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RATE OF GROWTH OF NONOSCILLATORY SOLUTIONS FOR THE DIFFERENTIAL EQUATION $\ddot{y} + q(t)|y|^\gamma \operatorname{sgn} y = 0$, $0 < \gamma < 1$*

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1. Introduction. The object of this paper is to study the rate of growth of real-valued nonoscillatory solutions of

$$\ddot{y} + q(t)\ |y|^\gamma \operatorname{sgn} y = 0 \tag{1.1}$$

where $q(t)$ is nonnegative and continuous on $[0, \infty)$ and γ is any real number satisfying $0 < \gamma < 1$. (These assumptions on $q(t)$ and γ will be implicitly assumed throughout.) There are necessary and sufficient conditions for the existence of nonoscillatory solutions $y(t)$ of (1.1) satisfying either $\lim_{t\to\infty} y(t) = \alpha \neq 0$ or $\lim_{t\to\infty} (y(t)/(t) = \alpha \neq 0$ (cf. Sec. 2). Our main purpose, then, is to discuss other possible types of nonoscillatory solutions, for example, solutions which grow like a *fractional* power of t. Very little is known about such solutions. We will give criteria for the existence of such solutions and also a criterion for a certain type of "dichotomy" which may occur.

All solutions of (1.1) exist on $[0, \infty)$ as follows from a theorem of Wintner (Hartman [5]). By definition, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros; i.e., if $T \geq 0$, then there is a $t > T$ such that $y(t) = 0$. If there is a $T \geq 0$ such that $y(t) \neq 0$ for $t \geq T$, then $y(t)$ is called nonoscillatory.

Note that if γ is the quotient of odd integers, Eq. (1.1) takes the form

$$\ddot{y} + q(t)y^\gamma = 0. \tag{1.2}$$

Even without this restriction Eq. (1.1) has the property that if $y(t)$ is a solution $-y(t)$ is also a solution. This clearly simplifies the discussion of nonoscillatory solutions of (1.1) because they can always be assumed to be nonnegative and then (1.1) takes the form (1.2).

2. Preliminaries. In this section we state, for future reference, results which are known or easily proved.

THEOREM 2.1. *All solutions of* (1.1) *are oscillatory if and only if*

$$\int^\infty s^\gamma q(s)\, ds = \infty.$$

This was first proved by Belohorec [3]. A very short proof of the sufficiency of $\int^\infty s^\gamma q(s)\, ds = \infty$ is given in [7]. Necessity will follow from the next theorem which is analogous to a result of Moore and Nehari [10] for $\gamma > 1$.

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THEOREM 2.2. *If* $\int^\infty s^\gamma q(s)\,ds < \infty$, *then Eq.* (1.1) *has a solution* $y(t)$ *with a positive zero which satisfies* $\lim_{t\to\infty} (y(t)/t) = \alpha$ *where* $0 < \alpha < \infty$.

Proof. Let $y(t)$ be a solution of (1.1) such that $y(t_0) = 0$ and $\dot y(t_0) > 0$. Then $y(t) \le \dot y(t_0)(t - t_0)$, $t \ge t_0$ as long as $y(t)$ is positive (since $y(t)$ is concave). Therefore in some interval to the right of t_0, we have, by integrating (1.1) (which simplifies to (1.2)),

$$\dot y(t_0) \le \dot y(t) + \dot y(t_0)^\gamma \int_{t_0}^{t} (s - t_0)^\gamma q(s)\,ds.$$

Hence

$$\dot y(t_0) \le \dot y(t) + \dot y(t_0)\left(\int_{t_0}^{\infty} s^\gamma q(s)\,ds \Big/ (\dot y(t_0))^{1-\gamma}\right).$$

If we now take $\dot y(t_0)$ so large that $2\int_{t_0}^\infty s^\gamma q(s)\,ds \le (\dot y(t_0))^{1-\gamma}$, we will have $\dot y(t_0) \le \dot y(t) + \frac{1}{2}\dot y(t_0)$ or $\frac{1}{2}\dot y(t_0) \le \dot y(t)$. This is true as long as $y(t) \ge 0$. Therefore it is true for $t \ge t_0$. Since $\dot y(t)$ is then nonincreasing, we see that $\lim_{t\to\infty} \dot y(t) \ge \frac{1}{2}\dot y(t_0) > 0$. This proves the theorem.

THEOREM 2.3. (1.1) *has a solution* $y(t)$ *satisfying* $\lim_{t\to\infty} y(t)/t = \alpha > 0$ *if and only if* $\int^\infty s^\gamma q(s)\,ds < \infty$.

THEOREM 2.4. (1.1) *has a solution* $y(t)$ *satisfying* $\lim_{t\to\infty} y(t) = \alpha > 0$ *if and only if* $\int^\infty sq(s)\,ds < \infty$.

Proof. Sufficiency of the condition $\int^\infty sq(s)\,ds < \infty$ is proved, for example, in [6]. Necessity is easily established as follows. Suppose that $y(t)$ is an eventually positive nonoscillatory solution of (1.1). Suppose that $y(t) > 0$, $\dot y(t) > 0$ for $t \ge t_0$. Note that $\dot y$ must be eventually positive since $\ddot y$ is eventually nonpositive. Multiplying (1.1) by t/y^γ and integrating by parts gives

$$\frac{t\dot y(t)}{(y(t))^\gamma} - \frac{(y(t))^{1-\gamma}}{1-\gamma} + \int_{t_0}^{t} \frac{sy'^2}{y^{\gamma+1}}\,ds + k + \int_{t_0}^{t} sq(s)\,ds = 0.$$

If $\int_{t_0}^t sq(s)\,ds \to \infty$ as $t \to \infty$, we must also have $y(t) \to \infty$ as $t \to \infty$. This proves the theorem.

Remark. Theorem 2.3 and Theorem 2.4 are actually true as stated for all $\gamma > 0$, including the linear case, although different proofs are required for $\gamma \ge 1$.

Remark. It was shown above that if $\int^\infty s^\gamma q(s)\,ds < \infty$, there are nonoscillatory solutions which grow like t and which moreover have a positive zero. Theorem 2.4 then raises the question, if $\int^\infty sq(s)\,ds < \infty$, do there exist bounded nonoscillatory solutions with a positive zero? For the Emden–Fowler equation, i.e., $q(t) = t^\sigma$, the answer is yes as was shown by Sansone [12]. For the general Eq. (1.1) this is an open question. For the case $\gamma > 1$, the question of nonoscillatory solutions with zeros is discussed in [10] and [11].

3. Rate of growth of nonoscillatory solutions. The preceding section was concerned with nonoscillatory solutions which either are bounded or grow like t. The present section is motivated by the observation (Bellman [2]) that the Emden–Fowler equation

$$\ddot y + t^\sigma y^\gamma = 0 \tag{3.1}$$

has the solution $y(t) = ct^w$ where

$$w = \frac{\sigma + 2}{1 - \gamma}, \qquad c = \left\{ -\frac{(\sigma + 2)(\sigma + \gamma + 1)}{(\gamma - 1)^2} \right\}^{1/(\gamma - 1)}.$$

Here we assume that $\sigma + \gamma + 1 < 0$ since otherwise all solutions of (3.1) are oscillatory. If $\sigma < -2$, all nontrivial solutions are either bounded or grow like t (as will be shown below). If $\sigma = -2$, then $c = 0$. Therefore we assume $\sigma > -2$. Thus $c > 0$ and $0 < w < 1$ and there exists a nonoscillatory solution which grows like a fractional power of t.

The question then which we want to consider is under what conditions Eq. (1.1) has nonoscillatory solutions which are neither bounded nor grow like t. The first step in this direction is a lower bound for the nonoscillatory solutions.

THEOREM 3.1. *If* (Belohorec [4])

$$\liminf_{t \to \infty} t^\beta \int_t^\infty q(s)\, ds > 0$$

for some $0 < \beta \leq 1$, *and if* $y(t)$ *is an eventually positive nonoscillatory solution of* (1.1), *then there is a* $c > 0$ *such that*

$$y(t) \geq ct^{(1-\beta)/(1-\gamma)} \tag{3.2}$$

for sufficiently large t.

The above discussion of the Emden–Fowler equation (3.1) shows that (3.2) is sharp. We know from Sec. 2 that if there are any nonoscillatory solutions at all, then there are nonoscillatory solutions which grow like t. Thus it is certainly impossible to find an upper bound for all nonoscillatory solutions which is analogous to the lower bound given by Theorem 3.1. However, it is possible to establish the existence of a nonoscillatory solution which grows no faster than a specified fractional power of t. To do this we use the theory of subfunctions. The following lemma is a special case of a theorem of Schrader [13] (see also Jackson [9, Theorem 7.4]).

LEMMA 3.1. *Suppose that there exist functions* $\phi(t)$, $\psi(t) \in C^2[a, \infty)$, $a \geq 0$, *such that* $\ddot{\phi} + q(t)\phi^\gamma \geq 0$ (*lower solution*) *and* $\ddot{\psi} + q(t)\psi^\gamma \leq 0$ (*upper solution*) *and also* $0 \leq \phi(t) \leq \psi(t)$ *for* $t \geq a$. *Then there is a solution* $y(t)$ *of* (1.1) *such that* $\phi(t) \leq y(t) \leq \psi(t)$ *for* $t \geq a$.

THEOREM 3.2. *If* $q(t)$ *satisfies* $0 \leq k_1 t^{\beta_1} \leq q(t) \leq k_2 t^{\beta_2}$ *for* $t \geq t_0$, *where* $k_2 > 0$, $-2 \leq \beta_1 \leq \beta_2 \leq -1 - \gamma$, *then there exists a positive nonoscillatory solution* $y(t)$ *of* (1.1) *which satisfies*

$$c_1 t^{\alpha_1} \leq y(t) \leq c_2 t^{\alpha_2}, \qquad t \geq t_0,$$

if $k_1 > 0$, *where*

$$\alpha_i = \frac{\beta_i + 2}{1 - \gamma}, \qquad c_i = \left(\frac{k_i}{\alpha_i (1 - \alpha_i)} \right)^{1/(1-\gamma)}, \qquad i = 1.\ 2.$$

If $k_1 = 0$, *then there exists a solution* $y(t)$ *of* (1.1) *satisfying* $c_2 t_0^{\alpha_2} \leq y(t) \leq c_2 t^{\alpha_2}$.

Proof. Let $\phi(t) = c_1 t^{\alpha_1}$ if $k_1 > 0$ and $\phi(t) = c_2 t_0^{\alpha_2}$ if $k = 0$, and let $\psi(t) = c_2 t^{\beta_2}$. Since $0 < \alpha_1 \leq \alpha_2$, $\phi(t) \leq \psi(t)$ for $t \geq t_0$. Clearly $\psi(t)$ is an upper solution and $\phi(t)$ a lower solution for (1.1). Therefore by Lemma 3.1 there exists a solution $y(t)$ of (1.1) such that $\phi(t) \leq y(t) \leq \psi(t)$ for $t \geq t_0$.

Remark. Better results can be obtained by assuming stronger conditions on $q(t)$, for example, if $q(t) = bt^\alpha(1 + a(t))$ where $|a(t)| + |t\dot{a}(t)| < At^{-\epsilon}$, $c > 0$. See [1].

COROLLARY 3.1. *If $q(t) \leq kt^{-(\gamma+3)/2}$, $k > 0$, $t \geq t_0$, then there is a nonoscillatory solution $y(t)$ of* (1.1) *such that $0 < y(t) \leq ct^{1/2}$ where c is given in the theorem.*

Nonoscillatory solutions $y(t)$ which satisfy $0 < y(t) \leq ct^{1/2}$ have a special significance as is indicated in the next theorem.

THEOREM 3.3. *Suppose $y_1(t)$ is a nonoscillatory solution of* (1.1) *which is positive on $[t_0, \infty)$ and such that*

$$\int^\infty \frac{1}{(y_1(s))^2}\, ds = \infty .$$

Let $y_2(t)$ be any other nonoscillatory solution of (1.1) *which is positive on $[t_0, \infty)$. Then either*

(i) $y_2(t) \leq k_1 y_1(t)$, $t \geq t_0$, *some* $k_1 > 0$

or

(ii) $y_2(t) \geq k_2 y_1(t) \int_{t_0}^t ds\, (y_1(s))^2$, *sufficiently large t, some* $k_2 > 0$.

Proof. The proof is based on the transformation

$$y(t) = y_1(t)u(x), \qquad x = \int_{t_0}^t \frac{1}{(y_1(s))^{-2}}\, ds, \tag{3.3}$$

which transforms (1.1) into

$$u'' + f(x)(|u|^\gamma \operatorname{sgn} u - u) = 0, \qquad ' = \frac{d}{dx}, \tag{3.4}$$

where $f(x) = (y_1(t))^{\gamma+3}q(t)$. Thus the t-interval $[t_0, \infty)$ corresponds to the x-interval $[0, \infty)$. Clearly $f(x) \geq 0$ for $x \geq 0$. For positive nonoscillatory solutions (3.4) is equivalent to

$$u'' + f(x)(u^\gamma - u) = 0.$$

The solution $y_1(t)$ of (1.1) corresponds to the solution $u_1(x) \equiv 1$ of (3.4). Let $y_2(t)$ correspond to $u_2(x)$. We will consider several different possibilities.

1. Suppose $0 < u_2(0) \leq 1$. If $u_2(x) \leq 1$ for $x \geq 0$ then (i) holds. Suppose $u_2(x_0) = 1$ where x_0 is the first such point. Then $u_2'(x_0) > 0$ (since $u(x) \equiv 1$ is the unique solution satisfying $u(x_0) = 1$, $u'(x_0) = 0$). Thus there is an $x_1 > x_0$ such that $u_2(x_1) > 1$ and $u_2'(x_1) > 0$. Then $u_2''(x_1) \geq 0$. Therefore $u_2'' \geq 0$, $u_2' > 0$, $u_2 > 1$ for $x \geq x_1$. Hence $u_2'(x) \geq u_2'(x_1) > 0$ for $x \geq x_1$. Therefore $u_2(x) \geq (x - x_1)u_2'(x_1)$ for $x \geq x_1$. Thus $u_2(x) \geq k_2 x$ for large x. Thus (ii) holds.

2. Now suppose that $u_2(0) > 1$. If $u_2'(x) \leq 0$ for $x \geq 0$, then $u_2(x) \leq u_2(x_0)$, $x \geq 0$, and (i) holds. Otherwise proceed as in Case 1.

COROLLARY 3.2. *If there is a positive nonoscillatory solution $y_1(t)$ of* (1.1) *satisfying $0 < ct^\alpha \leq y_1(t) \leq dt^\alpha$, $0 \leq \alpha \leq \frac{1}{2}$ then every other positive nonoscillatory solution $y_2(t)$ satisfies either $y_2(t) \leq k_1 t^\alpha$ or*

(i) $y_2(t) \geq k_2 t^{1-\alpha}$, $\qquad \alpha \neq \frac{1}{2}$,

(ii) $y_2(t) \geq k_2 t^{1/2} \log t$, $\qquad \alpha = \frac{1}{2}$.

Corollary 3.3. *If there are two positive nonoscillatory solutions $y_1(t)$, $y_2(t)$ of (1.1) such that $0 < c_i t^{\alpha_i} \leq y_i(t) \leq d_i t^{\alpha_i}$, $0 \leq \alpha_i \leq \frac{1}{2}$, $i = 1, 2$ then there are positive constants k_1 and k_2 such that*

$$k_1 \leq y_1(t)/y_2(t) \leq k_2$$

for large t.

Corollary 3.4 (Belohorec [4]). *If* (1.1) *has a bounded nonoscillatory solution, then every other positive nonoscillatory solution is either bounded or else satisfies* $\lim_{t\to\infty} y(t)/t = \alpha > 0$ *(and finite).*

Proof. Recall that every positive nonoscillatory solution is nondecreasing. Let $y_1(t)$ be a positive, bounded, nonoscillatory solution. If $y_2(t)$ is any other positive nonoscillatory solution, then by Corollary 3.2 either $y_2(t)$ is bounded or $y_2(t) \geq k_2 t$ for some $k_2 > 0$. Suppose the latter. Since $\lim_{t\to\infty} \dot{y}(t)$ exists and is finite for every nonoscillatory solution, it is clear that $\lim_{t\to\infty} \dot{y}_2(t) > 0$. Therefore, by l'Hospital's rule $\lim_{t\to\infty} y_2(t)/t = \alpha$ where $0 < \alpha < \infty$.

Theorem 3.4. *If* $\int^\infty sq(s)\, ds < \infty$ *and* $\dot{q}(t) \leq 0$, *then every nontrivial solution* $y(t)$ *of* (1.1) *satisfies either* $\lim_{t\to\infty} y(t) = \alpha$ *where* $\alpha \neq 0, \pm\infty$ *or* $\lim_{t\to\infty} y(t)/t = \beta$ *where* $\beta \neq 0, \pm\infty$.

Proof. This follows immediately from Corollary 3.4 and a nonoscillation theorem [6] which states that under the hypotheses of Theorem 3.4 all nontrivial solutions of (1.1) are nonoscillatory.

Remark. Many of the results in this paper have analogues for the case $\gamma > 1$ (c.f. Moore and Nehari [10]). However, analogues to Theorems 3.2 and 3.3 for the case $\gamma > 1$ are not known. The techniques used to prove these two theorems do not work for $\gamma > 1$. However, the Emden–Fowler equation (3.1) has a solution of the form $y(t) = ct^w$, $0 < w < 1$, for $\gamma > 1$ as well as $0 < \gamma < 1$. Thus there should be an analogue at least to Theorem 3.2, for the case $\gamma > 1$.

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