Rational General Solutions of Algebraic Ordinary Differential Equations*

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ABSTRACT

We give a necessary and sufficient condition for an algebraic ODE to have a rational type general solution. For an autonomous first order ODE, we give an algorithm to compute a rational general solution if it exists. The algorithm is based on the relation between rational solutions of the first order ODE and rational parametrizations of the plane algebraic curve defined by the first order ODE and Padé approximants.

Categories and Subject Descriptors

I.1.2 [**SYMBOLIC AND ALGEBRAIC MANIPULATION**]: Algorithms—*Algebraic algorithms*

General Terms

Algorithms, Theory

Keywords

Rational general solution, algebraic differential equation, autonomous first order ODE, algebraic curve, rational parametrization, Padé approximants

1. INTRODUCTION

In a pioneering paper [24], Risch gave an algorithm to find elementary function solutions for the simplest differential equation $y' = f(x)$, that is, to find elementary function solutions to integration $\int f(x)\mathrm{d}x$. In [20], Kovacic presented an effective method to find Liouvillian solutions for second order linear homogeneous differential equations and Riccati equations. In [28], Singer established the general framework for finding Liouvillian solutions for general linear homogeneous ODEs. Many other interesting results on finding Liouvillian solutions of linear ODEs were reported in [1, 4, 5, 6, 7, 8, 12, 22, 30, 31, 33, 34]. In [21], Li and Schwarz gave the first method to find rational solutions for a class of partial differential equations.

Most of these results are limited to the linear case or some special type nonlinear equations. There seems exist no general methods to find closed form solutions for nonlinear differential equations. With respect to the particular ODEs of the form $y' = R(x, y)$ where $R(x, y)$ is a rational function, Darboux and Poincaré made important contributions [23]. More recently, Cerveau, Neto and Carnicer also made important progresses [9, 10]. In [11], Cano proposed an algorithm to find their polynomial solutions. In [29], Singer studied the Liouvillian first integrals of differential equations. In [3], Bronstein gave an effective method to compute rational solutions of the Ricatti equations. In [18], Hubert gave a method to compute a basis of the general solutions of first order ODES and applied it to study the local behavior of the solutions.

In this paper, we try to find rational type general solutions to the autonomous first order ODEs (with constant coefficients). For example, the general solution for $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0$ is $y = \frac{1}{x+c}$, where c is an arbitrary constant. The motivation of finding the rational general solutions to algebraic ODEs is as follows. Converting between implicit representation and parametric representation of (differential) varieties is one of the basic topics in (differential) algebraic geometry. In the differential case, implicitization algorithms were given in [14]. As far as we know, there exist no results on parametrization of differential varieties. The results in this paper could be considered as a first step to the rational parametrization problem for differential varieties.

Three main results are given in this paper. In Section 2, we give a sufficient and necessary condition for an algebraic ODE to have a rational general solutions, by constructing a differential equation whose solutions are the rational functions.

In Section 3, by treating the variable and its derivative as independent variables, an autonomous first order ODE defines an algebraic plane curve. We show that a nontrivial rational solution of the autonomous first order ODE and its derivative provides a proper parametrization of the corresponding curve. From this observation, we may obtain a degree bound for the rational solutions of it. We also show how to obtain a rational solution to the autonomous first order ODE from the a rational parametrization of the corresponding plane curve.

In Section 4, based on the above results and Padé ap-

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proximants we give an algorithm to find a rational general solution for an autonomous first order ODE. The algorithm is implemented and experimental results are also reported.

2. RATIONAL GENERAL SOLUTIONS OF ALGEBRAIC ODES

2.1 Definition of rational general solutions

In the following, let $\mathbf{K} = \mathbf{Q}(x)$ be the differential field of rational functions in x with differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ and y an indeterminate over $\mathbf{K}$. We denote by y_i the i-th derivative of y. We use $\mathbf{K}\{y\}$ to denote the ring of differential polynomials over differential field $\mathbf{K}$, which consists of the polynomials in the y_i with coefficients in $\mathbf{K}$. All differential polynomials in this paper are in $\mathbf{K}\{y\}$. Let Σ be a system of differential polynomials in $\mathbf{K}\{y\}$. A *zero* of Σ is an element in a universal extension field of $\mathbf{K}$, which vanishes every differential polynomial in Σ [25]. The totality of the zeros in $\mathbf{K}$ is denoted by Zero(Σ).

Let $P \in \mathbf{K}\{y\}/\mathbf{K}$. We denote by $\mathrm{ord}(P)$ the highest derivative of y in P, called the *order* of P. Let $o = \mathrm{ord}(P) > 0$. We may write P as follows

$$P = a_d y_o^d + a_{d-1} y_o^{d-1} + \ldots + a_0$$

where a_i are polynomials in $y, y_1, \ldots, y_{o-1}$ for $i = 0, \ldots, d$ and $a_d \neq 0$. a_d is called the *initial* of P and $S = \frac{\partial P}{\partial y_o}$ is called the *separant* of P. The k-th derivative of P is denoted by $P^{(k)}$. Let S be the separant of P, $o = \mathrm{ord}(P)$ and $k > 0$. Then we have

$$P^{(k)} = S y_{o+k} - R_k \tag{1}$$

where R_k is of lower order than $o + k$.

Let P be a differential polynomial of order o. A differential polynomial Q is said to be *reduced* with respect to P if $\mathrm{ord}(Q) < o$ or $\mathrm{ord}(Q) = o$ and $\deg(Q, y_o) < \deg(P, y_o)$. For two differential polynomials P and Q, let $R = \mathrm{prem}(P, Q)$ be the differential pseudo-remainder of P with respect to Q. We have the following *differential remainder formula* for R (see [19, 25])

$$JP = \sum_i B_i Q^{(i)} + R$$

where J is a product of certain powers of the initial and separant of Q and B_i, R are differential polynomials. Moreover, R is reduced with respect to Q. For a differential polynomial P with order o, we say that P is *irreducible* if P is irreducible when P is treated as a polynomial in $\mathbf{K}[y, y_1, \ldots, y_o]$.

Let $P \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial and

$$\Sigma_P = \{A \in \mathbf{K}\{y\} | SA \equiv 0 \, mod \, \{P\}\}. \tag{2}$$

where$\{P\}$ is the differential ideal generated by P [19, 25]. Ritt proved that [25]

LEMMA 1. *Σ_P is a prime differential ideal and a differential polynomial Q belongs to Σ_P iff prem$(Q, P) = 0$.*

Let Σ be a non-trivial prime ideal in $\mathbf{K}\{y\}$. A zero η of Σ is called *a generic zero* of Σ if for any differential polynomial P, $P(\eta) = 0$ implies that $P \in \Sigma$. It is well known that an ideal Σ is prime iff it has a generic zero [25].

A *universal constant extension* of $\mathbf{Q}$ is obtained by first adding an infinite number of arbitrary constants to $\mathbf{Q}$ and then taking the algebraic closure.

Definition 1. Let $F \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial. A general solution of $F = 0$ is defined as a generic zero of Σ_F. A rational general solution of $F = 0$ is defined as a general solution of $F = 0$ of the form

$$\hat{y} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{x^m + b_{m-1} x^{m-1} + \ldots + b_0} \tag{3}$$

where a_i, b_j are in a universal constant extension of $\mathbf{Q}$.

Notation 1. $\deg_x(\hat{y})$:=max$\{n, m\}$ where $\hat{y}$ is as in (3) and $a_n \neq 0$.

As a consequence of Lemma 1, we have

LEMMA 2. *Let $F \in \mathbf{K}\{y\}/\mathbf{K}$ be an irreducible differential polynomial with a generic solution η. Then for a differential polynomial P we have $P(\eta) = 0$ iff prem$(P, F) = 0$.*

In the literature in general, a *general solution* of $F = 0$ is defined as a family of solutions with o independent parameters in a loose sense where $o = \mathrm{ord}(F)$. The definition given by Ritt is more precise. Theorem 6 in [19] (Chapter 2, section 12) tells us that Ritt's definition of general solution is equivalent to the definition in classical literature.

2.2 A Criterion for existence of rational general solutions

Let $\mathcal{D}_{n,m}$ be the following differential polynomial in y:

$$\begin{vmatrix} \binom{n+1}{0} y_{n+1} & \binom{n+1}{1} y_n & \ldots & \binom{n+1}{m} y_{n+1-m} \\ \binom{n+2}{0} y_{n+2} & \binom{n+2}{1} y_{n+1} & \ldots & \binom{n+2}{m} y_{n+2-m} \\ \vdots & \vdots & \ldots & \vdots \\ \binom{n+m+1}{0} y_{n+m+1} & \binom{n+m+1}{1} y_{n+m} & \ldots & \binom{n+m+1}{m} y_{n+1} \end{vmatrix}$$

where $\binom{n}{k}$ are binomial coefficients and $\binom{n}{k} = 0$ for $k > n$.

Note that when $m = 0$, $\mathcal{D}(n, 0) = y_{n+1}$, whose solutions are $c_n x^n + c_{n-1} x^{n-1} + \ldots + c_0$ where c_i are arbitrary constants.

LEMMA 3. *The solutions $\hat{y}$ of $\mathcal{D}_{n,m} = 0$ have the following form:*

$$\hat{y} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0}$$

where a_i, b_j are constants.

PROOF. We prove it by induction on m. If $m = 0$, then $\mathcal{D}_{n,0} = y_{n+1}$. It is clear. We suppose that for $m < k + 1$ the theorem is true. Now we will prove the theorem for $m = k + 1$. If $\mathcal{D}_{n,k}(\hat{y}) = 0$, by the induction hypothesis it is true. Now we suppose that $\mathcal{D}_{n,k}(\hat{y}) \neq 0$. Since $\mathcal{D}_{n,k+1}(\hat{y}) = 0$, there exist $Q_0, Q_1, \ldots, Q_{k+1}$ in $\mathbf{K}$ (not all Q_i are zero) such that

$$\begin{pmatrix} \binom{n+1}{0} \hat{y}_{n+1} & \ldots & \binom{n+1}{k+1} \hat{y}_{n-k} \\ \binom{n+2}{0} \hat{y}_{n+2} & \ldots & \binom{n+2}{k+1} \hat{y}_{n+1-k} \\ \vdots & \ldots & \vdots \\ \binom{n+k+2}{0} \hat{y}_{n+k+2} & \ldots & \binom{n+k+2}{k+1} \hat{y}_{n+1} \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{k+1} \end{pmatrix} = 0$$

Without loss of generality, we can assume that $Q_{k+1} = 0$ or 1. Then we have $\sum_{i=0}^{k+1} \binom{j}{i} \hat{y}_{j-i} Q_i = 0$ for $j = n+1, \ldots, n+k+2$. Differentiating $\sum_{i=0}^{k+1} \binom{j}{i} \hat{y}_{j-i} Q_i = 0$, we have

$$\left(\sum_{i=0}^{k+1} \binom{j}{i} \hat{y}_{j-i} Q_i\right)'$$

$$= \sum_{i=0}^{k+1} \binom{j}{i} \hat{y}_{j-i+1} Q_i + \sum_{i=0}^{k+1} \binom{j}{i} \hat{y}_{j-i} Q_i^{'} = 0 \quad (4)$$

Using the equation $\binom{j+1}{i} = \binom{j}{i} + \binom{j}{i-1}$, we have

$$\begin{aligned} &\sum_{i=0}^{k+1} \binom{j+1}{i} \hat{y}_{j+1-i} Q_i \\ = &\sum_{i=0}^{k+1} \binom{j}{i} \hat{y}_{j+1-i} Q_i + \sum_{i=0}^{k} \binom{j}{i} \hat{y}_{j-i} Q_{i+1} = 0 \quad (5) \end{aligned}$$

Then (4) – (5) implies that

$$\sum_{i=0}^{k} \binom{j}{i} \hat{y}_{j-i} (Q_i^{'} - Q_{i+1}) + \binom{j}{k+1} \hat{y}_{j-k-1} Q_{k+1}^{'} = 0$$

for $j = n+1, \ldots, n+k+1$ where $\binom{j}{k+1} = 0$ if $j > k+1$. Since $Q_{k+1}^{'} = 0$, we have $\sum_{i=0}^{k} \binom{j}{i} \hat{y}_{j-i} (Q_i^{'} - Q_{i+1}) = 0$ for $j = n+1, \ldots, n+k+1$ which can be written as the matrix form:

$$A \begin{pmatrix} Q_0^{'} - Q_1 \\ Q_1^{'} - Q_2 \\ \vdots \\ Q_k^{'} - Q_{k+1} \end{pmatrix} = 0$$

where

$$A = \begin{pmatrix} \binom{n+1}{0}\hat{y}_{n+1} & \binom{n+1}{1}\hat{y}_{n} & \cdots & \binom{n+1}{k}\hat{y}_{n-k} \\ \binom{n+2}{0}\hat{y}_{n+2} & \binom{n+2}{1}\hat{y}_{n+1} & \cdots & \binom{n+2}{k}\hat{y}_{n+1-k} \\ \vdots & \vdots & \cdots & \vdots \\ \binom{n+k+1}{0}\hat{y}_{n+k+1} & \binom{n+k+1}{1}\hat{y}_{n+k} & \cdots & \binom{n+k+1}{k}\hat{y}_{n+1} \end{pmatrix}$$

Since $\mathcal{D}_{n,k}(\hat{y}) \neq 0$, we have $Q_{i+1} = Q_i^{'}$ for $i = 0, 1, \ldots, k$. Hence $Q_0 = b_{k+1}x^{k+1} + \ldots + b_0$ where b_i are arbitrary constants and $b_{k+1} = 0$ or $\frac{1}{(k+1)!}$ (since $Q_{k+1} = 0$ or 1). Then

$$\begin{aligned} \sum_{i=0}^{k+1} \binom{n+1}{i} \hat{y}_{n+1-i} Q_i &= \sum_{i=0}^{n+1} \binom{n+1}{i} \hat{y}_{n+1-i} Q_i = 0 \\ &\Longrightarrow (\hat{y}Q_0)^{(n+1)} = 0 \\ &\Longrightarrow \hat{y} = \frac{a_n x^n + \ldots + a_0}{Q_0} \end{aligned}$$

where a_i are arbitrary constants. The proof is complete. □

By Lemma 3, we can prove the following theorem easily.

Theorem 1. *Let F be an irreducible differential polynomial. Then the differential equation $F = 0$ has a rational general solution $\hat{y}$ iff there exist non-negative integers n and m such that* $\mathrm{prem}(\mathcal{D}_{n,m}, F) = 0$

Proof. ($\Rightarrow$) Let $\hat{y} = \frac{P(x)}{Q(x)}$ be a rational general solution of $F = 0$. Let $n \geq deg(P(x))$ and $m \geq deg(Q(x))$. Then from Lemmas 2 and 3

$$\mathcal{D}_{n,m}(\hat{y}) = 0 \Rightarrow \mathcal{D}_{n,m} \in \Sigma_F \Rightarrow \mathrm{prem}(\mathcal{D}_{n,m}, F) = 0$$

($\Leftarrow$) By Lemma 1, $\mathrm{prem}(\mathcal{D}_{n,m}, F) = 0$ implies that $\mathcal{D}_{n,m} \in \Sigma_F$. Assume that m is the least integer such that $\mathcal{D}_{n,m} \in \Sigma_F$. Then all the zeros of Σ_F must have the form

$$\bar{y} = \frac{\bar{a}_n x^n + \bar{a}_{n-1}x^{n-1} + \ldots + \bar{a}_0}{\bar{b}_m x^m + \bar{b}_{m-1}x^{m-1} + \ldots + \bar{b}_0}$$

In particularly, the generic zero of Σ_F has the following form

$$\hat{y} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0}$$

Moreover, $b_m \neq 0$. Otherwise, we would have $\mathcal{D}_{n,m-1}(\hat{y}) = 0$ which implies that $\mathcal{D}_{n,m-1} \in \Sigma_F$, a contradiction. So the generic zero has the form (3). The proof is complete. □

3. RATIONAL GENERAL SOLUTION OF AUTONOMOUS FIRST ORDER ODE

In this and the next sections, let $\bar{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$. F will always be a first order non-zero differential polynomial with coefficients in $\mathbf{Q}$ and irreducible over $\bar{\mathbf{Q}}$. We call a rational solution $\bar{y}$ of $F = 0$ *nontrivial* if $\deg_x \bar{y} > 0$.

It is a trivial fact that for an algebraic ODE with constant coefficients, the solution set is invariant by the translation of the independent variable x. Moreover, we have the following fact.

Lemma 4. *Let $\bar{y} = \frac{\bar{a}_n x^n + \ldots + \bar{a}_0}{x^m + \ldots + \bar{b}_0}$ be a nontrivial solution of $F = 0$, where $\bar{a}_i$, $\bar{b}_j \in \mathbf{Q}$, and $\bar{a}_n \neq 0$. Then*

$$\hat{y} = \frac{\bar{a}_n (x+c)^n + \ldots + \bar{a}_0}{(x+c)^m + \ldots + \bar{b}_0}$$

is a rational general solution of $F = 0$, where c is an arbitrary constant.

Proof. It is easy to show that $\hat{y}$ is still a zero of Σ_F. For any $G \in \mathbf{K}\{y\}$ satisfying $G(\hat{y}) = 0$, let $R = \mathrm{prem}(G, F)$. Then $R(\hat{y}) = 0$. Suppose that $R \neq 0$. Since F is irreducible and $\deg(R, y_1) < \deg(F, y_1)$, there are two differential polynomials $P, Q \in \mathbf{K}\{y\}$ such that $PF + QR \in \mathbf{K}[y]$ and $PF + QR \neq 0$. Thus $(PF + QR)(\hat{y}) = 0$. Because c is an arbitrary constant which is transcendental over $\mathbf{K}$, we have $PF + QR = 0$, a contradiction. Hence $R = 0$ which means that $G \in \Sigma_F$. So $\hat{y}$ is a generic zero of Σ_F. The proof is complete. □

The above lemma reduces the problem of finding a rational general solution to the problem of finding a nontrivial rational solution. In what below, we will show how to find a nontrivial rational solution.

3.1 Parametrization of algebraic curves

In this subsection, we will introduce some basic conceptions about the parametrization of an algebraic plane curve. Let $F(x, y)$ be an polynomial in $\mathbf{Q}[x, y]$ and irreducible over $\bar{\mathbf{Q}}$

Definition 2. $(x, y) = (r(t), s(t))$ is called a *parametrization* of $F(x, y) = 0$ if $F(r(t), s(t)) = 0$ where $r(t), s(t) \in \bar{\mathbf{Q}}(t)$ and not all of them are in $\bar{\mathbf{Q}}$. A parametrization $(r(t), s(t))$ is called proper if $\bar{\mathbf{Q}}(r(t), s(t)) = \bar{\mathbf{Q}}(t)$.

Lüroth's Theorem guarantees that there always exists a proper parametrization if a parametrization exists [16, 35].

Lemma 5. *A proper parametrization has the following properties [26]:*

1. $\deg_t(r(t)) = deg(F, y)$
2. $\deg_t(s(t)) = deg(F, x)$
3. *If $(p(t), q(t))$ is another proper parametrization of $F(x, y)$, then there exists $f(t) = \frac{at+b}{ct+d}$ such that $p(t) = r(f(t))$, $q(t) = s(f(t))$.*

3.2 Autonomous first order ODEs

Since F has order one and constant coefficients, we can consider it as an algebraic polynomial in y, y_1.

Notation 2. We use $F(y, y_1)$ to denote F as an algebraic polynomial in y and y_1 which defines an algebraic curve.

If $\bar{y} = r(x)$ is a nontrivial rational solution of $F = 0$, then $(r(x), r'(x))$ can be regarded as a parametrization of $F(y, y_1) = 0$. Moreover, we will show that $(r(x), r'(x))$ is a proper parametrization of $F(y, y_1) = 0$.

LEMMA 6. *Let $f(x) = \frac{p(x)}{q(x)} \notin \bar{\mathbf{Q}}$ be a rational function in x such that $\gcd(p(x), q(x)) = 1$. Then $\bar{\mathbf{Q}}(f(x)) \neq \bar{\mathbf{Q}}(f'(x))$.*

PROOF. If $f'(x) \in \bar{\mathbf{Q}}$ then the result is clearly true. Otherwise, since $f(x), f'(x)$ are transcendental over $\bar{\mathbf{Q}}$, if $\bar{\mathbf{Q}}(f(x)) = \bar{\mathbf{Q}}(f'(x))$, from the Theorem in Section 63 of [35], we have

$$f(x) = \frac{af'(x) + b}{cf'(x) + d}$$

where $a, b, c, d \in \bar{\mathbf{Q}}$. Then

$$\frac{p(x)}{q(x)} = \frac{a(p'(x)q(x) - p(x)q'(x)) + bq(x)^2}{c(p'(x)q(x) - p(x)q'(x)) + dq(x)^2}$$

which implies that $q(x)|cp(x)q'(x)$ because $\gcd(p(x), q(x)) = 1$. So $c = 0$ or $q'(x) = 0$ which implies that $f(x) = (\frac{a}{d})f'(x) + \frac{b}{d}$ or $p(x) = c_1p'(x) + c_2$ where $c_1, c_2 \in \bar{\mathbf{Q}}$. This is impossible, because $f(x)$ is a rational function and $p(x)$ is a nonconstant polynomial if $q(x) \in \bar{\mathbf{Q}}$. $\Box$

THEOREM 2. *Let $f(x)$ be the same as in Lemma 6. Then $\bar{\mathbf{Q}}(f(x), f'(x)) = \bar{\mathbf{Q}}(x)$.*

PROOF. From Lüroth's Theorem, there exists $g(x) = \frac{u(x)}{v(x)}$ such that $\bar{\mathbf{Q}}(f(x), f'(x)) = \bar{\mathbf{Q}}(g(x))$, where $u(x), v(x) \in \bar{\mathbf{Q}}[x]$, $\gcd(u(x), v(x)) = 1$. We may assume that $\deg(u) > \deg(v)$. Otherwise, we have $u/v = c + w/v$ where $c \in \bar{\mathbf{Q}}$ and $\deg(w) < \deg(v)$, and v/w is also a generator of $\bar{\mathbf{Q}}(g(x))$. Then we have

$$\begin{aligned} f(x) &= \frac{p_1(g(x))}{q_1(g(x))} \\ f'(x) &= \frac{p_2(g(x))}{q_2(g(x))} = \frac{g'(x)(p_1'q_1 - p_1q_1')}{q_1^2} \end{aligned}$$

which implies that $g'(x) \in \bar{\mathbf{Q}}(g(x))$. If $g'(x) \notin \bar{\mathbf{Q}}$, we have

$$[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g'(x))] = [\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g(x))][\bar{\mathbf{Q}}(g(x)) : \bar{\mathbf{Q}}(g'(x))]$$

However, we have $[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g(x))] = \deg(u)$ and $[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g'(x))] \leq 2\deg(u) - 1$. Hence $[\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g(x))] = [\bar{\mathbf{Q}}(x) : \bar{\mathbf{Q}}(g'(x))]$. That is, $\bar{\mathbf{Q}}(g'(x)) = \bar{\mathbf{Q}}(g(x))$, a contradiction by Lemma 6. Hence, $g'(x) \in \bar{\mathbf{Q}}$ which implies that $g(x) = ax + b$. The proof is complete. $\Box$

LEMMA 7. *Let $f(x)$ be the same as in Lemma 6. Then $\deg_x(f(x)) - 1 \leq \deg_x(f'(x)) \leq 2\deg_x(f(x))$.*

PROOF. Inequality $\deg_x(f'(x)) \leq 2\deg_x(f(x))$ comes directly from the definition. If $q(x) \in \bar{\mathbf{Q}}$, then $\deg_x((\frac{p(x)}{q(x)})') = \deg_x(\frac{p(x)}{q(x)}) - 1$. Assume that $q(x) \notin \bar{\mathbf{Q}}$. Then we can assume that

$$q(x) = (x - a_1)^{\alpha_1}(x - a_2)^{\alpha_2} \dots (x - a_r)^{\alpha_r}$$

Then $(\frac{p(x)}{q(x)})' = \frac{U(x)}{V(x)}$ where

$$\begin{aligned} U(x) &= p' \prod (x - a_i) - p(\sum_{i=1}^{r} \prod_{j \neq i} \alpha_i(x - a_j)) \\ V(x) &= (x - a_1)^{\alpha_1+1}(x - a_2)^{\alpha_2+1} \dots (x - a_r)^{\alpha_r+1} \end{aligned}$$

Since $U(x)$ and $V(x)$ have no common divisors, we have $\deg_x((\frac{p(x)}{q(x)})') = max\{\deg(p) + r - 1, \deg(q) + r\}$ which is greater than $\deg_x(\frac{p(x)}{q(x)}) - 1$. The proof is complete. $\Box$

Theorem 2 implies that $(\bar{y}, \bar{y}_1)$ is a proper parametrization of $F(y, y_1) = 0$ if $\bar{y}$ is a nontrivial rational solution of $F = 0$. From Lemmas 5 and 7, we have proved the following theorem.

THEOREM 3. *If $F = 0$ has a rational general solution $\hat{y}$, then we have*

$$\begin{cases} \deg_x(\hat{y}) = \deg(F, y_1) \\ \deg(F, y_1) - 1 \leq \deg(F, y) \leq 2\deg(F, y_1) \end{cases}$$

As a direct consequence of Theorems 1 and 3, we could decide whether F has a rational general solution as follows.

THEOREM 4. *Let F be an irreducible autonomous first order ODE in $\bar{\mathbf{Q}}\{y\}$ and $d = \deg(F, y_1)$. Then $F = 0$ has a rational general solution iff $prem(\mathcal{D}_{d,d}, F) = 0$.*

By Theorem 4, we may find a rational solution to $F = 0$ as follows. Let $d = \deg(F, y_1)$. Substituting an arbitrary rational function (3) of degree d into $F = 0$, we have $F = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials in x whose coefficients are polynomials in a_i, b_j. Let PS and DS be the coefficients of $P(x)$ and $Q(x)$. Then (3) is a rational solution to $F = 0$ iff a_i, b_j are zeros of the polynomial equations in PS that do not vanish the polynomial equations in DS. This method is not efficient for large d since it involves the solution of a nonlinear algebraic equation system in $2d$ variables. We will give a more efficient algorithm below.

3.3 An algorithm based on parametrization

From Lemma 5, we can construct a nontrivial rational solution of $F = 0$ from a proper parametrization of $F(y, y_1) = 0$.

THEOREM 5. *Let $y = r(x), y_1 = s(x)$ be a proper rational parametrization of $F(y, y_1) = 0$, where $r(x), s(x) \in \bar{\mathbf{Q}}(x)$. Then $F = 0$ has a rational general solution iff we have the following relations*

$$ar'(x) = s(x) \quad or \quad a(x - b)^2r'(x) = s(x) \tag{6}$$

where $a, b \in \bar{\mathbf{Q}}$ and $a \neq 0$. If one of the above relations is true, then replacing x by $a(x + c)$ (or $b - \frac{1}{a(x+c)}$) in $y = r(x)$, we obtain a rational general solution of $F = 0$, where c is an arbitrary constant.

PROOF. Let $\bar{y} = q(x)$ be a nontrivial rational solution of $F = 0$. By Theorem 2, $(q(x), q'(x))$ is also a proper parametrization of $F(y, y_1) = 0$. Hence there exists $f(x) = \frac{c_1x + c_2}{c_3x + c_4}$ where $c_1c_4 - c_2c_3 \neq 0$ such that

$$\begin{aligned} q(x) &= r(f(x)), \\ q'(x) &= s(f(x)) = (r(f(x)))' = f'(x)r'(f(x)). \end{aligned} \tag{7}$$

If $c_3 = 0$, then $f'(x) = \frac{c_1}{c_4}$. From (7), we have $s(f(x)) = ar'(f(x))$ where $f(x) = ax + c, a = \frac{c_1}{c_4}, c = \frac{c_2}{c_4}$. If $c_3 \neq 0$,

$f(x) = c_1/c_3 + \frac{c_2c_3 - c_1c_4}{c_3(c_3x - x_4)}$. Then $f(x)' = (c_1c_4 - c_2c_3)/(c_3x + c_4)^2 = \frac{c_3^2(f(x) - c_1/c_3)^2}{c_1c_4 - c_2c_3}$. As a consequence of (7) , we have $a(x-b)^2 r'(x) = s(x)$ where $a = \frac{c_3^2}{c_1c_4 - c_2c_3}$ and $b = \frac{c_1}{c_3}$.

In both cases, we obtain a rational solution of $F = 0$: $q(x) = r(f(x))$. From Lemma 4, the general solution of $F = 0$ can be obtained by replacing x by $x + c$. The other direction of the theorem is easy. If (6) is valid, let $q(x) = r(f(x))$. From (7), we have $q'(x) = (r(f(x)))' = s(f(x))$, which implies that $F(q(x), q'(x)) = 0$. That is, $q(x)$ is a rational solution to $F = 0$. □

In paper [27], Sendra and Winkler proved that for a rational algebraic curve defined by a polynomial over $\mathbf{Q}$ which is irreducible over $\bar{\mathbf{Q}}$, it can be parametrized over an extension field of $\mathbf{Q}$ with degree at most two. Theorem 6 will tell us that $F(y, y_1) = 0$ is a special rational curve which can always be parametrized over $\mathbf{Q}$.

THEOREM 6. *If $F = 0$ has a rational general solution, then the coefficients of the rational general solution can be chosen in $\mathbf{Q}$.*

PROOF. We need only to prove that the coefficients of a nontrivial rational solution of $F = 0$ can be chosen in $\mathbf{Q}$. From Theorem 3.1 in the paper [27] and Theorem 5, we know that there exists a nontrivial rational solution $r(x)$ of $F = 0$ whose coefficients belong to $\mathbf{Q}(\alpha)$ where $\alpha^2 \in \mathbf{Q}$. We can assume that $r(x) = \frac{\alpha p_1(x) + p_2(x)}{x^m + \alpha q_1(x) + q_2(x)}$ where $p_i(x), q_j(x) \in \mathbf{Q}(x)$. Assume that $\alpha p_1(x) + p_2(x)$ and $x^m + \alpha q_1(x) + q_2(x)$ have no common divisors over $\mathbf{Q}(\alpha)[x]$. We further assume that $\deg(q_j(x)) \le m - 2$ by Lemma 4 and if $m = 1$ then $q_j(x) = 0$. Now if $\alpha \in \mathbf{Q}$ then it is nothing need to be proved. We suppose that $\alpha \notin \mathbf{Q}$. It is easy to check that $\bar{r}(x) = \frac{-\alpha p_1(x) + p_2(x)}{x^m - \alpha q_1(x) + q_2(x)}$ is also a nontrivial rational solution of $F = 0$. Since both $r(x)$ and $\bar{r}(x)$ are proper parametrizations of $F(y, y_1) = 0$, there exists an $f(x)$ such that $r(x) = \bar{r}(f(x))$ and $r'(x) = \bar{r}'(f(x))$. Since $r'(x) = f'(x)\bar{r}'(f(x))$, we have $f'(x) = 1$ which implies that $f(x) = x + c$ where $c \in \mathbf{Q}(\alpha)$. Thus

$$\frac{\alpha p_1(x) + p_2(x)}{x^m + \alpha q_1(x) + q_2(x)} = \frac{-\alpha p_1(x+c) + p_2(x+c)}{(x+c)^m - \alpha q_1(x+c) + q_2(x+c)}$$

Since $\alpha p_1(x) + p_2(x)$ and $x^m + \alpha q_1(x) + q_2(x)$ have no common divisors, we have

$$x^m + \alpha q_1(x) + q_2(x) = (x+c)^m - \alpha q_1(x+c) + q_2(x+c)$$

If $m > 0$, we have $c = 0$ because $\deg(q_j(x)) \le m - 2$, which implies that $p_1(x) = q_1(x) = 0$. If $m = 0$, then $r(x)$ is a polynomial. We can assume that $r(x) = (a_n\alpha + \tilde{a}_n)x^n + \alpha p_1(x) + p_2(x)$ where $p_i(x) \in \mathbf{Q}(x)$, $\deg(p_i(x)) \le n - 2$ and $a_n, \tilde{a}_n \in \mathbf{Q}$, at least one of a_n and $\tilde{a}_n$ is not 0. In a similar way, we have $a_n = 0$ and $p_1(x) = 0$. The proof is complete. □

As a consequence of Theorem 6, if the curve defined by the autonomous first order ODE is not a continuous curve over the real Euclidean plane, then it must not have rational solutions.

ALGORITHM 1. *The input is a first order irreducible (over $\bar{\mathbf{Q}}$) differential polynomial F with coefficients in $\mathbf{Q}$. The output is a rational general solution of $F = 0$ if it exists.*

1. Let $d = \deg(F, y_1)$ and $e = \deg(F, y)$. If $e < d - 1$ or $e > 2d$, then by Theorem 3, the algorithm terminates and $F = 0$ has no rational general solutions.

2. Compute a proper parametrization $(r(x), s(x))$ of $F(y, y_1)$ with algorithms in [2, 15, 26, 27, 32]. Using the method in [27], we may find a parametrization in $\mathbf{Q}(x)$, since the curve has such a parametrization by Theorem 6.

3. Let $A = s(x)/r'(x)$

   (a) If $A = a \in \mathbf{Q}$, then substituting x by $a(x + c)$ in $r(x)$, we get a rational general solution $\hat{y} = r(a(x + c))$ for $F = 0$.

   (b) If $A = a(x - b)^2$ for $a, b \in \mathbf{Q}$, then substituting x by $\frac{ab(x+c)-1}{a(x+c)}$ in $r(x)$, we get $\hat{y} = r(\frac{ab(x+c)-1}{a(x+c)})$.

   (c) Otherwise, by Theorem 5, the algorithm terminates and $F = 0$ has no rational general solutions.

From Theorem 5, we know that the above algorithm is correct. The complexity of the above algorithm depends entirely on the complexity of the parametrization algorithm.

Now we give an example.

EXAMPLE 1. *Let*

$$F = y_1^3 + 4y_1^2 + (27y^2 + 4)y_1 + 27y^4 + 4y^2$$

1. $d = 3, e = 4$. We have $d - 1 < e < 2d$.

2. $F(y, y_1) = 0$ has three double points: $(0, -2)$, $(\frac{2\sqrt{15}i}{9}, \frac{4}{3})$, $(-\frac{2\sqrt{15}i}{9}, \frac{4}{3})$. Then we have a proper parametrization:

$$\begin{cases} r = 216x^3 + 6x \\ s = -3888x^4 - 36x^2 \end{cases}$$

3. Since $r' = 648x^2 + 6$, we have

$$A = s/r' = -6x^2.$$

That is, $a = -6, b = 0$.

4. Let

$$\hat{y} = 216(\frac{1}{6(x+c)})^3 + 6(\frac{1}{6(x+c)}) = \frac{(x+c)^2 + 1}{(x+c)^3}$$

Then $\hat{y} = \frac{(x+c)^2+1}{(x+c)^3}$ is a rational general solution of $F = 0$.

4. AN EFFICIENT ALGORITHM FOR AUTONOMOUS FIRST ORDER ODES

Algorithm 1 depends on the rational parametrization of plane algebraic curve, which is computationally difficult. In this section, we will give a more effective method by Padé approximants.

4.1 Padé Approximants

The Padé approximants are a particular type of rational fraction approximation to the value of a function. It constructs the rational fraction from the Taylor series expansion of the original function. Its definition is given below [17]:

Definition 3. For the formal power series $A(x) = \sum_0^\infty a_j x^j$ and two non-negative integers L and M, the (L, M) Padé approximant to $A(x)$ is the rational fraction

$$[L \setminus M] = \frac{P_L(x)}{Q_M(x)}$$

such that

$$A(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1})$$

where $P_L(x)$ is a polynomial with degree not greater than L and $Q_M(x)$ is a polynomial with degree not greater than M. Moreover, $P_L(x)$ and $Q_M(x)$ are relatively prime and $Q_M(0) = 1$.

Let $P_L(x) = \sum_0^L p_i x^i$ and $Q_M(x) = \sum_0^M q_i x^i$. We can compute $P_L(x)$ and $Q_M(x)$ with the following equations:

$$\begin{gathered} a_0 = p_0 \\ a_1 + a_0 q_1 = p_1 \\ \cdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \\ a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ \cdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{gathered} \tag{8}$$

where $a_n = 0$ if $n < 0$ and $q_j = 0$ if $j > M$.

For the Padé approximation, we have the following theorems (see [17]).

Theorem 7. (*Frobenius and Padé*) *When it exists, the Padé approximant $[L \setminus M]$ to any formal power series $A(x)$ is unique.*

Theorem 8. (*Padé*) *The function $f(x)$ is of the form*

$$f(x) = \frac{p_l x^l + p_{l-1} x^{l-1} + \cdots + p_0}{q_m x^m + q_{m-1} x^{m-1} + \cdots + 1}$$

iff the Padé approximants are given by $[L \setminus M] = f(x)$ for all $L \geq l$ and $M \geq m$.

4.2 An Algorithm based on Padé Approximants

For a nontrivial rational solution $r(x)$ of $F = 0$, $(r(x), r'(x))$ is a proper parametrization of $F(y, y_1) = 0$. Hence for most of the points (z_0, z_1) on $F(y, y_1) = 0$, there exists a single c_0 such that $z_0 = r(c_0), z_1 = r'(c_0)$. By performing a linear transformation $x = x + a$ for $a \in \mathbf{Q}$ if necessary, we may always assume that $c_0 = 0$. Note that after a linear transformation if necessary, the constant term of the denominator of $r(x)$ will not vanish. If the point (z_0, z_1) on $F(y, y_1) = 0$ does not vanish the separant $S(y, y_1)$ of $F(y, y_1)$, we can compute $y_i = z_i$ step by step from (1). Then $z_i/i!$ will be the coefficients of the Taylor series expansion of $r(x)$ at $x = 0$. Hence we can construct the Padé approximants from it. From Theorem 8, the Padé approximants satisfying $L = M = deg(F, y_1)$ will equal to $r(x)$, since from Theorem 3, $deg_x(r(x)) = deg(F, y_1)$.

In the following, if we regard a differential polynomial G with order k as an algebraic polynomial, we denote it by $G(y, y_1, \ldots, y_k)$. From the above analysis, we have the following algorithm.

Algorithm 2. *The inputs are a first order irreducible (over $\bar{\mathbf{Q}}$) differential polynomial F with coefficients in $\mathbf{Q}$ and a point (z_0, z_1) on $F(y, y_1) = 0$. The outputs are a rational general solution of $F = 0$ if it exists or "failure" which means that we need to choose another point on $F(y, y_1) = 0$.*

1. Let $n = \deg(F, y_1)$ and $d = \deg(F, y)$. If $d < n - 1$ or $d > 2n$, then by Theorem 3, the algorithm terminates and $F = 0$ has no rational general solutions.

2. If $y = z_0, y_1 = z_1$ vanish the separant $S(y, y_1)$ of $F(y, y_1) = 0$, then return "failure". Otherwise, from (1) we have $F^{(i-1)} = S(y, y_1) y_i - R_i(y, y_1, \ldots, y_{i-1})$. Let $z_i = R_i(z_0, \ldots, z_{i-1})/S(z_0, z_1)$ for $i = 2, \ldots, 2n$.

3. Let $a_i = z_i/i!$ for $i = 0 \cdots 2n$. In (8), let $L = M = n$. Then we can find q_i by solving the following linear equations (note that we have $q_0 = 1$):

$$A \begin{pmatrix} q_n \\ q_{n-1} \\ \vdots \\ q_1 \end{pmatrix} = - \begin{pmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{pmatrix}$$

where

$$A = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & a_{n+1} \\ \vdots & \vdots & \ldots & \vdots \\ a_n & a_{n+1} & \ldots & a_{2n-1} \end{pmatrix}$$

(Note that the matrix A may be singular, from Theorem 7, we only need to select one of the solutions of the above linear equations.)

4. Let $p_i = a_0 q_i + a_1 q_{i-1} + \cdots + a_i q_0$ for $i = 0 \cdots n$. From Theorem 3,Theorem 8 and (8), if $\sum_0^{2n} a_i x^i$ is the first $2n + 1$ terms of the Taylor series expansion of some nontrivial rational solution of $F = 0$, then the nontrivial rational solution will equal to

$$\bar{y} = \frac{p_n x^n + p_{n-1} x^{n-1} + \cdots + p_0}{q_n x^n + q_{n-1} x^{n-1} + \cdots + 1}$$

5. If $\bar{y}$ is a constant, then return "failure". Otherwise, substituting $\bar{y}$ to F, if $F(\bar{y}) = 0$ then return $\hat{y} = \frac{p_n (x+c)^n + p_{n-1} (x+c)^{n-1} + \ldots + p_0}{q_n (x+c)^n + q_{n-1} (x+c)^{n-1} + \ldots + 1}$. Otherwise, $F = 0$ has no rational general solution.

From the proof of Lemma 3, we can see that Lemma 3 is still true in the field of formal Laurent series. Then by Lemma 3 and Theorem 1, we know that if $F = 0$ has a nontrivial rational solution, then every nontrivial formal power series solutions of $F = 0$ must have rational form. Hence the above algorithm is true.

Theorem 9. *Except a finite number of points on curve $F(y, y_1) = 0$, Algorithm 2 will find a rational solution for $F = 0$ or decide that $F = 0$ has no rational solutions.*

Proof. Let $n = \deg(F, y_1)$. The points which lead to the failure of our algorithm include two parts. One of them is the common points of $F(y, y_1) = 0$ and $S(y, y_1) = 0$, the other is the points such that the Padé approximants is a constant. By Bezout's Theorem, the number of the points in the first parts equals to $4n^4 - 2n^2$ at most. By the definition of the Padé approximants, if $z_1 \neq 0$, then the Padé approximants could not be a constant. Hence the number of the points in the second parts equals to $2n$ at most. So the number of the points on $F(y, y_1) = 0$ which make our algorithm fail is finite. $\square$

	degree	term	point	time(s)	solution
F_1	4	22	$(1, -2+2*I)$	1.344	yes
F_2	5	17	$(4, \frac{4}{7}+\frac{6*\sqrt{2}}{7})$	5.999	no
F_3	5	31	$(0, \frac{-7+\sqrt{3}I}{2})$	6.421	no
F_4	6	24	$(0, 6*I)$	6.797	yes
F_5	7	35	$(1, 1)$	22.610	yes

Table 1: Timings for solving autonomous first order ODEs

Theorem 9 ensures the termination of Algorithm 2 since we can always select an infinite number of points on $F(y, y_1) = 0$ as follows: set $z_0 = 0, \pm 1, \pm 2, \ldots$ and z_1 is an algebraic number determined by $F(z_0, z_1) = 0$. In most cases, z_1 will be an algebraic number since it is generally very difficult to find a point with rational coordinates on a plane curve.

We implement Algorithm 2 in Maple. Table 1 shows the computing times of the program for five examples. Times are collected on a PC with a 2.66G CPU and 256M memory and are given in seconds. In the table, "degree" means $\deg(F_i, y_1)$, "term" means the number of terms in F_i, "point" means (z_0, z_1) in the input, "solution" means whether F_i has rational general solutions. The differential equations $F_i = 0$ are given below.

$F_1 = -71y_1y^2 - 71y^4 - 62y_1^2y + 42y^3 - 12y_1^3y - 220y_1y^3 - 31y_1 - 31y^2 + 42y_1y + 188y_1^2y^2 + 11y_1^3 + 648y_1y^4 + 648y^6 - 220y^5 + 144y^6y_1 - 156y^3y_1^2 - 528y_1y^5 - 528y^7 + 12y_1^3y^2 + 48y^4y_1^2 + 144y^8 + 3y_1^4$

$F_2 = -2y_1^4y + y_1^5y^4 + 12y_1^3y^4 + 12y_1^4y^2 - y^3y_1 + 11y^3y_1^2 - 21y_1^3y^3 - 4y^4y_1 + 2y^4y_1^2 - 6y_1^4y^4 + y^5 - 3y_1^2y^5 + y_1^3y^5 - 3y_1^5y - 2y_1^3y^2 + y_1^4y^3 + y_1^5$

$F_3 = -3939 + 2661y^4y_1 - 5694y^3y_1 - 1372y^6 + 124y^7 + 343y_1^2y^5 + 126y_1^3y^3 + 108y_1^4y + 12585y - 2511y_1 + 739y^4y_1^2 - 649y^3y_1^2 - 15463y^2 + 8506y^4 + 1146y^3 - 1137y_1^2 - 54y_1^3 + 120y_1^4 - 1933y^5 + 1038y_1^3y - 2959y_1^2y^2 + 1779y_1y^5 + 6096yy_1 + 3813yy_1^2 - 2814y^2y_1 - 186y^6y_1 - 31y^6y_1^2 + 62y_1^3y^4 - 8y_1^4y^2 + 24y_1^5 - 520y_1^3y^2$

$F_4 = -672y^7y_1^2 - 19072y^6 + 6016y^7 - 44352y^5 + 33696y_1^2y + 233280y - 404352y^2 - 11664y_1^2 + 245376y^3 - 864y_1^4 - 22464y_1^2y^2 + 864y_1^4y + 5424y^4y_1^2 - 6912y^3y_1^2 - 128y_1^4y^3 + 48y_1^4y^4 - 16y_1^6 - 832y^6y_1^2 + 3968y^5y_1^2 + 25920y^4 + 3264y^8 - 144y_1^2y^8 + 576y^9 - 46656$

$F_5 = -870199 + 48y_1^6y + y_1^7 + 256y_1^3y^6 + 3336568y^5 - 924496y^6 + 339557y^2y_1^3 - 55752y^4y_1^2 - 18527499y^4 + 140154y_1^2y^3 + 38016y^7 + 3660594y + 457074y_1^2 - 16729917y^2 - 1033424yy_1^2 + 231921y_1^3 - 70101y^2y_1^2 - 405468y_1^3y + 30410226y^3 + 76914y_1^4 - 1536y_1^2y^6 - 1408y_1^3y^5 + 768y_1^4y^4 + 32y_1^5y^3 - 70744y_1^3y^3 + 7584y_1^2y^5 + 22512y_1^3y^4 - 6912y^8 + 27109y^2y_1^4 + 14238y_1^5 - 60660yy_1^4 - 2046y_1^5y - 3904y_1^4y^3 + 504y_1^5y^2 - 10y_1^6$

5. CONCLUSION

In this paper, we give a necessary and sufficient condition for an ODE to have a rational general solution and an algorithm to compute the rational general solution of an autonomous first order ODE if it exists.

As mentioned in Section 1, this work is motivated by the parametrization of differential algebraic varieties, which is still wide open. A problem of particular interests is to find conditions for a differential curve $f(y, z) = 0$ to have rational differential parameterizations. We may further ask whether we can define a differential genus for a differential curve similar to the genus of algebraic curves.

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