IJMMS 2004:9, 459–468
PII. S0161171204205038
http://ijmms.hindawi.com

RATE OF CONVERGENCE OF BOUNDED VARIATION FUNCTIONS BY A BÉZIER-DURRMEYER VARIANT OF THE BASKAKOV OPERATORS

VIJAY GUPTA and ULRICH ABEL

Received 6 May 2002

We consider a Bézier-Durrmeyer integral variant of the Baskakov operators and study the rate of convergence for functions of bounded variation.

2000 Mathematics Subject Classification: 41A36, 41A25, 26A45.

1. Introduction. Let $W(0,\infty)$ be the class of functions f which are locally integrable on $(0,\infty)$ and are of polynomial growth as $t\to\infty$, that is, for some positive r, there holds $f(t)=O(t^r)$ as $t\to\infty$. The Durrmeyer variant $\tilde{V}_n$ of the Baskakov operators associates to each function $f\in W(0,\infty)$ the series

$$\tilde{V}_n(f;x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)dt,\quad x\in[0,\infty), \tag{1.1}$$

where

$$p_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k} \tag{1.2}$$

is the Baskakov basis function. Note that (1.1) is well defined, for $n\geq r+2$, provided that $f(t)=O(t^r)$ as $t\to\infty$. The operators (1.1) were first introduced by Sahai and Prasad [9]. They termed these operators as modified Lupaş operators. In 1991, Sinha et al. [10] improved and corrected the results of [9] and denoted $\tilde{V}_n$ as modified Baskakov operators. The rate of convergence of the operators (1.1) on functions of bounded variation was studied in [8, 11].

We mention that Agrawal and Thamer [2] considered the variant

$$M_n(f;x)=(n-1)\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-1}(t)f(t)dt+(1+x)^{-n}f(0) \tag{1.3}$$

of the operators (1.1) and studied its properties in subsequent papers [3, 4, 5]. See also [1]. The rate of convergence of the operators discussed by Agrawal and Thamer was studied by the first author in [7].

For each function $f \in W(0,\infty)$ and $\alpha \geq 1$, we consider the Bézier–type Baskakov-Durrmeyer operators $\widetilde{V}_{n,\alpha}$ as

$$\widetilde{V}_{n,\alpha}(f;x) = (n-1)\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, \tag{1.4}$$

where

$$\begin{aligned} Q_{n,k}^{(\alpha)}(x) &= J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x), \\ J_{n,k}(x) &= \sum_{j=k}^{\infty} p_{n,j}(x). \end{aligned} \tag{1.5}$$

It is obvious that $\widetilde{V}_{n,\alpha}$ are positive linear operators and $\widetilde{V}_{n,\alpha}(1;x) = 1$. In the special case $\alpha = 1$, the operators $\widetilde{V}_{n,\alpha}$ reduce to the operators $\widetilde{V}_n \equiv \widetilde{V}_{n,1}$. Some basic properties of $J_{n,k}$ are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x)$ $(k = 0,1,2,\ldots)$;
(ii) $J'_{n,k}(x) = np_{n+1,k-1}(x)$ $(k = 1,2,3,\ldots)$;
(iii) $J_{n,k}(x) = n\int_0^x p_{n+1,k-1}(t)dt$ $(k = 1,2,3,\ldots)$;
(iv) $0 < \cdots < J_{n,k+1}(x) < J_{n,k}(x) < \cdots < J_{n,1}(x) < J_{n,0}(x) \equiv 1$ $(x > 0)$;
(v) $J_{n,k}$ is strictly increasing on $[0,\infty)$.

In this paper, we study the rate of convergence for the new sequence of operators (1.4), for functions f of bounded variation. Our result essentially generalizes and improves the results of [8, 11]. Furthermore, we find the limit of the sequence $\widetilde{V}_{n,\alpha}(f;x)$ for bounded locally integrable functions f having a discontinuity of the first kind at $x \in (0,\infty)$.

2. The main results.

As a main result, we derive the following estimate on the rate of convergence.

Theorem 2.1. *Assume that $f \in W(0,\infty)$ is a function of bounded variation on every finite subinterval of $(0,\infty)$. Furthermore, let $\alpha \geq 1$, $\lambda > 2$, and $x \in (0,\infty)$ be given. Then, for each $r \in \mathbb{N}$, there exists a constant $M(f,\alpha,r,x)$ such that for sufficiently large n, the Bézier–type Baskakov-Durrmeyer operators $\widetilde{V}_{n,\alpha}$ satisfy the estimate*

$$\begin{aligned} &\left|\widetilde{V}_{n,\alpha}(f;x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right| \\ &\quad \leq \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}\left|f(x+)-f(x-)\right| \\ &\qquad + \frac{2\alpha\lambda(1+x)+x}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{M(f,\alpha,r,x)}{n^r}, \end{aligned} \tag{2.1}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x-) & (0 \leq t < x), \\ 0 & (t = x), \\ f(t) - f(x+) & (x < t < \infty), \end{cases} \tag{2.2}$$

and $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

Remark 2.2. The exponent r in the O-term of (2.1) can be chosen arbitrary large.

As an immediate consequence of Theorem 2.1, we obtain in the special case $\alpha = 1$ the following estimate which improves the results of [8, 11].

Corollary 2.3. *Under the assumptions of Theorem 2.1, there holds, for sufficiently large n,*

$$\begin{aligned}
&\left|\tilde{V}_n(f;x) - \frac{1}{2}[f(x+)+f(x-)]\right| \\
&\quad \le \frac{(10+11x)}{2\sqrt{nx(1+x)}}\,|f(x+)-f(x-)| \\
&\qquad + \frac{2\lambda(1+x)+x}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{M(f,1,r,x)}{n^r},
\end{aligned} \tag{2.3}$$

where g_x is defined as in Theorem 2.1.

Theorem 2.4. *Let $x \in (0,\infty)$. If $f \in L(0,\infty)$ has a discontinuity of the first kind at x, then*

$$\lim_{n\to\infty}\tilde{V}_{n,\alpha}(f;x) = \frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-). \tag{2.4}$$

3. Auxiliary results.

In order to prove our main result, we will need the following lemmas. Throughout the paper, for each real x, let $\psi_x(t) = t - x$.

Lemma 3.1 (see [6]). *Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables with finite variance such that the expectation $E(\xi_i) = a_1 \in \mathbb{R} \equiv (-\infty,\infty)$, and the variance $V(\xi_i) = b_1^2 > 0$. Assume that $E|\xi_i - a_1|^3 < \infty$. Then there exists a constant c with $1/\sqrt{2\pi} < c < 0.82$ such that, for all $n = 1,2,3,\ldots$ and all $t \in \mathbb{R}$,*

$$\left|P\left(\frac{1}{b_1\sqrt{n}}\sum_{k=1}^{n}(\xi_i - a_1) \le t\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-u^2/2}du\right| \le c\,\frac{E|\xi_i - a_1|^3}{\sqrt{n}b_1^3}. \tag{3.1}$$

Lemma 3.2 (see [10]). *For each fixed $x \in [0,\infty)$ and $m \in \mathbb{N}_0$, the central moments $\tilde{V}_n(\psi_x^m;x)$ of the Baskakov-Durrmeyer operators (1.1) satisfy*

$$\tilde{V}_n(\psi_x^m;x) = O\left(n^{-\lfloor (m+1)/2\rfloor}\right) \quad (n \longrightarrow \infty). \tag{3.2}$$

In particular,

$$\tilde{V}_n(1;x) = 1, \qquad \tilde{V}_n(\psi_x^2;x) = \frac{2(n-1)x(1+x)}{(n-2)(n-3)} + \frac{2(1+2x)^2}{(n-2)(n-3)}. \tag{3.3}$$

Remark 3.3. Note that, given any $\lambda > 2$ and any $x > 0$, for all n sufficiently large, we have the estimate

$$\tilde{V}_n(\psi_x^2;x) < \frac{\lambda x(1+x)}{n}. \tag{3.4}$$

Lemma 3.4 (see [13]). *For all $x > 0$ and $n, k \in \mathbb{N}$, there holds*

$$J^{\alpha}_{n,k}(x)p_{n,k}(x) \le Q^{(\alpha)}_{n,k}(x) \le \alpha p_{n,k}(x) < \frac{\alpha\sqrt{1+x}}{\sqrt{2enx}}. \tag{3.5}$$

Throughout, let

$$K_{n,\alpha}(x,t) = (n-1)\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x)p_{n,k}(t), \tag{3.6}$$

$$\lambda_{n,\alpha}(x,y) = \int_0^y K_{n,\alpha}(x,t)\,dt. \tag{3.7}$$

With this definition, for each function $f \in W(0,\infty)$, there holds, for all sufficiently large n,

$$\tilde{V}_{n,\alpha}(f;x) = \int_0^\infty K_{n,\alpha}(x,t)f(t)\,dt. \tag{3.8}$$

Note that, in particular,

$$\lambda_{n,\alpha}(x,\infty) = \int_0^\infty K_{n,\alpha}(x,u)\,du = 1. \tag{3.9}$$

Lemma 3.5. *For each $\lambda > 2$ and, for all sufficiently large n, there exist, for all $x \in (0,\infty)$,*

$$\lambda_{n,\alpha}(x,y) = \int_0^y K_{n,\alpha}(x,t)\,dt \le \frac{\lambda\alpha x(1+x)}{n(x-y)^2} \quad (0 \le y < x), \tag{3.10}$$

$$1-\lambda_{n,\alpha}(x,z) = \int_z^\infty K_{n,\alpha}(x,t)\,dt \le \frac{\lambda\alpha x(1+x)}{n(z-x)^2} \quad (x < z < \infty). \tag{3.11}$$

Proof. First we prove (3.10). There holds

$$\begin{aligned}\int_0^y K_{n,\alpha}(x,t)\,dt &\le \int_0^y K_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}\,dt \\ &\le (x-y)^{-2}\tilde{V}_{n,\alpha}(\psi_x^2;x) \\ &\le \alpha(x-y)^{-2}\tilde{V}_{n,1}(\psi_x^2;x),\end{aligned} \tag{3.12}$$

where we applied Lemma 3.4. Now (3.10) is a consequence of Remark 3.3. The proof of (3.11) is similar. □

Lemma 3.6 (see [13]). *Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of independent random variables with the same geometric distribution*

$$P(\xi_1 = k) = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} \quad (k \in \mathbb{N}), \tag{3.13}$$

where $x > 0$ is a parameter. Then,

$$E(\xi_1) = x, \qquad E(\xi_1 - E\xi_1)^2 = x(1+x), \qquad E\left|\xi_1 - E\xi_1\right|^3 \le 3x(1+x)^2. \tag{3.14}$$

Lemma 3.7. *For all $x \in (0,\infty)$ and $k = 0,1,2,\dots,$ there hold*

$$\left|J_{n,k}^{\alpha}(x) - J_{n-1,k+1}^{\alpha}(x)\right| \le \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}, \tag{3.15}$$

$$\left|J_{n,k}^{\alpha}(x) - J_{n-1,k}^{\alpha}(x)\right| \le \frac{\alpha(10+11x)}{2\sqrt{nx(1+x)}}. \tag{3.16}$$

Proof. First we prove (3.15). Proceeding along the lines of [8, Lemma 2.8] and [12], it is easily checked that

$$\begin{aligned}\left|J_{n,k}(x) - J_{n-1,k+1}(x)\right| &\le \frac{2(0.82)E\left|\xi_1 - E\xi_1\right|^3}{\sqrt{n}(x(1+x))^{3/2}} + \frac{x}{\sqrt{2\pi nx(1+x)}}\\ &\le \frac{2(0.82)\cdot 3x(1+x)^2}{\sqrt{n}(x(1+x))^{3/2}} + \frac{x}{2\sqrt{nx(1+x)}}\\ &\le \frac{10+11x}{2\sqrt{nx(1+x)}},\end{aligned} \tag{3.17}$$

where we used Lemmas 3.1 and 3.6. Application of the inequality $|a^{\alpha} - b^{\alpha}| \le \alpha|a-b|$, for $0 \le a,\ b \le 1$, and $\alpha \ge 1$ yields (3.15). The proof of (3.16) is similar. □

4. Proofs of the main results

Proof of Theorem 2.1. Our starting point is the identity

$$\begin{aligned} f(t) = {}& \frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-) + \left(\operatorname{sign}(t-x) + \frac{\alpha-1}{\alpha+1}\right)\frac{f(x+)-f(x-)}{2}\\ &+ g_x(t) + \delta_x(t)\left(f(x) - \frac{f(x+)+f(x-)}{2}\right),\end{aligned} \tag{4.1}$$

where $\delta_x(t) = 1\ (t = x)$ and $\delta_x(t) = 0\ (t \ne x)$ (see [12, Equation (28)]). Since $\tilde{V}_{n,\alpha}(\delta_x;x) = 0$, we conclude that

$$\begin{aligned}&\left|\tilde{V}_{n,\alpha}(f;x) - \left[\frac{1}{\alpha+1}f(x+) + \frac{\alpha}{\alpha+1}f(x-)\right]\right|\\ &\quad \le \frac{1}{2}\left|\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x) + \frac{\alpha-1}{\alpha+1}\right|\left|f(x+)-f(x-)\right| + \left|\tilde{V}_{n,\alpha}(g_x;x)\right|.\end{aligned} \tag{4.2}$$

First, we obtain

$$\begin{aligned}\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x) &= (n-1)\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x)\left(\int_x^{\infty} p_{n,j}(t)\,dt - \int_0^x p_{n,j}(t)\,dt\right)\\ &= (n-1)\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x)\left(\int_0^{\infty} p_{n,j}(t)\,dt - 2\int_0^x p_{n,j}(t)\,dt\right)\\ &= 1 - 2(n-1)\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x)\int_0^x p_{n,j}(t)\,dt.\end{aligned} \tag{4.3}$$

Using

$$\sum_{j=0}^{k} p_{n-1,j}(x) = (n-1)\int_{x}^{\infty} p_{n,k}(t)\,dt, \tag{4.4}$$

we conclude that

$$\begin{aligned}\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x) &= 1-2\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x)\left(1-\sum_{k=0}^{j} p_{n-1,k}(x)\right)\\ &= -1+2\sum_{k=0}^{\infty} p_{n-1,k}(x)\sum_{j=k}^{\infty} Q_{n,j}^{(\alpha)}(x)\\ &= -1+2\sum_{k=0}^{\infty} p_{n-1,k}(x)J_{n,k}^{\alpha}(x)\end{aligned} \tag{4.5}$$

since $\sum_{j=0}^{\infty} Q_{n,j}^{(\alpha)}(x) = 1$. Therefore, we obtain

$$\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x)+\frac{\alpha-1}{\alpha+1} = 2\sum_{k=0}^{\infty} p_{n-1,k}(x)J_{n,k}^{\alpha}(x)-\frac{2}{\alpha+1}\sum_{k=0}^{\infty} Q_{n-1,k}^{(\alpha+1)}(x) \tag{4.6}$$

since $\sum_{k=0}^{\infty} Q_{n-1,k}^{(\alpha+1)}(x) = 1$. By the mean value theorem, it follows that

$$Q_{n-1,k}^{(\alpha+1)}(x) = J_{n-1,k}^{\alpha+1}(x)-J_{n-1,k+1}^{\alpha+1}(x) = (\alpha+1)p_{n-1,k}(x)\gamma_{n,k}^{\alpha}(x), \tag{4.7}$$

where $J_{n-1,k+1}(x) < \gamma_{n,k}(x) < J_{n-1,k}(x)$. Hence,

$$\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x)+\frac{\alpha-1}{\alpha+1} = 2\sum_{k=0}^{\infty} p_{n-1,k}(x)\left(J_{n,k}^{\alpha}(x)-\gamma_{n,k}^{\alpha}(x)\right), \tag{4.8}$$

where

$$J_{n,k}^{\alpha}(x)-J_{n-1,k}^{\alpha}(x) < J_{n,k}^{\alpha}(x)-\gamma_{n,k}^{\alpha}(x) < J_{n,k}^{\alpha}(x)-J_{n-1,k+1}^{\alpha}(x). \tag{4.9}$$

Lemma 3.7 implies that

$$\left|\tilde{V}_{n,\alpha}(\operatorname{sign}(t-x);x)+\frac{\alpha-1}{\alpha+1}\right| \le \frac{\alpha(10+11x)}{\sqrt{nx(1+x)}} \quad \text{for } x\in(0,\infty). \tag{4.10}$$

In order to complete the proof of the theorem, we need an estimate of $\tilde{V}_{n,\alpha}(g_x;x)$. We use the integral representation (3.8) and decompose $[0,\infty)$ into three parts as follows:

$$\begin{aligned}\tilde{V}_{n,\alpha}(g_x;x) &= \left(\int_{0}^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}+\int_{x+x/\sqrt{n}}^{\infty}\right)K_{n,\alpha}(x,t)g_x(t)\,dt\\ &= I_1+I_2+I_3, \quad \text{say.}\end{aligned} \tag{4.11}$$

We start with I_2. For $t\in[x-x/\sqrt{n},x+x/\sqrt{n}]$, we have

$$|g_x(t)| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x), \tag{4.12}$$

and therefore

$$|I_2| \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{4.13}$$

Next we estimate I_1. Let $y = x - x/\sqrt{n}$. Using integration by parts with (3.7), we have

$$I_1 = \int_0^y g_x(t)\,d_t\lambda_{n,\alpha}(x,t) = g_x(y)\lambda_{n,\alpha}(x,y) - \int_0^y \lambda_{n,\alpha}(x,t)\,d_t g_x(t). \tag{4.14}$$

Since $|g_x(y)| = |g_x(y) - g_x(x)| \le \bigvee_y^x(g_x)$, we conclude that

$$|I_1| \le \bigvee_y^x(g_x)\lambda_{n,\alpha}(x,y) + \int_0^y \lambda_{n,\alpha}(x,t)\,d_t\left(-\bigvee_t^x(g_x)\right). \tag{4.15}$$

Since $y = x - x/\sqrt{n} \le x$, (3.10) implies that

$$|I_1| \le \frac{\lambda\alpha x(1+x)}{n(x-y)^2}\bigvee_y^x(g_x) + \frac{\lambda\alpha x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}\,d_t\left(-\bigvee_t^x(g_x)\right). \tag{4.16}$$

Integrating the last term by parts, we get

$$|I_1| \le \frac{\lambda\alpha x(1+x)}{n}\left(x^{-2}\bigvee_0^x(g_x) + 2\int_0^y \frac{\bigvee_t^x(g_x)}{(x-t)^3}\,dt\right). \tag{4.17}$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we obtain

$$\begin{aligned}\int_0^{x-x/\sqrt{n}}\bigvee_t^x(g_x)(x-t)^{-3}\,dt &= \sum_{k=1}^{n-1}\int_{x/\sqrt{k+1}}^{x/\sqrt{k}}\bigvee_{x-t}^{x}(g_x)t^{-3}\,dt \\ &\le \frac{1}{2x^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x).\end{aligned} \tag{4.18}$$

Hence,

$$|I_1| \le \frac{2\lambda\alpha(1+x)}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x). \tag{4.19}$$

Finally, we estimate I_3. We let

$$\tilde{g}_x(t) = \begin{cases} g_x(t) & (0 \le t \le 2x), \\ g_x(2x) & (2x < t < \infty) \end{cases} \tag{4.20}$$

and divide $I_3 = I_{31} + I_{32}$, where

$$\begin{aligned} I_{31} &= \int_{x+x/\sqrt{n}}^{\infty} K_{n,\alpha}(x,t)\tilde{g}_x(t)\,dt, \\ I_{32} &= \int_{2x}^{\infty} K_{n,\alpha}(x,t)\left[g_x(t) - g_x(2x)\right]dt. \end{aligned} \tag{4.21}$$

With $y = x + x/\sqrt{n}$, the first integral can be written in the form

$$\begin{aligned} I_{31} = \lim_{R\to+\infty} \Big\{ & g_x(y)\left[1 - \lambda_{n,\alpha}(x,y)\right] + \tilde{g}_x(R)\left[\lambda_{n,\alpha}(x,R) - 1\right] \\ & + \int_y^R \left[1 - \lambda_{n,\alpha}(x,t)\right] d_t \tilde{g}_x(t) \Big\}. \end{aligned} \tag{4.22}$$

By (3.11), we conclude that

$$\begin{aligned} |I_{31}| &\le \frac{\lambda\alpha x(1+x)}{n} \lim_{R\to+\infty} \left\{ \frac{\bigvee_x^y(g_x)}{(y-x)^2} + \frac{|\tilde{g}_x(R)|}{(R-x)^2} + \int_y^R \frac{1}{(t-x)^2} d_t \left(\bigvee_x^t (\tilde{g}_x) \right) \right\} \\ &= \frac{\lambda\alpha x(1+x)}{n} \left\{ \frac{\bigvee_x^y(g_x)}{(y-x)^2} + \int_y^{2x} \frac{1}{(t-x)^2} d_t \left(\bigvee_x^t (g_x) \right) \right\}. \end{aligned} \tag{4.23}$$

In a similar way as above we obtain

$$\int_y^{2x} \frac{1}{(t-x)^2} d_t \left(\bigvee_x^t (g_x) \right) \le x^{-2} \bigvee_x^{2x} (g_x) - \frac{\bigvee_x^y(g_x)}{(y-x)^2} + x^{-2} \sum_{k=1}^{n-1} \bigvee_x^{x+x/\sqrt{k}} (g_x) \tag{4.24}$$

which implies the estimate

$$|I_{31}| \le \frac{2\lambda\alpha(1+x)}{nx} \sum_{k=1}^{n} \bigvee_x^{x+x/\sqrt{k}} (g_x). \tag{4.25}$$

We proceed with I_{32}. By assumption, there exists an integer r such that $f(t) = O(t^{2r})$ as $t \to \infty$. Thus, for a certain constant $M > 0$, depending only on f, x, and r, we have

$$\begin{aligned} |I_{32}| &\le M(n-1) \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} p_{n,k}(t) t^{2r}\,dt \\ &\le \alpha M(n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} p_{n,k}(t) t^{2r}\,dt, \end{aligned} \tag{4.26}$$

where we used Lemma 3.4. Obviously, $t \ge 2x$ implies $t \le 2(t-x)$ and it follows that

$$|I_{32}| \le 2^{2r}\alpha M(n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t)(t-x)^{2r}\,dt = 2^{2r}\alpha M \tilde{V}_n(\psi_x^{2r}; x). \tag{4.27}$$

By Lemma 3.2, the central moments of the Baskakov-Durrmeyer operators (1.1) satisfy $\tilde{V}_n(\psi_x^{2r};x) = O(n^{-r})(n \to \infty)$, and we obtain

$$I_{32} = O(n^{-r}) \quad (n \longrightarrow \infty). \tag{4.28}$$

Collecting the estimates (4.13), (4.19), (4.25), and (4.28) yields with regard to (4.11)

$$|\tilde{V}_{n,\alpha}(g_x;x)| \leq \frac{2\lambda\alpha(1+x)+x}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(n^{-r}) \quad (n \longrightarrow \infty). \tag{4.29}$$

Finally, combining (4.2), (4.10), and (4.29), we obtain (2.1). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4. Since the function ψ_x^2 given by $\psi_x^2(t) = (t-x)^2$ is of bounded variation on every finite subinterval of $[0,\infty)$, we deduce from Theorem 2.1 that, for all $x \in (0,\infty)$,

$$\lim_{n\to\infty} \tilde{V}_{n,\alpha}(\psi_x^2;x) = 0. \tag{4.30}$$

If $f \in L_\infty(0,\infty)$, then g_x defined as in (2.2) is also bounded and is continuous at the point x. By the Korovkin theorem, we conclude that

$$\lim_{n\to\infty} \tilde{V}_{n,\alpha}(g_x;x) = g_x(x) = 0. \tag{4.31}$$

Therefore, the right-hand side of inequality (4.2) tends to zero as $n \to \infty$. This completes the proof of Theorem 2.4. □

Acknowledgment. The authors are thankful to the referee for the valuable comments that helped to improve this paper. Moreover, the referee provided further important references.

References

[1] P. N. Agrawal and A. J. Mohammad, *On convergence of derivatives of a new sequence of linear positive operators*, to appear in Rev. Un. Mat. Argentina.

[2] P. N. Agrawal and K. J. Thamer, *Approximation of unbounded functions by a new sequence of linear positive operators*, J. Math. Anal. Appl. **225** (1998), no. 2, 660–672.

[3] ______, *Inverse theorem for a new sequence of linear positive operators on L_p-spaces*, Rev. Un. Mat. Argentina **42** (2000), no. 1, 1–8.

[4] ______, *A new sequence of linear positive operators for higher order L_p-approximation*, Rev. Un. Mat. Argentina **41** (2000), no. 4, 9–18.

[5] ______, *Degree of approximation by a new sequence of linear operators*, Kyungpook Math. J. **41** (2001), no. 1, 65–73.

[6] W. Feller, *An Introduction to Probability Theory and Its Applications. Vol. II*, John Wiley & Sons, New York, 1966.

[7] V. Gupta, *Rate of approximation by a new sequence of linear positive operators*, Comput. Math. Appl. **45** (2003), no. 12, 1895–1904.

[8] V. Gupta and G. S. Srivastava, *An estimate of the rate of convergence of modified Baskakov operators for functions of bounded variation*, Kyungpook Math. J. **36** (1996), no. 2, 237–247.

[9] A. Sahai and G. Prasad, *On simultaneous approximation by modified Lupas operators*, J. Approx. Theory **45** (1985), no. 2, 122–128.

[10] R. P. Sinha, P. N. Agrawal, and V. Gupta, *On simultaneous approximation by modified Baskakov operators*, Bull. Soc. Math. Belg. Sér. B **43** (1991), no. 2, 217–231.

[11] Y. Wang and S. Guo, *Rate of approximation of functions of bounded variation by modified Lupas operators*, Bull. Austral. Math. Soc. **44** (1991), no. 2, 177–188.

[12] X.-M. Zeng and W. Chen, *On the rate of convergence of the generalized Durrmeyer type operators for functions of bounded variation*, J. Approx. Theory **102** (2000), no. 1, 1–12.

[13] X.-M. Zeng and V. Gupta, *Rate of convergence of Baskakov-Bézier type operators for locally bounded functions*, Comput. Math. Appl. **44** (2002), no. 10-11, 1445–1453.

Vijay Gupta: School of Applied Sciences, Netaji Subhas Institute of Technology, Azad Hind Fauj Marg, Sector-3, Dwarka, New Delhi-110045, India
E-mail address: vijay@nsit.ac.in

Ulrich Abel: Fachbereich MND, Fachhochschule Giessen-Friedberg, University of Applied Sciences, Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany
E-mail address: Ulrich.Abel@mnd.fh-friedberg.de

Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	February 1, 2009
First Round of Reviews	May 1, 2009
Publication Date	August 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Department of Physics, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk