BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 1, July 1992

RATIONAL FUNCTION CERTIFICATION OF MULTISUM/INTEGRAL/" q " IDENTITIES

HERBERT S. WILF AND DORON ZEILBERGER

ABSTRACT. The method of rational function certification for proving terminating hypergeometric identities is extended from single sums or integrals to multi-integral/sums and " q " integral/sums.

1. INTRODUCTION

Special functions have been defined by Richard Askey [As2] as "functions that occur often enough to merit a name," while Turán [As1, p. 47] defined them as "useful functions." The impact of these special functions on classical mathematics and physics can be gauged by the stature of those whose names they bear: Bessel, Gauss, Hermite, Jacobi, Legendre, Tschebycheff, to name a few. It turns out that most special functions are of *hypergeometric type*, which is to say that they can be written as a sum in which the summand is a hypergeometric term. Also of great interest are the so-called q-analogs of special functions and hypergeometric series, called q-series. These have many applications to number theory, combinatorics, physics, group theory [An], and other areas of science and mathematics.

There are countless identities relating special functions (e.g., [PBM, R, An, As1]). In addition to their intrinsic interest, some of them imply important properties of these special functions, which in turn sometimes imply deep theorems elsewhere in mathematics (e.g., [deB, Ap]). Just as important for mathematics are the extremely successful attempts to instill meaning and insight, both representation-theoretic (e.g., [Mi]) and combinatorial (e.g., [Fo]), into these identities.

The general theory of multivariate hypergeometric functions is currently a very active field, rooted in multivariate statistics and the physics of angular momentum. A very novel and fruitful approach is currently being pursued by Gelfand, Kapranov, Zelevinsky (e.g., [GKZ]) and their collaborators.

We now know [Z1, WZ1] that terminating identities involving sums and integrals of products of special functions of hypergeometric type can be proved by a finite algorithm, viz., find recurrence or differential equations that are satisfied by the left and the right sides of the claimed identity (they always exist), and then compute enough initial values of the two sides to assure that the two recurrence or differential equations have the same solution. We know

Received by the editors December 2, 1991 and, in revised form, January 28, 1992.

1991 *Mathematics Subject Classification*. Primary 05A19, 05A10, 11B65, 33A30, 33A99; Secondary 05A30, 33A65, 35N, 39A10.

Key words and phrases. Identities, hypergeometric, holonomic, recurrence relation, q-sums, multisums, constant term identities, Selberg and Mehta-Dyson integrals.

The first author was supported in part by the U. S. Office of Naval Research.

The second author was supported in part by the U. S. National Science Foundation.

further that for *one-variable* hypergeometric sums [WZ1, WZ2, Z2, Z3] and single "hyperexponential" integrals [AZ] there are *efficient* algorithms for doing this, so computers really can do the job.

Here we announce fast, efficient algorithms for such identities that involve *multiple* sums and integrals of products of special functions of hypergeometric type. We also announce the algorithmic provability of single- and multivariate q-identities and, furthermore, that the algorithms are fast and efficient. The proofs that are generated by these programs are extremely short, and human beings can verify them easily.

Further we show how to extend to "explicitly-evaluable" *multiple* sums and integrals the notions of *rational function certification*, *WZ-pair*, and *companion identity* that were introduced in [WZ1] and [WZ2] for single sums.

The present theory applies directly only to terminating identities, by which we mean that for any specific numerical assignment of the auxiliary parameters the integral-sum is trivially evaluable. However, very often *nonterminating* identities are limiting cases or "analytic continuations" of terminating ones, so our results also bear on them.

Our method proves (or refutes) any such given conjectured identity, but humans are still needed to conjecture interesting ones.

Full details and many examples will appear elsewhere [WZ3]. Our Maple programs are available from < `zeilberg@euclid.math.temple.edu` >.

2. Hypergeometric multivariate identities

For a function $F(\mathbf{k}, \mathbf{y})$ of r discrete variables $\mathbf{k}$ and s continuous variables $\mathbf{y}$ let K_i be the operator defined by $K_i F(\mathbf{k}, \mathbf{y}) = F(k_1, \ldots, k_i + 1, \ldots, k_r, \mathbf{y})$, and let $D_j = \partial/\partial y_j$, for $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

Definition. *A function* $F(\mathbf{k}, \mathbf{y})$ *is a hypergeometric term if* $K_i F/F$ $(i = 1, \ldots, r)$ *and* $D_j F/F$ $(j = 1, \ldots, s)$ *are all rational functions of* $(\mathbf{k}, \mathbf{y})$.

A sequence of special functions of hypergeometric type (in one variable) is a sequence that is of the form $P_n(x) := \sum_k F(n, k) x^k$ where F is a hypergeometric term.

A typical identity in the theory of special functions involves multiple integral-sums of products of polynomials of hypergeometric type. After a full expansion, such an identity is of the form "left side"="right side" where both sides are of the form $\sum_{\mathbf{k}} \int_{\mathbf{y}} F(\mathbf{k}, \mathbf{n}, \mathbf{x}, \mathbf{y}) d\mathbf{y}$. Here F is a hypergeometric term in the discrete multivariables $(\mathbf{k}, \mathbf{n})$ and the continuous multivariables $(\mathbf{x}, \mathbf{y})$. It might happen that one of the sides, say the right side, has no $\sum$'s or $\int$'s in it, i.e., it is already hypergeometric, in which case we speak of *explicit evaluation*.

Finally, we also require that our integrand/summands be holonomic. This is certainly the case for what we call proper-hypergeometric terms, which for purely discrete summands, look like

$$F(n, \mathbf{k}) = \frac{\prod (a_i n + \mathbf{b}_i \cdot \mathbf{k} + c_i)!}{\prod (u_i n + \mathbf{v}_i \cdot \mathbf{k} + w_i)!} P(n, \mathbf{k}) \boldsymbol{\xi}^{\mathbf{k}}, \qquad \text{(P-H)}$$

where the a's and u's are specific integers, $\mathbf{b}$ and $\mathbf{v}$ are vectors of specific integer entries, the c's and the w's are complex numbers that may depend upon additional parameters, P is a polynomial in $\mathbf{k}$, and $\boldsymbol{\xi}$ is a vector of parameters.

This allows us to use the holonomic theory of [Z1]. However for pure multi-sums, we give proofs of our results, with effective bounds, that are self-contained and independent of the theory of holonomic functions.

3. The fundamental theorem

Let $\Delta_i = K_i - 1$ be the forward difference operator in k_i, and let N be the forward shift in n: $Nf(n) = f(n+1)$.

Theorem 1. *Let $F(n, \mathbf{k}, \mathbf{y})$ (resp. $F(x, \mathbf{k}, \mathbf{y})$) be a proper-hypergeometric term, or more generally, a holonomic hypergeometric term, in $(\mathbf{k}, \mathbf{y})$ and n (resp. x), where $n, \mathbf{k}$ are discrete and $x, \mathbf{y}$ are continuous variables. Then there exist a linear ordinary recurrence (resp. differential) operator $P(N, n)$ (resp. $P(D_x, x)$) with polynomial coefficients and rational functions $R_1, \dots, R_r$, $S_1, \dots, S_s$ such that*

$$(1)\qquad P(N, n)F \quad (\textit{resp. } P(D_x, x)F) = \sum_{i=1}^{r} \Delta_{k_i}(R_i F) + \sum_{j=1}^{s} D_{y_j}(S_j F).$$

For proper-hypergeometric F we give explicit a priori bounds for the order of the operator P. By summation-integration of (1) over $\mathbf{k}, \mathbf{y}$ we obtain the following

Corollary A. *If $F(n, \mathbf{k}, \mathbf{y})$ (resp. $F(x, \mathbf{k}, \mathbf{y})$) is as above and of compact support in $(\mathbf{k}, \mathbf{y})$ for every fixed n (resp. x) then*

$$(2)\qquad f(n)\ (\textit{resp. } f(x)) := \sum_{\mathbf{k}} \int_{\mathbf{y}} F\, d\mathbf{y}$$

satisfies a linear recurrence (resp. differential) equation with polynomial coefficients

$$(3)\qquad P(N, n)f(n) \equiv 0 \quad (\textit{resp. } P(D_x, x)f(x) \equiv 0).$$

The denominators of the rational functions $\mathbf{R}, \mathbf{S}$ can be predicted for any given F, and an upper bound for the order of the operator P can be given in advance. Hence by assuming the operator and the rational functions in the most general form with those denominators and with that order, the determination of the unknown operator and rational functions quickly reduces to solving a system of linear equations with *symbolic* coefficients. By [W] this reduces to solving such a system with *numerical* coefficients, and that in turn reduces to solving linear systems with *integer* coefficients, a problem for which fast parallelizable algorithms exist [CC].

Using the terminology of [WZ1, WZ2] one can say that the rational functions $(\mathbf{R}, \mathbf{S})$ *certify* the recurrence (resp. the differential equation) (3).

Corollary B. *Any identity of the form "left side"="right side," where both sides have the form (2) and F is a proper-hypergeometric or holonomic-hypergeometric term, has a two-line elementary proof, constructible by a computer and verifiable by a human or a computer.*

4. Sketch of the proof of the theorem

From the general theory of [Z1] we can find a linear partial recurrence-differential operator $T(n, N, \mathbf{K}, \mathbf{D_y})$, independent of $\mathbf{k}, \mathbf{y}$, that annihilates

F. For proper-hypergeometric functions we have a completely elementary proof that also gives explicit upper bounds for the orders in N and $\mathbf{K}$. This elementary proof was obtained by extending to multisums-integrals Sister Celine Fasenmyer's technique [Fa, R] as systematized by Verbaeten [V].

Once the operator T has been found, we write

$$(4)\quad T(n, N, \mathbf{K}, \mathbf{D_y}) = P(N, n) + \sum_{i=1}^{r}(K_i - 1)T_i(N, \mathbf{K}, \mathbf{D}) + \sum_{j=1}^{s} D_j \widehat{T}_j(N, \mathbf{K}, \mathbf{D}).$$

It is easy to see that this is always possible. Next set

$$(5)\quad \begin{aligned} G_i(n, \mathbf{k}, \mathbf{y}) &:= -T_i F(n, \mathbf{k}, \mathbf{y}) \qquad (i = 1, \dots, r), \\ \widehat{G}_j(n, \mathbf{k}, \mathbf{y}) &:= -\widehat{T}_j F(n, \mathbf{k}, \mathbf{y}) \qquad (j = 1, \dots, s). \end{aligned}$$

Since F is hypergeometric, the G_i and the $\widehat{G}_j$ are rational multiples of F: $G_i = R_i F$, $\widehat{G}_j = S_j F$. Now apply (4) to $F(n, \mathbf{k}, \mathbf{y})$, remembering that $TF = 0$, to get (1). □

5. Discrete and continuous q-analogues

The above extends to multivariate q-hypergeometric identities. Let Q_j be the operator that acts on F by replacing y_j by qy_j wherever it appears. Then we say that a function $F(\mathbf{k}, \mathbf{y})$ is a q-hypergeometric term if for each $i = 1, \dots, r$ and $j = 1, \dots, s$, it is true that $K_i F/F$ and $Q_j F/F$ are rational functions of $(q, q^{k_1}, \dots, q^{k_r}, y_1, \dots, y_s)$. There is also a natural definition of *q-proper-hypergeometric*, which is given in [WZ3].

The fundamental theorem still holds, where the integration is replaced either by Jackson's q-integration [An] or by an ordinary contour integral, or, in the case of a formal Laurent series, by the action of taking "constant term of." Macdonald's q-constant term conjectures ([Ma], see [Gu, GG] for a recent update) for every *specific* root system, fall under the present heading.

6. Explicit closed-form identities: WZ-tuples and companion identities

In the case where the identity "left side"="right side" is such that the right side does not contain any '$\sum$' or '$\int$' signs, i.e., is of closed form, one has an *explicit identity.* If the right side is nonzero one can divide through by it to get an identity of the form

$$\sum_{\mathbf{k}} \int_{\mathbf{y}} F(n, \mathbf{k}, \mathbf{y})\, d\mathbf{y} = 1.$$

Since the summand satisfies (1), the left side, call it $L(n)$, satisfies some linear recurrence $P(N, n)L(n) = 0$, by Corollary A. Often the operator P turns out to be the minimal order recurrence that is satisfied by the sequence that is identically 1, viz. $(N - 1)L(n) = 0$. If that happens then if we let $G_i := -R_i F$ and $H_j := -S_j F$ we find that (1) becomes

$$(6)\quad \Delta_n F + \sum_{i=1}^{r} \Delta_i G_i + \sum_{j=1}^{s} D_{y_j} H_j = 0.$$

We call $(F, \mathbf{G}, \mathbf{H})$ a *WZ-tuple.* It generalizes the idea of *WZ-pair* developed in [WZ1, WZ2]. Recall that a WZ pair gave, as a bonus, a new identity, the *companion identity.* Here, if we sum-integrate (6) w.r.t. all of the variables except *one*, we get a new identity for each choice of that one variable, for a total of $r+s$ new companion identities altogether!

7. Example: The Hille-Hardy Bilinear Formula for Laguerre Polynomials

As an example we will now show the computer proof of the Hille-Hardy formula [R, Theorem 69, p. 212], namely,

$$\frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = \frac{1}{2\pi i}\int_{|u|=\epsilon} u^{-n-1}(1-u)^{-\alpha-1}\exp\left\{-\frac{(x+y)u}{(1-u)}\right\} \times \left(\sum_m \frac{i}{m!(\alpha+1)_m}\left(\frac{xyu}{(1-u)^2}\right)^m\right) du.$$

Many other examples appear in [WZ3].

To this end, it is enough to prove that the right side is annihilated by the well-known second-order differential operator annihilating the Laguerre polynomials, both w.r.t. x and y. Of course, by symmetry, it suffices to do it only for x, but the computer does not mind doing it for both x and y. We still need to prove that the initial conditions match, but they are just the usual defining generating function for the Laguerre polynomials. The computer output was as follows.

Theorem. *Let*

$$F(u, m, x) := \frac{(1-u)^{-\alpha-1}\exp(-(x+y)u/(1-u))(xyu/(1-u)^2)^m}{u^{n+1} m! \Gamma(\alpha+1+m)},$$

and let $a(x)$ be its contour integral w.r.t. u and sum w.r.t. m. Let D_x be differentiation w.r.t. x. The function $a(x)$ satisfies the differential equation

$$(n + (\alpha + 1 - x)D_x + xD_x^2)a(x) = 0.$$

Proof. It is routinely verifiable that

$$(n + (\alpha + 1 - x)D_x + xD_x^2)F(u, m, x) = D_u(-uF(u, m, x)) + \Delta_m(-(m(\alpha+m)/x)F(u, m, x))$$

and the result follows by integrating w.r.t. u and summing w.r.t. m. □

Remark. The phrase "routinely verifiable" in the above means that after carrying out the indicated differentiation and differencing, and after dividing through by F and clearing denominators, what will remain will be a trivially verifiable *polynomial* identity.

Acknowledgment

Many thanks are due to Dr. James C. T. Pool, head of Drexel University's Department of Mathematics and Computer Science, for his kind and generous permission to use his department's computer facilities.

References

[AZ] G. Almkvist and D. Zeilberger, *The method of differentiating under the integral sign*, J. Symbolic Comp. **10** (1990), 571–591.

[An] G. E. Andrews, *q-Series: their development and applications in analysis, number theory, combinatorics, physics and computer algebra*, CBMS Series, vol. 66, Amer. Math. Soc., Providence, RI, 1986.

[Ap] R. Apéry, *Irrationalité de* $\zeta(3)$, Asterisque **61** (1979), 11–13.

[As1] R. A. Askey, *Orthogonal polynomials and special functions*, Regional Conference Series in Appl. Math., vol. 21, SIAM, 1975.

[As2] ——, *Special functions: Group theoretical aspects and applications* (*preface*) (R. A. Askey, T. H. Koornwinder, and W. Schempp, eds.), D. Reidel Publ. Co., Dordrecht, 1984.

[deB] L. deBranges, *A proof of the Bieberbach conjecture*, Acta. Math. **154** (1985), 137–152.

[CC] T.-W. Chou and G. E. Collins, *Algorithms for the solution of systems of linear Diophantine equations*, SIAM J. Comp. **11** (1982), 687–708.

[Fa] Sister M. C. Fasenmyer, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. **53** (1947), 806–812.

[Fo] D. Foata, *Combinatoire des identites sur les polynomes orthogonaux*, in Proc. ICM, Warsaw, Aug. 16-24, 1983, Varsovie, 1983, pp. 1541–1553.

[GG] F. G. Garvan and G. Gonnet, *Macdonald's constant term identities for exceptional root systems*, Bull. Amer. Math. Soc. **24** (1991), 343–347.

[GKZ] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), 255–271.

[Gu] R. A. Gustafson, *A generalization of Selberg's beta integral*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 97–105.

[Ma] I. G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 91–143.

[Mi] W. Miller, *Symmetry and separation of variables*, Encyclopedia Math. Appl., vol. 4, Addison-Wesley, London, 1977.

[PBM] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series*, vol. 3 (translated from Russian by G. G. Gould), Gordon and Breach, New York, 1990.

[R] E. D. Rainville, *Special functions* (reprinted by Chelsea, Bronx, 1971), Macmillan Co., New York, 1960.

[V] P. Verbaeten, *The automatic construction of pure recurrence relations*, Proc. EUROSAM '74, ACM-SIGSAM Bull. **8** (1974), 96–98.

[W] H. S. Wilf, *Finding the kernel of a symbolic matrix*, preprint.

[WZ1] H. S. Wilf and D. Zeilberger, *Rational functions certify combinatorial identities*, J. Amer. Math. Soc. **3** (1990), 147–158.

[WZ2] ——, *Towards computerized proofs of identities*, Bull. Amer. Math. Soc. **23** (1990), 77–84.

[WZ3] ——, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Inventiones Math. (to appear).

[Z1] D. Zeilberger, *A holonomic systems approach to special functions identities*, J. Comp. Appl. Math. **32** (1990), 321–368.

[Z2] ——, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80** (1990), 207–211.

[Z3] ——, *The method of creative telescoping*, J. Symbolic Comp. **11** (1991), 195–204.

University of Pennsylvania, Philadelphia, Pennsylvania 19104-6395
E-mail address: wilf@central.cis.upenn.edu

Temple University, Philadelphia, Pennsylvania 19122-2585
E-mail address: zeilberg@euclid.math.temple.edu