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RATES OF CONVERGENCE IN CENTRAL LIMIT THEOREM FOR MARTINGALE DIFFERENCES

By

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In this note we shall give an estimate to the rate of convergence in central limit theorem for bounded martingale difference sequence, which is a better estimate than Ibragimov's one [3]. Our method is based on Skorohod representation theorem (straightforward probabilistic methods).

Let $\{X_i; i=1, 2, \dots\}$ be a sequence of martingale differences with $P\{|X_i| \leq C\} = 1$ for all i . We define the random variables

$$s_0^2 = E\{X_1^2\}$$
$$s_j^2 = E\{X_{j+1}^2 | X_j, \dots, X_1\}, \quad j=1, 2, \dots$$

and we also define the random indexes ν_n by the inequalities

$$(1) \quad s_0^2 + \dots + s_{\nu_n-1}^2 < n \leq s_0^2 + \dots + s_{\nu_n}^2, \quad n=1, 2, \dots$$

Finally, we define

$$S_n = X_1 + \dots + X_{\nu_n}.$$

I. A. Ibragimov [3] showed that if $\sum_{j=0}^{\infty} s_j^2$ diverges with probability 1, then

$$\sup_x \left| P\left\{ \frac{S_n}{\sqrt{n}} \leq x \right\} - \Phi(x) \right|$$
$$\leq 2 \frac{C^{1/2}}{\sqrt[4]{n}} \left(1 + \frac{3}{2} \frac{C}{\sqrt{n}} + \frac{1}{3} \frac{C^2}{n} \right),$$

where $\Phi(x)$ is a standard normal distribution function. We give the theorem which deals with the suggestions given to me by Ibragimov [4] in private conversations.

THEOREM: Let $\{X_i; i=1, 2, \dots\}$ be a sequence of martingale differences with $P\{|X_i| \leq C\} = 1$. If

$$(2) \quad \sum_{j=0}^n s_j^2(\omega) \asymp f(n) \quad \text{uniformly}$$

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for some monotone increasing function $f(n)$ such that

$$(3) \quad \frac{f^{-1}(n)}{n} = o(\log n), \quad \frac{f(n)}{n} = o(1),$$

then there exists a constant M such that for sufficiently large n

$$\sup_x \left| P \left\{ \frac{S_n}{\sqrt{n}} \leq x \right\} - \Phi(x) \right| \leq M \frac{(\log n)^3}{\sqrt{n}}.$$

The constant M depends only on the constants C , β and γ ,

where $\beta = \liminf_{n \rightarrow \infty} \operatorname{ess. inf}_{\omega} \frac{1}{f(n)} \sum_{j=0}^n s_j^2(\omega)$, $\gamma = \limsup_{n \rightarrow \infty} \operatorname{ess. sup}_{\omega} \frac{1}{f(n)} \sum_{j=0}^n s_j^2(\omega)$.

REMARK: The condition (2) of Theorem means that

$$\begin{aligned} 0 &< \liminf_{n \rightarrow \infty} \operatorname{ess. inf}_{\omega} \frac{1}{f(n)} \sum_{j=0}^n s_j^2(\omega) \\ &\leq \limsup_{n \rightarrow \infty} \operatorname{ess. sup}_{\omega} \frac{1}{f(n)} \sum_{j=0}^n s_j^2(\omega) < \infty. \end{aligned}$$

In proving the theorem, we shall use the following three lemmas. The first one is a Skorohod representation theorem for martingale differences which was given by Strassen [1].

LEMMA 1: Let

$$(4) \quad X_1, X_2, \dots$$

be random variables such that for all n , $E\{X_n^2 \| X_{n-1}, \dots, X_1\}$ exists and $E\{X_n \| X_{n-1}, \dots, X_1\} = 0$ a.e. Then, without loss of generality, there is a Brownian motion $w(t)$ together with a sequence of non-negative random variables T_1, T_2, \dots such that

$$\sum_{i=1}^n X_i = w \left(\sum_{i=1}^n T_i \right) \quad \text{a.e.,}$$

for all n . Moreover, if \mathcal{B}_n is generated by X_1, \dots, X_n and $w(t)$ for $0 \leq t \leq \sum_{i=1}^n T_i$, then the following hold.

- (i) T_n is \mathcal{B}_n -measurable.
- (ii) For any $s > 0$, $w \left(\sum_{i=1}^n T_i + s \right) - w \left(\sum_{i=1}^n T_i \right)$ is independent of \mathcal{B}_n .
- (iii) $E\{T_n \| \mathcal{B}_{n-1}\}$ exists and $E\{T_n \| \mathcal{B}_{n-1}\} = E\{X_n^2 \| \mathcal{B}_{n-1}\} = E\{X_n^2 \| X_{n-1}, \dots, X_1\}$ a.e.
- (iv) If k is a real number > 1 and $E\{X_n^{2k} \| X_{n-1}, \dots, X_1\}$ exists, then $E\{T_n^k \| \mathcal{B}_{n-1}\}$ exists too, and further $E\{T_n^k \| \mathcal{B}_{n-1}\} \leq L_k E\{X_n^{2k} \| \mathcal{B}_{n-1}\} \leq L_k E\{X_n^{2k} \| X_{n-1}, \dots, X_1\}$ a.e., where each L_k is a constant which depends only on k .

The phrase ‘without loss of generality’ in the above lemma is used in a specific sense, namely: there is a new probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$ on which random variables

$$(5) \quad \hat{X}_1, \hat{X}_2, \dots$$

are defined in such a way that the both sequences (4) and (5) have the same distribution, and the assertions in the above lemma (where now ‘without loss of generality’ is to be omitted) remain valid for the \hat{X}_n ($w(t)$ and the T_n are of course defined on the new space). In this note, we shall assume that the new probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$ satisfies the following conditions: (I) There exists a sequence of independent Brownian motions $\{B_n(t), n \geq 1\}$ on $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$ such that $B_n(t)$ converges with probability 1 to Brownian motion $B(t)$. (II) There exist random variables independent of all the $B_n(t)$, Y_1, Y_2, \dots on $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$ which are pairwise independent and distributed uniformly over the interval $[0, 1]$.

Hereafter, we shall discuss on the new probability space, and we shall write X_1, X_2, \dots in place of (5) and (Ω, \mathcal{B}, P) in place of $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$.

The next two lemmas are the generalizations of results of Skorohod [2] on sums of independent random variables to martingales. The proofs can be obtained in a way similar to Skorohod’s one and are omitted.

LEMMA 2: For each n , suppose that $\mathcal{B}_{n,1}, \mathcal{B}_{n,2}, \dots$ are non-decreasing sequence of σ -algebras and that $\eta_{n,1}, \eta_{n,2}, \dots$ are random variables such that for each i , $\eta_{n,i}$ is $\mathcal{B}_{n,i}$ -measurable, $E\{\eta_{n,i} | \mathcal{B}_{n,i-1}\} = 0$ a. e., $E\{\eta_{n,i}^2 | \mathcal{B}_{n,i-1}\} \leq H_1/n$ and $E\{\eta_{n,i}^4\} \leq H_2/n^2$, where $\mathcal{B}_{n,0}$ is a trivial σ -algebra. If $\nu_n(\omega)$ is a non-negative integer-valued random variable such that $\text{ess. sup}_\omega \nu_n(\omega) \leq g(n)$ for some positive integer-valued function $g(n)$, then

$$\begin{aligned} P\left\{\left|\sum_{i=1}^{\nu_n} \eta_{n,i}\right| > 2 \log n\right\} \\ \leq \frac{g(n)}{n^2} \left(2 \exp\left\{\left(1 + \frac{e}{2}\right) H_1 \frac{g(n)}{n}\right\} + H_2\right). \end{aligned}$$

LEMMA 3: For each n , suppose that $\mathcal{B}_{n,1}, \mathcal{B}_{n,2}, \dots$ are non-decreasing sequence of σ -algebras and that $\eta_{n,1}, \eta_{n,2}, \dots$ are random variables such that for each i , $\eta_{n,i}$ is $\mathcal{B}_{n,i}$ -measurable, $E\{\eta_{n,i} | \mathcal{B}_{n,i-1}\} = 0$ a. e. and $E\{\eta_{n,i}^k | \mathcal{B}_{n,i-1}\} \leq H_k / (\log n)^k$, $k=2, 3, \dots$, where $H_k \leq A^k$ for some constant A and $\mathcal{B}_{n,0}$ is a trivial σ -algebra. Then we have

$$\begin{aligned} P\left\{\left|\sum_{i=1}^{\lceil (\log n)^8 \rceil} \eta_{n,i}\right| > \left(\left(1 + \frac{e}{2}\right) H_2 + 3\right) \log n\right\} \\ \leq \frac{2}{n^3} + \frac{A^{3n}}{(\log n)^{3n-3}}. \end{aligned}$$

We can now prove our theorem as follows. It follows from Lemma 1 that there is a Brownian motion $w(t)$ together with a sequence of non-negative random variables T_1, T_2, \dots such that for each k

$$\sum_{i=1}^k \xi_{n,i} = w\left(\sum_{i=1}^k T_i\right) \quad \text{a. e.},$$

where $\xi_{n,i} = \frac{1}{\sqrt{n}} X_i$. Moreover, if $\mathcal{B}_{n,i} = \mathcal{B}\left\{\xi_{n,1}, \dots, \xi_{n,i}, w(t), 0 \leq t \leq \sum_{j=1}^i T_j\right\}$, then we have

$$E\{T_i^m \mid \mathcal{B}_{n,i-1}\} \leq L_m \left(\frac{C}{\sqrt{n}}\right)^{2m} \quad \text{a. e.},$$

since, by hypothesis, $P\left\{\left|\xi_{n,i}\right| \leq \frac{C}{\sqrt{n}}\right\} = 1$.

We denote by $Q_n(x)$ the probability

$$\begin{aligned} (6) \quad Q_n(x) &= P\left\{\frac{S_n}{\sqrt{n}} \leq x\right\} \\ &= P\{\xi_{n,1} + \dots + \xi_{n,\nu_n} \leq x\} \\ &= P\left\{w\left(\sum_{i=1}^{\nu_n} T_i\right) \leq x\right\} \end{aligned}$$

and by $Q(x)$ the probability

$$(7) \quad Q(x) = P\{w(1) \leq x\}.$$

We define $\eta_{n,i} = n^{1/2}(T_i - E\{T_i \mid \mathcal{B}_{n,i-1}\})$, and $\zeta_{n,i} = \sum_{j=1}^i \eta_{n,j}$. Then we have

$$w(T_1 + \dots + T_{\nu_n}) = w\left(\frac{\zeta_{n,\nu_n}}{\sqrt{n}} + \sum_{i=1}^{\nu_n} E\{T_i \mid \mathcal{B}_{n,i-1}\}\right).$$

Thus

$$Q_n(x) = P\left\{w\left(\frac{\zeta_{n,\nu_n}}{\sqrt{n}} + \sum_{i=1}^{\nu_n} E\{T_i \mid \mathcal{B}_{n,i-1}\}\right) \leq x\right\}.$$

From the property of the stopping time T_{k+1} , it follows that for any $s \in \left[\sum_{i=1}^k T_i, \sum_{i=1}^{k+1} T_i\right]$

$$\left|w(s) - w\left(\sum_{i=1}^k T_i\right)\right| \leq C/n^{1/2}.$$

Therefore, we have

$$\begin{aligned} (8) \quad P\left\{w(s) \leq x - \frac{(\log n)^3}{\sqrt{n}} C, \text{ for some } s \in \left[\sum_{i=1}^{\nu_n} T_i, \sum_{i=1}^{\nu_n + [(\log n)^3]} T_i\right]\right\} &\leq Q_n(x) \\ &\leq P\left\{w(s) \leq x + \frac{(\log n)^3}{\sqrt{n}} C, \text{ for any } s \in \left[\sum_{i=1}^{\nu_n} T_i, \sum_{i=1}^{\nu_n + [(\log n)^3]} T_i\right]\right\}. \end{aligned}$$

From the conditions (2) and (3) of Theorem, it follows that for sufficiently large n

$$\sum_{i=\nu_{n+1}}^{\nu_n+[(\log n)^3]} E\{T_i \mid \mathcal{B}_{n,i-1}\} \geq \frac{\beta}{6} \frac{(\log n)^3}{n \log n},$$

where $\beta = \liminf_{n \rightarrow \infty} \operatorname{ess. inf}_{\omega} \frac{1}{f(n)} \sum_{j=0}^n s_j^2(\omega)$. We define $\bar{\eta}_{n,i} = \frac{n^{\nu/2}}{\log n} \eta_{n,i}$, then the variables $\bar{\eta}_{n,i}$ satisfy the conditions of Lemma 3. Furthermore, we observe that event $\{\nu_n = k\}$ is $\mathcal{B}_{n,k}$ -measurable, and hence

$$E\{\chi_k \bar{\eta}_{n,j} \mid \mathcal{B}_{n,j-1}\} = \chi_k E\{\bar{\eta}_{n,j} \mid \mathcal{B}_{n,j-1}\} = 0 \quad \text{a. e.}$$

for all $k \leq j-1$, where χ_k is a indicator function of $\{\nu_n = k\}$. Thus there exists a set B such that, for sufficiently large n , $P(B^c) \leq 3/n$ and

$$\sup_{\omega \in B} \left| \frac{1}{\sqrt{n}} \sum_{i=\nu_{n+1}}^{\nu_n+[(\log n)^3]} \eta_{n,i} \right| \leq \frac{(\log n)^2}{n} \left(\left(1 + \frac{\epsilon}{2}\right) C^4 + 3 \right).$$

Therefore, we have

$$\delta_n = \inf_{\omega \in B} \sum_{i=\nu_{n+1}}^{\nu_n+[(\log n)^3]} T_i > \frac{(\log n)^2}{n},$$

so that, it follows from (8) that

$$Q_n(x) \leq \frac{3}{n} + P\left\{w(s) \leq x + \frac{(\log n)^3}{\sqrt{n}} C, \quad \text{for any } s \in \left[\sum_{i=1}^{\nu_n} T_i, \sum_{i=1}^{\nu_n} T_i + \delta_n \right] \right\}.$$

Furthermore, from the property (I) and (II) of the new probability space, there exists a Brownian motion $w_1(t)$ independent of $w(t)$ such that

$$P\left\{ \left| \sum_{i=1}^{\nu_n} T_i - \sum_{i=1}^{\nu_n} T_i^{(1)} \right| > \frac{1}{n} \right\} < \frac{1}{n},$$

where random variables $T_1^{(1)}, T_2^{(1)}, \dots$ are the stopping times for $w_1(t)$ determined by Skorohod representation theorem (Lemma 1), see Basu [5]. Consequently, we have for sufficiently large n

$$Q_n(x) \leq P\left\{w(\tau) \leq x + \frac{(\log n)^3}{\sqrt{n}} C\right\} + \frac{4}{n},$$

where the random variable $\tau = \left(\sum_{i=1}^{\nu_n} T_i^{(1)} + \frac{\delta_n}{2} \right)$ is independent of Brownian motion $w(t)$ and

$$(9) \quad \tau \in \left[\sum_{i=1}^{\nu_n} T_i, \sum_{i=1}^{\nu_n} T_i + \delta_n \right].$$

Since $w(\tau)$ and $\sqrt{\tau} w(1)$ have the same distribution for fixed τ , it follows that

$$(10) \quad Q_n(x) \leq P\left\{\sqrt{\tau} w(1) \leq x + \frac{(\log n)^3}{\sqrt{n}} C\right\} + \frac{4}{n}.$$

From the condition (2) of Theorem, it follows that for sufficiently large n

$$\operatorname{ess. sup}_{\omega} \nu_n(\omega) \leq \left[f^{-1}\left(\frac{2}{\beta} n\right) \right] + 1 \leq 2f^{-1}\left(\frac{2}{\beta} n\right).$$

Furthermore, the variables $\eta_{n,i}$ satisfy the conditions of Lemma 2, then we have

$$(11) \quad \begin{aligned} P\{|\zeta_{n,\nu_n}| > 2 \log n\} \\ \leq \frac{2}{n^2} f^{-1}\left(\frac{2}{\beta} n\right) \cdot \left(\exp\left\{\frac{C_1}{n} f^{-1}\left(\frac{2}{\beta} n\right)\right\} + C_2\right), \end{aligned}$$

where $C_1 = 2\left(1 + \frac{\theta}{2}\right)C^4$ and $C_2 = C^8$. From the definition of ν_n and Lemma 1, we have

$$1 - \frac{C^2}{n} \leq \sum_{i=1}^{\nu_n} E\{T_i \mid \mathcal{B}_{n,i-1}\} < 1.$$

Therefore, we have the following inequality on $B \cap \{|\zeta_{n,\nu_n}| \leq 2 \log n\}$

$$1 - M_1 \frac{\log n}{\sqrt{n}} \leq \tau \leq 1 + M_1 \frac{\log n}{\sqrt{n}},$$

where M_1 is a constant depending only on C and γ , $\gamma = \limsup_{n \rightarrow \infty} \operatorname{ess. sup}_{\omega} \frac{1}{f(n)} \sum_{j=0}^n s_j^2(\omega)$.

Then, from (10) and (11), we have for sufficiently large n

$$\begin{aligned} Q_n(x) \leq P\left\{w(1) \leq x + 2M_1|x| \frac{\log n}{\sqrt{n}} + M_2 \frac{(\log n)^3}{\sqrt{n}}\right\} \\ + \frac{2}{n^2} f^{-1}\left(\frac{2}{\beta} n\right) \cdot \left(\exp\left\{\frac{C_1}{n} f^{-1}\left(\frac{2}{\beta} n\right)\right\} + C_2\right) + \frac{8}{n}, \end{aligned}$$

where M_2 is a constant depending only on C and γ .

In an analogous fashion, we can show that

$$\begin{aligned} Q_n(x) \geq P\left\{w(1) \leq x - 2M_1|x| \frac{\log n}{\sqrt{n}} - M_2 \frac{(\log n)^3}{\sqrt{n}}\right\} \\ - \frac{2}{n^2} f^{-1}\left(\frac{2}{\beta} n\right) \cdot \left(\exp\left\{\frac{C_1}{n} f^{-1}\left(\frac{2}{\beta} n\right)\right\} + C_2\right) - \frac{8}{n}. \end{aligned}$$

Therefore, we have for sufficiently large n

$$(12) \quad \begin{aligned} |Q_n(x) - Q(x)| \\ \leq P\left\{x - 2M_1|x| \frac{\log n}{\sqrt{n}} - M_2 \frac{(\log n)^3}{\sqrt{n}} < w(1) \leq x\right\} \end{aligned}$$

$$\begin{aligned}
 &+ P\left\{x < w(1) \leq x + 2M_1|x| \frac{\log n}{\sqrt{n}} + M_2 \frac{(\log n)^3}{\sqrt{n}}\right\} \\
 &+ \frac{4}{n^2} f^{-1}\left(\frac{2}{\beta}n\right) \cdot \left(\exp\left\{\frac{C_1}{n} f^{-1}\left(\frac{2}{\beta}n\right)\right\} + C_2\right) + \frac{16}{n}.
 \end{aligned}$$

But, it follows that for sufficiently large n

$$\begin{aligned}
 (13) \quad &P\left\{x - 2M_1|x| \frac{\log n}{\sqrt{n}} - M_2 \frac{(\log n)^3}{\sqrt{n}} < w(1) \leq x\right\} \leq M_3 \frac{(\log n)^3}{\sqrt{n}} \\
 &P\left\{x < w(1) \leq x + 2M_1|x| \frac{\log n}{\sqrt{n}} + M_2 \frac{(\log n)^3}{\sqrt{n}}\right\} \leq M_3 \frac{(\log n)^3}{\sqrt{n}},
 \end{aligned}$$

where M_3 is a constant depending only on C and γ . Consequently, it follows from (6), (7), (12) and (13) that

$$\begin{aligned}
 &\sup_x \left| P\left\{\frac{S_n}{\sqrt{n}} \leq x\right\} - \Phi(x) \right| \\
 &\leq 2M_3 \frac{(\log n)^3}{\sqrt{n}} + \frac{16}{n} \\
 &+ \frac{4}{n^2} f^{-1}\left(\frac{2}{\beta}n\right) \cdot \left(\exp\left\{\frac{C_1}{n} f^{-1}\left(\frac{2}{\beta}n\right)\right\} + C_2\right).
 \end{aligned}$$

Therefore, it follows from the condition (3) of Theorem that there exists a constant M (depending only on C , β and γ) such that

$$\sup_x \left| P\left\{\frac{S_n}{\sqrt{n}} \leq x\right\} - \Phi(x) \right| \leq M \frac{(\log n)^3}{\sqrt{n}}.$$

The proof of Theorem is now complete.

REMARKS: (1) We can prove in the same manner that

$$\sup_x \left| P\left\{\frac{S_n}{\sqrt{n}} \leq x\right\} - \Phi(x) \right| = o\left(\frac{(\log n)^{2+\delta}}{\sqrt{n}}\right)$$

for any $\delta > 0$.

(2) In the same manner, we can give an estimate to the rate of convergence in central limit theorem for a class of dependent R^k -valued random variables, see Y. Kato [6].

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