

Ratio-Cum-Product Estimators of Population Mean Using Known Population Parameters of Auxiliary Variates

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Abstract

This paper suggests two ratio-cum-product estimators of finite population mean using known coefficient of variation and co-efficient of kurtosis of auxiliary characters. The bias and mean squared error of the proposed estimators with large sample approximation are derived. It has been shown that the estimators suggested by Upadhyaya and Singh (1999) are particular case of the suggested estimators. Almost ratio-cum product estimators of suggested estimators have also been obtained using Jackknife technique given by Quenouille (1956). An empirical study is also carried out to demonstrate the performance of the suggested estimators.

Keywords: Ratio-cum-product estimator, population mean, coefficient of variation, coefficient of kurtosis, bias, mean squared error.

1. Introduction

Use of auxiliary information has been in practice to increase the efficiency of the estimators. When the population mean of an auxiliary variate is known, so many estimators for population parameter(s) of study variate have been discussed in the literature. When correlation between study variate and auxiliary variate is positive (high) ratio method of estimation (Cochran, 1940) is used. On the other hand if the correlation is negative, product method of estimation (Robson, 1957; Murthy, 1967) is preferred. In practice information on coefficient of variation(CV) of an auxiliary variate is seldom known. Sisodia and Dwivedi (1981) suggested a modified ratio estimator for population mean of the study variate. Later on Upadhyaya and Singh (1999), derived another ratio and product type estimators using coefficient of variation and coefficient of kurtosis of the auxiliary variate. Singh (1967) utilized information on two auxiliary variates x_1 and x_2 and suggested a ratio-cum-product estimator for population mean. Singh and Tailor (2005) utilized known correlation coefficient between auxiliary variates ($\rho_{x_1x_2}$) x_1 and x_2 .

Singh and Tailor (2005) motivates authors to suggest ratio-cum-product estimators of population mean utilizing the information on co-efficient of variation of auxiliary variates *i.e.* C_{x_1} and C_{x_2} and co-efficient of kurtosis of auxiliary variates $\beta_2(x_1)$ and $\beta_2(x_2)$ besides the population means (\bar{X}_1 and \bar{X}_2) of auxiliary variates x_1 and x_2 .

Let $U = \{U_1, U_2, \dots, U_N\}$ be a finite population of N units. Suppose two auxiliary variates x_1 and x_2 are observed on U_i ($i = 1, 2, \dots, N$), where x_1 is positively and x_2 is negatively correlated with the study variate y . A simple random sample of size n with $n < N$, is drawn using simple random sampling without replacement(SRSWOR) from the population U to estimate the population mean(\bar{Y})

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of study character y , when the population means $\bar{X}_1 = \sum_{i=1}^N x_{1i}/N$ and $\bar{X}_2 = \sum_{i=1}^N x_{2i}/N$ of x_1 and x_2 respectively are known.

Usual ratio and product estimators given by Cochran (1940) and Robson (1957) respectively for estimating the population mean \bar{Y} respectively are defined as

$$\bar{y}_R = \bar{y} \left(\frac{\bar{X}_1}{\bar{x}_1} \right), \quad (1.1)$$

$$\bar{y}_P = \bar{y} \left(\frac{\bar{x}_2}{\bar{X}_2} \right). \quad (1.2)$$

Utilizing the information on co-efficient of variations (C_{x_1} and C_{x_2}) and co-efficient of kurtosis ($\beta_2(x_1)$ and $\beta_2(x_2)$), Upadhyaya and Singh (1999) suggested ratio and product estimators as

$$\hat{Y}_1 = \bar{y} \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{x}_1 C_{x_1} + \beta_2(x_1)} \right), \quad (1.3)$$

$$\hat{Y}_2 = \bar{y} \left(\frac{\bar{x}_2 C_{x_2} + \beta_2(x_2)}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right), \quad (1.4)$$

$$\hat{Y}_3 = \bar{y} \left(\frac{\bar{X}_1 \beta_2(x_1) + C_{x_1}}{\bar{x}_1 \beta_2(x_1) + C_{x_1}} \right), \quad (1.5)$$

$$\hat{Y}_4 = \bar{y} \left(\frac{\bar{x}_2 \beta_2(x_2) + C_{x_2}}{\bar{X}_2 \beta_2(x_2) + C_{x_2}} \right). \quad (1.6)$$

To estimate \bar{Y} , Singh (1967) suggested a ratio-cum-product estimator as

$$\hat{Y}_5 = \bar{y} \left(\frac{\bar{X}_1}{\bar{x}_1} \right) \left(\frac{\bar{x}_2}{\bar{X}_2} \right). \quad (1.7)$$

Assuming that the correlation coefficient ($\rho_{x_1 x_2}$) between auxiliary characters x_1 and x_2 is known, Singh and Tailor (2005) suggested a ratio-cum-product estimator of \bar{Y}

$$\hat{Y}_6 = \bar{y} \left(\frac{\bar{X}_1 + \rho_{x_1 x_2}}{\bar{x}_1 + \rho_{x_1 x_2}} \right) \left(\frac{\bar{x}_2 + \rho_{x_1 x_2}}{\bar{X}_2 + \rho_{x_1 x_2}} \right). \quad (1.8)$$

To the first degree of approximation the mean squared error(MSE) of the estimators \bar{y}_R , \bar{y}_P , \hat{Y}_1 , \hat{Y}_2 , \hat{Y}_3 , \hat{Y}_4 , \hat{Y}_5 and \hat{Y}_6 respectively are

$$\text{MSE}(\bar{y}_R) = \theta \bar{Y}^2 \left[C_y^2 + C_{x_1}^2 - 2\rho_{yx_1} C_y C_{x_1} \right], \quad (1.9)$$

$$\text{MSE}(\bar{y}_P) = \theta \bar{Y}^2 \left[C_y^2 + C_{x_2}^2 + 2\rho_{yx_2} C_y C_{x_2} \right], \quad (1.10)$$

$$\text{MSE}(\hat{Y}_1) = \theta \bar{Y}^2 \left[C_y^2 + \lambda_1 C_{x_1}^2 - 2\rho_{yx_1} \lambda_1 C_y C_{x_1} \right], \quad (1.11)$$

$$\text{MSE}(\hat{Y}_2) = \theta \bar{Y}^2 \left[C_y^2 + \lambda_2 C_{x_2}^2 + 2\rho_{yx_2} \lambda_2 C_y C_{x_2} \right], \quad (1.12)$$

$$\text{MSE}(\hat{Y}_3) = \theta \bar{Y}^2 \left[C_y^2 + \gamma_1^2 C_{x_1}^2 - 2\rho_{yx_1} \gamma_1 C_y C_{x_1} \right], \quad (1.13)$$

$$\text{MSE}(\hat{Y}_4) = \theta \bar{Y}^2 \left[C_y^2 + \gamma_2^2 C_{x_2}^2 + 2\rho_{yx_2} \gamma_2 C_y C_{x_2} \right], \quad (1.14)$$

$$\text{MSE}(\hat{Y}_5) = \theta \bar{Y}^2 \left[C_y^2 + C_{x_1}^2 (1 - 2K_{yx_1}) + C_{x_2}^2 \{1 + 2(K_{yx_2} - K_{x_1 x_2})\} \right] \quad (1.15)$$

and

$$\text{MSE}(\hat{Y}_6) = \theta \bar{Y}^2 \left[C_y^2 + \mu_1^* C_{x_1}^2 (\mu_1^* - 2K_{yx_1}) + \mu_2^* C_{x_2}^2 \{ \mu_2^* + 2(K_{yx_2} - \mu_1^* K_{x_1 x_2}) \} \right], \quad (1.16)$$

where

$$\begin{aligned} K_{yx_1} &= \rho_{yx_1} \left(\frac{C_y}{C_{x_1}} \right), \quad K_{yx_2} = \rho_{yx_2} \left(\frac{C_y}{C_{x_2}} \right), \quad K_{x_1 x_2} = \rho_{x_1 x_2} \left(\frac{C_{x_1}}{C_{x_2}} \right), \quad C_y = \frac{S_y}{\bar{Y}}, \quad \lambda_i = \frac{\bar{X}_i C_{x_i}}{\bar{X}_i C_{x_i} + \beta_2(x_i)}, \\ \gamma_i &= \frac{\bar{X}_i \beta_2(x_i)}{\bar{X}_i \beta_2(x_i) + C_{x_i}}, \quad \mu_1^* = \frac{\bar{X}_i}{\bar{X}_i + \rho_{x_1 x_2}}, \quad \theta = \left(\frac{1}{n} - \frac{1}{N} \right), \quad C_{x_i} = \frac{S_{x_i}}{\bar{X}_i}, \quad \rho_{yx_i} = \frac{S_{yx_i}}{S_y S_{x_i}}, \\ S_y^2 &= \frac{\sum_{j=1}^N (y_j - \bar{Y})^2}{N-1}, \quad S_{x_i}^2 = \frac{\sum_{j=1}^N (x_{ij} - \bar{X}_i)^2}{N-1} \quad \text{and} \quad S_{yx_i}^2 = \frac{\sum_{j=1}^N (y_j - \bar{Y})(x_{ij} - \bar{X}_i)}{N-1}, \end{aligned}$$

where $(i = 1, 2)$.

2. Proposed Estimator

Assuming that the information on coefficient of variation (C_{x_1} and C_{x_2}) and co-efficient of kurtosis ($\beta_2(x_1)$ and $\beta_2(x_2)$) of auxiliary variate x_1 and x_2 , are known, the proposed estimators are

$$\hat{Y}_7 = \bar{y} \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{x}_1 C_{x_1} + \beta_2(x_1)} \right) \left(\frac{\bar{x}_2 C_{x_2} + \beta_2(x_2)}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right), \quad (2.1)$$

$$\hat{Y}_8 = \bar{y} \left(\frac{\bar{X}_1 \beta_2(x_1) + C_{x_1}}{\bar{x}_1 \beta_2(x_1) + C_{x_1}} \right) \left(\frac{\bar{x}_2 \beta_2(x_2) + C_{x_2}}{\bar{X}_2 \beta_2(x_2) + C_{x_2}} \right). \quad (2.2)$$

To obtain the bias and mean squared error of the proposed estimators, we assume that $\bar{y} = \bar{Y}(1 + e_0)$, $\bar{x}_1 = \bar{X}_1(1 + e_1)$, $\bar{x}_2 = \bar{X}_2(1 + e_2)$ such that $E(e_0) = E(e_1) = E(e_2) = 0$ and $E(e_0^2) = \theta C_y^2$, $E(e_1^2) = \theta C_{x_1}^2$, $E(e_2^2) = \theta C_{x_2}^2$, $E(e_0 e_1) = \theta \rho_{yx_1} C_y C_{x_1}$, $E(e_0 e_2) = \theta \rho_{yx_2} C_y C_{x_2}$ and $E(e_1 e_2) = \theta \rho_{x_1 x_2} C_{x_1} C_{x_2}$.

Expressing the \hat{Y}_7 in terms of e_i 's, we get

$$\begin{aligned} \hat{Y}_7 &= \bar{Y}(1 + e_0) \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{X}_1(1 + e_1) C_{x_1} + \beta_2(x_1)} \right) \left(\frac{\bar{X}_2(1 + e_2) C_{x_2} + \beta_2(x_2)}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right) \\ &= \bar{Y}(1 + e_0) \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{X}_1 C_{x_1} + \beta_2(x_1) + \bar{X}_1 C_{x_1} e_1} \right) \left(\frac{\bar{X}_2 C_{x_2} + \beta_2(x_2) + \bar{X}_2 C_{x_2} e_2}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right) \\ &= \bar{Y}(1 + e_0) (1 + \lambda_1 e_1)^{-1} (1 + \lambda_2 e_2) \\ &= \bar{Y}(1 + e_0) (1 - \lambda_1 e_1 + \lambda_1^2 e_1^2) (1 + \lambda_2 e_2) \\ &= \bar{Y}(1 + e_0) (1 - \lambda_1 e_1 + \lambda_1^2 e_1^2 + \lambda_1 \lambda_2 e_1 e_2 + \lambda_2 e_2) \\ \hat{Y}_7 &= \bar{Y} (1 - \lambda_1 e_1 + \lambda_1^2 e_1^2 + \lambda_1 \lambda_2 e_1 e_2 + \lambda_2 e_2 + e_0 - \lambda_1 e_0 e_1 + \lambda_2 e_0 e_2) \\ (\hat{Y}_7 - \bar{Y}) &= \bar{Y} (-\lambda_1 e_1 + \lambda_1^2 e_1^2 + \lambda_2 e_2 - \lambda_1 \lambda_2 e_1 e_2 + e_0 - \lambda_1 e_0 e_1 + \lambda_2 e_0 e_2). \end{aligned} \quad (2.3)$$

Taking expectation of both sides of (2.3)

$$E(\hat{Y}_7 - \bar{Y}) = \bar{Y} E(-\lambda_1 e_1 + \lambda_1^2 e_1^2 + \lambda_2 e_2 - \lambda_1 \lambda_2 e_1 e_2 + e_0 - \lambda_1 e_0 e_1 + \lambda_2 e_0 e_2).$$

Substituting the values of $E(e_0)$, $E(e_1)$, $E(e_2)$, $E(e_1^2)$, $E(e_0e_1)$, $E(e_0e_2)$ and $E(e_1e_2)$ we get the bias of \hat{Y}_7 as

$$B(\hat{Y}_7) = \theta\bar{Y} \left[\lambda_1 C_{x_1}^2 (\lambda_1 - K_{yx_1}) + \lambda_2 C_{x_2}^2 (K_{yx_2} - \lambda_1 K_{x_1x_2}) \right]. \quad (2.4)$$

To find the mean squared error of the suggested estimator \hat{Y}_7 up to first degree of approximation, squaring and taking expectation of (2.3)

$$\begin{aligned} E(\hat{Y}_7 - \bar{Y})^2 &= \bar{Y}^2 E(e_0 - \lambda_1 e_1 + \lambda_2 e_2)^2, \\ \text{MSE}(\hat{Y}_7) &= \bar{Y}^2 E(e_0^2 + \lambda_1^2 e_1^2 + \lambda_2^2 e_2^2 - 2\lambda_1 e_0 e_1 + 2\lambda_2 e_0 e_2 - 2\lambda_1 \lambda_2 e_1 e_2). \end{aligned}$$

After substituting the values of $E(e_0^2)$, $E(e_1^2)$, $E(e_2^2)$, $E(e_0e_1)$, $E(e_0e_2)$ and $E(e_1e_2)$ we have mean squared error of \hat{Y}_7 as

$$\text{MSE}(\hat{Y}_7) = \theta\bar{Y}^2 \left[C_y^2 + \lambda_1 C_{x_1}^2 (\lambda_1 - 2K_{yx_1}) + \lambda_2 C_{x_2}^2 \left\{ \lambda_2 + 2(K_{yx_2} - \lambda_1 K_{x_1x_2}) \right\} \right]. \quad (2.5)$$

Similarly bias and mean squared error of \hat{Y}_8 can be obtained as

$$B(\hat{Y}_8) = \theta\bar{Y} \left[\gamma_1 C_{x_1}^2 (\gamma_1 - K_{yx_1}) + \gamma_2 C_{x_2}^2 (K_{yx_2} - \gamma_1 K_{x_1x_2}) \right], \quad (2.6)$$

$$\text{MSE}(\hat{Y}_8) = \theta\bar{Y}^2 \left[C_y^2 + \gamma_1 C_{x_1}^2 (\lambda_1 - 2K_{yx_1}) + \gamma_2 C_{x_2}^2 \left\{ \gamma_2 + 2(K_{yx_2} - \gamma_1 K_{x_1x_2}) \right\} \right]. \quad (2.7)$$

3. Efficiency Comparison

We know that the variance of sample mean \bar{y} in simple random sampling without replacement(SRSWOR) is

$$V(\bar{y}) = \left(\frac{1}{n} - \frac{1}{N} \right) S_y^2. \quad (3.1)$$

From (1.9) to (1.16), (2.5), (2.7) and (3.1) we have

(i) $\text{MSE}(\hat{Y}_7) < \text{MSE}(\bar{y})$ if

$$K_{yx_1} > \frac{\lambda_1}{2} \quad \text{and} \quad K_{yx_2} > \left(\lambda_1 K_{x_1x_2} - \frac{\lambda_2}{2} \right) \quad (3.2)$$

(ii) $\text{MSE}(\hat{Y}_7) < \text{MSE}(\bar{y}_R)$ if

$$K_{yx_1} < \left(\frac{1 + \lambda_1}{2} \right) \quad \text{and} \quad K_{yx_2} < \left(\lambda_1 K_{x_1x_2} - \frac{\lambda_2}{2} \right) \quad (3.3)$$

(iii) $\text{MSE}(\hat{Y}_7) < \text{MSE}(\bar{y}_p)$ if

$$K_{yx_1} > \left(\frac{\lambda_1}{2} - \lambda_2 K_{x_2x_1} \right) \quad \text{and} \quad K_{yx_2} > - \left(\frac{1 + \lambda_2}{2} \right) \quad (3.4)$$

(iv) $MSE(\hat{Y}_7) < MSE(\hat{Y}_1)$ if

$$K_{yx_2} < \left(\lambda_1 K_{x_1x_2} - \frac{\lambda_2}{2} \right) \quad (3.5)$$

(v) $MSE(\hat{Y}_7) < MSE(\hat{Y}_2)$ if

$$K_{yx_1} > -\lambda_2 K_{x_2x_1} + \frac{\lambda_2}{2} \quad (3.6)$$

(vi) $MSE(\hat{Y}_7) < MSE(\hat{Y}_3)$ if

$$\text{either } K_{yx_1} > \left(\frac{\gamma_1 + \lambda_1}{2} \right) \text{ if } \gamma_1 < \lambda_1 \text{ and } K_{yx_2} < \left(\lambda_1 K_{x_1x_2} - \frac{\lambda_2}{2} \right) \quad (3.7)$$

$$\text{or } K_{yx_1} < \left(\frac{\gamma_1 + \lambda_1}{2} \right) \text{ if } \gamma_1 > \lambda_1 \text{ and } K_{yx_2} < \left(\lambda_1 K_{x_1x_2} - \frac{\lambda_2}{2} \right) \quad (3.8)$$

(vii) $MSE(\hat{Y}_7) < MSE(\hat{Y}_4)$ if

$$\text{either } K_{yx_2} > -\left(\frac{\gamma_2 + \lambda_2}{2} \right) \text{ if } \gamma_2 > \lambda_2 \text{ and } K_{yx_1} > \left(\frac{\lambda_1}{2} - \lambda_2 K_{x_2x_1} \right) \quad (3.9)$$

$$\text{or } K_{yx_2} < -\left(\frac{\gamma_2 + \lambda_2}{2} \right) \text{ if } \gamma_2 < \lambda_2 \text{ and } K_{yx_1} > \left(\frac{\lambda_1}{2} - \lambda_2 K_{x_2x_1} \right) \quad (3.10)$$

(viii) $MSE(\hat{Y}_7) < MSE(\hat{Y}_5)$ if

$$K_{yx_1} > -\left(\frac{1 + \lambda_2}{2} \right) \text{ if } \lambda_2 < 1 \text{ and } K_{yx_1} > \left\{ \frac{1 + \lambda_1}{2} - \frac{K_{x_2x_1}(\lambda_1 \lambda_2 - 1)}{\lambda_1 - 1} \right\} \quad (3.11)$$

$$K_{yx_1} < -\left(\frac{1 + \lambda_2}{2} \right) \text{ if } \lambda_2 > 1 \text{ and } K_{yx_1} > \left\{ \frac{1 + \lambda_1}{2} - \frac{K_{x_2x_1}(\lambda_1 \lambda_2 - 1)}{\lambda_1 - 1} \right\} \quad (3.12)$$

(ix) $MSE(\hat{Y}_7) < MSE(\hat{Y}_6)$ if one of the following conditions is satisfied

$$K_{yx_1} < \left(\frac{\mu_1^* + \lambda_1}{2} \right) \text{ if } \lambda_1 < \mu_1^* \text{ and } K_{yx_2} > \left\{ \frac{K_{x_1x_2}(\lambda_1 \lambda_2 - \mu_1^* \mu_2^*)}{\lambda_2 - \mu_2^*} - \frac{\lambda_2 + \mu_2^*}{2} \right\} \text{ if } \lambda_2 < \mu_2^* \quad (3.13)$$

$$K_{yx_1} < \left(\frac{\mu_1^* + \lambda_1}{2} \right) \text{ if } \lambda_1 < \mu_1^* \text{ and } K_{yx_2} < \left\{ \frac{K_{x_1x_2}(\lambda_1 \lambda_2 - \mu_1^* \mu_2^*)}{\lambda_2 - \mu_2^*} - \frac{\lambda_2 + \mu_2^*}{2} \right\} \text{ if } \lambda_2 > \mu_2^* \quad (3.14)$$

$$K_{yx_1} > \left(\frac{\mu_1^* + \lambda_1}{2} \right) \text{ if } \lambda_1 > \mu_1^* \text{ and } K_{yx_2} > \left\{ \frac{K_{x_1x_2}(\lambda_1 \lambda_2 - \mu_1^* \mu_2^*)}{\lambda_2 - \mu_2^*} - \frac{\lambda_2 + \mu_2^*}{2} \right\} \text{ if } \lambda_2 < \mu_2^* \quad (3.15)$$

$$K_{yx_1} > \left(\frac{\mu_1^* + \lambda_1}{2} \right) \text{ if } \lambda_1 > \mu_1^* \text{ and } K_{yx_2} < \left\{ \frac{K_{x_1x_2}(\lambda_1 \lambda_2 - \mu_1^* \mu_2^*)}{\lambda_2 - \mu_2^*} - \frac{\lambda_2 + \mu_2^*}{2} \right\} \text{ if } \lambda_2 > \mu_2^* \quad (3.16)$$

(x) $MSE(\hat{Y}_8) < MSE(\hat{Y}_7)$ if of the following conditions is satisfied

$$\frac{C_{x_1}^2}{C_{x_2}^2} < \frac{\gamma_2 C_{x_2}^2 \left\{ \gamma_2 + 2(K_{yx_2} - \gamma_1 K_{x_1x_2}) \right\} - \lambda_2 C_{x_2}^2 \left\{ \lambda_2 + 2(K_{yx_2} - \lambda_1 K_{x_1x_2}) \right\}}{\lambda_1 - \gamma_1} \quad (3.17)$$

It is observed that the proposed estimators \hat{Y}_j ($j = 7, 8$) are biased. Bias is disadvantageous in many situations. Keeping this in view, a family of almost unbiased estimators is also proposed using Random Group technique envisaged by Quenouille (1956).

4. A Family of Unbiased Estimators of Population Mean \bar{Y} Using Random Group Method

Suppose a simple random sample of size $n = gm$ is drawn without replacement and split at random into g sub-samples, each of size m . Then Jack-knife type ratio-cum-product estimator for population mean \bar{Y} , using \hat{Y}_7 is given as

$$\hat{Y}_{7J} = \frac{1}{g} \sum_{j=1}^g \bar{y}'_j \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{x}'_{1j} C_{x_1} + \beta_2(x_1)} \right) \left(\frac{\bar{x}'_{2j} C_{x_2} + \beta_2(x_2)}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right), \quad (4.1)$$

where $\bar{y}'_j = (n\bar{y} - m\bar{y}_j)/(n - m)$ and $\bar{x}'_{ij} = (n\bar{x}_i - m\bar{x}_{ij})/(n - m)$, $i = 1, 2$; are the sample means based on a sample of $(n - m)$ units obtained by omitting the j^{th} group and \bar{y}_j and \bar{x}_{ij} ($i = 1, 2$; $j = 1, 2, \dots, g$) are the sample means based on the j^{th} sub samples of size $m = n/g$.

The bias of \hat{Y}_{7J} , upto the first degree of approximation can be easily obtained as

$$B(\hat{Y}_{7J}) = \frac{N - n + m}{N(n - m)} \bar{Y} \left[\lambda_1 C_{x_1}^2 (\lambda_1 - K_{y_{x_1}}) + \lambda_2 C_{x_2}^2 (K_{y_{x_2}} - \lambda_1 K_{x_1 x_2}) \right]. \quad (4.2)$$

From (2.4) and (4.2) we have

$$\frac{B(\hat{Y}_7)}{B(\hat{Y}_{7J})} = \frac{(N - n)(n - m)}{n(N - n + m)} \quad \text{or} \quad B(\hat{Y}_7) - \frac{(N - n)(n - m)}{n(N - n + m)} B(\hat{Y}_{7J}) = 0 \quad (4.3)$$

$$\Rightarrow \lambda^* B(\hat{Y}_7) - \delta^* \lambda^* B(\hat{Y}_{7J}) = 0 \quad (4.4)$$

for any scalar λ^* , where

$$\delta^* = \frac{(N - n)(n - m)}{n(N - n + m)}. \quad (4.5)$$

From (4.4), we have

$$\begin{aligned} \lambda^* E(\hat{Y}_7 - \bar{Y}) - \delta^* \lambda^* E(\hat{Y}_{7J} - \bar{Y}) &= 0 \quad \text{or} \quad \lambda^* E(\hat{Y}_7 - \bar{y}) - \delta^* \lambda^* E(\hat{Y}_{7J} - \bar{y}) = 0 \quad \text{or} \\ E[\lambda^* \hat{Y}_7 - \lambda^* \delta^* \hat{Y}_{7J} - \bar{y} \{\lambda^* (1 - \delta^*) - 1\}] &= \bar{Y}. \end{aligned}$$

Thus we get a general family of almost unbiased ratio-cum-product estimators of \bar{Y} as

$$\hat{Y}_{7u} = [\bar{y} \{1 - \lambda^*(1 - \delta^*)\} + \lambda^* \hat{Y}_7 - \lambda^* \delta^* \hat{Y}_{7J}]. \quad (4.6)$$

Remark 1. For $\lambda^* = 0$, \hat{Y}_{7u} yields the usual unbiased estimator \bar{y} while $\lambda^* = (1 - \delta^*)^{-1}$, gives an almost unbiased estimator for \bar{Y} as

$$\begin{aligned} \hat{Y}_{7u}^* &= \frac{(N - n + m)}{N} g \bar{y} \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{x}_1 C_{x_1} + \beta_2(x_1)} \right) \left(\frac{\bar{x}_2 C_{x_2} + \beta_2(x_2)}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right) \\ &\quad - \frac{(N - n)(g - 1)}{Ng} \sum_{j=1}^g \bar{y}'_j \left(\frac{\bar{X}_1 C_{x_1} + \beta_2(x_1)}{\bar{x}'_{1j} C_{x_1} + \beta_2(x_1)} \right) \left(\frac{\bar{x}'_{2j} C_{x_2} + \beta_2(x_2)}{\bar{X}_2 C_{x_2} + \beta_2(x_2)} \right). \end{aligned} \quad (4.7)$$

Which is Jack-knifed version of the proposed estimator \hat{Y}_7 .

Different suites values of λ^* provides many almost unbiased estimators in (4.6).

5. An Optimum Estimator In Family \hat{Y}_{7u}

The family of almost unbiased estimator \hat{Y}_{7u} at (4.6) can be expressed as

$$\hat{Y}_{7u} = \bar{y} - \lambda^* \bar{y}_1, \quad (5.1)$$

where $\bar{y}_1 = [(1 - \delta^*)\bar{y} - \bar{y}_2]$ and $\bar{y}_2 = \hat{Y}_7 - \delta^* \hat{Y}_{7J}$.

The variance of \hat{Y}_{7u} is given by

$$V(\hat{Y}_{7u}) = V(\bar{y}) + \lambda^{*2} V(\bar{y}_1) - 2\lambda^* \text{Cov}(\bar{y}, \bar{y}_1) \quad (5.2)$$

which is minimized for

$$\lambda^* = \text{Cov}(\bar{y}, \bar{y}_1) / V(\bar{y}_1). \quad (5.3)$$

Substitution of (5.3) in (5.2) yields minimum variance of \hat{Y}_{7u} as

$$\min . V(\hat{Y}_{7u}) = V(\bar{y}) - \frac{\{\text{Cov}(\bar{y}, \bar{y}_1)\}^2}{V(\bar{y}_1)} = V(\bar{y}) (1 - \rho_{01}^2), \quad (5.4)$$

where ρ_{01} is the correlation coefficient between \bar{y} and \bar{y}_1 .

From (5.4) it is clear that $\min . V(\hat{Y}_{7u}) < V(\bar{y})$.

To obtain the explicit expression of the variance of \hat{Y}_{7u} , we write the following results upto terms of order n^{-1} , as

$$\text{MSE}(\hat{Y}_{7J}) = \text{Cov}(\hat{Y}_7, \hat{Y}_{7J}) = \text{MSE}(\hat{Y}_7) \quad (5.5)$$

and

$$\text{Cov}(\bar{y}, \hat{Y}_7) = \text{Cov}(\bar{y}, \hat{Y}_{7J}) = \theta \bar{Y}^2 [C_y^2 - \lambda_1 \rho_{yx_1} C_y C_{x_1} + \lambda_2 \rho_{yx_2} C_y C_{x_2}], \quad (5.6)$$

where $\text{MSE}(\hat{Y}_7)$ is given by (2.5).

Using (2.5), (3.1) and (5.6) in (5.2), the variance of \hat{Y}_{7u} upto the terms of order n^{-1} is given as

$$V(\hat{Y}_{7u}) = \theta \bar{Y}^2 [C_y^2 + \lambda^{*2} (1 - \delta^*)^2 (\lambda_1^2 C_{x_1}^2 + \lambda_2^2 C_{x_2}^2 - 2\rho_{x_1 x_2} C_{x_1} C_{x_2} \lambda_1 \lambda_2) - 2\lambda^* (1 - \delta^*) (\lambda_1 \rho_{yx_1} C_y C_{x_1} - \lambda_2 \rho_{yx_2} C_y C_{x_2})], \quad (5.7)$$

which is minimized for

$$\lambda^* = \frac{\lambda_1 \rho_{yx_1} C_y C_{x_1} - \lambda_2 \rho_{yx_2} C_y C_{x_2}}{(1 - \delta^*) (\lambda_1^2 C_{x_1}^2 + \lambda_2^2 C_{x_2}^2 - 2\lambda_1 \lambda_2 \rho_{x_1 x_2} C_{x_1} C_{x_2})} = \lambda_{opt}^*. \quad (5.8)$$

Substitution of the value of λ_{opt}^* in \hat{Y}_{7u} yields the optimum estimator $\hat{Y}_{7u(opt)}$ (say). Thus the resulting minimum variance of \hat{Y}_{7u} is given by

$$\min . V(\hat{Y}_{7u}) = \theta \bar{Y}^2 C_y^2 \left[1 - \frac{(\lambda_1 \rho_{yx_1} C_{x_1} - \lambda_2 \rho_{yx_2} C_{x_2})^2}{\lambda_1^2 C_{x_1}^2 + \lambda_2^2 C_{x_2}^2 - 2\lambda_1 \lambda_2 \rho_{x_1 x_2} C_{x_1} C_{x_2}} \right] = V(\hat{Y}_{7u(opt)}). \quad (5.9)$$

The optimum value λ_{opt}^* of λ^* can be obtained quite accurately through past data or experience.

Adopting the similar procedure, using proposed estimator \hat{Y}_8 , we can obtain an almost unbiased family of estimators \hat{Y}_{8u} .

Further the variance of the proposed almost unbiased family of estimators \hat{Y}_{8u} to the first degree of approximation is given by

$$V(\hat{Y}_{8u}) = \theta \bar{Y}^2 \left[C_y^2 + \lambda^{*2} (1 - \delta^*)^2 (\gamma_1^2 C_{x_1}^2 + \gamma_2^2 C_{x_2}^2 - 2\rho_{x_1 x_2} C_{x_1} C_{x_2} \gamma_1 \gamma_2) \right. \\ \left. - 2\gamma^* (1 - \delta^*) (\gamma_1 \rho_{yx_1} C_y C_{x_1} - \gamma_2 \rho_{yx_2} C_y C_{x_2}) \right] \\ \lambda^* = \frac{\gamma_1 \rho_{yx_1} C_y C_{x_1} - \gamma_2 \rho_{yx_2} C_y C_{x_2}}{(1 - \delta^*) (\gamma_1^2 C_{x_1}^2 + \gamma_2^2 C_{x_2}^2 - 2\gamma_1 \gamma_2 \rho_{x_1 x_2} C_{x_1} C_{x_2})} = \lambda_{opt}^*$$

and resulting $\min.V(\hat{Y}_{8u})$ is obtained as

$$\min.V(\hat{Y}_{8u}) = \theta \bar{Y}^2 C_y^2 \left[1 - \frac{(\gamma_1 \rho_{yx_1} C_{x_1} - \gamma_2 \rho_{yx_2} C_{x_2})^2}{\gamma_1^2 C_{x_1}^2 + \gamma_2^2 C_{x_2}^2 - 2\gamma_1 \gamma_2 \rho_{x_1 x_2} C_{x_1} C_{x_2}} \right] = V(\hat{Y}_{8u(opt)}).$$

6. Empirical Study

To observe the relative performance of different estimators of \bar{Y} , a natural population data sets is being considered

- **Population [Source: Steel and Torrie (1960, p.282)]**

y : Log of leaf burn in sec.,

x_1 : Potassium percentage,

x_2 : Chlorine percentage.

The required population parameters are

$$\begin{aligned} \bar{Y} &= 0.6860, & C_y &= 0.4803, & \rho_{yx_1} &= 0.1794, & N &= 30, \\ \bar{X}_1 &= 4.6537, & C_{x_1} &= 0.2295, & \rho_{yx_2} &= -0.4996, & \beta_2(x_1) &= 1.56, & n &= 6, \\ \bar{X}_2 &= 0.8077, & C_{x_2} &= 0.7493, & \rho_{x_1 x_2} &= 0.4074, & \beta_2(x_2) &= 1.40. \end{aligned}$$

To see the performance of the various estimators in comparison to \bar{y} , we calculate the percent relative efficiency of all estimators with respect to \bar{y} which is the ratio of the variance of \bar{y} to the mean squared error of the estimator multiplied by 100. The Percent relative efficiency(%) of the estimators \bar{y} , \bar{y}_R , \bar{y}_P , \hat{Y}_1 , \hat{Y}_2 , \hat{Y}_3 , \hat{Y}_4 , \hat{Y}_5 , \hat{Y}_6 , \hat{Y}_7 , $\hat{Y}_7^{(opt)}$, \hat{Y}_8 and $\hat{Y}_8^{(opt)}$ have been computed and presented in Table 1. Formulae for percent relative efficiencies of different estimators are given below:

$$\begin{aligned} \text{PRE}(\bar{y}_R, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\bar{y}_R)} = \frac{C_y^2}{C_y^2 + C_{x_1}^2 - 2\rho_{yx_1} C_y C_{x_1}} \times 100 \\ \text{PRE}(\bar{y}_P, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\bar{y}_P)} \times 100 = \frac{V(\bar{y})}{C_y^2 + C_{x_1}^2 + 2\rho_{yx_1} C_y C_{x_1}} \times 100 \\ \text{PRE}(\hat{Y}_1, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_1)} \times 100 = \frac{C_y^2}{C_y^2 + \lambda_1 C_{x_1}^2 - 2\rho_{yx_1} \lambda_1 C_y C_{x_1}} \times 100 \end{aligned}$$

Table 1: Percent relative efficiencies of different estimators of with respect to

Estimators	\bar{y}	\bar{y}_R	\bar{y}_P	\hat{Y}_1	\hat{Y}_2	\hat{Y}_3	\hat{Y}_4
PREs	100.00	94.62	53.33	97.21	58.74	95.39	106.06
Estimators	\hat{Y}_5	\hat{Y}_6	\hat{Y}_7	$\hat{Y}_7^{(opt)}$	\hat{Y}_8	$\hat{Y}_8^{(opt)}$	
PREs	75.50	142.17	155.10	169.81	156.96	165.14	

$$\begin{aligned} \text{PRE}(\hat{Y}_2, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_2)} \times 100 = \frac{C_y^2}{C_y^2 + \lambda_2 C_{x_2}^2 + 2\rho_{yx_2} \lambda_2 C_y C_{x_2}} \times 100 \\ \text{PRE}(\hat{Y}_3, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_3)} \times 100 = \frac{C_y^2}{C_y^2 + \gamma_1^2 C_{x_1}^2 - 2\rho_{yx_1} \gamma_1 C_y C_{x_1}} \times 100 \\ \text{PRE}(\hat{Y}_4, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_4)} \times 100 = \frac{C_y^2}{C_y^2 + \gamma_2^2 C_{x_2}^2 + 2\rho_{yx_2} \gamma_2 C_y C_{x_2}} \times 100 \\ \text{PRE}(\hat{Y}_5, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_5)} \times 100 = \frac{C_y^2}{C_y^2 + C_{x_1}^2 (1 - 2K_{yx_1}) + C_{x_2}^2 \{1 + 2(K_{yx_2} - K_{x_1x_2})\}} \times 100 \\ \text{PRE}(\hat{Y}_6, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_6)} \times 100 = \frac{C_y^2}{C_y^2 + \mu_1^* C_{x_1}^2 (\mu_1^* - 2K_{yx_1}) + \mu_2^* C_{x_2}^2 \{\mu_1^* + 2(K_{yx_2} - \mu_1^* K_{x_1x_2})\}} \times 100 \\ \text{PRE}(\hat{Y}_7, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_7)} \times 100 = \frac{C_y^2}{C_y^2 + \lambda_1 C_{x_1}^2 (\lambda_1 - 2K_{yx_1}) + \lambda_2 C_{x_2}^2 \{\lambda_2 + 2(K_{yx_2} - \lambda_1 K_{x_1x_2})\}} \times 100 \\ \text{PRE}(\hat{Y}_7^{(opt)}, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_7^{(opt)})} \times 100 = \frac{1}{1 - \frac{(\lambda_1 \rho_{yx_1} C_{x_1} - \lambda_2 \rho_{yx_2} C_{x_2})^2}{\lambda_1^2 C_{x_1}^2 + \lambda_2^2 C_{x_2}^2 - 2\lambda_1 \lambda_2 \rho_{x_1x_2} C_{x_1} C_{x_2}}} \times 100 \\ \text{PRE}(\hat{Y}_8, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_8)} \times 100 = \frac{C_y^2}{C_y^2 + \gamma_1 C_{x_1}^2 (\gamma_1 - 2K_{yx_1}) + \gamma_2 C_{x_2}^2 \{\gamma_2 + 2(K_{yx_2} - \gamma_1 K_{x_1x_2})\}} \times 100 \\ \text{PRE}(\hat{Y}_8^{(opt)}, \bar{y}) &= \frac{V(\bar{y})}{\text{MSE}(\hat{Y}_8^{(opt)})} \times 100 = \frac{1}{1 - \frac{(\gamma_1 \rho_{yx_1} C_{x_1} - \gamma_2 \rho_{yx_2} C_{x_2})^2}{\gamma_1^2 C_{x_1}^2 + \gamma_2^2 C_{x_2}^2 - 2\gamma_1 \gamma_2 \rho_{x_1x_2} C_{x_1} C_{x_2}}} \times 100. \end{aligned}$$

Table 1 shows that the suggested estimators \hat{Y}_7 (or $\hat{Y}_7^{(opt)}$) and \hat{Y}_8 (or $\hat{Y}_8^{(opt)}$) with $\lambda^* = \lambda_{(opt)}^*$ and $\alpha^* = \alpha_{(opt)}^*$ are more efficient than usual unbiased estimator \bar{y} , ratio estimator \bar{y}_r , product estimator \bar{y}_p , ratio-cum-product estimators suggested by Singh (1967) and Singh and Tailor (2005) with considerable gain in efficiency. Thus, if coefficient of variation (C_{x_1} and C_{x_2}) and coefficient of kurtosis ($\beta_2(x_1)$ and $\beta_2(x_2)$) are known of auxiliary variates x_1 and x_2 , both estimators are recommended for use in practice.

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