RATIONAL BLOWDOWNS OF SMOOTH 4-MANIFOLDS

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1. Introduction

The invariants of Donaldson and of Seiberg and Witten are power-ful tools for studying smooth 4-manifolds. A fundamental problem is to determine procedures which relate smooth 4-manifolds in such a fashion that their effect on both the Donaldson and Seiberg-Witten invariants can be computed. The purpose of this paper is to initiate this study by introducing a surgical procedure, called rational blowdown, and to determine how this procedure affects these two sets of invariants. The technique of rationally blowing down and its effect on the the Don-aldson invariant were first announced at the 1993 Georgia International Topology Conference and represents the bulk of the mathematics in this paper. We fell upon this surgical procedure while we were investigating the behavior of the Donaldson invariant in the presence of embedded spheres and while investigating methods for producing a topological logarithmic transform. The rational blowdown procedure completes the full computation of the Donaldson series (and Seiberg-Witten invari-ants) of all elliptic surfaces with $p_g \geq 1$ by showing that the Donaldson series of elliptic surfaces is that conjectured by Kronheimer and Mrowka in [23]:

Theorem 1.1. *Let $E(n; p, q)$, $n \geq 2$, be the simply connected ellip-tic surface with $p_g = n - 1$ and multiple fibers of relatively prime orders*

$p, q \geq 1$. *Then*

$$\mathbf{D}_{E(n;p,q)} = \exp(Q/2) \frac{\sinh^n(f)}{\sinh(f_p)\sinh(f_q)}.$$

Thus, the diffeomorphism type of the elliptic surfaces $E(n; p, q)$, $n \geq 2$, is determined by the integer n and the unordered pair of integers $\{p, q\}$. This diffeomorphism classification is also established in [12], [26], [27], [30], [31], [35]. The advantage to our proof is that, at bottom, it only requires as input the 0 degree Donaldson invariant of the K3 surface $E(2)$. This diffeomorphism classification of elliptic surfaces is now most easily established by computing the Seiberg-Witten invariants of the $E(n; p, q)$ (cf. [42] or §8). However, as we will demonstrate, this rational blowdown procedure goes further and routinely computes the Donaldson series and Seiberg-Witten invariants for many 4-manifolds and for most of the currently known examples of 4-manifolds which are not even homotopy equivalent to complex surfaces.

The ideas presented in this paper have led to rather easy proofs of the blowup formulas for the Donaldson invariants for arbitrary smooth 4-manifolds [9] and alternate proofs and generalizations (to spheres of negative self-intersections) [10] of some of the results of Kronheimer and Mrowka ([23],[24]). While we chose to first write up these later results, another major delay in the appearance of this paper was the introduction of the Seiberg-Witten invariants.

From the beginning, Witten has conjectured how the Seiberg-Witten invariants and the Donaldson invariants determine each other (cf. [42]). Some progress in proving this relationship has been announced by Pidstrigach and Tyurin [34] and also by Feehan and Leness [7]. Our techniques verify Witten's conjecture for elliptic surfaces and for a large class of manifolds obtained from them by rational blowdowns. (See §8.)

Here is an outline of the paper: In §2 we introduce the concept of a rational blowdown and discuss relevant topological issues. Our main analytical result, Theorem 5.1, gives a universal formula which relates the Donaldson invariants of a manifold with those of its rational blowdown. Three examples of the effect of a rational blowdown are given in §3 and these examples are used in subsequent sections to compute the universal quantities given in Theorem 5.1. In §4 we give the fundamental definitions of the Donaldson series, and §5 presents our key analytical results. Here we shall take advantage of our later results and techniques ([9],[10]) to streamline our earlier arguments. In particular, we will utilize the ' pullback — pushforward ' point of view introduced

and developed by Cliff Taubes in [36], [37], [38], [39] (or, alternatively the thesis of Wieczorek [41]) to prove our basic universal formula (Theorem 5.1) and show that the formula takes on a particularly simple form (Theorem 5.11). Starting with the computations of the Donaldson series for elliptic surfaces without multiple fibers given in [23],[10] and [26], we apply Theorem 5.11 and some of the examples presented in §3 to compute the Donaldson series of the elliptic surfaces with multiple fibers in §6. Under the assumption of simple type and the additional assumption that the configuration of curves that is blown down is 'taut', Theorem 5.11 yields a very simple formula relating the basic classes of X with those of its rational blowdowns (cf. Theorem 7.1). This, as well as applications to the computations of the Donaldson series of other manifolds, is discussed in §7. Theorem 5.11 has a straightforward analogue relating the Seiberg-Witten invariants of X and those of its rational blowdowns. We conclude this paper with a statement and proof of this relationship in §8.

2. The topology of rational blowdowns

In this section we define what is meant by a rational blowdown. Let C_p denote the simply connected smooth 4-manifold obtained by plumbing the $(p-1)$ disk bundles over the 2-sphere according to the linear diagram:

$$
\begin{array}{cccc}
-(p+2) & -2 & & -2 \\
\bullet & \bullet & \cdots & \bullet \\
u_{p-1} & u_{p-2} & & u_1
\end{array}
$$
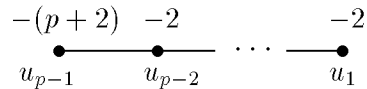
Here, each node denotes a disk bundle over S^2 with Euler class indicated by the label; an interval indicates that the endpoint disk bundles are plumbed, i.e., identified fiber to base over the upper hemisphere of each S^2. Label the homology classes represented by the spheres in C_p by $u_1, \ldots, u_{p-1}$ so that the self-intersections are $u_{p-1}^2 = -(p+2)$ and $u_j^2 = -2$ for $j = 1, \ldots, p-2$. Further, orient the spheres so that $u_j \cdot u_{j+1} = +1$. Then C_p is a 4-manifold with negative definite intersection form and with boundary the lens space $L(p^2, 1-p)$.

Lemma 2.1. *The lens space $L(p^2, 1-p) = \partial C_p$ bounds a rational ball B_p with $\pi_1(B_p) = \mathbf{Z}_p$ and a surjective inclusion-induced homomorphism $\pi_1(L(p^2, 1-p) = \mathbf{Z}_{p^2} \to \pi_1(B_p)$.*

Proof. There are several constructions of B_p; we present three here. The first construction is perhaps amenable to showing that if the configuration of spheres C_p are symplectically embedded in a symplectic 4-manifold X, then the rational blowdown X_p is also symplectic (cf. [16]). For this construction let $\mathbf{F}_{p-1}$, $p \geq 2$, be the simply connected ruled surface whose negative section s_- has square $-(p-1)$. Let s_+ be a positive section (with square $(p-1)$) and f a fiber. Then the homology classes $s_+ + f$ and s_- are represented by embedded 2-spheres which intersect each other once and have intersection matrix

$$\begin{pmatrix} p+1 & 1 \\ 1 & -(p-1) \end{pmatrix}.$$

It follows that the regular neighborhood of this pair of 2-spheres has boundary $L(p^2, p-1)$. Its complement in $\mathbf{F}_{p-1}$ is the rational ball B_p.

The second construction begins with the configuration of $(p-1)$ 2-spheres

$$\overset{\text{p+2}}{\bullet}\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!\overset{2}{\bullet}\!\!\!\!\!-\!\!\!\!\! \cdots \!\!\!\!\!-\!\!\!\!\!\overset{2}{\bullet}$$
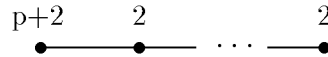
in $\#(p-1)\mathbf{CP}^2$ where the spheres (from left to right) represent

$$2h_1 - h_2 + \cdots \pm h_{p-1}, \ h_1 + h_2, \ h_2 + h_3, \ldots, h_{p-2} + h_{p-1},$$

h_i being the hyperplane class in the ith copy of $\mathbf{CP}^2$. The boundary of the regular neighborhood of the configuration is $L(p^2, p-1)$, and the classes of the configuration span $H_2(\mathbf{CP}^2; \mathbf{Q})$. The complement is the rational ball B_p.

The third construction is due to Casson and Harer [3]. It utilizes the fact that any lens space is the double cover of S^3 branched over a 2-bridge knot. The 2-bridge knot $K((1-p)/p^2)$ corresponding to $L(p^2, 1-p)$ is slice, and B_p is the double cover of the 4-ball branched over the slice disk.   q.e.d.

That all these constructions produce the same rational ball B_p is an exercise in Kirby calculus. However, for the purposes of this paper, it is the third construction that is the most useful, since it allows us to quickly prove:

Corollary 2.2. *Each diffeomorphism of $L(p^2, 1 - p)$ extends over the rational ball B_p.*

Proof. It is a theorem of Bonahon [2] that $\pi_0(\mathrm{Diff}(L(p^2, 1-p))) = \mathbf{Z}_2$, and is generated by the deck transformation τ of the double branched cover of $K((1 - p)/p^2)$. The extension of τ to B_p is given by the deck transformation of the double cover of B^4 branched over the slice disk.

<div align="right">q.e.d.</div>

Suppose that C_p embeds in a closed smooth 4-manifold X. Then let X_p be the smooth 4-manifold obtained by removing the interior of C_p and replacing it with B_p. Corollary 2.2 implies that this construction is well-defined. We call this procedure a **rational blowdown** and say that X_p is obtained by **rationally blowing down** X. Note that $b_X^+ = b_{X_p}^+$ so that rationally blowing down increases the signature while keeping b^+ fixed. An algebro-geometric analogue of rationally blowing down is discussed in [21].

With respect to the basis $\{u_1, \ldots, u_{p-1}\}$ for $H_2(C_p)$, the plumbing matrix for C_p is given by the symmetric $(p - 1) \times (p - 1)$ matrix

$$P = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ 0 & 1 & -2 & 1 & & & \\ & & & & \ddots & & \\ & 0 & & & & -2 & 1 \\ & & & & & 1 & -(p+2) \end{pmatrix}$$

with inverse given by $(P^{-1})_{i,j} = -j + (ij)(p+1)/p^2$ for $j \le i$.

Let $Q : H_2(C_p, \partial C_p; \mathbf{Z}) \times H_2(C_p; \mathbf{Z}) \to \mathbf{Z}$ be the (relative) intersection form of C_p and let $\{\gamma_1, \ldots, \gamma_{p-1}\}$ be the basis of $H_2(C_p, \partial C_p; \mathbf{Z})$ dual to the basis $\{u_1, \ldots, u_{p-1}\}$ of $H_2(C_p; \mathbf{Z})$ with respect to Q, i.e., $\gamma_k \cdot u_\ell = \delta_{k\ell}$. Let $i_* : H_2(C_p; \mathbf{Z}) \to H_2(C_p, \partial C_p; \mathbf{Z})$ be the inclusion-induced homomorphism. Then the intersection form of $H_2(C_p, \partial C_p; \mathbf{Q})$ is defined by

$$\gamma_k \cdot \gamma_\ell = \frac{1}{p^2} \gamma_k \cdot \gamma_\ell',$$

where $\gamma_\ell' \in H_2(C_p; \mathbf{Z})$ is chosen such that $i_*(\gamma_\ell') = p^2 \gamma_\ell$. Since $\gamma_\ell' = p^2 P^{-1}(\gamma_\ell)$, the intersection matrix for $H_2(C_p, \partial C_p; \mathbf{Q})$ is $(\gamma_k \cdot \gamma_\ell) = P^{-1}$. Note also that using the sequence

$$0 \to H_2(C_p; \mathbf{Z}) \xrightarrow{\ P\ } H_2(C_p, \partial C_p; \mathbf{Z}) \xrightarrow{\ \partial\ } H_1(L(p^2, 1 - p; \mathbf{Z}) \to 0$$

we may identify $H_1(L(p^2, 1-p; \mathbf{Z})$ with $\mathbf{Z}_{p^2}$ so that ∂ is given by $\partial(\gamma_j) = j$.

There is an alternative choice of dual bases for $H_2(C_p; \mathbf{Z})$ and $H_2(C_p, \partial C_p; \mathbf{Z})$ that we shall find useful because of its symmetry. Define the basis $\{v_i\}$ of $H_2(C_p; \mathbf{Z})$ by

$$v_i = u_{p-1} + \cdots + u_i, \qquad u_j = v_j - v_{j+1},$$

so $v_i^2 = -(p+2)$ for each i, and if $i \neq j$ then $v_i \cdot v_j = -(p+1)$. The dual basis $\{\delta_i\}$ of $H_2(C_p, \partial C_p; \mathbf{Z})$ is given in terms of $\{\gamma_i\}$ by

$$\begin{aligned}
\delta_i &= \gamma_i - \gamma_{i-1}, \quad i \neq 1, \\
\delta_1 &= \gamma_1.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_i \cdot \delta_j &= \frac{(p+1)}{p^2}, \quad i \neq j, \\
\delta_i^2 &= -\frac{(p^2 - p - 1)}{p^2},
\end{aligned}$$

and

$$\partial\left(\sum a_i \delta_i\right) = \sum a_i.$$

Let the character variety of $SO(3)$ representations of $\pi_1(L(p^2, 1-p))$ mod conjugacy be denoted by $\chi_{SO(3)}(L(p^2, 1-p))$. Let η be the generator of $\chi_{SO(3)}(L(p^2, 1-p))$ satisfying

$$\eta(1) = \begin{pmatrix} \cos(2\pi/p^2) & \sin(2\pi/p^2) & 0 \\ -\sin(2\pi/p^2) & \cos(2\pi/p^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we identify $\pi_1(L(p^2, 1-p))$ with $\mathbf{Z}_{p^2}$ as above, then we have an identification

$$(1) \qquad \begin{aligned} \chi_{SO(3)}(L(p^2, 1-p)) &\cong \mathbf{Z}_{p^2}/\{\pm 1\} \\ &\cong H_1(L(p^2, 1-p); \mathbf{Z})/\{\pm 1\}. \end{aligned}$$

Fix a metric on C_p which gives a Riemannian collar $L(p^2, 1-p) \times [0, \infty)$ with a flat metric on the lens space. Let $e \in H_2(C_p, \partial C_p; \mathbf{Z})$; so ∂e is some $n_e \in \mathbf{Z}_{p^2}$. Since $b^+_{C_p} = 0$, it is a standard fact that e defines an anti-self-dual connection A_e on the complex line bundle L_e over C_p whose first Chern class is the Poincaré dual of e. (E.g. see [6].)

Throughout this paper we shall identify $H_2(C_p, \partial C_p; \mathbf{Z}) \equiv H^2(C_p; \mathbf{Z})$; so we may write $c_1(L_e) = e$. The connection A_e has an asymptotic value as $t \to \infty$, and this is a flat connection on $L(p^2, 1 - p)$. Dividing out by gauge equivalence, we obtain the element $\partial A_e \in \chi_{SO(3)}(L(p^2, 1 - p))$ which is identified with ∂e via (1). For later use, we define

$$\partial' : H_2(C_p, \partial C_p; \mathbf{Z}) \to \chi_{SO(3)}(L(p^2, p - 1))$$
$$= \mathbf{Z}_{p^2}/\{\pm 1\} = \{0, 1, \ldots, [p/2]\}$$

by $\partial'(e) = \bar{n}_e$, the equivalence class of ∂e.

3. Examples of rational blowdowns

In this section we present four examples of the effect of rational blowdowns. These are essential for our later computations.

Example 1. Logarithmic transform as a rational blowdown.

This first example, whose discovery motivated our interest in this procedure, shows that a logarithmic transform of order p can be obtained by a sequence of $(p - 1)$ blowups (i.e., connected sum with $(p - 1)$ copies of $\overline{\mathbf{CP}}^2$) and one rational blowdown of a natural embedding of the configuration C_p. First, we need some terminology. Recall that simply connected elliptic surfaces without multiple fibers are classified up to diffeomorphism by their holomorphic Euler characteristic $n = e(X)/12 = p_g(X) + 1$. The underlying smooth 4-manifold is denoted $E(n)$. The tubular neighborhood of a torus fiber is a copy of $T^2 \times D^2 = S^1 \times (S^1 \times D^2)$. By a **log transform** on $E(n)$ we mean the result of removing this $T^2 \times D^2$ from $E(n)$ and regluing it by a diffeomorphism

$$\varphi : T^2 \times \partial D^2 \to T^2 \times \partial D^2.$$

The **order** of the log transform is the absolute value of the degree of

$$\mathrm{pr}_{\partial D^2} \circ \varphi : \mathrm{pt} \times \partial D^2 \to \partial D^2.$$

Let $E(n)_\varphi$ denote the result of this operation on $E(n)$. It follows from work of Moishezon [28] that if φ and φ' have the same order, there is a diffeomorphism from $E(n)_\varphi$ to $E(n)_{\varphi'}$. What is needed here is the existence of a cusp neighborhood (cf.[8]). Let $E(n; p)$ denote any $E(n)_\varphi$ where the multiplicity of φ is p.

In $E(n;p)$ there is again a copy of the fiber F, but there is also a new torus fiber, the **multiple fiber**. Denote its homology class by f_p; so in $H_2(E(n;p);\mathbf{Z})$ we have $f = p\,f_p$. We can continue this process on other torus fibers; to insure that the resulting manifold is simply connected we can take at most two log transforms with orders that are pairwise relatively prime. Let the orders be p and q and denote the result by $E(n;p,q)$. We sometimes write $E(n;p,q)$ in general, letting p or q equal 1 if there are fewer than 2 multiple fibers. Of course one can take arbitrarily many log transforms (which we shall sometimes do), and we denote the result of taking r log transforms of orders $p_1, \ldots, p_r$ by $E(n;p_1, \ldots, p_r)$.

The homology class f of the fiber of $E(n)$ can be represented by an immersed sphere with one positive double point (a nodal fiber). Figure 1 represents a handlebody (Kirby calculus) picture for a cusp neighborhood N which contains this nodal fiber. (See [20] for an explanation of such pictures and how to manipulate them.) Blow up this double point (i.e., take the proper transform of f) so that the class $f - 2e_1$ (where e_1 is the homology class of the exceptional divisor) is represented by an embedded sphere with square -4 (cf. Figure 2). This is just the configuration C_2. Now the exceptional divisor intersects this sphere in two positive points. Blow up one of these points, i.e., again take a proper transform. One obtains the homology classes $u_2 = f - 2e_1 - e_2$ and $u_1 = e_1 - e_2$ which form the configuration C_3. Continuing in this fashion, C_p naturally embeds in $N\#_{p-1}\overline{\mathbf{CP}}^2 \subset E(n)\#_{p-1}\overline{\mathbf{CP}}^2$ as in Figure 3. Our first important example of a rational blowdown is:

Theorem 3.1. *The rational blowdown of the above configuration*

$$C_p \subset E(n)\#(p-1)\overline{\mathbf{CP}}^2$$

is diffeomorphic $E(n;p)$.

Proof. As proof, we offer a sequence of Kirby calculus moves in Figures 4 through 8. In Figure 4 we add to Figure 3 the handle (with framing -1) which has the property that when added to ∂C_p one obtains $S^2 \times S^1$ (so that when a further 3 and 4-handle are attached B_p is obtained). Then we blow down the added handle, keeping track of the dual 2-handle (which is labelled in Figure 5 with 0-framing), to obtain Figure 6. Blow down the leftmost -1 curve in Figure 6; so the -2 curve becomes a -1 curve. Continue this process $p - 2$ times to obtain Figure 7. In this picture replace a (unknotted) handle with O-framing

by a 1-handle to obtain Figure 8. This is handlebody picture given by Gompf in [15] for N_p, the order p log-transformed cusp neighborhood.

<div align="right">q.e.d.</div>

For the case $p = 2$, this theorem was first observed by Gompf [16].

Here is a useful observation: To perform a log transform of order pq, first perform a log transform of order p and then perform a log transform of order q on the resulting multiple fiber f_p. This can also be obtained via a rational blowdown procedure. Figure 9 is a handlebody decomposition of $N_p \#_{q-1} \overline{\mathbf{CP}}^2$ with an easily identified copy of C_q. The proof that the result of blowing down C_q results in $E(n; pq)$ is to again follow through the steps of the proof of Theorem 3.1.

Proposition 3.2. *Let f_p be the multiple fiber in $E(n; p)$. Then there is an immersed (nodal) 2-sphere $S \subset E(n; p)$ representing the homology class of f_q. Let q be a positive integer relatively prime to p. If the process of Theorem 3.1 is applied to S, i.e., if Y is the rational blowdown of the configuration C_q in $E(n; p) \#(q - 1)\overline{\mathbf{CP}}^2$ obtained from blowing up S, then $Y \cong E(n; pq)$, the result of a multiplicity pq log transform on E.*

Example 2. In $E(2)$ there is an embedded sphere with self-intersection -4 such that its blowdown is diffeomorphic to $3\mathbf{CP}^2 \# 18\overline{\mathbf{CP}}^2$.

For this, any -4 curve suffices; however, to verify that the rational blowdown decomposes, requires more Kirby calculus manipulations. The Milnor fiber $M(2, 3, 5)$ for the Poincaré homology 3-sphere $P = \Sigma(2, 3, 5)$ embeds in $E(2)$ so that $E(2) = M(2, 3, 5) \cup W$ for some 4-manifold W (cf. [8]). Now $\partial M(2, 3, 5) = P$ also bounds another negative definite 4-manifold S which is the trace of -1 surgery on the left hand trefoil knot. It is known that $S \cup W$ is diffeomorphic to $3\mathbf{CP}^2 \# 11\overline{\mathbf{CP}}^2$. Thus, to construct the example, it suffices to find a -4 curve in $M(2, 3, 5)$ whose rational blowdown produces $S \# 7\overline{\mathbf{CP}}^2$. Recall that $M(2, 3, 5)$ is just the E_8 plumbing manifold given in Figure 10. Slide the handle labeled h over the handle labeled k to obtain the -4 curve $h + k$ in Figure 11. Blow down this -4 curve to obtain Figure 12. Now slide the handle labeled h' over the handle labeled k' to obtain Figure 13. Now succesively blow down the -1 curves to obtain Figure 14. Cancelling the $1-$ handle with the $2-$handle with framing -2 yields $S \# 7\overline{\mathbf{CP}}^2$.

Example 3. Given any smooth 4-manifold X, there is an embedding of the configuration $C_p \subset X \#(p - 1)\overline{\mathbf{CP}}^2 = Y$ with $u_i =$

$e_{p-(i+1)} - e_{p-i}$ for $i = 1, \ldots, p - 2$, and $u_{p-1} = -2e_1 - e_2 - \cdots - e_{p-1}$ such that the rational blowdown Y_p of Y is diffeomorphic to $X \# H_p$ where H_p is the homology 4-sphere with $\pi_1 = \mathbf{Z}_p$ which is the double of the rational ball B_p.

In fact $C_p \subset \#(p-1)\overline{\mathbf{CP}}^2 = Y$, and, from the proof of Lemma 2.1, the result of blowing down this configuration is just the double of B_p.

Note that Example 3 points out that although a smooth 4-manifold Y may have a symplectic structure, it need not be the case that a rational blowdown Y_p of Y must also have a symplectic structure. For in this example $X \# H_p$ will never have a symplectic structure since its p–fold cover can be written as a connected sum of two 4-manifolds with positive b^+ so has vanishing Seiberg-Witten invariants and hence, by Taubes [40], is not symplectic. Of course, in this example, the configuration C_p is not symplectically embedded. This brings up the possibility that any smooth 4-manifold can be rationally blown up to a symplectic 4-manifold. For other examples of (nonsimply connected) smooth 4-manifolds with nontrivial Seiberg-Witten invariants but without symplectic structures see [22]. [1]

4. The Donaldson series

In this section we outline the definition of the Donaldson invariant. We refer the reader to [5] and [6] for a more complete treatment. Given an oriented simply connected 4-manifold with $b_X^+ > 1$ and with a generic Riemannian metric and an $SU(2)$ or $SO(3)$ bundle P over X, the moduli space of gauge equivalence classes of anti-self-dual connections on P is a manifold $\mathcal{M}_X(P)$ of dimension

$$8\,c_2(P) - 3\,(1 + b_X^+)$$

if P is an $SU(2)$ bundle, and

$$-2p_1(P) - 3\,(1 + b_X^+)$$

if P is an $SO(3)$ bundle. It will often be convenient to treat these two cases together by identifying $\mathcal{M}_X(P)$ and $\mathcal{M}_X(\mathrm{ad}(P))$ for an $SU(2)$

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[1]For more recent, simply connected, examples, see the preprints, 'Simply-connected irreducible 4-manifolds with no symplectic structures' and 'Exotic 4-manifolds with $b_2^+ = 1$', by Z. Szabo, and also the authors' preprint, 'Knots, links, and 4-manifolds'.

bundle P. Over the product $\mathcal{M}_X(P) \times X$ there is a universal $SO(3)$ bundle $\mathbf{P}$ which gives rise to a homomorphism

$$\mu : H_i(X; \mathbf{R}) \to H^{4-i}(\mathcal{M}_X(P); \mathbf{R})$$

obtained by decomposing the class $-\frac{1}{4}p_1(\mathbf{P}) \in H^4(\mathcal{M}_X \times X)$.

The moduli space $\mathcal{M}_X(P)$ is orientable, and a specific orientation is determined by a choice of homology orientation of X, that is, an orientation of the vector space $H_0(X; \mathbf{R}) \oplus H_2^+(X; \mathbf{R})$ [5]. When either $w_2(P) \neq 0$ or when $w_2(P) = 0$ and $-p_1(P) > 3(1 + b_X^+)$, the Uhlenbeck compactification $\overline{\mathcal{M}}_X(P)$ of the moduli space carries a fundamental class. In practice, one is able to get around the restriction on $p_1(P)$ by blowing up X and considering bundles over $X \# \overline{\mathbf{CP}}^2$ which are nontrivial when restricted to the exceptional divisor [29]. In [14] it is shown that for $\alpha \in H_2(X; \mathbf{Z})$, the classes $\mu(\alpha) \in H^2(\mathcal{M}_X(P))$ extend over $\overline{\mathcal{M}}_X(P)$. When b_X^+ is odd, $\dim \mathcal{M}_X(P)$ is even, say equal to $2d$. In fact, a class $c \in H_2(X; \mathbf{Z})$ and a nonnegative integer $d \equiv -c^2 + \frac{1}{2}(1 + b^+)$ (mod 4) determine an $SO(3)$ bundle $P_{c,d}$ over X with $w_2(P_{c,d}) \equiv c$ (mod 2) and formal dimension $\dim \mathcal{M}_X(P_{c,d}) = 2d$. For $\bar{\alpha} = (\alpha_1, \ldots, \alpha_d) \in H_2(X; \mathbf{Z})^d$, write $\mu(\bar{\alpha}) = \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_d)$. Then one has

$$\langle \mu(\bar{\alpha}), [\overline{\mathcal{M}}_X(P_{c,d})] \rangle = \int_{\overline{\mathcal{M}}_X(P_{c,d})} \mu(\bar{\alpha}),$$

when $\mu(\bar{\alpha})$ is viewed as a $2d$-form. Alternatively, Donaldson has described a method for obtaining codimension-2 divisors V_α in $\mathcal{M}_X(P_{c,d})$ so that $\langle \mu(\bar{\alpha}), [\overline{\mathcal{M}}_X(P_{c,d})] \rangle$ is the signed number of points in the intersection $V_{\alpha_1} \cap \cdots \cap V_{\alpha_d}$.

If $[1] \in H_0(X; \mathbf{Z})$ is the generator, set $\nu = \mu([1]) = -\frac{1}{4}p_1(\beta) \in H^4(\mathcal{M}_X(P))$ where β is the basepoint fibration $\tilde{\mathcal{M}}_X(P) \to \mathcal{M}_X(P)$ with $\tilde{\mathcal{M}}_X(P)$ the manifold of anti-self-dual connections on P modulo based gauge transformations, i.e., those that are the identity on the fiber over a fixed basepoint. The class ν extends over the Uhlenbeck compactification $\overline{\mathcal{M}}_X(P)$ if $w_2(P) \neq 0$, and in case P is an $SU(2)$ bundle, the class will extend under certain dimension restrictions. Once again, these restrictions can be done away with via the tricks mentioned above [29].

Consider the graded algebra

$$\mathbf{A}(X) = \mathrm{Sym}_*(H_0(X) \oplus H_2(X)),$$

where $H_i(X)$ has degree $\frac{1}{2}(4 - i)$. The Donaldson invariant $D_c = D_{X,c}$ is then an element of the dual algebra $\mathbf{A}^*(X)$, i.e., a linear function $D_c :$ $\mathbf{A}(X) \to \mathbf{R}$. This is a homology orientation-preserving diffeomorphism invariant for manifolds X satisfying $b_X^+ \geq 3$.

Throughout the rest of this paper we assume $b_X^+ \geq 3$ and odd.

We let x be the degree-2 element of $\mathbf{A}(X)$ corresponding to $[1] \in H_0(X; \mathbf{Z})$. In case $a + 2b = d > \frac{3}{2}(1 + b_X^+)$ and $\alpha \in H_2(X)$, we have

$$D_c(\alpha^a x^b) = \langle \mu(\alpha)^a \nu^b, [\overline{\mathcal{M}}_X(P_{c,d})] \rangle.$$

We may extend μ over $\mathbf{A}(X)$, and write for $z \in \mathbf{A}(X)$ of degree d, $D_c(z) = \langle \mu(z), [\overline{\mathcal{M}}_X(P_{c,d})] \rangle$. Since such moduli spaces $\mathcal{M}_X(P_{c,d})$ exist only for $d \equiv -c^2 + \frac{1}{2}(1 + b_X^+)$ (mod 4), the Donaldson invariant D_c is defined only on elements of $\mathbf{A}(X)$ whose total degree is congruent to $-c^2 + \frac{1}{2}(1 + b_X^+)$ (mod 4). By definition, D_c is 0 on all elements of other degrees. When P is an $SU(2)$ bundle, one simply writes D or D_X.

If Y is a simply connected 4-manifold with boundary a lens space $L(p, q)$, one can similarly construct relative Donaldson invariants (see [41])

$$D_{Y,c}[\lambda_i] : \mathbf{A}(Y) \to \mathbf{R}$$

corresponding to each $\lambda_i \in \chi_{SO(3)}(L(p, q))$.

Following [23], one considers the invariant

$$\hat{D}_{X,c} : \mathrm{Sym}_*(H_2(X)) \to \mathbf{R}$$

defined by $\hat{D}_{X,c}(u) = D_{X,c}((1 + \frac{x}{2})u)$. Whereas $D_{X,c}$ can be nonzero only in degrees congruent to $-c^2 + \frac{1}{2}(1 + b^+)$ (mod 4), $\hat{D}_{X,c}$ can be nonzero in degrees congruent to $-c^2 + \frac{1}{2}(1 + b^+)$ (mod 2). The Donaldson series, $\mathbf{D}_c = \mathbf{D}_{X,c}$, is defined by

$$\mathbf{D}_{X,c}(\alpha) = \hat{D}_{X,c}(\exp(\alpha)) = \sum_{d=0}^{\infty} \hat{D}_{X,c} \frac{(\alpha^d)}{d!}$$

for all $\alpha \in H_2(X)$. This is a formal power series on $H_2(X)$.

A simply connected 4-manifold X is said to have *simple type* if the relation $D_{X,c}(x^2 z) = 4 D_{X,c}(z)$ is satisfied by its Donaldson invariant for all $z \in \mathbf{A}(X)$ and all $c \in H_2(X; \mathbf{Z})$. This important definition is due to Kronheimer and Mrowka [23] and was observed to hold for many

4-manifolds [24], [10]. In terms of $\hat{D}_{X,c}$, the simple type condition is that $\hat{D}_{X,c}(zx) = 2\hat{D}_{X,c}(z)$ for all $z \in \mathbf{A}(X)$ and all $c \in H_2(X;\mathbf{Z})$. The assumption of simple type assures that for each c, the complete Donaldson invariant $D_{X,c}$ is determined by the Donaldson series $\mathbf{D}_{X,c}$. It is still an open question whether all 4-manifolds with $b^+ > 1$ are of simple type.

The structure theorem is:

Theorem 4.1. (Kronheimer and Mrowka [24], [10]) *Let X be a simply connected 4-manifold of simple type with $b_X^+ > 1$. Then, there exist finitely many 'basic' classes $\kappa_1, \ldots, \kappa_p \in H_2(X, \mathbf{Z})$ and nonzero rational numbers $a_1, \ldots, a_p$ such that*

$$\mathbf{D}_X = \exp(Q/2) \sum_{s=1}^{p} a_s e^{\kappa_s}$$

as analytic functions on $H_2(X)$. Each of the 'basic classes' κ_s is characteristic, i.e., $\kappa_s \cdot x \equiv x \cdot x \pmod 2$ for all $x \in H_2(X;\mathbf{Z})$.

Further, suppose $c \in H_2(X;\mathbf{Z})$. Then

$$\mathbf{D}_{X,c} = \exp(Q/2) \sum_{s=1}^{p} (-1)^{\frac{c^2 + \kappa_s \cdot c}{2}} a_s e^{\kappa_s}.$$

Here, the homology class κ_s acts by intersection, i.e., $\kappa_s(u) = \kappa_s \cdot u$. The basic classes κ_s satisfy certain inequalities analogous to the adjunction formula in a complex surface [24], [10]. We shall need

Theorem 4.2. ([10]) *Let X be a simply connected 4-manifold of simple type with $b_X^+ > 1$, and let $\{\kappa_s\}$ be the set of basic classes as above. If $u \in H_2(X;\mathbf{Z})$ is represented by an immersed 2-sphere with $p \geq 1$ positive double points, then for each s*

$$(2) \qquad\qquad 2p - 2 \geq u^2 + |\kappa_s \cdot u|.$$

Theorem 4.3. ([10]) *Let X be a simply connected 4-manifold of simple type with $b_X^+ > 1$ and with basic classes $\{\kappa_s\}$ as above. For the nontrivial class $u \in H_2(X;\mathbf{Z})$ represented by an immersed 2-sphere with no positive double points, let*

$$\{\kappa_s \,|\, s = 1, \ldots, 2m\}$$

be the collection of basic classes which violate the inequality (2). Then $\kappa_s \cdot u = \pm u^2$ *for each such* κ_s. *Order these classes so that*

$$\kappa_s \cdot u = -u^2 \; (> 0)$$

for $s = 1, \ldots, m$. *Then*

$$\sum_{s=1}^{m} a_s e^{\kappa_s + u} - (-1)^{\frac{1+b_X^+}{2}} \sum_{s=1}^{m} a_s e^{-\kappa_s - u} = 0.$$

5. The Basic Computational Theorem

Recall that for $u \in H_2(X)$ and $F \in \mathbf{A}(X)$, interior product

$$\iota_u F(v) = (\deg(v) + 1) F(uv)$$

defines a derivation which we denote by ∂_u and call 'partial derivation'. Our basic theorem is:

Theorem 5.1. *Let* X *be a simply connected 4-manifold of simple type with* $b_X^+ > 1$. *Suppose that* X *contains the configuration* C_p, *and let* X_p *be the result of rationally blowing down* C_p. *Then, restricted to* $X^* = X_p \setminus B_p = X \setminus C_p$:

$$\mathbf{D}_{X_p} = \sum_{i=1}^{m(p)} \alpha_i(p) \partial^{n_i(p)} \mathbf{D}_{X, c_i(p)},$$

where $\alpha_i(p) \in \mathbf{Q}$, $c_i(p) \in H_2(C_p; \mathbf{Z})$, $\partial^{n_i(p)}$ *is an* n_i*th order partial derivative with respect to classes in* $H_2(C_p; \mathbf{Z})$, *and these quantities depend only on* p, *not on* X.

As motivation, and for use in the next section, we begin with a 'by hand' calculation.

Lemma 5.2. *Let* X *be a simply connected 4-manifold with* $b^+ > 1$ *containing an embedded 2-sphere* Σ *of square* -4 *representing the homology class* σ. *Let* X_2 *be the result of rationally blowing down* Σ. *Then*

$$\mathbf{D}_{X_2}|_{X^*} = \mathbf{D}_X - \mathbf{D}_{X,\sigma}.$$

Proof. For the definition of **D** it is important to have the Donaldson invariant defined without restriction on p_1. As we mentioned in the previous section, this is accomplished by blowing up X at a point of X^*, and using the fact that for e, the exceptional class of the blowup and for $z \in \mathbf{A}(X)$, that $D_{X \# \overline{\mathbf{CP}}^2, e}(ze) = D_X(z)$, whenever the latter invariant is defined. This is proved in [29].

Hence we work with $SO(3)$ connections over $Y_2 = X_2 \# \overline{\mathbf{CP}}^2 = Y^* \cup B_2$ and $Y = X \# \overline{\mathbf{CP}}^2 = Y^* \cup C_2$ with $w_2 = e$. Suppose that z is an element of degree d in $\mathbf{A}(X^*)$. Let $\mathcal{M}_{Y_2}(P; g)$ denote the $2d + 2$-dimensional moduli space of anti-self-dual connections on Y_2 corresponding to the $SO(3)$ bundle P in question. Consider a collar neighborhood (neck) $L(4, -1) \times [-1, 1]$ in Y_2, and suppose that we have a sequence of generic metrics $\{g_n\}$ on Y_2 which stretch the length of the neck to infinity, and whose limit is the disjoint union of generic metrics g^* on $X_+^* = X^* \cup (L(4, 1) \times [0, \infty))$, g_B on $B_{2,+} = B_2 \cup (L(4, -1) \times [0, \infty))$, and a product metric on $L(-4, 1) \times \mathbf{R}$, restricting to the standard flat metric on each $L(4, -1) \times \{t\}$.

Corresponding to ze, there is a divisor V_{ze} in the space of gauge equivalence classes of connections on Y_2. By the Uhlenbeck compactness theorem, any sequence $\{A_n \in \mathcal{M}_{Y_2}(P; g_n) \cap V_{ze}\}$ has a weak limit, $A^* \coprod A_T \coprod A_B$, in the disjoint union of moduli spaces for Y^*, for $L(4, 1) \times \mathbf{R}$ and for B_2. The anti-self-dual connection A^* limits exponentially to some boundary value $\lambda^* \in \chi_{SO(3)}(L(4, 1))$, and similarly, A_B has a boundary value λ_B. We write $A^* \in \mathcal{M}_{Y^*}[\lambda^*]$, $A_B \in \mathcal{M}_{B_2}[\lambda_B]$ and $A_T \in \mathcal{M}_{L(4,1) \times \mathbf{R}}[\lambda^*, \lambda_B]$. We note that the conjugacy classes of $SO(3)$ representations of $L(4, 1)$ are $\{\eta^i | 0 \le i \le 2\}$. We denote $\dim(\mathrm{stab}_{SO(3)}(\lambda))$ by $\delta(\lambda)$. Then $\delta(1) = 3$ and $\delta(i) = 1$, $i \ne 0$.

The Atiyah-Patodi-Singer Theorem allows the calculation of the formal dimensions of the moduli spaces $\mathcal{M}_{Y^*}[\lambda^*]$ and $\mathcal{M}_{B_2}[\lambda_B]$. If $P(A^*)$ and $P(A_B)$ are the bundles over Y^* and B_2 on which the connections A^* and A_B live, then they have Pontryagin numbers in $\frac{1}{4}\mathbf{Z}$ and

$$
\begin{aligned}
\dim \mathcal{M}_{Y^*}[\lambda^*] = &- 2p_1(P(A^*)) - \frac{3}{2}(\mathrm{e}(Y^*) + \mathrm{sign}(Y^*)) \\
&- \frac{1}{2}(\delta(\lambda^*) + \rho(\lambda^*)) \\
= &- 2p_1(P(A^*)) - 3(1 + b_{Y_2}^+) + \frac{3}{2} - \frac{1}{2}(\delta(\lambda^*) + \rho(\lambda^*)),
\end{aligned}
$$

$$\dim \mathcal{M}_{B_2}[\lambda_B] = - 2p_1(P(A_B)) - \frac{3}{2}(\mathrm{e}(B_2) + \mathrm{sign}(B_2))$$
$$- \frac{1}{2}(\delta(\lambda_B) - \rho(\lambda_B))$$
$$= - 2p_1(P(A_B)) - \frac{3}{2} - \frac{1}{2}(\delta(\lambda_B) - \rho(\lambda_B)).$$

If there are bubble instantons occuring in the limit process $A_n \rightharpoonup A^* \coprod A_T \coprod A_B$, then denote the points at which they occur by $x_1, \ldots, x_m \in Y^*$ and $y_1, \ldots, y_n \in B_2$. Furthermore, let T denote the dimension of $\mathcal{M}_{L(4,1) \times \mathbf{R}}[\lambda^*, \lambda_B]$. Then

$$2d \geq \dim \mathcal{M}_{Y^*}[\lambda^*] + \dim \mathcal{M}_{B_2}[\lambda_B] + 4(m + n) + T + \delta(\lambda^*) + \delta(\lambda_B).$$

Suppose that $ze = \alpha_1 \cdots \alpha_r \cdot x^s$ where each $\alpha_i \in H_2(Y^*; \mathbf{Z})$. Represent the α_i by oriented surfaces Σ_i in general position in Y^*; then no x_j lies on more than two of the Σ_i. It is a basic fact that for any of the surfaces Σ_i, either Σ_i contains one of the x_j or $A_{Y^*} \in V_{\alpha_i}$. (See [6].) Let $z' = \alpha_{i_1} \cdots \alpha_{i_b} \cdot x^s$ where $\Sigma_{i_1}, \ldots, \Sigma_{i_b}$ are the surfaces containing none of the x_j. Since no x_j lies on more than two of the Σ_i, it follows that $2m \geq r - b$.

Since $w_2(P(A^*)) = e$, we have $p_1(P(A^*)) \neq 0$. This means that A^* is not a flat connection. Also, since $b_{Y^*}^+ = b_X^+ > 0$, the only reducible anti-self-dual connections with respect to g^* are flat; so there are none. It is then a consequence of the fact that g^* is a generic metric that $\mathcal{M}_{Y^*}[\lambda^*]$ and $V_{z'}$ intersect transversely at A^*. Hence $\dim \mathcal{M}_{Y^*}[\lambda^*] \geq 2b + 4s$, and we get the dimension estimate

$$2d \geq (2b + 4s) + 8(\frac{r - b}{2} + n) + \dim \mathcal{M}_{B_2}[\lambda_B] + T + \delta(\lambda^*) + \delta(\lambda_B).$$

Since $r + 2s = d$ and $r - b \geq 0$,

$$0 \geq 8n + \dim \mathcal{M}_{B_2}[\lambda_B] + \delta(\lambda_B) + T + \delta(\lambda^*).$$

Because we are working with generic metrics, if the connection A_T is irreducible, then $T \geq 0$, and if A_B is irreducible, then $\dim \mathcal{M}_{B_2}[\lambda_B] \geq 0$. However, the only possible values for $\delta(\lambda^*)$ and $\delta(\lambda_B)$ are 1 and 3. Thus if A_T is irreducible, then A_B is reducible, hence flat. Since $w_2(P_{A_B}) = 0$, this means that A_B is trivial, $\dim \mathcal{M}_{B_2}[\lambda_B] = -3$ and $\delta(\lambda_B) = 3$. Hence $0 \geq 8n + T + \delta(\lambda^*)$, which implies that $n = 0$ and A_T is also reducible, a contradiction.

Thus A_T is flat; so $\lambda_B = \lambda^*$ and $T = -\delta(\lambda^*)$, which gives that $0 \geq 8n + \dim \mathcal{M}_{B_2}[\lambda_B] + \delta(\lambda_B)$. Therefore again we see that A_B is flat, hence trivial, and $n = 0$.

Each occurrence of an instanton bubble increases the Pontryagin number by 4. Since A_B and A_T are flat, and $n = 0$, we have $p_1(P(A^*)) = p_1(P) + 4m$. Now $\delta(1) = 3$ and $\rho(1) = 0$; so from the index calculation above it follows that

$$\dim \mathcal{M}_{Y^*}[1] = -2p_1(P) - 8m - 3(1 + b_{Y_2}^+) = 2d - 8m.$$

Since g^* is a generic metric, and $A^* \in \mathcal{M}_{Y^*}[1] \cap V_{z'}$, transversality implies that $\dim(\mathcal{M}_{Y^*}[1] \cap V_{z'}) \geq 0$, i.e., $2d - 8m - (2b + 4s) \geq 0$. Because $d = r + 2s$ and $b \geq r - 2m$, this implies in turn that $m = 0$.

The upshot of this dimension counting argument is that for generic metrics on Y_2 with a long enough neck, each connection $A \in \mathcal{M}_{Y_2}(P) \cap V_{ze}$ has the form of a perturbed glued-up connection $A^* \# A_B$ where A^* is an irreducible connection in $\mathcal{M}_{Y^*}(P^*)[1] \cap V_{ze}$, P^* being the restriction of P over Y^*, and A_B is the trivial $SO(3)$ connection over B_2. Conversely, this construction always builds elements of $\mathcal{M}_{Y_2}(P) \cap V_{ze}$. Thus we have

$$D_{Y_2,e}(ze) = \pm D_{Y^*,e}[1](ze).$$

The invariant $D_{Y^*,e}[1](z')$ is the relative Donaldson invariant of Y^* with the trivial boundary value.

Each $A^* \in \mathcal{M}_{Y^*}(P^*)[1]$ restricts to a trivial connection over $L(4,1)$, thus framing the $\mathbf{R}^3$-bundle $P|_{L(4,1)}$, and it is easily seen that there are two such framings. If we we use these framings to glue together the bundle P^* over Y^* with the trivial bundle over C_2, we get two different bundles over Y. Each has $w_2|_{Y^*}$ equal to the Poincaré dual of e mod 2, and $w_2|_{C_2} \equiv 0 \pmod 2$. The Mayer-Vietoris sequence shows that the first has w_2 equal to the Poincaré dual of e mod 2, and the second has w_2 equal to the Poincaré dual of $e + \sigma$ mod 2. (Note that the Mayer-Vietoris sequence for $Y_2 = Y^* \cup B_2$ shows that there is just one class in $H^2(Y_2; \mathbf{Z}_2)$ which restricts to e on Y^* and to 0 on B_2. This is why all the connections $A^* \# A_B$ above lie in the same bundle.) It now follows that

$$D_{Y_2,e}(ze) = \pm D_{Y,e}(ze) \pm D_{Y,e+\sigma}(ze).$$

This is almost obvious by applying an argument like the one above. We need to know that there are no nontrivial reducible connections on the

neighborhood C_2 of Σ with boundary value 1 and in a moduli space of negative dimension. This follows simply from the fact that if ξ is the complex line bundle whose first Chern class generates $H^2(C_2; \mathbf{Z})$, then the moduli space of anti-self-dual connections on $\xi^{2m} + \mathbf{R}$ has dimension $4m - 3$ (see [10]). Thus (by definition) we have

$$(3) \qquad\qquad D_{X_2}(z) = \pm D_X(z) \pm D_{X,\sigma}(z).$$

Finally, we need to determine signs. A key point following from our discussion is that they are independent of X. Recall from Example 2 that there is a sphere Σ of square -4 in the $K3$-surface X which has a rational blowdown X_2 with $\mathbf{D}_{X_2} = 0$. Since $\mathbf{D}_{X,\sigma} = \exp(Q/2) = \mathbf{D}_X$, our formula must read

$$D_{X_2}(z) = \pm(D_X(z) - D_{X,\sigma}(z)).$$

To compute the overall sign, we must compare the way that signs are attached to $A^* \# A_B$, and $A^* \# A_C$ where A^* is an anti-self-dual connection on X^* with boundary value 1, and A_B and A_C are the trivial connections on B_2 and C_2. This is done in a way similar to the proof of [9, Theorem 2.1], and the sign is easily seen to be '+'.   q.e.d.

The first step in the proof of Theorem 5.1 is to understand reducible connections over C_p. It will be convenient here to use the symmetric dual bases $\{v_i\}$ and $\{\delta_i\}$ of §2. Using these coordinates, we express elements of $H_2(C_p, \partial C_p; \mathbf{Z})$ as

$$\beta = \sum t_i \delta_i = \langle t_1, \ldots, t_{p-1} \rangle.$$

Classes of the form $\langle t, \ldots, t, s, \ldots, s \rangle$ will play a special role. We shall use the abbreviation

$$\langle t, \ldots, t, s, \ldots, s \rangle = \langle t, s; b \rangle$$

if the number of s's is $1 \leq b \leq p - 1$. If $e \in H_2(C_p, \partial C_p; \mathbf{Z})$, write $\mathcal{M}_e$ for the $SO(3)$-moduli space of anti-self-dual connections on C_p which contains the reducible connection in the bundle $L_e \oplus \mathbf{R}$ where $c_1(L_e) = e$, and which are asymptotically flat with boundary value $\partial' e \in \chi_{SO(3)}(L(p^2, 1 - p))$. Note that $\partial \langle t, t + 1; b \rangle = (p - 1)t + b$.

Lemma 5.3. *Let* $e = \langle t, t + 1; b \rangle$ *with* $0 \leq t \leq p$. *Then* $\dim \mathcal{M}_e = 2t - 1$.

Proof. With respect to the basis $\{\delta_i\}$, the intersection form of $H_2(C_p, \partial)$ is

$$(4) \qquad Q = -\frac{(p^2 - p - 1)}{p^2} \sum x_i^2 + 2\,\frac{p+1}{p^2} \sum_{i<j} x_i x_j$$

and

$$e^2 = (b(t+1)^2 + (p - b - 1)t^2)(-\frac{(p^2 - p - 1)}{p^2})$$
$$+ 2((p - b - 1)bt(t+1) + \binom{p - b - 1}{2}t^2 + \binom{b}{2}(t+1)^2)\frac{p+1}{p^2}.$$

Hence

$$(5) \qquad e^2 = \frac{1}{p^2}(b^2 + b^2 p - bp^2 - 2bt + t^2 - pt^2).$$

By hypothesis, $\partial e = (p - 1)t + b \neq 0$. From [25] we have

$$\frac{\rho}{2}(\partial e) = -\frac{1}{p^2}(-2b^2 - 2b^2 p - p^2 + 2bp^2 + 4bt - 2p^2 t - 2t^2 + 2pt^2)$$

and by the index theorem [1]:

$$\dim \mathcal{M}_e = -2e^2 - \frac{3}{2} - \frac{1}{2}(h + \rho)(\partial e) = -2e^2 - 2 - \frac{\rho}{2}(\partial e) = 2t - 1.$$

<div align="right">q.e.d.</div>

Lemma 5.4. *Let $e = \langle t, t+1; b \rangle$ with $t \geq 0$ and $(p - 1)t + b \leq p^2/2$. Suppose also that $e' = \langle \alpha_1, \ldots, \alpha_{p-1} \rangle$ with $\sum \alpha_i = (p - 1)t + b + rp^2$, $r \neq 0, -1$. Then $\dim \mathcal{M}_{e'} > \dim \mathcal{M}_e$.*

Proof. Using (4), it follows from symmetry that for fixed $s = \sum x_i$, the minimum absolute value of $Q\langle x_1, \ldots, x_{p-1} \rangle$ occurs at

$$\mu(s) = \langle s/(p+1), \ldots, s/(p+1) \rangle,$$

and

$$\mu(s)^2 = -\frac{(p^2 - p - 1)}{p^2}(p - 1)\frac{s^2}{(p - 1)^2}$$
$$+ 2\,\frac{p+1}{p^2}\binom{p - 1}{2}\frac{s^2}{(p - 1)^2} = \frac{s^2}{p^2 - p^3}.$$

On the other hand by (5), $e^2 = \frac{1}{p^2}(b^2 + b^2 p - bp^2 - 2bt + t^2 - pt^2)$.
    Set $s = (p-1)t + b + rp^2$. Then

$$\mu(s)^2 - e^2 = -\frac{1}{p-1}(b + b^2 + 2br + p^2 r^2 + 2rt(p-1) - bp).$$

Since $1 \le b \le p - 1$, we have $bp \le p^2 - p \le p^2 r^2$. So

$$\mu(s)^2 - e^2 \le -\frac{1}{p-1}(b + b^2 + 2br + 2rt(p-1)),$$

and if we assume $r \ge 1$, then $\mu(s)^2 < e^2(\; < 0)$. By the index theorem, we obtain

$$\begin{aligned}
\dim \mathcal{M}_{e'} &= -2e'^2 - \frac{3}{2} - \frac{1}{2}(h + \rho)(\partial e') \\
&\ge -2\mu(s)^2 - \frac{3}{2} - \frac{1}{2}(h + \rho)(\partial e) \\
&\ge \dim \mathcal{M}_e
\end{aligned}$$

since $(h + \rho)(\partial e') = (h + \rho)(\partial e)$. Notice that we have not yet used the hypothesis that $(p-1)t + b \le p^2/2$.
    If $r < -1$, set $\bar{e} = \langle t', t' + 1; c \rangle$ with t', c chosen such that

$$(p-1)t' + c = p^2 - ((p-1)t + b) \ge p^2/2.$$

By Lemma 5.3, $\dim \mathcal{M}_{\bar{e}} \ge \dim \mathcal{M}_e$ with equality only if $t' = t$. Note that $\dim \mathcal{M}_{-e'} = \dim \mathcal{M}_{e'}$, and $-\sum \alpha_i = (p-1)t' + c - (r+1)p^2$. Since $-(r+1) \ge 1$, the case we have already handled shows that $\dim \mathcal{M}_{-e'} \ge \dim \mathcal{M}_{\bar{e}}$.   q.e.d.

Lemma 5.5. *Let $e = \langle t, t + 1; b \rangle$ with $t \ge 0$. Suppose that $e' = \langle \alpha_1, \dots, \alpha_{p-1} \rangle \ne e$ but $\sum \alpha_i = (p-1)t + b$. Then $\dim \mathcal{M}_{e'} > \dim \mathcal{M}_e$ unless e' is a permutation of e.*

   Proof. It suffices to show that $e'^2 < e^2$. Write $e' = e + \nu$ where

$$\nu = \langle n_1, \dots, n_{p-b-1}, n_{p-b}, \dots, n_{p-1} \rangle.$$

Since the sum of the coordinates of e and e' is the same, $\sum n_i = 0$. Let

$$N_L = \sum_{i=1}^{p-b-1} n_i, \qquad N_R = \sum_{i=p-b}^{p-1} n_i.$$

$$e'^2 = e^2 + 2(N_L((p-2)t+b) + N_R((p-2)t+b-1))(\frac{p+1}{p^2})$$

$$- 2(N_L t + N_R(t+1))(\frac{p^2-p-1}{p^2}) + \nu^2$$

$$= e^2 - 2N_R + \nu^2$$

since $N_L + N_R = 0$. Hence

$$\frac{1}{2}(\dim \mathcal{M}_{e'} - \dim \mathcal{M}_e) = e^2 - e'^2 = -\nu^2 + 2N_R.$$

However, if y is the result of adding $+1$ to x_{i_0} and -1 to x_{i_1} in $x = \langle x_1, \ldots, x_{p-1} \rangle$, then $y^2 - x^2 = 2(x_{i_1} - x_{i_0} - 1)$. Starting with $x = \langle 0, \ldots, 0 \rangle$ and making these ± 1 moves with constant sign in each coordinate until reaching ν, we see that the minimum change in the square is -2. This is achieved only if each coordinate operated on is originally 0. Thus, if N_+ is the sum of the positive coordinates n_i, we have $-\nu^2 \geq 2N_+$. Equality occurs only if each n_i is ± 1 or 0. In this case there are N_+ such -1's. If $|N_R| < N_+$ then

$$-\nu^2 + 2N_R \geq 2(N_+ - |N_R|) > 0.$$

If $|N_R| = N_+$ then each -1 occurs in a coordinate n_i, $i = p-b, \ldots, p-1$, and so e' is a permutation of e. If $-\nu^2 > 2N_+$ then we have $-\nu^2 + 2N_R > 0$ since $|N_R| \leq N_+$.   q.e.d.

Proposition 5.6. *Let $e = \langle t, t+1; b \rangle$ with $t \geq 0$ and $(p-1)t + b \leq p^2/2$. If $e' = \langle \alpha_1, \ldots, \alpha_{p-1} \rangle$ with $e' \equiv e$ (mod 2) and $\dim \mathcal{M}_{e'} \leq \dim \mathcal{M}_e$, then $\partial e' \leq \partial e$ as elements of $\mathbf{Z}_{p^2}$.*

Proof. Let $\bar{e} = \langle s, s+1; c \rangle$ with $s \geq 0$ be the unique class of this form with $0 \leq \partial \bar{e} \leq p^2/2$ satisfying $\partial e' = \partial \bar{e}$. Lemmas 5.4 and 5.5 imply that unless $-p^2/2 \leq \sum a_i < 0$, we have $\dim \mathcal{M}_{\bar{e}} \leq \dim \mathcal{M}_{e'}$; so $s \leq t$. This holds in any case, since we can always work with $-e'$. If $s = t$ then $\bar{e} = e$ since no class $\langle t, t+1; b' \rangle$ with $b' \neq b$ is congruent to e (mod 2). This means that $\partial e' \leq \partial e$.   q.e.d.

Corollary 5.7. *Let $e = \langle t, t+1; b \rangle$ with $t \geq 0$ and $(p-1)t+b \leq p^2/2$. Suppose that $e' = \langle \alpha_1, \ldots, \alpha_{p-1} \rangle$ with $\partial'e' = \partial'e \in \chi_{SO(3)}(L(p^2, 1-p))$ and $e' \equiv e$ (mod 2). Then $\dim \mathcal{M}_{e'} = \dim \mathcal{M}_e + 4k$, $k \geq 0$.*

Proof. As above, $\dim \mathcal{M}_e \leq \dim \mathcal{M}_{e'}$. But $e' \equiv e$ (mod 2) implies that $e'^2 = e^2$ (mod 4); so the corollary follows from the index theorem.

q.e.d.

We need one more simple fact. Let $\iota : (C_p, \emptyset) \to (C_p, \partial)$ be the inclusion.

Lemma 5.8. *Let* $e \in H_2(C_p, \partial; \mathbf{Z})$, *and suppose* $\partial e \equiv 0$ *(mod 2) in case* p *is even. Then there is a* $c \in H_2(C_p; \mathbf{Z})$ *such that* $\iota_*(c) \equiv e$ *(mod 2).*

Proof. This follows directly from the exact sequence

$$0 \to H_2(C_p; \mathbf{Z}) \to H_2(C_p, \partial; \mathbf{Z}) \to \mathbf{Z}_{p^2} \to 0 . \quad q.e.d.$$

We now proceed toward the proof of Theorem 5.1. We shall work always with structure group $SO(3)$ and identify $SU(2)$ connections with $SO(3)$ connections on $w_2 = 0$ bundles. We wish to calculate $D_{X_p}(z)$ for $z \in \mathbf{A}(X^*)$. The basic outline of our argument is the same as the proof of Lemma 5.2. The only extra bit of information we need to point out is that of the characters $\eta^j \in \chi_{SO(3)}(L(p^2, 1-p))$, only the p-powers, η^{np} extend over the rational ball B_p. Furthermore, since the only reducible connections on B_p are flat, we get in the same fashion as (3):

$$(6) \qquad D_{X_p}(z) = \sum_{n=0}^{[p/2]} \pm D_{X^*}[\eta^{np}](z).$$

The notation $D_{X^*}[\eta^{np}]$ stands for the relative Donaldson invariant on X^* constructed from the moduli space of anti-self-dual connections over X^* (with a cylindrical end) which decay exponentially to a flat connection whose gauge equivalence class corresponds to the conjugacy class of the representation η^{np}.

To calculate the terms $D_{X^*}[\eta^{np}](z)$, we must utilize techniques of Taubes [36], [37], [38], [39] or Wieczorek [41] as in [10, §4]. Consider the calculation of $D_X(zy)$, where $z \in \mathbf{A}(X^*)$ has degree d, and $y \in \mathbf{A}(C_p)$ has degree t. Following the proof of Lemma 5.2, a neck-stretching argument shows that any connection in $\mathcal{M}_X(P) \cap V_z \cap V_y$ is obtained by gluing together a connection A^* on X^* with a reducible connection A_C on C_p. Now let L_ϵ denote the $SO(2)$ bundle over C_p with Euler class ϵ, and let $\tilde{\mathcal{M}}_{C_p, \epsilon, \ell}$ be the based moduli space of exponentially decaying asymptotically flat anti-self-dual connections on the $SO(3)$ bundle $E_{\epsilon, \ell}$ which is obtained from the reducible bundle $L_\epsilon \oplus \mathbf{R}$ by

grafting in ℓ instanton bundles. By letting $\partial \epsilon = j$, there are $SO(3)$-equivariant boundary value maps

$$\partial_{C_p,\epsilon,\ell} : \tilde{\mathcal{M}}_{C_p,\epsilon,\ell} \to G[j], \qquad \partial_{X^*}[j] : \tilde{\mathcal{M}}_{X^*}[\eta^j] \to G[j],$$

where $G[j] \subset SO(3)$ is the conjugacy class η^j of representations of $\pi_1(L(p^2, 1-p))$ to $SO(3)$. If $j \neq 0, p^2/2$, then $G[j]$ is a 2-sphere, $G[0] = \{I\}$, and, in case p is even, $G[p^2/2] \cong \mathbf{RP}^2$. We thus have the fibered product

$$(7) \qquad \tilde{\mathcal{M}}_{X^*}[\eta^j] \times_j \tilde{\mathcal{M}}_{C_p,\epsilon,\ell}$$

constructed by means of the boundary value maps $\partial_{C_p,\epsilon,\ell}$ and $\partial_{X^*}[j]$. Let τ denote a 3-form which integrates to 1 over the fibers of the basepoint fibration $\beta_{X^*,j} : \tilde{\mathcal{M}}_{X^*}[\eta^j] \to \mathcal{M}_{X^*}[\eta^j]$. The principal $SO(3)$ bundle $\beta_{X^*,j}$ has a reduction to a bundle with structure group S^1. As in [10, §4], we let $\varepsilon \in H^2(\mathcal{M}_{X^*}[\eta^j])$ denote the Euler class of this S^1 bundle.

Taubes' theory implies that $D_X(zy)$ is a sum of terms of the form

$$\int_{\tilde{\mathcal{M}}_{X^*}[\eta^j] \times_j \tilde{\mathcal{M}}_{C_p,\epsilon,\ell}} \tau \wedge \tilde{\mu}(z) \wedge \tilde{\mu}(y)$$

where $\tilde{\mu}(z)$ and $\tilde{\mu}(y)$ come from lifts of $\mu(z)$ and $\mu(y)$ to the based moduli spaces. The form $\tilde{\mu}(y)$ is supported near the orbit of the reducible connection corresponding to ϵ. (If $\ell > 0$, this reducible connection lies in the Uhlenbeck compactification of $\mathcal{M}_{C_p,\epsilon,\ell}$.) In the terms of (7), the Euler classes ϵ must be congruent to $w_2(P|_{C_p})$ mod 2, and $\dim \mathcal{M}_{C_p,\epsilon,\ell} = \dim \mathcal{M}_{C_p,\epsilon,0} + 8\ell \leq 2t - 1$.

Taubes has shown further that there is a form $\tilde{\mu}(y)$ representing a class $\mu_{SO(3)}(y)$ in the $SO(3)$-equivariant cohomology of an enlargement of $\tilde{\mathcal{M}}_{C_p,\epsilon,\ell}$. The lift $\tilde{\mu}(z)$ defines an element of the equivariant cohomology $H^{2d}_{SO(3)}(\tilde{\mathcal{M}}_{X^*}[\eta^j])$, and furthermore, the push-forward $(\partial_{C_p,\epsilon,\ell})_*$ is well-defined, and

$$
\begin{aligned}
(8) \qquad & \int_{\tilde{\mathcal{M}}_{X^*}[\eta^j] \times_j \tilde{\mathcal{M}}_{C_p,\epsilon,\ell}} \tau \wedge \tilde{\mu}(z) \wedge \tilde{\mu}(y) \\
& = \int_{\tilde{\mathcal{M}}_{X^*}[\eta^j]} \tau \wedge \tilde{\mu}(z) \wedge (\partial_{X^*}[j])^*(\partial_{C_p,\epsilon,\ell})_*(\tilde{\mu}(y)),
\end{aligned}
$$

where $(\partial_{X^*}[j])^*$ denotes the pullback in equivariant cohomology.

The pushforward defines a cohomology class

$$[(\partial_{C_p,\epsilon,\ell})_*(\tilde{\mu}(y))] \in H^{2t-f}_{SO(3)}(G[j]; \mathbf{R}),$$

where $f = f(p, \epsilon, \ell)$ is the fiber dimension of $\partial_{C_{p,\epsilon,\ell}}$. For $j = 0$, $f(p, \epsilon, \ell) = \dim \tilde{\mathcal{M}}_{C_{p,\epsilon,\ell}} = -2\epsilon^2 + 8\ell$. Thus

$$[(\partial_{C_{p,\epsilon,\ell}})_*(\tilde{\mu}(y))] \in H_{SO(3)}^{2t-f}(\{1\}; \mathbf{R}) = H^{2t-f}(BSO(3); \mathbf{R}),$$

and $H^*(BSO(3); \mathbf{R})$ is a polynomial algebra on the 4-dimensional class $\wp$, which pulls back over $\tilde{\mathcal{M}}_{X^*}[1]$ as $p_1(\beta_{X^*,0})$. For $j \neq 0, p^2/2$,

$$H_{SO(3)}^{2t-f}(G[j]; \mathbf{R}) = H_{SO(3)}^{2t-f}(S^2; \mathbf{R}) = H^{2t-f}(\mathbf{CP}^\infty; \mathbf{R}).$$

Let v be the 2-dimensional generator of $H^*(\mathbf{CP}^\infty; \mathbf{R})$. The pullback $(\partial_{X^*}[j])^*(v) = \varepsilon$, and $\varepsilon^2 = p_1(\beta_{X^*,j})$.

The simplest application of this technology is the calculation of the term $D_{X^*}[\eta^0](z) = D_{X^*}[1](z)$ in (6). Consider $D_X(z)$. The standard counting argument shows that the terms (8) that contribute (with $y = 1$) must have $\dim(\mathcal{M}_{C_{p,\epsilon,\ell}}) < 0$. Furthermore, since $w_2(P|C_p) = 0$, any such ϵ must have the form $\epsilon = 2\zeta$. It then follows from Lemma 5.5 that $\dim(\mathcal{M}_{C_{p,2\zeta,\ell}}) > 0$. This means that the only contribution is from $j = 0$. We must have $\epsilon = 0$, $\ell = 0$, and $f(p, \epsilon, \ell) = 0$. Thus $(\partial_{C_{p,\epsilon,0}})_*(\tilde{\mu}(1)) = \pm 1 \in H^0(BSO(3); \mathbf{R})$. By integrating out τ in (7), we get $D_X(z) = \pm D_{X^*}[1](z)$, and the sign is independent of X.

Now our plan is to evaluate all the $D_{X^*}[\eta^m](z)$ inductively. (In case p is even, we only need to calculate this for m even.) First, for each positive integer j, define t_j and b_j by $j = (p-1)t_j + b_j$. We evaluate $D_{X^*}[\eta^m](z)$ by computing $D_{X,c_m}(z\,w_m)$ where $c_m \in H_2(X; \mathbf{Z})$ is supported in C_p and $w_m \in \text{Sym}_{t_m}(H_2(C_p; \mathbf{Z}))$ depending only on m and p. First we obtain c_m. Let $e_m \in H^2(C_p; \mathbf{Z})$ be the Poincaré dual of $\langle t_m, t_m + 1; b_m \rangle$. By Lemma 5.8 we can find $c_m \in H_2(C_p; \mathbf{Z}) \subset H_2(X; \mathbf{Z})$ such that $\iota_*(c_m) \equiv \langle t_m, t_m + 1; b_m \rangle \pmod 2$. Thus the Poincaré dual of c_m in $H^2(X; \mathbf{Z})$ restricts to C_p congruent to $e_m \pmod 2$ and restricts trivially to X^*.

To calculate $D_{X,c_m}(z\,w_m)$ we must first consider the $2t_m$-form $\tilde{\mu}(w_m)$ on moduli spaces $\tilde{\mathcal{M}}_{C_{p,\epsilon,\ell}}$ where $\epsilon \equiv e_m \pmod 2$ and $\dim \mathcal{M}_{C_{p,\epsilon,\ell}} = \dim \mathcal{M}_{C_{p,\epsilon,0}} + 8\ell \leq 2t_m - 1$. Lemmas 5.3 and 5.5 imply such an $\epsilon = e_j + 2v$ where $j \leq m$ and $v \in H^2(C_p, \partial C_p; \mathbf{Z})$ (so that $\partial \epsilon = j$). We then have $\dim \tilde{\mathcal{M}}_{C_{p,\epsilon,\ell}} = 2t_j + 2 + 8\ell - 4v^2 = 2t_j + 2 + 8\ell + 4k$, where $\ell \geq 0$, $k \geq 0$. If $j \neq 0$ the fiber dimension of $\partial_{C_{p,\epsilon,\ell}}$ is $f(p, \epsilon, \ell) = 2t_j + 8\ell + 4k$; thus $(\partial_{C_{p,\epsilon,\ell}})_*\tilde{\mu}(w_m) \in H^{2(t_m-t_j)-8\ell-4k}(\mathbf{CP}^\infty; \mathbf{R})$. Hence, if $j \neq 0$ then $(\partial_{C_{p,\epsilon,\ell}})_*\tilde{\mu}(w_m)$ is a multiple of $v^{t_m-t_j-4\ell-2k}$. Similarly, for $j = 0$, $\partial_{C_{p,\epsilon,\ell}}$, has fiber dimension equal to $\dim \tilde{\mathcal{M}}_{C_{p,\epsilon,\ell}} = 4k + 8\ell$ for some $k \geq 0$, and

$(\partial_{C_p, \epsilon, \ell})_* \tilde{\mu}(w_m)$ is a multiple of $\wp^{(t_m/2) - 2k - 4\ell}$. (Therefore t_m must be even for such contributions.) Now $(\partial_{X*}[j])^*(v) = \varepsilon$ where $\varepsilon^2 = p_1(\beta_{X*,j})$ and $(\partial_{X*}[0])^*(\wp) = p_1(\beta_{X*,0})$. However, $\tilde{\mu}(x) = -\frac{1}{4} p_1(\beta_{X*,j})$; so it follows from (7) that

$$
(9) \quad
\begin{aligned}
D_{X, c_m}(z\, w_m) = {} & \sum_{\substack{\{j \mid t_j \equiv t\,(2)\} \\ 0 \le j \le m}} \sum_q r_{m,j,q} D_{X*}[\eta^j](z x^q) \\
& + \sum_{\substack{\{j \mid t_j \not\equiv t\,(2)\} \\ 1 \le j \le m}} \sum_q r'_{m,j,q} D_{X*}[\eta^j](z x^q \varepsilon).
\end{aligned}
$$

The notation $D_{X*}[\eta^j](z x^q \varepsilon)$ is not standard, but its meaning is clear. We emphasize that in order to obtain $r_{m,j,q}$ or $r'_{m,j,q} \ne 0$ we must have an $\epsilon \in H^2(C_p; \mathbf{Z})$ satisfying $\partial' \epsilon = j$, $\epsilon \equiv e_m \pmod 2$, and

$$
\dim \mathcal{M}_{C_p, \epsilon, 0} + 8q \le 2t - 1.
$$

Assume inductively that:

a) For each $j < m$ ($j \equiv 0 \pmod 2$ if p is even) there are classes $w_{j,i} \in \mathrm{Sym}_*(H_2(C_p; \mathbf{Z}))$ and rational numbers $a_{j,i}$ satisfying

$$
(10) \qquad D_{X*}[\eta^j](z) = \sum_{i=1}^{j} a_{j,i} D_{X, c_i}(z w_{j,i}).
$$

b) For each j with $t_j < t_m - 1$ (and $j \equiv 0 \pmod 2$ if p is even) there are classes $w'_{j,i} \in \mathrm{Sym}_*(H_2(C_p; \mathbf{Z}))$ and rational numbers $a'_{j,i}$ satisfying

$$
(11) \qquad D_{X*}[\eta^j](z \varepsilon) = \sum_{i=1}^{j} a'_{j,i} D_{X, c_i}(z w'_{j,i})
$$

for all $z \in \mathbf{A}(X^*)$, and the coefficients $a_{j,i}$ and $a'_{j,i}$ are independent of z and X.

Before defining w_m we first recall that m is even if p is even. Now set

$$
w_m = (u_{p-1} - (t_m - 1) u_{p-1-b}) \cdot (u_{p-1})^{t_m - 1} \in \mathbf{A}(C_p).
$$

We wish to calculate $D_{X,c_m}(z\,w_m)$ using (9). For $j = m$ in this formula, we need to compute $(\partial_{C_p,e_m,0})_*(\tilde{\mu}(w_m)) \in H^0_{SO(3)}(G[m];\mathbf{R}) = \mathbf{R}$. In fact,

$$
\begin{aligned}
(\partial_{C_p,e_m,0})_*&(\tilde{\mu}(w_m)) \\
&= r_{m,m,0} \\
&= -\frac{1}{2}\langle u_{p-1} - (t_m - 1)u_{p-1-b}, e_m \rangle \, (-\frac{1}{2}\langle u_{p-1}, e_m \rangle)^{t_m-1} \\
&= (-\frac{1}{2})^{t_m}(2t_m)(t_m+1)^{t_m-1} \neq 0,
\end{aligned}
$$

(cf. [6, p.187]). In (9), $r_{m,m,0}D_{X*}[\eta^m](z)$ is the only term which involves the boundary value η^m.

Suppose that j is the boundary value of an ϵ with $\epsilon \equiv e_m \pmod 2$, and $\dim \mathcal{M}_{C_p,\epsilon,0} + 8q \leq 2t_m - 1$. If $t_j = t_m - 1$, then by Corollary 5.7 and Lemma 5.5, $\dim \mathcal{M}_{C_p,\epsilon,0} = 2(t_m - 1) + 1 + 4k$; so $k = 0$, $q = 0$ and ϵ is a permutation of $\langle t_m - 1, t_m; b \rangle = \langle t_m - 1, \ldots, t_m - 1, t_m, \ldots, t_m \rangle$ which is congruent mod 2 to

$$
e_m = \langle t_m, t_m + 1; b_m \rangle = \langle t_m, \ldots, t_m, t_m + 1, \ldots, t_m + 1 \rangle.
$$

Thus $b = p - 1 - b_m$. This means that $j = (t_m - 1)(p - 1) + (p - 1 - b_m)$, and $\epsilon = \langle t_m, t_m - 1; b_m \rangle = (t_m - 1)\gamma_{p-1} + \gamma_{p-1-b_m}$, so that

$$
\langle u_{p-1} - (t_m - 1)u_{p-1-b}, \epsilon \rangle = 0.
$$

Hence, no such j occurs in the second sum of the expansion (9) for $D_{X,c_m}(z\,w_m)$. (I.e., for such j, necessarily $q = 0$ and $r'_{m,j,q} = 0$.) If p is even, then we are assuming that m is also even, and if $r_{m,i,q}$ or $r'_{m,i,q} \neq 0$ then as above there is an ϵ with $\partial\epsilon = i$ and $\epsilon \equiv e_m \pmod 2$; so for

$$
\partial_2 : H_2(C_p, \partial; \mathbf{Z}_2) \to H_1(L(p^2, 1 - p); \mathbf{Z}_2) = \mathbf{Z}_2
$$

$j \equiv \partial_2(\epsilon) \equiv \partial_2 e_m \equiv m \pmod 2$. Accordingly, all the other terms in (9) are given inductively by (10) and (11), and the powers of x can be removed using the hypothesis that X has simple type. Since the coefficient of $D_{X*}[\eta^m](z)$ is nonzero, we may solve for it, completing the induction step for (10).

For (11), we show how to compute

$$
D_{X*}[\eta^{m'}](z\varepsilon) \text{ for } m' = (t_m - 1)(p - 1) + (p - 1 - b)
$$

as required. Thus after completing the inductive step for each $t_m(p-1) + c$, $1 \leq c \leq p-1$, we will have completed the calculation of $D_{X*}[\eta^j](z\varepsilon)$ for all $j = (t_m - 1)p + a$, $1 \leq a \leq p - 1$. So to calculate $D_{X*}[\eta^{m'}](z\varepsilon)$ and therefore complete the induction, we calculate $D_{X,c_m}(z\,w'_{m'})$ where

$$w'_{m'} = (u_{p-1} + (t_m + 1)u_{p-1-b}) \cdot (u_{p-1} + (t_m - 1)u_{p-1-b}) \cdot (u_{p-1})^{t_m-2}.$$

Using (9), we obtain

$$
(12) \qquad
\begin{aligned}
D_{X,c_m}(z\,w'_{m'}) &= \sum_{\substack{\{j|t_j \equiv t\,(2)\} \\ 0 \leq j \leq m}} \sum_q s_{m',j,q} D_{X*}[\eta^j](zx^q) \\
&+ \sum_{\substack{\{j|t_j \not\equiv t\,(2)\} \\ 1 \leq j \leq m}} \sum_q s'_{m',j,q} D_{X*}[\eta^j](zx^q\varepsilon).
\end{aligned}
$$

Computing as above, we see that $s_{m',m',0} = 0$. What we need to see is that $s'_{m',m',0} \neq 0$. By the argument of the above paragraph, m' is the only possible boundary value not covered by the induction step. Let $\epsilon = (t_m - 1)\gamma_{p-1} + \gamma_{p-1-b}$. This is the only Euler class that can give boundary value m' in (12). Then

$$(\partial_{C_p,\epsilon,0})_*(\tilde{\mu}(w'_m)) \in H^2_{SO(3)}(S^2; \mathbf{R}) \cong H^2_{SO(3)}(G[m']; \mathbf{R}) = \mathbf{R}$$

and $(\partial_{C_p,\epsilon,0})_*(\tilde{\mu}(w'_m)) = (-\frac{1}{2})^{t_m-1} 2t_m (2t_m - 2)(t_m - 1)^{t_m-2} v$ which pulls back over $\tilde{\mathcal{M}}_{X*}[\eta^j]$ as $(-\frac{1}{2})^{t_m-3} t(t_m - 1)(t_m - 1)^{t_m-2}\varepsilon$. This means that we can solve (12) for $D_{X*}[\eta^{m'}](z\varepsilon)$, completing the induction and the proof of Theorem 5.1.

The argument above shows that all of the relative invariants $D_{X*}[\eta^{np}]$ can be expressed in terms of absolute invariants of X. Since we are assuming that X has simple type, it follows that each of the relative invariants satisfies the formula

$$D_{X*}[\eta^{np}](z\,x^2) = 4\,D_{X*}[\eta^{np}](z).$$

Hence from (6) we have:

Corollary 5.9. *Let X_p be the result of rationally blowing down $C_p \subset X$. If X has simple type, then so does X_p.*

Now we shall make stronger use of the hypothesis that X has simple

type. By [23], [10] we can write

$$
\begin{aligned}
\mathbf{D}_X &= \exp(Q_X/2) \sum_{s=1}^{n} a_s e^{\kappa_s}, \\
\mathbf{D}_{X,c} &= \exp(Q_X/2) \sum_{s=1}^{n} (-1)^{\frac{1}{2}(c^2 + c \cdot \kappa_s)} a_s e^{\kappa_s}
\end{aligned}
$$

for nonzero rational numbers a_s and basic classes $\kappa_1, \ldots, \kappa_n \in H_2(X; \mathbf{Z})$. Here Q_X is the intersection form of X. Now

$$
\partial_u (\exp(Q_X/2) e^{\kappa}) = \exp(Q_X/2)(\tilde{u} + \kappa \cdot u) e^{\kappa},
$$

where $\tilde{u} : H_2(X) \to \mathbf{R}$ is $\tilde{u}(\alpha) = u \cdot \alpha$ and $\partial_v \tilde{u} = v \cdot u$. Apply Theorem 5.1: since all derivatives are taken with respect to classes $u \in H_2(C_p; \mathbf{Z})$, after all derivatives are taken, the remaining $\tilde{u}$'s restricted to X^* vanish. Hence,

(13)
$$
\begin{aligned}
\mathbf{D}_{X_p}|_{X^*} &= \exp(Q_{X^*}/2) \sum_{s=1}^{n} a_s b_s e^{\kappa_s}|_{X^*} \\
&= \exp(Q_{X^*}/2) \sum_{s=1}^{n} a_s b_s e^{\kappa'_s}.
\end{aligned}
$$

Here $\kappa'_s = \kappa_s|_{X^*} = \mathrm{PD}(i^*(\mathrm{PD}(\kappa_s))) \in H_2(X^*, \partial; \mathbf{Z})$, where PD denotes Poincaré duality, i is the inclusion $X^* \subset X$, and b_s depends only on the intersection numbers of κ_s with the generators u_i of $H_2(C_p; \mathbf{Z})$.

Lemma 5.10. *If $b_s \neq 0$ in* (13), *then*

$$
\partial \kappa'_s \in p\mathbf{Z}_{p^2} \subset H_1(L(p^2, 1 - p); \mathbf{Z}) = \mathbf{Z}_{p^2}.
$$

Proof. Corollary 5.9 implies that X_p has simple type. We thus have

(14)
$$
\mathbf{D}_{X_p} = \exp(Q_{X_p}/2) \sum_{r=1}^{m} c_r e^{\lambda_r},
$$

where the basic classes of X_p are $\lambda_1, \ldots, \lambda_m$. Restrict $\mathbf{D}_{X_p}$ to X^* and compare the restrictions of $\exp(Q_{X_p}/2)^{-1} \mathbf{D}_{X_p}$ in (13) and (14). Since for distinct $\alpha \in H_2(X^*, \partial; \mathbf{Z})$ the functions $e^\alpha : H_2(X^*) \to \mathbf{R}$ are linearly independent, it follows that if $b_s \neq 0$, then $\kappa'_s = \lambda_i|_{X^*}$ for some i. Thus κ'_s extends over B_p, and hence $\partial \kappa'_s \in p\mathbf{Z}_{p^2}$.   q.e.d.

As a result, we have the following restatement of Theorem 5.1.

Theorem 5.11. *Suppose that X has simple type and*

$$\mathbf{D}_X = \exp(Q_X/2) \sum_{s=1}^{n} a_s e^{\kappa_s}.$$

Let $C_p \subset X$ and let X_p be its rational blowdown. Let $\{\kappa_t | t = 1, \ldots, m\}$ be the basic classes of X which satisfy $\partial \kappa_t' \in p\mathbf{Z}_{p^2}$, and for each t, let $\bar{\kappa}_t$ be the unique extension of κ_t'. Then

$$\mathbf{D}_{X_p} = \exp(Q_{X_p}/2) \sum_{t=1}^{m} a_t b_t e^{\bar{\kappa}_t},$$

where b_t depend only on the intersection numbers $u_i \cdot \kappa_t$, $i = 1, \ldots, p-1$.

6. The Donaldson invariant of elliptic surfaces

In this section we shall compute the effect on the Donaldson series of performing log transforms. The Donaldson invariants of the elliptic surfaces $E(n), n \geq 2$, without multiple fibers have been known for some time. There is a complete calculation in [23] and [10], for example. For $n \geq 2$:

$$\mathbf{D}_{E(n)} = \exp(Q/2) \sinh^{n-2}(f),$$

where f is the class of a fiber. In this notation, the $K3$ surface is $E(2)$. As in Theorem 3.1, let $X = E(2) \# (p-1)\overline{\mathbf{CP}}^2$, and let X_p be the rational blowdown of $C_p \subset X$, so that $X_p \cong E(2; p)$. Since $\mathbf{D}_{E(2)} = \exp(Q/2)$, the blowup formula [9] yields

$$(15) \qquad \mathbf{D}_X = \frac{1}{2^{p-1}} \exp(Q/2) \sum_J \exp(\sum_{i=1}^{p-1} \epsilon_{J,i} e_i),$$

where the outer sum is taken over all $J = (\epsilon_{J,1}, \ldots, \epsilon_{J,p-1}) \in \{\pm 1\}^{p-1}$. The basic classes of X are $\{\kappa_J = \sum \epsilon_{J,i} e_i\}$, and applying Theorem 5.11 we get

$$(16) \qquad \mathbf{D}_{X_p} = \frac{1}{2^{p-1}} \exp(Q_{X_p}/2) \sum_J b_J e^{\bar{\kappa}_J},$$

where $\bar{\kappa}_J \in H_2(X_p; \mathbf{Z})$ is the unique extension of $\kappa_J|_{X*}$. Recall that the spheres of the configuration C_p represent homology classes $u_i = e_{p-(i+1)} - e_{p-i}$ for $1 \leq i \leq p - 2$, and $u_{p-1} = f - 2e_1 - e_2 - \cdots - e_{p-1}$. In X_p we have the multiple fiber $f_p = f/p$.

Proposition 6.1. $\bar\kappa_J = |J| \cdot f_p$ *where* $|J| = \sum_{i=1}^{p-1} \epsilon_{J,i}$.

Proof. First we find a class $\zeta \in H_2(C_p; \mathbf{Q})$ so that $(\kappa_J + \zeta) \cdot u_i = 0$ for each i. This means that $\kappa_J + \zeta \in H_2(X^*; \mathbf{Q})$, and as dual forms: $H_2(X^*; \mathbf{Z}) \to \mathbf{Z}$, $\kappa_J|_{X^*} = \kappa_J + \zeta$. To find ζ we need to solve the linear system

$$\left(\kappa_J + \sum x_i u_i\right) \cdot u_j = 0, \quad j = 1, \ldots, p-1.$$

We begin by rewriting these equations. Let $\{\omega_i\}$ be a standard basis for $\mathbf{Q}^{p-1}$, and let A be the $(p-1) \times (p-1)$ matrix whose ith row vector is

$$
\begin{aligned}
A_i &= \omega_{p-(i+1)} - \omega_{p-i}, \quad i = 1, \ldots, p-2, \\
A_{p-1} &= -2\omega_1 - \omega_2 - \cdots - \omega_{p-1}.
\end{aligned}
$$

We have $u_i = A^t(\omega_i) \cdot \mathbf{e}$ and $u_{p-1} = f + A^t(\omega_{p-1}) \cdot \mathbf{e}$, where $\mathbf{e} = (e_1, \ldots, e_{p-1})$. Our linear system is equivalent to

$$P\mathbf{x} = A\epsilon_J,$$

where $\mathbf{x} = (x_1, \ldots, x_{p-1})$ and $\boldsymbol\epsilon_J = (\epsilon_{J,1}, \ldots, \epsilon_{J,p-1})$. (The matrix P is the plumbing matrix for C_p.) Hence $\mathbf{x} = P^{-1} A \epsilon_J$.

We claim that $P(A^t)^{-1} = -A$. This can be checked on the basis

$$\{\omega_2 - \omega_1, \ldots, \omega_{p-1} - \omega_{p-2}, \omega_{p-1}\}$$

using

$$
\begin{aligned}
A(\omega_i) &= -\omega_{p-1} - \omega_{p-(i+1)} + \omega_{p-i}, \ 2 \le i \le p-1 \ \ (\omega_0 = 0), \\
A(\omega_1) &= -2\omega_{p-1} + \omega_{p-2}, \\
P(\omega_i) &= \omega_{i+1} - 2\omega_i + \omega_{i-1}, \ i \ne p-1, \\
P(\omega_{p-1}) &= -(p+2)\omega_{p-1} + \omega_{p-2}.
\end{aligned}
$$

It follows that $A^t P^{-1} A = -I$. Thus

$$
\begin{aligned}
\kappa_J + \zeta &= \kappa_J + \sum x_i u_i = (\epsilon_J + A^t \mathbf{x}) \cdot \mathbf{e} + x_{p-1} f \\
&= (\epsilon_J - \epsilon_J) \cdot \mathbf{e} + x_{p-1} f = x_{p-1} f.
\end{aligned}
$$

To compute x_{p-1} note that

$$A\epsilon_J = (\epsilon_{J,p-2} - \epsilon_{J,p-1}, \epsilon_{J,p-3} - \epsilon_{J,p-2}, \ldots, \epsilon_{J,1} - \epsilon_{J,2}, -2\epsilon_{J,1} - \epsilon_{J,2} - \cdots - \epsilon_{J,p-1})$$

so that if $(P^{-1})_{p-1}$ denotes the bottom row of P^{-1}, then

$$
\begin{aligned}
x_{p-1} &= (P^{-1})_{p-1}(A\epsilon_J) \\
&= -\frac{1}{p^2}(1, 2, \ldots, p-1) \cdot (A\epsilon_J) \\
&= \frac{1}{p}\sum \epsilon_{J,i} = \frac{1}{p}|J|.
\end{aligned}
$$

Thus $\kappa_J|_{X^*} = \kappa_J + \zeta = \frac{1}{p}|J|f$ as forms: $H_2(X^*; \mathbf{Z}) \to \mathbf{Z}$. The homology class $\kappa_J + \zeta$ is in fact an integral class $\bar{\kappa}_J = |J|f_p \in H_2(X_p; \mathbf{Z})$ which is the unique extension of $\kappa_J|_{X^*}$.   q.e.d.

In an arbitrary smooth 4-manifold X, define a **nodal fiber** to be an immersed 2-sphere S with one singularity, a positive double point, such that the regular neighborhood of S is diffeomorphic to the regular neighborhood of a nodal fiber in an elliptic surface. (There need not be any associated ambient fibration of X.) Given such a nodal fiber S, one can perform a 'log transform' of multiplicity p by blowing up to get $C_p \subset X \#(p-1)\overline{\mathbf{CP}}^2$ with $u_{p-1} = S - 2e_1 - e_2 - \cdots - e_{p-1}$, and then blowing down C_p. (See Theorem 3.1.) We denote the result of this process by X_p.

Throughout, we use the following notation. If X has simple type, and

$$
\mathbf{D}_X = \exp(Q/2)\sum a_s e^{\kappa_s},
$$

then we write $\mathbf{K}_X = \sum a_s e^{\kappa_s}$.

Proposition 6.2. *Let S be a nodal fiber which satisfies $S \cdot \lambda_j = 0$ for each basic class λ_j of X. Then*

$$
\mathbf{D}_{X_p} = \begin{cases}
\exp(Q_{X_p}/2)\mathbf{K}_X \cdot (b_{p,0} + \sum\limits_{i=1}^{\frac{p-1}{2}} b_{p,2i}(e^{2iS/p} + e^{-2iS/p})), & p \text{ odd}, \\[3mm]
\exp(Q_{X_p}/2)\mathbf{K}_X \cdot (\sum\limits_{i=1}^{\frac{p}{2}} b_{p,2i-1}(e^{(2i-1)S/p} + e^{-(2i-1)S/p})), & p \text{ even},
\end{cases}
$$

where the coefficients $b_{p,j}$ depend only on p, not on X.

Proof. The Donaldson series of $X \#(p-1)\overline{\mathbf{CP}}^2$ is

$$
\frac{1}{2^{p-1}}\mathbf{D}_X \cdot \exp(Q_{(p-1)\overline{\mathbf{CP}}^2}/2)\sum_J \exp(\sum_{i=1}^{p-1} \epsilon_{J,i}e_i).
$$

Theorem 5.1 states that $\mathbf{D}_{X_p}$ is obtained from this by applying a differential operator which by hypothesis evaluates trivially on $\mathbf{D}_X$. The proposition now follows from (16) and Propostion 6.1 by the Leibniz rule. (That the coefficients of e^{mp} and e^{-mp} are equal follows from the fact that $\mathbf{D}_{E(2;p)}$ is an even function.)   q.e.d.

Proposition 6.3. *The Donaldson series of the simply connected elliptic surface $E(n;2)$ with $p_g = n - 1$ (> 0) and one multiple fiber of multiplicity 2 is*

$$\mathbf{D}_{E(n;2)} = \exp(Q/2)\frac{\sinh^{n-1}(f)}{\sinh(f_2)}.$$

Proof. According to Theorem 3.1, we obtain $E(n;2)$ from $E(n)\#\overline{\mathbf{CP}}^2$ by blowing down the sphere of square -4 representing $f - 2e$. We have $\mathbf{D}_{E(n)\#\overline{\mathbf{CP}}^2} = \exp(Q/2)\sinh^{n-2}(f)\cosh(e)$. Lemma 5.2 gives

$$\begin{aligned}
\mathbf{D}_{E(n;2)}\big|_{X^*} &= (\mathbf{D}_{E(n)\#\overline{\mathbf{CP}}^2} - \mathbf{D}_{E(n)\#\overline{\mathbf{CP}}^2, f-2e})\big|_{X^*} \\
&= 2\exp(Q/2)\sinh^{n-2}(f)\cosh(e)\big|_{X^*},
\end{aligned}$$

(cf.[24], [10, Thm.5.13]). By Proposition 6.1, we obtain

$$\mathbf{D}_{E(n;2)} = 2\exp(Q/2)\sinh^{n-2}(f)\cosh(f_2) = \exp(Q/2)\frac{\sinh^{n-1}(f)}{\sinh(f_2)}.$$

<div align="right">q.e.d.</div>

Proposition 6.2 now implies:

Corollary 6.4. *If S is a nodal fiber in X orthogonal to all basic classes, and X_2 is the multiplicity 2 log transform of X formed from S, then*

$$\mathbf{D}_{X_2} = \exp(Q_{X_2}/2)\mathbf{K}_X \cdot (e^{S/2} + e^{-S/2}).$$

Lemma 6.5. *Let X contain a nodal fiber S orthogonal to all basic classes. Then the sum of the coefficients $b_{p,j}$ in the expression for $\mathbf{D}_{X_p}$ in Proposition 6.2 is equal to p.*

Proof. In Example 3 we showed that there is a configuration $C'_p \subset X\#(p-1)\overline{\mathbf{CP}}^2 = Y$ where $u'_i = e_{p-(i+1)} - e_{p-i}$ for $i = 1,\ldots,p-2$, and $u'_{p-1} = -2e_1 - e_2 - \cdots - e_{p-1}$ such that the rational blowdown

$Y_p = X \# H_p$ where H_p is a rational homology 4-sphere with $\pi_1 = \mathbf{Z}_p$. It follows easily that $\mathbf{D}_{Y_p} = p \cdot \mathbf{D}_X$.

As above, we let $\kappa_J = \sum \epsilon_{J,i} e_i$, $J \in \{\pm 1\}^{p-1}$. Then

$$\mathbf{D}_Y = \frac{1}{2^{p-1}} \mathbf{D}_X \cdot \exp(Q_{(p-1)\overline{\mathbf{CP}}^2}/2) \sum_J e^{\kappa_J}.$$

All partial derivatives of $\mathbf{D}_X$ with respect to classes in $H_2(C'_p)$ are trivial; so

$$p\mathbf{D}_X = \mathbf{D}_{Y_p} = \mathbf{D}_X \cdot \sum_J b_J e^{\bar{\kappa}_J}.$$

The proof of Proposition 6.1 shows that each $\bar{\kappa}_J = 0$; hence $\sum_J b_J = p$.

We can also form the configuration $C_p \subset Y$ whose blowdown is the p-log transform of the nodal fiber $S \subset X$. The configurations C_p, C'_p agree, $u_i = u'_i$, except that $u_{p-1} = u'_{p-1} + S$. However, since S is orthogonal to all the basic classes of X, for all i, all intersections of u_i and u'_i with all basic classes of $Y = X \# (p-1)\overline{\mathbf{CP}}^2$ agree. Thus, according to Theorem 5.11, the coefficients b_J are the same coefficients that arise in the formula

$$\mathbf{D}_{X_p} = \exp(Q_{X_p}/2)\mathbf{K}_X \sum_J c_J e^{|J|S/p}.$$

Hence the sum of the coefficients of the expression for $\mathbf{D}_{X_p}$ in Proposition 6.2 is $\sum_J b_J = p$.   q.e.d.

We next invoke Proposition 3.2 to see that if p is any positive odd integer, then a multiplicity $2p$ log transform can be obtained as the result of either a multiplicity p log transform on a nodal fiber of multiplicity 2, or by a multiplicity 2 log transform on a nodal fiber of multiplicity p. Therefore

$$\mathbf{D}_{E(n;2p)} = \exp(Q/2)(e^{f_2} + e^{-f_2})\Big(b_{p,0} + \sum_{i=1}^{(p-1)/2} b_{p,2i}(e^{2if_2/p} + e^{-2if_2/p})\Big)$$

$$= \exp(Q/2)\Big(b_{p,0} + \sum_{i=1}^{(p-1)/2} b_{p,2i}(e^{2if_p} + e^{-2if_p})\Big)(e^{f_p/2} + e^{-f_p/2})$$

since we already know the formula for a log transform of multiplicity 2. We compare the coefficients using $f_2 = pf_{2p}$ and $f_p = 2f_{2p}$.

Assume for the sake of definiteness that $p \equiv 1 \pmod 4$ and let $r = (p-1)/4$. In the top expansion, the coefficient of $e^{\pm p f_{2p}}$ is $b_{p,0}$, and $b_{p,2j}$ is the coefficient of $e^{\pm(p+2j)f_{2p}}$ and $e^{\pm(p-2j)f_{2p}}$. In the second expansion, the coefficient of $e^{\pm f_{2p}}$ is $b_{p,0}$, and $b_{p,2j}$ is the coefficient of $e^{\pm(4j-1)f_{2p}}$ and $e^{\pm(4j+1)f_{2p}}$. To simplify notation, let $(m)_1$ be the coefficient of $e^{m f_{2p}}$ in the top expansion, and $(m)_2$ its coefficient in the bottom expansion. Then,

$$
\begin{aligned}
b_{p,0} = (p)_1 \; &= \; (p)_2 = b_{p,2r} = (p-2)_2 = (p-2)_1 \\
&= \; b_{p,2} = (p+2)_1 = (p+2)_2 \\
&= \; b_{p,2(r+1)} = (p+4)_2 = (p+4)_1 \\
&= \; b_{p,4} = (p-4)_1 = (p-4)_2 \\
&= \; b_{p,2(r-1)} = (p-6)_2 = (p-6)_1 = b_{p,6} = \cdots,
\end{aligned}
$$

and we see inductively that when p is odd, all the $b_{p,2i}$ are equal. But by Lemma 6.5

$$
b_{p,0} + 2 \sum_{i=1}^{(p-1)/2} b_{p,2i} = p.
$$

It follows that each $b_{p,2i} = 1$, $i = 0, \ldots, (p-1)/2$.

Similarly, if p is even, let $q = p - 1$. Expanding $\mathbf{D}_{E(n;pq)}$ we see that all $b_{p,2i-1}$, $i = 1, \ldots, p/2$ are equal; and so again each $b_{p,2i-1} = 1$.

Theorem 6.6. *Let X be a 4-manifold of simple type and suppose that X contains a nodal fiber S orthogonal to all its basic classes. Then*

$$
\mathbf{D}_{X_p} = \exp(Q_{X_p}/2)\mathbf{K}_X \cdot \frac{\sinh(S)}{\sinh(S/p)}.
$$

Proof. If, e.g., p is odd, then

$$
\begin{aligned}
\mathbf{D}_{X_p} =\; &\exp(Q_{X_p}/2)\mathbf{K}_X \cdot (1 + 2\cosh(2S/p) \\
&\qquad + 2\cosh(4S/p) + \cdots + 2\cosh((p-1)S/p)) \\
=\; &\exp(Q_{X_p}/2)\mathbf{K}_X \cdot \frac{\sinh(S)}{\sinh(S/p)}.
\end{aligned}
$$

q.e.d.

As a result we have the calculation of the Donaldson series for all simply connected elliptic surfaces with $p_g \geq 1$.

Theorem 6.7. *If $n \geq 2$, and $p, q \geq 1$ are relatively prime, then*

$$\mathbf{D}_{E(n;p,q)} = \exp(Q/2) \frac{\sinh^n(f)}{\sinh(f_p)\sinh(f_q)}.$$

This formula was originally conjectured by Kronheimer and Mrowka [23] and shows that the diffeomorphism type of the elliptic surfaces $E(n; p, q)$ is determined by the integer n and the unordered pair of integers $\{p, q\}$. This is also established in [12], [26], [27], [30], [31], [35]. The advantage to our proof is that, at bottom, it only requires as input the 0 degree Donaldson invariant of $E(2)$.

As an example of Theorem 6.6 consider $E(n)$. It follows from [17] and [8] that in $E(n)$ there are 3 pairs of disjoint nodal fibers such that the nodal fibers in each pair are homologous, but give three linearly independent homology classes. Form $E(n; p_1, q_1; p_2, q_2; p_3, q_3)$ by performing log transforms with each pair $\{p_i, q_i\}$ relatively prime. The resulting manifold is simply connected, and we have

Proposition 6.8.

$$\mathbf{D}_{E(n;p_1,q_1;p_2,q_2;p_3,q_3)} = \exp(Q/2) \sinh^{n-2}(f) \prod_{i=1}^{3} \frac{\sinh^2(f_i)}{\sinh(f_{i,p_i})\sinh(f_{i,q_i})}.$$

Applying Theorem 6.7 and Proposition 6.8 to the manifolds $E(n; p_1, q_1; p_2, q_2; p_3, q_3)$, we see that they do not admit complex structures with either orientation (cf.[17],[8, Theorem 8.3]). The manifolds $E(2; p_1, q_1; p_2, q_2; p_3, q_3)$ are the Gompf-Mrowka fake K3-surfaces [17].

7. Tautly embedded configurations

Consider a 4-manifold X of simple type containing the configuration C_p. By Theorem 4.2 for each 2-sphere u_i in C_p and each basic class κ of X, we have

$$(17) \qquad\qquad -2 \geq u_i^2 + |u_i \cdot \kappa|$$

except in the special case described in Theorem 4.3 where $0 \geq u_i^2 + |u_i \cdot \kappa|$. The only examples known where the special case arises are in blowups. This was the situation in the previous section where we studied log transforms. In this section, we assume that we are not in the special case. We say that a configuration is **tautly embedded** if

(17) is satisfied for each u_i of the configuration and each basic class κ of X. Thus, if C_p is tautly embedded, then for every basic class κ, $u_i \cdot \kappa = 0$ for $i = 1, \ldots, p - 2$ and $|u_{p-1} \cdot \kappa| \leq p$.

Theorem 7.1. *Suppose that X is of simple type and contains the tautly embedded configuration C_p. If*

$$\mathbf{D}_X = \exp(Q_X/2) \sum a_s e^{\kappa_s},$$

then the rational blowdown X_p satisfies

$$\mathbf{D}_{X_p} = \exp(Q_{X_p}/2) \sum \bar{a}_s e^{\bar{\kappa}_s},$$

where

$$\bar{a}_s = \begin{cases} 2^{p-1} a_s, & |u_{p-1} \cdot \kappa_s| = p, \\ 0, & |u_{p-1} \cdot \kappa_s| < p. \end{cases}$$

Furthermore, if $|u_{p-1} \cdot \kappa_s| = p$, then $\bar{\kappa}_s^2 = \kappa_s^2 + (p - 1)$.

Proof. If $\kappa_s \cdot u_{p-1} \neq 0, \pm p$ then $\bar{a}_s = 0$ by Lemma 5.10. For $\kappa_s \cdot u_{p-1} = 0$, $\kappa_s \neq 0$, note that since the κ_s are characteristic, p must be even. But then $\bar{\kappa}_s$ cannot even be characteristic in X_p, since $\bar{\kappa}_s^2 = \kappa^2$ is not mod 4 congruent to $(3\text{sign} + 2e)(X_p)$. Thus, Theorem 5.11 implies that $\bar{a}_s = 0$.

In case $\kappa_s \cdot u_{p-1} = \pm p$, we compare with the model for the order p log transforms of $E(2)$; $C_p'' \subset Y = E(2)\#(p - 1)\overline{\mathbf{CP}}^2$ which is blown down to obtain $Y_p = E(2; p)$. Again let $\lambda_0 = \pm(e_1 + \cdots + e_{p-1})$; then by Lemma 6.1, $\pm\lambda_0$ are the unique basic classes of Y_p satisfying $\pm\bar{\lambda}_0 = \pm(p - 1)f_p \in H_2(Y_p; \mathbf{Z})$. Now

$$\begin{aligned}
\mathbf{D}_Y &= \frac{1}{2^{p-1}} \exp(Q/2) \sum \exp(\pm e_1 \pm \cdots \pm e_{p-1}) \\
&= \exp(Q/2) \sum_J \frac{1}{2^{p-1}} e^{\lambda_J}, \\
\mathbf{D}_{Y_p} &= \exp(Q/2) \sum_{\substack{|\ell| \leq p-1 \\ \ell \equiv p \ (\text{mod } 2)}} e^{\ell f_p} = \exp(Q/2) \sum \frac{1}{2^{p-1}} b_J e^{\lambda_J}.
\end{aligned}$$

Since $\pm\lambda_0$ are the unique λ_J with $\bar{\lambda}_0 = \pm(p - 1)f_p$, the corresponding coefficient is $b_0 = 2^{p-1}$. We may now apply Theorem 5.11 to obtain our result since $\kappa_s \cdot u_i = \lambda_0 \cdot u_i''$ for each i.

In order to compute $\bar{\kappa}_s^2$, we find $x_i \in \mathbf{Q}$, $i = 1, \ldots, p-1$, such that

$$\kappa_s + \zeta = \kappa_s + \sum_{i=1}^{p-1} x_i u_i \in H_2(X^*; \mathbf{Q})$$

as in the proof of Proposition 6.1. We can solve for the x_i using the model $C_p'' \subset Y''$, and $\epsilon_J = \pm(1, \ldots, 1)$ in the proof of Proposition 6.1. Referring there, we get

$$\mathbf{x} = P^{-1} A \epsilon_J = -(A^t)^{-1} \epsilon_J = \pm \frac{1}{p}(1, 2, \ldots, p-1).$$

So $\zeta = \pm \sum \frac{i}{p} u_i$, and $\zeta^2 = \mathbf{x} \cdot P\mathbf{x} = 1 - p$. Hence

$$\bar{\kappa}_s^2 = (\kappa_s + \zeta)^2 = \kappa_s^2 + 2\kappa_s \cdot \zeta + \zeta^2 = \kappa_s^2 + (p-1).$$

<div align="right">q.e.d.</div>

Now consider the elliptic surface $E(1)$. It can be constructed by blowing up $\mathbf{CP}^2$ at the nine intersection points of a generic pencil of cubic curves. The fiber class of $E(1)$ is $f = 3h - e_1 - \cdots e_9$ where $3h$ is the class of the cubic in $H_2(\mathbf{CP}^2; \mathbf{Z})$. The nine exceptional curves are disjoint sections of the elliptic fibration. The elliptic surface $E(n)$ can be obtained as the fiber sum of n copies of $E(1)$, and these sums can be made so that the sections glue together to give nine disjoint sections of $E(n)$, each of square $-n$. In particular, consider $E(4)$ with 9 disjoint sections of square -4. The basic classes of $E(4)$ are 0 and $2f$; so we see that each of the 9 sections gives us a tautly embedded configuration C_2. Let W_n be the rational blowdown of n of these sections, $1 \leq n \leq 9$. For $n \leq 8$, W_n is simply connected. Gompf has shown that all these manifolds admit symplectic structures, and it is not hard to see that W_2 is the 2-fold branched cover of $\mathbf{CP}^2$ branched over the octic curve [16, §5.2].

Proposition 7.2. $\mathbf{D}_{W_n} = 2^{n-1} \exp(Q/2) \cosh(\kappa_n)$ *where* $\kappa_n^2 = n$.

Proof. We have

$$\mathbf{D}_{E(4)} = \exp(Q/2) \sinh^2(f) = \exp(Q/2)(\frac{1}{2}\cosh(2f) + \frac{1}{2}).$$

The basic classes $\pm 2f$ intersect each section twice; so Theorem 7.1 implies that each X_n has only the basic classes, $\pm \kappa_n$, and that each blowdown multiplies its coefficient by 2 and increases its square by 1. (We start with coefficient $\frac{1}{2}$ and square 0.)   q.e.d.

To further illustrate the utility of Theorem 7.1 we compute the Donaldson invariants of a family of Horikawa surfaces $\{H(n)\}$ with $c_1(H(n))^2 = 2n - 6$. To obtain $H(n)$, start with the simply connected ruled surface $\mathbf{F}_{n-3}$ whose negative section s_- has square $-(n-3)$. We have seen in the proof of Lemma 2.1 that the classes $s_+ + f$ and s_- form a configuration in $\mathbf{F}_{n-3}$ whose regular neighborhood D_{n-2} has complement the rational ball B_{n-2}. The Horikawa surface $H(n)$ is defined to be the 2-fold branched cover of $\mathbf{F}_{n-3}$ branched over a smoothing of $4(s_+ + f) + 2s_-$.

Lemma 7.3. *For $n \geq 4$, the elliptic surface $E(n)$ contains a pair of disjoint configurations C_{n-2} in which the spheres u_{n-1} are sections of $E(n)$ and for $1 \leq j \leq n - 2$, $u_j \cdot f = 0$. Furthermore, the rational blowdown of this pair of configurations is the Horikawa surface $H(n)$.*

Proof. It follows from our description of $H(n)$ that there is a decomposition

$$H(n) = B_{n-2} \cup \tilde{D}_{n-2} \cup B_{n-2},$$

where $\tilde{D}_{n-2}$ is the branched cover of D_{n-2}. Rationally blow up each B_{n-2}; this is then the 2−fold branched cover of $\mathbf{F}_{n-3}$ with B_{n-2} blown up. The result is the minimal complex surface $\tilde{X} = C_{n-2} \cup \tilde{D}_{n-2} \cup C_{n-2}$. By the Kodaira classification of surfaces and the computation of characteristic numbers, $\tilde{X} = E(n)$.   q.e.d.

The first case, $n = 4$, gives the example $H(4) = W_2$ above. The Horikawa surfaces $H(n)$ lie on the Noether line $5c_1^2 - c_2 + 36 = 0$, and of course the elliptic surfaces $E(n)$ lie on the line $c_1^2 = 0$ in the plane of coordinates (c_1^2, c_2). Let $Y(n)$ be the simply connected 4-manifold obtained from $E(n)$ by blowing down just one of the configurations C_{n-2}. Then $c_1(Y(n))^2 = n - 3$ and $c_2(Y(n)) = 11n + 3$; so $Y(n)$ lies on the bisecting line $11c_1^2 - c_2 + 36 = 0$. The calculation of Donaldson invariants of $Y(n)$ and $H(n)$ follows directly from Theorem 7.1.

Proposition 7.4. *The Donaldson invariants of $Y(n)$ and $H(n)$ are:*

$$\mathbf{D}_{Y(n)} = \begin{cases} \exp(Q/2)\sinh(\lambda_n), & n \ odd, \\ \exp(Q/2)\cosh(\lambda_n), & n \ even, \end{cases}$$

$$\mathbf{D}_{H(n)} = \begin{cases} 2^{n-3}\exp(Q/2)\sinh(\kappa_n), & n \ odd, \\ 2^{n-3}\exp(Q/2)\cosh(\kappa_n), & n \ even, \end{cases}$$

where $\lambda_n^2 = n - 3$ and $\kappa_n^2 = 2n - 6$.

Corollary 7.5. *The simply connected 4-manifolds $Y(n)$ are not homotopy equivalent to any complex surface.*

Proof. If $Y(n)$ were homeomorphic to a complex surface, this computation shows that it would have to be minimal, since the formula for $\mathbf{D}_{Y(n)}$ does not contain a factor $\cosh(e)$ where $e^2 = -1$. Certainly the surface in question could not be elliptic since $c_1(Y(n))^2 \neq 0$. But neither could the surface be of general type since $Y(n)$ violates the Noether inequality. Thus $Y(n)$ is not homeomorphic to any complex surface.   q.e.d.

D. Gomprecht [18] has computed the value of the Donaldson invariant $D_X(F^k)$ for any Horikawa surface X and k large, where F is the branched cover of the fiber f of F_{n-3}. Furthermore, F. Catanese [4] has computed the zero degree Donaldson invariants for these Horikawa surfaces.

8. Seiberg-Witten invariants of rational blowdowns

In this section we will determine the effect of a rational blowdown on the Seiberg-Witten invariants. First we recall the definition of these invariants. Suppose we are given a spin^c structure on an oriented closed Riemannian 4-manifold X. Let W^+ and W^- be the associated spin^c bundles with $L = \det W^+ = \det W^-$ the associated determinant line bundle. Since $c_1(L) \in H^2(X; \mathbf{Z})$ is a characteristic cohomology class, i.e., has mod 2 reduction equal to $w_2(X) \in H^2(X; \mathbf{Z}_2)$, we refer to L as a characteristic line bundle. We will confuse a characteristic line bundle L with its first Chern class $c_1(L) \in H^2(X; \mathbf{Z})$. For simplicity we assume that $H^2(X; \mathbf{Z})$ has no 2-torsion so that the set $Spin^c(X)$ of spin^c structures on X is precisely the set of characteristic line bundles on X.

Clifford multiplication, c, maps T^*X into the skew adjoint endomorphisms of $W^+ \oplus W^-$ and is determined by the requirement that $c(v)^2$ is multiplication by $-|v|^2$. Thus c induces a map

$$c : T^*X \to \mathrm{Hom}(W^+, W^-).$$

The 2-forms $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ of X then act on W^+ leading to a map $\rho : \Lambda^+ \to su(W^+)$. A connection A on L together with the Levi-Civita connection on the tangent bundle of X induces a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$ on W^+. This connection, followed by Clifford

multiplication, induces the Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$. (Thus D_A depends both on the connection A and the Riemannian metric on X.) Given a pair $(A, \psi) \in \mathcal{A}_X(L) \times \Gamma(W^+)$, i.e., A a connection in L and ψ a section of W^+, the monopole equations of Seiberg and Witten [42] are

$$
\begin{aligned}
D_A\psi &= 0, \\
\rho(F_A^+) &= (\psi \otimes \psi^*)_o,
\end{aligned}
\tag{18}
$$

where $(\psi \otimes \psi^*)_o$ is the trace-free part of the endomorphism $\psi \otimes \psi^*$.

The gauge group $\mathrm{Aut}(L) = \mathrm{Map}(X, S^1)$ acts on the space of solutions, and its orbit space is the moduli space $M_X(L)$ whose formal dimension is

$$
\dim M_X(L) = \frac{1}{4}(c_1(L)^2 - (3\,\mathrm{sign}(X) + 2\,e(X)).
\tag{19}
$$

If this formal dimension is nonnegative and $b^+ > 0$, then for a generic metric on X the moduli space $M_X(L)$ contains no reducible solutions (solutions of the form $(A, 0)$ where A is an anti-self-dual connection on L), and for a generic perturbation of the second equation of (18) by the addition of a self-dual 2-form of X, the moduli space $M_X(L)$ is a compact manifold of the given dimension ([42]).

The Seiberg-Witten invariant for X is the function

$$
SW_X : Spin^c(X) \to \mathbf{Z}
$$

defined as follows. Let L be a characteristic line bundle. If $\dim M_X(L)$ is negative or odd, then $SW_X(L)$ is defined to be 0. If $\dim M_X(L) = 0$, the moduli space $M_X(L)$ consists of a finite collection of points and $SW_X(L)$ is defined to be the number of these points counted with signs. These signs are determined by an orientation on $M_X(L)$, which in turn is determined by an orientation on the determinant line $\det(H^0(X; \mathbf{R})) \otimes \det(H^1(X; \mathbf{R})) \otimes \det(H^2_+(X; \mathbf{R}))$. If $\dim M_X(L) > 0$, then we consider the basepoint map

$$
\tilde{M}_X(L) = \{\text{solutions } (A, \psi)\}/\mathrm{Aut}^0(L) \to M_X(L),
$$

where $\mathrm{Aut}^0(L)$ consists of gauge transformations which are the identity on the fiber of L over a fixed basepoint in X. If there are no reducible solutions, the basepoint map is an S^1 fibration, and we denote its Euler

class by $\beta \in H^2(M_X(L); \mathbf{Z})$. The moduli space $M_X(L)$ represents an integral cycle in the configuration space

$$B_X(L) = (\mathcal{A}_X(L) \times \Gamma(W^+))/\mathrm{Aut}(L),$$

and if $\dim M_X(L) = 2n$, the Seiberg-Witten invariant is defined to be the integer

$$SW_X(L) = \langle \beta^n, [M_X(L)] \rangle.$$

A fundamental result is that if $b_X^+ \geq 2$, the map

$$SW_X : Spin^c(X) \to \mathbf{Z}$$

is a diffeomorphism invariant ([42]); i.e., $SW_X(L)$ does not depend on the (generic) choice of Riemannian metric on X nor the choice of generic perturbation of the second equation of (18).

It is often convenient to observe that the space $\mathcal{A}_X(L) \times \Gamma(W^+)$ is contractible, and $\mathrm{Aut}(L) \cong \mathrm{Map}(X, S^1)$ acts freely on

$$\mathcal{A}_X(L) \times (\Gamma(W^+) \setminus \{0\}).$$

Since S^1 is a $K(\mathbf{Z}, 1)$, if we further assume that $H^1(X; \mathbf{R}) = 0$, then the quotient

$$B_X^*(L) = \left(\mathcal{A}_X(L) \times (\Gamma(W^+) \setminus \{0\}) \right) / S^1$$

of this action is homotopy equivalent to $\mathbf{CP}^\infty$. So if there are no reducible solutions, we may view $M_X(L) \subset \mathbf{CP}^\infty$. Under these identifications, the class β becomes the standard generator of $H^2(\mathbf{CP}^\infty; \mathbf{Z})$.

Call a characteristic line bundle with nontrivial Seiberg-Witten invariant a **Seiberg-Witten class**. The assumption in Seiberg-Witten theory which is analogous to the assumption of simple type in Donaldson theory is

- For each Seiberg-Witten class L, $\dim M_X(L) = 0$.

If this condition is satisfied, X is said to have **Seiberg-Witten simple type**. This condition often follows directly from the existence of an embedding of C_p as the next lemma indicates.

Lemma 8.1. *Suppose that there is a class $u \in H_2(X; \mathbf{Z})$ with $u^2 = -2$ which is represented by an embedded sphere. If L is a Seiberg-Witten class such that $L \cdot u = 0$ and neither of $L \pm 2u$ is a Seiberg-Witten class, then $\dim M_X(L) = 0$.*

Proof. According to [11, Theorem 1.3], if $\dim M_X(L) > 0$, then provided that neither of $L \pm 2u$ is a Seiberg-Witten class, L obeys an enhanced adjunction inequality with respect to u; namely

$$-2 \geq u^2 + |u \cdot L| + 2r,$$

where $r > 0$. This gives an immediate contradiction.   q.e.d.

Lemma 8.2. *Let L' be any characteristic line bundle on B_p and let*

$$B_p^+ = B_p \cup (L(p^2, 1 - p) \times [0, \infty)).$$

Then for each connection A' on L' which is asymptotically flat, the index of the Dirac operator $indD_{A'} = 0$.

Proof. Let L'_∂ denote the restriction of L' to $L(p^2, 1 - p)$. Then $c_1(L'_\partial)$ is the restriction of $c_1(L') \in H^2(B_p; \mathbf{Z}) = \mathbf{Z}_p$. Thus $c_1(L'_\partial) \in p\mathbf{Z}_{p^2}$, which is the image of $H^2(B_p; \mathbf{Z})$ in $H^2(L(p^2, 1 - p) = \mathbf{Z}_{p^2}$. Write $c_1(L'_\partial) = mp$. Then we identify $c_1(L') = m \pmod{p}$. If p is even then B_p is not spin, and since L' is characteristic, m is odd. In case p is odd an argument given shortly will also show that m is odd.

One computes from the index theorem [1] that

$$indD_A = -\frac{p_1(B_p)}{24} + \xi(p, m),$$

where

(20) $$\xi(p, m) = \frac{1}{4p^2} \sum_{k=1}^{p^2-1} \csc(\frac{\pi k}{p^2}) \csc(\frac{\pi(1 - p)k}{p^2}) e^{\pi imk/p}.$$

The signature theorem gives

$$0 = \text{sign}(B_p) = \frac{p_1(B_p)}{3} - \frac{1}{p^2} \sum_{k=1}^{p^2-1} \cot(\frac{\pi k}{p^2}) \cot(\frac{\pi(1 - p)k}{p^2})$$

$$= \frac{p_1(B_p)}{3} - 4s(p - 1, p^2),$$

where $s(p - 1, p^2)$ is the Dedekind sum (cf. [19]). From Dedekind reciprocity and $s(1, p - 1) = \frac{(p-2)(p-3)}{12(p-1)}$ ([19, p.93]) it follows that $s(p - 1, p^2) = \frac{(p^2-1)}{6p^2}$. Thus

$$\sum_{k=1}^{p^2-1} \cot(\frac{\pi k}{p^2}) \cot(\frac{\pi(1 - p)k}{p^2}) = \frac{2}{3}(1 - p^2)$$

and $p_1(B_p) = \frac{2}{p^2}(1 - p^2)$.

A simple calculation shows that

$$
(21) \quad \xi(p, m) = \begin{cases} \dfrac{1}{2p^2} \displaystyle\sum_{k=1}^{\frac{p^2-1}{2}} \csc\left(\frac{\pi k}{p^2}\right) \csc\left(\frac{\pi(1-p)k}{p^2}\right) \cos\left(\frac{\pi m k}{p}\right), \\ \qquad\qquad\qquad\qquad\qquad p \text{ odd}, \\ \dfrac{1}{2p^2}\left(\dfrac{1}{2} + \displaystyle\sum_{k=1}^{\frac{p^2}{2}-1} \csc\left(\frac{\pi k}{p^2}\right) \csc\left(\frac{\pi(1-p)k}{p^2}\right) \cos\left(\frac{\pi m k}{p}\right)\right), \\ \qquad\qquad\qquad\qquad\qquad p \text{ even}. \end{cases}
$$

unless both p and m are odd, in which case the sum in (20) is purely imaginary. Thus when p is odd, the 'rotation number' m is also odd. By this last expression for $\xi(p, m)$ it can be shown that for m odd, all $\xi(p, m)$ agree. (See the thesis of J. Park [33].) We compute $\xi(p, 1)$ for p odd. The calculation for p even is similar. Using

$$\pi k/p = \pi k/p^2 - (1 - p)\pi k/p^2$$

we get

$$
\begin{aligned}
\xi(p, 1) &= \frac{1}{2p^2} \sum_{k=1}^{\frac{p^2-1}{2}} \csc\left(\frac{\pi k}{p^2}\right) \csc\left(\frac{\pi(1 - p)k}{p^2}\right) \\
&\quad \cdot \left(\cos\left(\frac{\pi k}{p^2}\right)\cos\left(\frac{\pi(1 - p)k}{p^2}\right) + \sin\left(\frac{\pi k}{p^2}\right)\sin\left(\frac{\pi(1 - p)k}{p^2}\right)\right) \\
&= \frac{1}{2p^2} \sum_{k=1}^{\frac{p^2-1}{2}} \left(\cot\left(\frac{\pi k}{p^2}\right)\cot\left(\frac{\pi(1 - p)k}{p^2}\right) + 1\right).
\end{aligned}
$$

From (21) it follows that for $m = 1$ (hence for all odd m)

$$\operatorname{ind} D_A = -\frac{p_1(B_p)}{24} + \xi(p, 1) = \frac{1 - p^2}{12p^2} + \frac{1}{2p^2}\left(\frac{1 - p^2}{3} + \frac{p^2 - 1}{2}\right) = 0.$$

<div align="right">q.e.d.</div>

Lemma 8.3. *Let L'' be any characteristic line bundle on C_p and let A'' be any asymptotically flat connection on L''. Suppose that the restriction L''_∂ to $L(p^2, 1 - p)$ extends to a characteristic line bundle L' on B_p. Then the index of the Dirac operator $\operatorname{ind} D_{A''} \leq 0$. Furthermore,*

given a fixed $m \in \mathbf{Z}_p$, with m odd if p is even, there is a line bundle L''_m on C_p with $c_1(L''_{m,\partial}) = mp \in H^2(L(p^2, 1-p); \mathbf{Z})$, and such that for any connection A'' on L''_m as above, $\operatorname{ind} D_{A''} = 0$.

Proof. We know from Lemma 2.1 that $Q = C_p \cup -B_p$ is diffeomorphic to $(p-1)\overline{\mathbf{CP}}^2$. Let L be the characteristic line bundle on Q obtained from L'' and $-L'$, and let A be a connection on L which restricts to A'' on C_p. Since L is characteristic, $c_1(L)^2 \leq 1 - p$, and

$$\operatorname{ind} D_A = \frac{c_1(L)^2 - \operatorname{sign}(Q)}{8} \leq 0.$$

But also

$$\operatorname{ind} D_A = \operatorname{ind} D_{A''} + h + \operatorname{ind} D_{A'} = \operatorname{ind} D_{A''}$$

by Lemma 8.2, since h is the dimension of the space of harmonic spinors on $L(p^2, 1-p)$. This dimension is 0 since the lens space has positive scalar curvature.

Now suppose that we are given $m \in \mathbf{Z}_p$ as above. In the proof of Lemma 8.2 we have seen that m must be odd when p is even and mp is to be the boundary value of a characteristic line bundle on B_p. When p is odd, we can assume that m is even and $-p < m < p$. Let $E_1, \ldots, E_{p-1}$ be the exceptional classes generating $H^2(Q; \mathbf{Z})$ with Poincaré duals $e_1 \ldots, e_{p-1}$. We can identify C_p in Q as consisting of 2-spheres representing $u_j = e_{p-j-1} - e_{p-j}$, $j = 1, \ldots, p-2$, and $u_{p-1} = -(2e_1 + e_2 + \cdots + e_{p-1})$. Let $j : L(p^2, 1-p) \subset C_p$. Then for each i, $j^*(E_i) = p \in H^2(L(p^2, 1-p) = \mathbf{Z}_{p^2}$. Let

$$\lambda_m = \sum_{i=1}^{(m+p-1)/2} E_i - \sum_{i=(m+p+1)/2}^{p-1} E_i,$$

and let L_m be the line bundle over Q with $c_1(L_m) = \lambda_m$. Let $L''_m = L_m|_{C_p}$. We have $c_1(L''_{m,\partial}) = mp$, and $0 = \operatorname{ind} D_A = \operatorname{ind} D_{A''}$, as required.
<div align="right">q.e.d.</div>

Note that $\dim M_{C_p}(L'') = 2\operatorname{ind} D_{A''} - 1$ for an asymptotically flat connection A''. Similarly $\dim M_{B_p}(L'') = 2\operatorname{ind} D_{A'} - 1 = -1$. If $\bar{L}$ is a line bundle on X_p and L is a line bundle on X satisfying $L|_{X^*} = \bar{L}|_{X^*}$, we say that L is a *lift* of $\bar{L}$.

Proposition 8.4. *Let $C_p \subset X$ and let X_p be its rational blowdown. For each characteristic line bundle $\bar{L}$ on X_p, there is a lift L of $\bar{L}$ which is characteristic on X and satisfies $\dim M_X(L) = M_{X_p}(\bar{L})$.*

Proof. Let $\bar{L}|_{X^*} = L^*$ and $\bar{L}|_{B_p} = L'$. The index theorem implies that

$$\dim M_{X_p}(\bar{L}) = \dim M_{X^*}(L^*) + h + \dim M_{B_p}(L').$$

Since $L(p^2, 1 - p)$ has positive scalar curvature and $\mathrm{ad}(L)$ is a trivial real line bundle, the term h is the sum of the dimensions:

$$h = \dim(H^0 \oplus H^1)(L(p^2, 1 - p); \mathrm{ad}(L)) = (b^0 + b^1)(L(p^2, 1 - p)) = 1.$$

Thus $\dim M_{X_p}(\bar{L}) = \dim M_{X^*}(L^*)$. Hence, by a similar calculation, it suffices to produce a line bundle L'' over C_p satisfying

1. $L''|_{\partial C_p} = -L^*|_{\partial X^*}$,

2. $L^* \cup L''$ is characteristic, and

3. $\dim M_{C_p}(L'') = -1$.

Since $L^*|_{L(p^2, 1-p)}$ extends over B_p, $c_1(L^*)|_{\partial X^*} = -mp$ for some m. Let L''_m be the line bundle over C_p constructed in Lemma 8.3. If p is odd, then any lift L of $\bar{L}$ is characteristic provided L'' is characteristic on C_p. Hence L''_m satisfies $(1) - (3)$.

Suppose that p is even, then the same argument works unless there are lifts of $\bar{L}$ which are not characteristic. From the mod 2 Mayer-Vietoris sequence for $X = X^* \cup C_p$ it follows that this can happen only if $H^1(X^*; \mathbf{Z}_2) = 0$, which we thus assume. Denote the nontrivial class in $H^1(L(p^2, 1 - p); \mathbf{Z}_2) = \mathbf{Z}_2$ by η. By the construction of Lemma 8.3, $c_1(L''_m) - c_1(L''_{p-m}) = 2\mu_m$ for some $\mu_m \in H^2(C_p; \mathbf{Z})$. Furthermore, $j^*(\mu_m) = p^2/2$, where $j : L(p^2, 1 - p) \subset C_p$. Let $\theta : H^2(X; \mathbf{Z}) \to H^2(X^*; \mathbf{Z}) \oplus H^2(C_p; \mathbf{Z})$ in the Mayer-Vietoris sequence. Then there is a ν such that $\theta(\nu) = (0, 2\mu_m)$.

We claim that the mod 2 reduction $\nu_{(2)} \neq 0$. Otherwise $\nu = 2\mu$, and $2\theta(\mu) = (0, 2\mu_m)$. Since we are assuming that $H^1(X^*; \mathbf{Z}_2) = 0$, it follows that $H^2(X^*; \mathbf{Z})$ has no 2-torsion, and the same is true for $H^2(C_p; \mathbf{Z}_2)$. Thus $\theta(\mu) = (0, \mu_m)$. This means that the restriction $j^*(\mu_m) = 0$, a contradiction. Working mod 2, $\theta_{(2)}(\nu_{(2)}) = (0, 0)$, and therefore $\nu_{(2)} = \delta\eta$. Hence if $c', c'' \in H^2(X; \mathbf{Z})$ are given by $\theta(c') = (c_1(L^*), c_1(L_m))$, $\theta(c'') = (c_1(L^*), c_1(-L_{p-m}))$, then one of these two classes is characteristic. Thus for one of the two choices L''_m, $-L_{p-m}$, $(1) - (3)$ are satisfied.   q.e.d.

Theorem 8.5. *Let $C_p \subset X$ and let X_p be its rational blowdown. Let $\bar{L}$ be a characteristic line bundle on X_p and let L be any lift of $\bar{L}$ which is characteristic on X. Suppose that $\dim M_X(L) \geq 0$. Then*

$$SW_{X_p}(\bar{L}) = SW_X(L).$$

Proof. We study the solutions on X_p for $\bar{L}$ by stretching the neck between X^* and B_p. Consider the manifolds X^* and B_p with cylindrical ends. We may assume that there is a positive scalar curvature metric on the neck $L(p^2, 1 - p) \times \mathbf{R}^+$. This means that any solution to the Seiberg-Witten equations on these manifolds must be asymptotically reducible. In a temporal gauge on the neck, the solutions have the form $(a, 0)$ where a is a flat $U(1)$-connection on $L(p^2, 1 - p)$.

Using Lemma 8.2, $\dim M_{B_p}(L') = 2 \operatorname{ind} D_{A'} - (1 + b_{B_p}^+) = -1$, and since this dimension is negative, the only equivalence class of solutions to the Seiberg-Witten equations on B_p is the reducible one $(A', 0)$, where A' is an anti-self-dual connection on L. As above, the index theorem implies that

$$\dim M_{X_p}(\bar{L}) = \dim M_{X^*}(L^*) + 1 + \dim M_{B_p}(L') = \dim M_{X^*}(L^*),$$

which we may assume is even, $2d_{\bar{L}}$ say. (If $\dim M_{X_p}(\bar{L})$ is odd, there is nothing to prove.) For metrics on X_p with a long enough cylindrical neck, all solutions to the Seiberg-Witten equations arise from gluing together solutions on X^* to $(A', 0)$, and the fact that $\dim M_{X^*}(L^*) = \dim M_{X_p}(\bar{L})$ means that there is no obstruction to perturbing to global solutions on X_p. Thus

$$SW_{X_p}(\bar{L}) = \langle \beta^{d_{\bar{L}}}, [M_{X_p}(\bar{L})] \rangle = \langle \beta^{d_{\bar{L}}}, [M_{X^*}(L^*)] \rangle.$$

Let L be a characteristic line bundle on X which is a lift of $\bar{L}$, and let $\dim M_X(L) = 2d_L$. Lemma 8.3 implies that $\dim M_{C_p}(L'') < 0$ for the restriction L'' of L to C_p. The discussion of the last paragraph applies in this case to show that for metrics on X with a long enough cylindrical neck, all solutions arise from gluing together solutions on X^* to $(A'', 0)$, where A'' lies in the unique equivalence class of anti-self-dual connections on L''. However, in this case we only have

$$2d_{\bar{L}} = \dim M_{X^*}(L^*) \geq \dim M_X(L) = 2d_L.$$

If $d_{\bar{L}}$ is actually greater than d_L, there is an obstruction to perturbing glued-up pairs of solutions $(A^*, \psi^*) \# (A'', 0)$ to a solution on X. As

in Donaldson theory, there is an obstruction bundle η over $M_{X^*}(L^*)$, and it is the complex vector bundle of rank $d_{\bar{L}} - d_L$ associated to the basepoint fibration. The zero set of a generic section of η is homologous to $M_X(L)$ in $B_X(L)$. Thus

$$SW_X(L) = \langle \beta^{d_L}, [M_X(L)] \rangle = \langle \beta^{d_L}, \beta^{d_L - d_L} \cap [M_{X^*}(L^*)] \rangle$$
$$= \langle \beta^{d_{\bar{L}}}, [M_{X^*}(L^*)] \rangle = SW_{X_p}(\bar{L}).$$

<div align="right">q.e.d.</div>

From this theorem and Proposition 8.4 it follows that the Seiberg-Witten invariants of X_p are completely determined by those of X and the embedding of C_p in X. Furthermore, we have

Corollary 8.6. *Let $C_p \subset X$ and let X_p be its rational blowdown. If X has Seiberg-Witten simple type, then so does X_p.*

In a fashion similar to the proof of Theorem 8.5, one can prove a blowup formula for Seiberg-Witten invariants (cf. [11]). The characteristic line bundles of $X \# \overline{\mathbf{CP}}^2$ are those of the form $L \otimes E^{2k+1}$ where L is characteristic on X and $c_1(E) = e$, and $\dim M_{L \otimes E^{2k+1}}(X \# \overline{\mathbf{CP}}^2) = \dim M_L(X) - k(k+1)$. It is shown in [11] that $SW_{X \# \overline{\mathbf{CP}}^2}(L \otimes E^{2k+1}) = SW_X(L)$ provided $\dim M_L(X) - k(k+1) \geq 0$. It follows that if X has Seiberg-Witten simple-type, then so does $X \# \overline{\mathbf{CP}}^2$.

Recall from § 6 that a nodal fiber in a smooth 4-manifold X is an immersed 2-sphere S with one singularity, a positive double point, such that the regular neighborhood of S is diffeomorphic to the regular neighborhood of a nodal fiber in an elliptic surface. (There need not be any associated ambient fibration of X.) Also recall that given such a nodal fiber S, one can perform a 'log transform' of multiplicity p by blowing up to get $C_p \subset X \# (p-1)\overline{\mathbf{CP}}^2$ with $u_{p-1} = S - 2e_1 - e_2 - \cdots - e_{p-1}$, and then blowing down C_p. Now suppose that X contains the nodal fiber S, and X_p is the result of performing an order p log transform on S. The characteristic line bundles on X_p are obtained from characteristic bundles $L \otimes E_1^{2k_1+1} \otimes \cdots \otimes E_{p-1}^{2k_{p-1}+1}$ on $Y = X \# (p-1)\overline{\mathbf{CP}}^2$ by restricting to $Y^* = Y \setminus C_p$ and then extending over B_p. If we assume that $\langle c_1(L), S \rangle = 0$, then for each $L \otimes E_1^{\pm 1} \otimes \cdots \otimes E_{p-1}^{\pm 1} = L(\epsilon_J)$ with $c_1(L(\epsilon_J)) = c_1(L) + \sum_J \epsilon_{J,i} e_i$, it follows from Proposition 6.1 that the unique extension $\bar{L}_J$ over X_p has $c_1(\bar{L}_J) = c_1(L) + |J|\sigma_p$, where σ_p is the Poincaré dual of S/p. (Note that when p is even, $|J|$ must be odd;

so the extension $\bar{L}_J$ is characteristic.) Hence

$$\dim M_{\bar{L}_J}(X_p) = \dim M_L(X),$$

and Theorem 8.5 implies:

Theorem 8.7. *Suppose that X has Seiberg-Witten simple type and contains the nodal fiber S. Let L be a characteristic line bundle on X with $\langle c_1(L), S \rangle = 0$. Let X_p be the result of performing an order p log transform on S. For each $J \in \{\pm 1\}^{p-1}$, we have $SW_{X_p}(\bar{L}_J) = SW_X(L)$. Suppose furthermore that $\langle c_1(L), S \rangle = 0$ for each characteristic L on X with $SW_X(L) \neq 0$. Then X_p also has Seiberg-Witten simple type and each line bundle Λ on X_p with $SW_{X_p}(\Lambda) \neq 0$ is of the form $\Lambda = \bar{L}_J$.*

By a **nodal configuration** we shall mean a configuration $C_p \subset X \# (p-1)\overline{\mathbf{CP}}^2$ as above, obtained from a nodal fiber S satisfying the condition $\langle c_1(L), S \rangle = 0$ for each characteristic L on X with $SW_X(L) \neq 0$.

Witten [42] has conjectured that (for manifolds with $b^+ > 1$) the Seiberg-Witten simple type condition is equivalent to the simple type condition of Kronheimer and Mrowka for Donaldson theory. Further, under this hypothess of simple type, Witten gives a precise conjecture for relating the Seiberg-Witten invariants and the Donaldson series, namely:

Conjecture 8.1. (Witten) *The set of basic classes in the two theories are the same, and*

$$\mathbf{D}_X = 2^{3sign+2e-(\frac{b^+-3}{2})}\exp(Q/2)\sum SW_X(\kappa_s)e^{\kappa_s}.$$

Theorem 8.8. *Witten's conjecture is true for simply connected elliptic surfaces.*

Proof. Witten has given a recipe for calculating SW_X for all Kähler manifolds X. So one could prove this theorem simply by comparing the answer obtained with that of Theorem 6.7. Alternatively, note that Witten's recipe yields the result that the nonzero Seiberg-Witten invariants of $E(n)$ are:

$$(22) \quad SW_{E(n)}((n-2-2r)f) = (-1)^r \binom{n-2}{r}, \quad r = 0, \ldots, n-2,$$

where f is the fiber class. Suppose we define

$$\mathbf{W}_X = 2^{3\mathrm{sign}+2e-(\frac{b^+-3}{2})} \sum SW_X(\kappa_s)e^{\kappa_s}, \quad \mathbf{SW}_X = \exp(Q_X/2)\mathbf{W}_X$$

Then (22) shows that $\mathbf{D}_{E(n)} = \mathbf{SW}_{E(n)}$. Suppose that X_p is the result of an order p log transform on a nodal fiber which is orthogonal to all classes in $H_2(X)$ with nontrivial Seiberg-Witten invariants. Then Theorem 8.7 implies that $\mathbf{W}_{X_p} = \mathbf{W}_X \cdot (\sinh(f)/\sinh(f_p))$. It follows that $\mathbf{SW}_{E(n;p,q)} = \mathbf{D}_{E(n;p,q)}$.   q.e.d.

Furthermore we have,

Theorem 8.9. *If X satisfies the Witten conjecture, then so do all blowups and blowdowns and any rational blowdown X_p of a nodal or taut configuration.*

V. Pidstrigach and A. Tyurin [34] have developed a technique which shows how the Seiberg-Witten basic classes for arbitrary smooth 4-manifolds determine the Donaldson invariants. Many of the requisite details have been worked out by P. Feehan and T. Leness (cf. [7]). In particular their work shows promise in proving this Witten conjecture.

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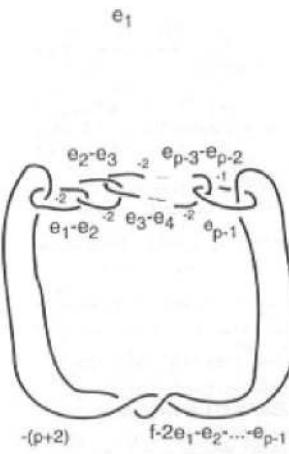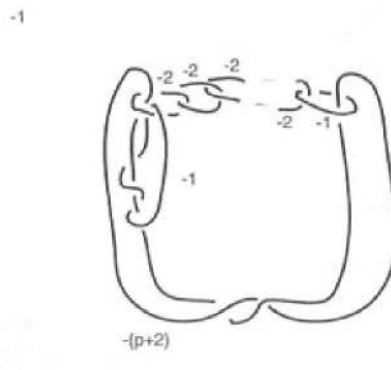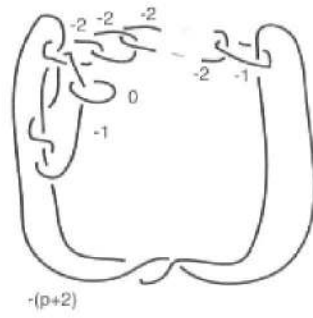MICHIGAN STATE UNIVERSITY
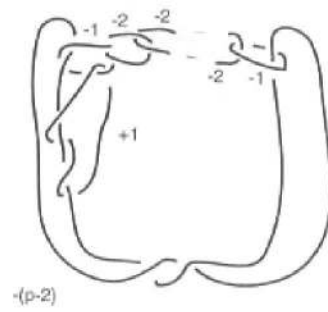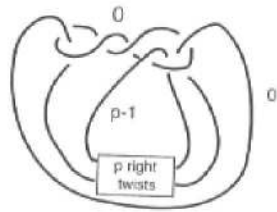UNIVERSITY OF CALIFORNIA, IRVINE
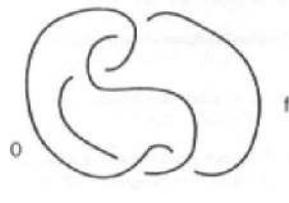
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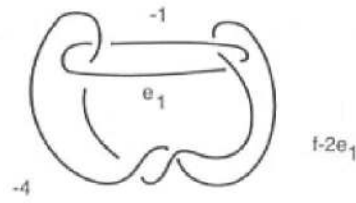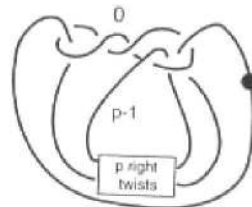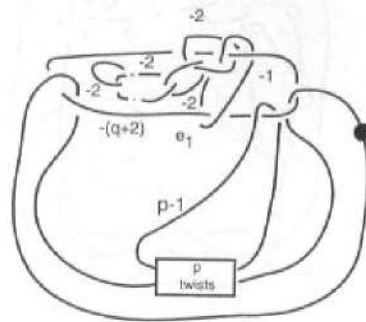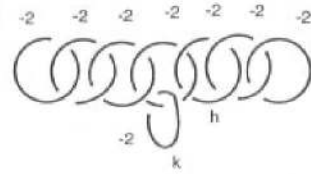
FIGURE 2

FIGURE 3

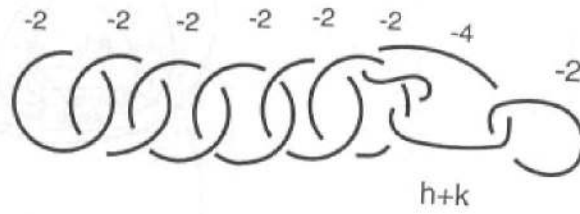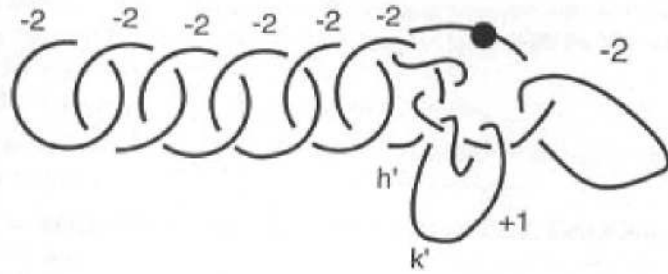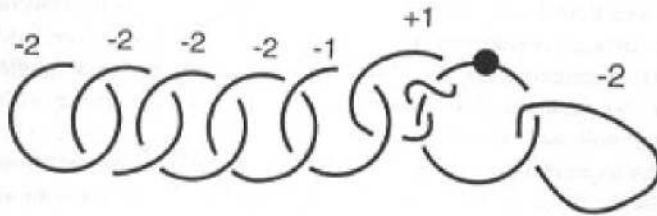FIGURE 4

FIGURE 5
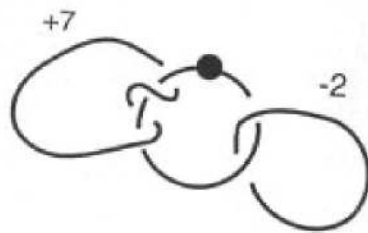
FIGURE 6

FIGURE 7

FIGURE 8

FIGURE 9

FIGURE 10

FIGURE 11

FIGURE 12

FIGURE 13

FIGURE 14