligned} \tag{7.6}$$

Analogously to Section 4, we obtain that

$$\int_{-\infty}^{\infty} \mathbb{E} \left| \frac{1}{n} (\operatorname{tr} \mathbf{R}(u + iV) - \mathbb{E} \operatorname{tr} \mathbf{R}(u + iV)) \right| du \leq Cn^{-1}. \tag{7.7}$$

From Lemma 6.1 it follows that

$$\begin{aligned}
& \int_{v_0}^V \mathbb{E} \left| \frac{1}{n} \operatorname{tr} \mathbf{R}(x_0 + iv) - \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{R}(x_0 + iv) \right| dv \\
& \leq C \int_{v_0}^V \mathbb{E}^{1/2} \left| \frac{1}{n} \operatorname{tr} \mathbf{R}(x_0 + iv) - \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{R}(x_0 + iv) \right|^2 dv \\
& \leq \int_{v_0}^V \left[ \frac{C\sqrt{M_8}}{nv|z_0 + y - 1 + 2yz_0s_y(z_0)|} \right] dv, \tag{7.8}
\end{aligned}$$

where  $z_0 = x_0 + iv$ . Using the fact that  $|z_0 + y - 1 + 2yz_0s_y(z_0)| \geq v$ , we obtain after integration in  $v$ ,

$$\int_{v_0}^V \mathbb{E} \left| \frac{1}{n} \operatorname{tr} \mathbf{R}(x_0 + iv) - \frac{1}{n} \mathbb{E} \operatorname{tr} \mathbf{R}(x_0 + iv) \right| dv \leq \frac{C\sqrt{M_8}}{nv_0} \leq \frac{CM_8^{1/4}}{n^{1/2}} \leq \frac{M_{12}^{1/6}}{n^{1/2}}.$$

Let  $z = x + iv$ . Note that, for  $v \geq v_0$ ,

$$\int_{x \in I_\varepsilon} \frac{1}{|y + z - 1 + 2yzs_y(z)|} du \leq C.$$

By Cauchy's theorem, we have

$$\left| \frac{1}{n} (\operatorname{tr} \mathbf{R}^2 - \mathbb{E} \operatorname{tr} \mathbf{R}^2) \right| \leq Cv^{-1} \sup_{\zeta \in \Gamma_v} \left| \frac{1}{n} \Delta_{\mathbb{E}}(\mathbf{R}) \right|,$$

where  $\Gamma_v = \{z : |\zeta - z| = v_0/2\}$ . Applying Cauchy's inequality and Proposition 6.1 gives

$$\begin{aligned}
\mathbb{E} \left| \frac{1}{n} (\operatorname{tr} \mathbf{R}^2 - \mathbb{E} \operatorname{tr} \mathbf{R}^2) \right| & \leq Cv^{-1} \sup_{\zeta \in \Gamma_v} \mathbb{E}^{1/2} \left| \frac{1}{n} \Delta_{\mathbb{E}}(\mathbf{R}) \right|^2 \\
& \leq Cv^{-1} \left[ \frac{C\sqrt{M_8}}{nv|z + y - 1 + 2yzs_y(z)|} \right].
\end{aligned}$$

After integration, we obtain

$$\int_{v_0}^V \int_{x \in I_\varepsilon} \mathbb{E} \left| \left( \frac{1}{n} \operatorname{tr} \mathbf{R}^2(x + iu) - \mathbb{E} \frac{1}{n} \operatorname{tr} \mathbf{R}^2(x + iu) \right) \right| dx du \leq \frac{CM_8^{1/4}}{n^{1/2}} \leq \frac{CM_{12}^{1/6}}{n^{1/2}}.$$

This proves Theorem 1.2. □

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