Rational Hedging and Valuation with Utility-Based Preferences

vorgelegt von Diplom-Mathematiker

Dirk Becherer

aus Lüdenscheid

Von der Fakultät II - Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

– Dr. rer. nat. –

genehmigte Dissertation

Vorsitzender des Promotionsausschusses:
Prof. Dr. Fredi Tröltzsch – Technische Universität Berlin

Berichter:
Prof. Dr. Mark H. A. Davis – Imperial College of Science and Technology, London
Prof. Dr. Peter Imkeller – Humboldt-Universität zu Berlin
Prof. Dr. Martin Schweizer – Ludwig-Maximilians-Universität München

Tag der wissenschaftlichen Aussprache: 8. Oktober 2001

Berlin 2001
D 83

Abstract

In this thesis, we study stochastic optimization problems in which concave functionals are maximized on spaces of stochastic integrals. Such problems arise in mathematical finance for a risk-averse investor who is faced with valuation, hedging, and optimal investment problems in incomplete financial markets. We are mainly concerned with utility-based methods for the valuation and hedging of non-replicable contingent claims which confront the issuer with some inevitable intrinsic risks.

We adopt the perspective of a rational investor who aims to maximize his expected exponential utility. Based on these preferences, the issuer's valuation process and hedging strategy are defined via utility indifference arguments. In a general semimartingale model, the solution to this problem is characterized by a stochastic representation problem. Solving the problem amounts to finding a martingale measure whose density process can be written in a particular form. We then specialize our analysis to two stochastic models which satisfy further structural assumptions. In a semi-complete product model, the valuation and hedging methods are shown to be additive when applied to an aggregate of "sufficiently independent" individual claims. We study the impact of diversification and derive a computation scheme. For a second model, we set up a Markovian system of stochastic differential equations which describes the dynamics of an Itô process and an additional finite-state process, and permits for various dependencies between both. In the financial market model, the Itô process models the price fluctuations of the risky assets while the second process represents some untradable factors of risk. The solution to the pricing and hedging problem is explicitly described by an interacting system of semi-linear partial differential equations – a so-called reaction-diffusion equation. Using Feynman-Kac results and the Picard-iteration method, we establish existence and uniqueness of a classical solution.

In a variation of the basic theme, a similar utility indifference approach is applied to quantify the value of additional investment information. On the mathematical side, this involves a martingale preserving measure transformation and martingale representation results for initially enlarged filtrations. Finally, we show that the so-called numeraire portfolio is related to another utility-based valuation method which relies on a marginal rate of substitution argument and can be seen as a limiting case of the utility indifference method.

Zusammenfassung

Die vorliegende Arbeit behandelt stochastische Optimierungsprobleme, in denen ein konkaves Funktional über einem Raum von stochastischen Integralen maximiert wird. In der Finanzmathematik treten derartige Probleme bei der Behandlung von Bewertungs-, Absicherungs-, und Anlageproblemen in unvollständigen Finanzmärkten auf. Wir beschäftigen uns vornehmlich mit nutzenbasierten Methoden zur Bewertung und Absicherung von zufallsbehafteten Finanzpositionen, welche unvermeidbare intrinsische Risiken beinhalten.
Wir betrachten das Problem aus der Perspektive eines rationalen Investors, dessen Ziel die Maximierung seines erwarteten exponentiellen Nutzens ist. Ausgehend von diesen Präferenzen, definieren wir mittels Nutzenindifferenz-Argumenten seinen Bewertungsprozess und eine Absicherungsstrategie. In einem Semimartingalmodell kann die Lösung durch ein stochastisches Darstellungsproblem charakterisiert werden. Um das Problem zu lösen, gilt es ein Martingalmaß zu finden, dessen Dichteprozess eine bestimmte Form hat. Im Weiteren untersuchen wir zwei Modelle, welche gewissen strukturellen Bedingungen genügen. In einem halbvollständigen Produktmodell wird gezeigt, dass die Nutzenindifferenz-Bewertungs- und Absicherungsmethode additiv ist, wenn sie auf ein Aggregat von “genügend unabhängigen” Positionen angewandt wird. Wir untersuchen Diversifikationseffekte und leiten ein Berechnungsschema her. Für das zweite Modell betrachten wir ein Markovsches System stochastischer Differentialgleichungen, welches einen Itô-Prozess und einen weiteren Prozess mit endlichem Zustandsraum beschreibt und verschiedene wechselseitige Abhängigkeiten zulässt. In unserem Marktmodell stellt der Itô-Prozess die Preise der riskanten Anlagen dar während der zweite Prozess irgendwelche nicht handelbaren Risikofaktoren repräsentiert. Die Lösung des Bewertungs- und Absicherungsproblems wird durch ein wechselwirkenendes System semilinearer partieller Differentialgleichungen, eine so genannte Reaktions-Diffusions Gleichung, beschrieben. Mittels Feynman-Kac Resultaten und der Iterationstechnik von Picard zeigen wir Existenz und Eindeutigkeit einer klassischen Lösung.
Ergänzend zu unserem Hauptthema nutzen wir ein ähnliches Indifferenzargument um den Wert von zusätzlichen Anlage-Informationen zu quantifizieren. Die wesentlichen Mittel sind eine Martingal erhaltende Maßtransformation und Martingal-Darstellungsresultate für anfangsvergrößerte Filtrationen. Schließlich zeigen wir, dass das so genannte Numeraire-Portfolio mit einer weiteren nutzenbasierten Bewertungsmethode zusammenhängt, welche sich auf ein Grenznutzenargument stützt und als Grenzfall der Indifferenz-Methode angesehen werden kann.

Contents

Introduction

In this thesis, we study stochastic optimization problems in which concave functionals are maximized on spaces of stochastic integrals. Such problems arise in mathematical finance for a risk-averse investor who is faced with valuation, hedging, risk management and optimal investment problems in incomplete financial markets. The emphasis of this thesis is on utility-based methods for the valuation and hedging of non-replicable contingent claims which confront the issuer with some inevitable intrinsic risks.

If the payoff of a contingent claim is replicable by dynamic trading, the claim can be perfectly hedged by the replicating strategy, and its price is determined as the replication costs. However, such claims are dynamically redundant and have apparently little reason to exist. In incomplete markets, a contingent claim may be not redundant but incorporate some unavoidable risk. Hence, the issuer's valuation and hedging strategy have to take into account his attitudes and preferences towards risk. We adopt the perspective of a rational investor who aims to maximize his expected utility according to his level of risk aversion. In doing so, the investor balances his two conflicting objectives to maximize return and minimize risk. Based on these preferences, the utility indifference price π of a contingent claim is defined as the amount of money, which makes a potential issuer indifferent – in terms of maximal expected utility – between the opportunity to earn the premium π and take the liability associated with the claim, and the alternative to skip the deal. The hedging strategy for the contingent claim is defined as the adjustment of the investor's optimal portfolio strategy, induced by the additional liability from the claim. Motivated by classical results from actuarial mathematics, we consider exponential utility which leads to desirable valuation properties.

In a general semimartingale model, we relate the solution to the utility indifference pricing and hedging problem to the solution of the utility maximization problem with an additional liability. For both problems, the respective solution is characterized by a stochastic representation problem which can be viewed as a verification theorem. Essentially, solving the problem amounts to finding a martingale measure whose density process can be written in a particular form. This gives rise to the introduction of a valuation process which turns out to be crucial for the construction and description of more explicit solutions. The martingale measure is the solution to a certain dual optimization problem.

To obtain more constructive and explicit results, we specialize our analysis to two classes of

financial market models which satisfy further structural assumptions. The first is a semi-complete product model which consists of a complete financial sub-market and additional independent sources of uncertainty and risk. Starting from a result on the solution to the dual problem, we show that our valuation and hedging methods are additive when applied to an aggregate of "sufficiently independent" individual claims. We study the impact of diversification among different factors of risk and derive a backward computation scheme for the utility indifference price as well as upper and lower bounds. As a second class of models, we set up a Markovian system of stochastic differential equations to describe the dynamics of an Itô process and an additional finite-state process. Our model allows for a variety of mutual dependencies between the Itô process modelling the price fluctuations of the risky assets and the second process which represents some untradable factors of risk. Nonetheless, the solution to the pricing and hedging problem can be explicitly described by an interacting system of semi-linear partial differential equations – a so-called reaction-diffusion equation. A similar solution is obtained for the underlying utility maximization problem with an additional liability. Combining Feynman-Kac results with a contraction argument, we establish existence and uniqueness of a classical solution under mild conditions on the coefficient functions. The scope of our results on utility-based hedging and pricing is illustrated by several applications, including index-linked life insurances, weather derivatives, stochastic volatility models and credit-risk derivatives.
In a variation of the basic theme, a similar utility indifference valuation approach is applied to quantify the monetary value of additional investment information for our rational investor. On the mathematical side, this involves a martingale preserving measure transformation and martingale representation results for initially enlarged filtrations. Finally, we show that pricing contingent claims by the so-called numeraire portfolio is related to another utility-based pricing method which relies on a marginal rate of substitution argument and can be seen as a limiting case of the utility indifference method. It is a delicate question whether the method leads to arbitrage-free valuations in the general. Based on a characterization of the numeraire portfolio in the general semimartingale case, we provide several examples to show which phenomena may occur.

Historical background in mathematical finance

By its very nature, the utility based pricing and hedging of a contingent claim integrates two classical problems which have been at the core of the evolution of mathematical finance in the last three decades: The optimal investment problem and the hedging and pricing problem. To put our results and methods into perspective, let us briefly review the historical background before we discuss the mathematical contributions of this thesis in more detail.

The wide adoption of the Itô calculus in the economic theory of finance and the banking industry can be traced back to the path-breaking contributions of Merton [Mer69, Mer71, Mer73] and Black and Scholes [BlS73]. Since then, the acceptance and routine use of probabilistic "high tech"-methods in finance – both in theory and practice – has grown rapidly. In turn, questions arising from applications have become an important source of intriguing mathematical problems and have initiated new developments in stochastic analysis and martingale theory. This interplay of theory and practice has enriched the new field of mathematical finance.

Black and Scholes [BlS73] and Merton [Mer69, Mer71, Mer73] assumed that the financial market is frictionless, the investor acts as a price taker and that trading takes place in continuous time. Using Samuelson's geometric Brownian motion model to describe financial price fluctuations of the stock, Black and Scholes [BlS73] and Merton [Mer73] derived their celebrated formula for the price of a European call option. Their fundamental insight was that the option payoff can be replicated by a dynamic trading strategy. Hence, the option can be perfectly hedged by the replicating strategy, and no-arbitrage arguments determine the option price as the replication cost. Besides proving the Black-Scholes formula by means of stochastic calculus, Merton [Mer69, Mer71] applied techniques from dynamic programming to the portfolio optimization problem for a rational investor whose objective is to maximize his expected utility. Founded on the axiomatization of choice under uncertainty introduced by von Neumann and Morgenstern [vNM44] and Herstein and Milnor [HerM53], this type of rational preferences constitutes the reference model in the theory of portfolio selection. In a diffusion framework, Merton derived the Hamilton-Jacobi-Bellman equation and solved the arising partial differential equations for special utility functions, provided asset dynamics were sufficiently homogeneous.

Mathematically, the work of Black, Scholes and Merton relies on the Itô calculus, on the theory of diffusion processes and on Bellman's dynamic programming approach. In the further development of the theory, the well-established theory of martingales turned out to be tailor made for problems in finance. This trend was already indicated by a feature of the pioneering formula by Black, Scholes and Merton which seems surprising at first sight: The price of a European call option involves the riskless rate, but not the rate of return for the underlying risky asset. Using a discrete approximation of the Black-Scholes model, Cox and Ross [CoR76] pointed out that the option price is the expected discounted payoff of the option under a measure which is risk-neutral in the sense that it assigns the same rate of return to the risky asset and the bank account. In modern terminology, this means that both the asset and the option prices, discounted at the riskless rate, are martingales under the risk-neutral measure. This insight was systematically developed by Harrison and Kreps [HK79] and Harrison and Pliska [HP81] who established the fundamental role of martingale theory in finance. Thanks to their work, many financial questions on the market model can be translated into dual questions about the set of equivalent martingale measures. First of all, the economic postulate of the absence of arbitrage opportunities is essentially equivalent to the existence of an equivalent martingale measure for the discounted price processes. This result is known as the fundamental theorem of asset pricing. It implies that no-arbitrage pricing of contingent claims amounts to taking conditional expectations of discounted payoffs under an equivalent martingale measure. On a general semimartingale level, the precise meaning of the word "essentially" involves some delicate mathematical problems, see Delbaen and Schachermayer [DeS94, DeS98]. Another important example for the connection between martingale theory and mathematical finance is the intimate duality relation between super-replicable contingent claims – i.e. payoffs which can be dominated by dynamic trading – and the equivalent martingale measures; see Kramkov [Kr96] and Föllmer and Kabanov [FK98] for a general account on the so-called optional decomposition. Actually, these results have been initiated by an approach of El Karoui and Quenez [ElKQ95] to the hedging problem in incomplete markets which we will discuss below in more detail. In complete markets, the equivalent martingale measure is unique and the duality relation reduces to the classical martingale representation theorem.

These duality results have provided the basis for the martingale approach to the utility maximization problem. In comparison to Merton's original approach, the martingale method avoids the assumption of Markovian asset price processes which is necessary for the classical Bellman approach. In the complete market case, the wealth process of the optimal portfolio can be explicitly specified in terms of the density of the unique equivalent martingale measure, see Cox and Huang [CoH89], Karatzas, Lehoczky and Shreve [KaLS87], and market completeness guarantees the existence of the optimal strategy. In the incomplete market case, the central idea is to introduce a dual variational problem over the set of martingale measures, and then to derive the solution to the primal utility maximization problem via the duality relation between equivalent martingale measures and wealth processes and methods from convex analysis; see Karatzas, Lehoczky, Shreve and Xu [KaLSX91] and – for a general semimartingale account – Schachermayer and Kramkov [KrS99] and Schachermayer [Scha01]. Besides existence and uniqueness results, the martingale approach yields a convex duality characterization between the two dual optimization problems. However, in comparison to the Bellman-approach there is also a price to pay for the increase in generality. The results are neither explicit nor constructive since the solution to the dual problem is typically not known, and little – except existence – is gained about the optimal investment strategy. Nevertheless, the general duality approach gives a universal tool to attack the optimal investment problem in concrete models, and it is fundamental for the study of more advanced problems involving an additional liability. The latter is central for the utility indifference approach to valuation and hedging problems in incomplete markets which brings us to the topic of this thesis. Before we explain this approach in more detail, we give a brief account on alternative concepts which address the pricing and hedging problem in incomplete markets.

Approaches to pricing and hedging in incomplete markets

In incomplete markets, contingent claims may be not redundant so that the replication argument from Black, Scholes and Merton does not apply. For such claims, perfect hedging – i.e. replication – is not possible, and no-arbitrage arguments alone do not define a unique price, but only an interval of possible valuations. This confronts an issuer of such contingent claims with two problems: How can he hedge his risk at least partially and which of the possible valuations is appropriate? There are several approaches which are often primarily concerned with only one of the two questions and barely touch the other.

Let us first consider those which mainly address the hedging problem: A prominent class of approaches aims to minimize the expectation of some quadratic functional of the hedging error, see Schweizer [Schw01a] for a survey. The concepts can be divided into *mean-variance hedging*, see Bouleau and Lamberton [BoL89] and Duffie and Richardson [DuR91], and *(local) risk-minimization* which was developed in Föllmer and Sondermann [FSo86], Föllmer and Schweizer [FSchw91], and Schweizer [Schw91]. By their nature, both ideas concentrate on the hedging purpose. The wealth process of the hedging strategy can be considered as a valuation of the contingent claim and can be calculated as the expectation of the discounted payoff under the variance optimal and the minimal martingale measures, respectively. In general, these measures are only signed measures and this leads to some deficiencies of the valuation which are not easy to accept. In fact, the methods may provide negative valuations for some positive payoffs.

Another approach which looks very attractive at first sight is the concept of *super-replication*. This was introduced by El Karoui and Quenez [ElKQ95] and has initiated more general results on the optional decomposition, cf. [Kr96, FK98], which are now at the heart of the martingale approach in incomplete markets. The objective for the hedging strategy here is to assure that the value of the portfolio dominates the payoff of the contingent claim. The smallest initial cost necessary for this purpose defines the super-replication price. In this approach, the issuer can always cover his liabilities from the claim whatever the eventual outcome may be. However, this leads to a price which is typically too high for practical purposes.

If only less capital is available, super-replication with probability one is no longer affordable. A natural idea is then to maximize the probability of a successful (super-) hedge. From a dual perspective, this amounts to finding the minimal capital that suffices to keep the probability that a loss occurs below some pre-specified level. This motivated the terminology *quantile hedging* in Föllmer and Leukert [FL99]. To address also the size of the shortfall, other approaches aim to *minimize the expected shortfall*, see Cvitanić-Karatzas [CvK99], or an expected convex functional of the loss. The latter concept, named *efficient hedging*, was studied in Föllmer and Leukert [FL00]. By their very nature, quantile hedging, efficient hedging and expected shortfall minimization only aim to limit the potential losses. Hence, the resulting amount of capital that is necessary to keep the respective risk-measure below some pre-specified level can be considered as a monetary reservation which is required by a risk manager, but it is not a valuation or pricing of the claim as a whole.

Let us now turn to approaches which mainly address the pricing problem. By the fundamental theorem of asset pricing, choosing an arbitrage-free valuation for a contingent claim is basically equivalent to choosing a pricing measure. In the literature, the latter typically is done according to some optimality criterion involving the original probability measure. Examples are the *minimal martingale measure* [FSchw91], the *variance optimal martingale measure* [Schw96] or the *minimal entropy martingale measure* [Ft00]. We note that choosing a particular pricing measure determines a valuation for the claim but does not settle the hedging problem – unless the pricing measure is derived from a hedging approach. It is sometimes argued that the only reasonable approach to the pricing problem is to extract the valuation for the contingent claim under consideration from observed prices of related assets in the real financial market, which amounts to finding the so-called *empirical pricing measure*. However, this argument necessitates that the arbitrage-free price of the claim is determined by real market prices of related assets. Consequently, it does not apply if the claim incorporates some inevitable intrinsic risk which is not traded in the financial market so far.

From an issuer's perspective, his valuation and his (partial) hedging strategy for such claims should take into account his preferences and attitudes towards risk. This leads in a natural way to (expected) utility arguments. Davis [D97] proposed an approach for the pricing of contingent claims which is based on the concept of a "zero marginal rate of substitution". A "fair price" of a contingent claim is defined in such a way that an investor cannot increase his expected utility by diverting an infinitesimal amount of his capital into a long or short position in the claim, i.e. his marginal expected utility for such trades is zero. However, hedging questions are not addressed by this method. We adopt a similar utility based approach to address both the pricing and the hedging problem. In comparison to the "fair price", our method relies on utility indifference arguments with respect to *non-infinitesimal* amounts of the claim.

The utility indifference approach to pricing and hedging

Our methods rely on indifference arguments that are formulated from the perspective of an issuer who is rational in the sense that he ranks risky payoffs by maximal expected utility, i.e. he prefers the payoff with the higher expected utility. Following Hodges and Neuberger [HodN89], the utility indifference price of a contingent claim is defined as the amount of money which just compensates the issuer for taking the liability in terms of his maximal expected utility. We define a corresponding utility indifference hedging strategy as the adjustment of the issuer's optimal portfolio strategy that is induced by the additional liability from the claim.

The valuation method essentially goes back to an idea of Daniel Bernoulli [B1738] in the 18th century, who postulated that an investor, say gambler, will rank risky ventures, say lotteries, by their expected utilities. In a natural way, this leads to a utility equivalence valuation of random payoffs which has been well-known for a long time e.g. in actuarial mathematics, see Gerber [Ge79]. Hodges and Neuberger [HodN89] adapted this approach to the setting of a dynamical financial market just by taking *maximal* expected utilities.
Mathematically, the problem is closely related with the utility maximization problem under an additional liability. This has been addressed in a general semimartingale framework by Frittelli and Bellini [FtB00, Ft00a] and Owen [Ow01] for general utility functions and by Delbaen, Grandits, Rheinländer, Samperi, Schweizer and Stricker [DGRS^{3}00] for the exponential utility function. Cvitanić, Schachermayer and Wang [CvSW01] treated the conceptually closely related problem of utility maximization with random endowment in an incomplete market. All these authors use the powerful martingale approach to relate the solution to the utility maximization problem and the corresponding value function to the solution of an associated dual problem over the set of equivalent martingale measures. By the very nature of the approach, the solution is neither constructive nor explicit – except in the case of a complete market which excludes non-redundant contingent claims. Frittelli [Ft00a] obtained several properties of the utility indifference price, most notably its consistence with the principle of no arbitrage. For the exponential utility function, the papers of Delbaen et al. [DGRS^{3}00] and – in a Brownian setting – Rouge and El Karoui [RouE00] resolved some technical obstacles and contributed new proofs and further properties of the valuation method. More constructive and explicit results for the combined hedging and investment problem are available so far only in very specific models. Hamilton-Jacobi-Bellman equations and PDE-type solutions for expected utility maximization in the presence of an additional random liability (or endowment) have been studied by many authors; see for instance Brown [Bw95], Davis [D00], Davis, Panas and Zariphopoulou [DPZ93], Delbaen et al. [DGRS^{3}00], Duffie, Fleming, Soner and Zariphopoulou [DuFSZ97], and Hobson and Henderson [HenH00, Hen01]. Throughout, these authors consider classical incomplete models with a Brownian filtration where the asset prices are driven by Brownian motions and the incompleteness comes from the fact that there are more Brownian motions – sources of risk – than assets (or transaction exists as in [DPZ93]). But reasonably explicit results, in particular for the indifference pricing *and* hedging problem, have been obtained only in specific models with constant coefficients and a single tradable asset, cf. [D00, HenH00, Hen01] and the overview in Section 4.3 of [DGRS^{3}00].

Contributions of this thesis

Chapters 1 through 4 are devoted to the utility indifference approach to the pricing and hedging problem in incomplete markets. We consider exponential utility functions, corresponding to a constant absolute level of risk aversion $\alpha \in (0, \infty)$ of our investor.

In Chapter 1, the utility indifference price and a corresponding hedging strategy are defined and studied in a general framework where the (discounted) prices of the tradable financial assets are given by locally bounded semimartingales. We review – on a slightly more general level – duality results on exponential utility maximization by Delbaen et al. [DGRS^{3}00] and Kabanov and Stricker [KS00b] which are fundamental for our treatment of the indifference approach. Following Hodges and Neuberger [HodN89], the utility indifference (selling) price of a contingent claim B is defined as the implicit solution of an equation which involves the maximal expected utility functions with and without additional liability B. It is shown that the utility indifference price tends to the expected payoff of the claim under the entropy minimal martingale measure if the risk aversion tends to zero. This generalizes a similar result in a Brownian framework by Rouge and El Karoui [RouE00], and complements results in Delbaen et al. [DGRS^{3}00]. The utility indifference hedging strategy ψ for a contingent claim B is defined as the difference of the optimal strategies ϑ^B and ϑ^0 for the utility maximization problem with and without the additional liability. Hence $\vartheta^B = \vartheta^0 + \psi$ holds and ψ may be viewed as the part of the optimal strategy ϑ^B which stems from the additional liability B.
We relate the solution of the utility indifference pricing and hedging problem to the solution of the utility maximization problem with an additional liability. Using duality results by Delbaen et al. [DGRS^{3}00] and a criterion for the minimal entropy martingale measure by Grandits and Rheinländer [GraR99, Rh99], each solution is characterized by a stochastic representation problem which amounts to finding an equivalent martingale measure whose *density* has a specific form. We extend this to an (dynamical) stochastic representation result for the density *process* of this measure which induces in a natural way the definition of a valuation process. This process has a clear economic interpretation, see Remark 1.4.7. In subsequent chapters, this characterization is used as a verification theorem and the valuation process turns out to be central for the description and construction of explicit solutions in more specific financial models. Examples include Theorem 2.4.1 (backward computation scheme), Theorems 3.4.2 and 3.4.3 (solution to the (dual) utility maximization problem with additional liabilities), and Theorem 3.5.2 (solution to the indifference pricing and hedging problem).

In Chapter 2, we consider a semi-complete product model which consists of a complete financial sub-market and additional independent sources of uncertainty and risk. We prove some structural results on the set of martingale measure and on the solution of the dual problem in particular, using a martingale representation result for enlarged filtrations from Chapter 5. Via the characterization results from Chapter 1, we then derive several results on the "primal" indifference problem. Theorem 2.3.1 shows that the indifference pricing and hedging method is additive when applied to a sum of sufficiently independent individual claims. Using this result, we show that diversification effects among a large number of risks can asymptotically lead to a utility based valuation which is risk neutral with respect to untradable factors of risk. Moving on, we provide upper and lower bounds for the utility indifference price. If the additional information is piecewise constant, the utility indifference price can be obtained by an explicit backward computation scheme, see Theorem 2.4.1.

In Chapter 3, we develop a Markov-type model that is driven by a system of stochastic differential equations. These describe the dynamics of an Itô process S modelling the asset prices and an additional finite-state process η which represents some untradable factors of risk. The model permits for various mutual dependencies between S and η. More precisely, η is a multivariate point process whose intensities depend on the current value of S, and both S and η enter the coefficients of the stochastic differential equation that describes the dynamics of S. This yields a non-standard model for an incomplete financial market which has – to the best of our knowledge – not been considered in this form so far. In a sense, the model is both a generalization and a fusion of Lando's [La98] random intensity model and a diffusion model which is modulated by an independent Markov process as in, e.g., Di Masi, Kabanov and Runggaldier [DiMKR94]; we refer to Chapter 4 for a more detailed comparison.
We obtain explicit solutions to the utility maximization problem under additional liability and to the utility indifference pricing and hedging problem, respectively, in terms of a partial differential equation (PDE); see Sections 3.4 and 3.5. More precisely, both solutions are determined by an interacting system of semi-linear partial differential equations, a so-called reaction diffusion equation. Each single PDE in the system corresponds to one possible state of the process η. The impact of the untradable factors of risk – i.e. the uncertainty about the evolution of η – on the optimization problems is reflected by the interaction between the individual PDEs in the corresponding system. In analogy to the classical Black-Scholes PDE in [BlS73], the utility indifference price of the contingent claim is basically given by the solution to the PDE-system, and the corresponding hedging strategy is described by the gradient of this function with respect to the tradable assets.
To prove these results, we use our characterization of the optimal solution from Chapter 1 and construct an martingale measure whose *density process* has the required certain form. The main tool for this construction are existence and uniqueness results for reaction diffusion equations. These are interesting in their own right and are presented in Chapter 7.

Chapter 4 illustrates the results and the range of possible applications for the model of the previous chapter. Moreover, Section 4.3 exemplifies results from Chapter 2 on additivity and diversification. For this purpose, we consider a variety of concrete financial products like equity linked life insurances, default-risk contracts and weather derivatives. Furthermore, we will discuss our results in the context of stochastic volatility models. To put our modelling framework and our contributions into perspective, we thereby also point out similarities and crucial differences to related work in the literature. For first reading, Chapter 4 should be accessible from the basis of Chapter 1 since the results of Chapter 3 are reviewed and moreover presented in a different parameterization which is more reminiscent of the familiar Black-Scholes situation.

In Chapter 5, we apply a similar utility indifference argument to quantify the value of additional investment information for an investor. We suppose that the investor aims to maximize his expected utility where the utility function need not be exponential. This investor faces the opportunity to acquire some additional initial information $\mathcal{G}$. The subjective fair value of this information for the investor is defined as the amount of money that he can pay for $\mathcal{G}$ such that this cost is balanced out by the informational advantage in terms of maximal expected utility. We calculate this value for common utility functions in the setting of a complete market modeled by general semimartingales, and provide closed form solutions in a concrete example where the additional infomation basically consists of a noisy signal about the future stock price.
Mathematically, the additional information is modeled by an initial enlargement $\mathbb{G}$ of the common filtration $\mathbb{F}$ by a sigma-field $\mathcal{G}$. Following Grorud and Pontier [GroP98] and Amendinger, Imkeller and Schweizer [AIS98] – on a slightly more general level – we suppose that the enlargement satisfies a certain decoupling or equivalence condition. This grants the existence of the so-called martingale preserving probability measure (MPPM) $\widetilde{P}$ corresponding to the original probability P. This measure was introduced by Föllmer and Imkeller [FI93] and extensively used in [AIS98, GroP98, GroP01, A00]. The main property of the MPPM is that it decouples $\mathbb{F}$ and $\mathcal{G}$ in such a way that $\mathbb{F}$-martingales under P remain $\mathbb{G}$-martingales under $\widetilde{P}$. By means of the MPPM, we then transfer in Section 5.2 the strong predictable representation property for local martingales from $\mathbb{F}$ to the initially enlarged filtration $\mathbb{G}$. This extends prior work of Pikovsky [Pi97], Grorud and Pontier [GroP98] and Amendinger [A99, A00] to the general unbounded semimartingale case and, moreover, contributes a conceptually new proof.

Chapter 6 is devoted to the so-called numeraire portfolio which was introduced by Long [Lo90] and subsequently studied in, e.g., [Ar97, BP97, KoSO01]. Although the concept does not involve utility arguments at first sight, the numeraire portfolio is intimately related to growth optimization and "pricing by the numeraire portfolio" in fact leads to Davis' [D97] utility based "fair price" of zero marginal rate of substitution with respect to the logarithmic utility function. The basic idea of the numeraire portfolio is to look for a tradable numeraire N^P such that the discounted asset prices become martingales with respect to the objective probability measure P. We study this idea under minimal assumptions on the model of the financial market: The prices of the tradable assets are given by unbounded semimartingales and the model satisfies the no-arbitrage type property of No Free Lunch with Vanishing Risk, introduced by Delbaen and Schachermayer [DeS94, DeS98]. Since the above martingale condition is too stringent to obtain a general existence result, we have to look for a weaker requirement. Our central tool is a general duality theorem by Kramkov and Schachermayer [KrS99] on utility maximization. Combining their result on the optimal solution to the dual problem for logarithmic utility with the myopic character of the growth optimal portfolio, we obtain that every tradable numeraire, discounted with the growth optimal wealth process, becomes a P-supermartingale. We use this property to define a (generalized) numeraire portfolio N^P. This gives a consistent extension of Long's [Lo90] original definition, and guarantees existence and uniqueness under no-arbitrage type conditions. Several examples show that $1/N^P$ (or S/N^P) is a martingale in several cases, but may also merely be a local martingale or even only a supermartingale. In fact, the complete market case and the case of a finite probability space are the only examples in which both processes are true martingales without further technical assumptions. In the general situation however, it seems rather delicate to consider the P-expected payoff of a contingent claim, discounted by the numeraire N^P, as an arbitrage-free valuation.

Chapter 7 is concerned with existence and uniqueness results for classical solutions of interacting systems of semi-linear partial differential equations. These systems are known as reaction-diffusion equations and play a central role for the results in Chapter 3. Our proofs rely on stochastic methods and combine Feynman-Kac arguments with the Picard-iteration technique. To obtain results which are applicable in typical financial models, we use a Feynman-Kac type theorem from Heath and Schweizer [HeaS00] which basically imposes only local smoothness conditions on the coefficient functions.

Acknowledgments

Foremost, I wish to express my deep gratitude to my advisor Martin Schweizer for introducing me to a fascinating area of probability theory and for his continual support throughout my work on this thesis. He generously shared with me his knowledge, experience, and his sense of good style in mathematics and moreover provided an excellent research environment. His numerous comments also have helped to essentially improve the presentation of this thesis.
I want to thank Mark H. A. Davis and Peter Imkeller for readily accepting the task of being co-examiners of this thesis and for various motivating inputs. Furthermore, I thank Mark Davis for coming from London to Berlin to attend the final examination of this thesis. Thanks also to Claudia Klüppelberg, Thomas Møller and Ragnar Norberg for the kind invitations to Munich and Copenhagen, respectively, to present my work at workshops and colloquia which lead to motivating and inspiring discussions.

Throughout my time here in Berlin, I very much appreciated the stimulating atmosphere in the (enlarged) stochastic community at Technical University, Humboldt University and the Weierstrass Institute. Thanks to all members of the group for the unforgettable and pleasant time I spend with them. My colleagues and friends Jürgen Amendinger, Peter Bank, Julia Brettschneider, Frank Döberlein, Felix Esche, Urs Gruber, Ulrich Horst, Anis Matoussi, Beat Niederhauser, Thorsten Rheinländer, Alexander Schied and Ingrid Schmidt merit a special note of thanks for inspiring discussions and their support on my thesis. Finally, my heartfelt thanks go to my parents and my brother for all their love and strong support over the years.
Financial support by the Deutsche Forschungsgemeinschaft via Sonderforschungsbereich 373 ("Quantification and Simulation of Economic Processes") and via Graduiertenkolleg ("Stochastic Processes and Probabilistic Analysis") is gratefully acknowledged.

Part I

RATIONAL PRICING AND HEDGING OF CONTINGENT CLAIMS

Chapter 1

General results for exponential utility

In this chapter, we define the exponential utility indifference price and a corresponding hedging strategy, and study these objects in a general framework where the prices of the tradable financial assets are given by locally bounded semimartingales. We introduce first some notation and summarize duality results on exponential utility maximization by Delbaen et al. [DGRS^{3}00] and Kabanov and Stricker [KS00b] for ease of future reference. We take this as an opportunity to include some straightforward improvements and choose a formulation that is tailored for our subsequent purposes. For closely related and complementary results, we refer to Schachermayer [Scha00, Scha00a, Scha01].
Following Hodges and Neuberger [HodN89], the utility indifference (sell) price π of a contingent claim is defined as a subjective fair valuation from the perspective of an investor with exponential utility preferences. An issuer who takes the uncertain liabilities from the claim is just compensated in terms of maximal expected utility if he receives the premium π. We summarize some properties of this valuation method, and show that the utility indifference price is a systematic interpolation between the super-replication price and Davis' "fair price" of zero marginal rate of substitution. This generalizes similar results by Rouge and El Karoui [RouE00] in a Brownian framework, and complements results in Delbaen et al. [DGRS^{3}00].
We define the utility indifference hedging strategy for a contingent claim as the optimal adjustment of the portfolio strategy for an investor who has issued the claim. Building on results from Grandits and Rheinländer [GraR99], the solution to the utility indifference pricing and hedging problem and also the solution to the utility maximization problem under additional liability are characterized by a stochastic representation problem. In both cases, solving the problem amounts to finding an equivalent martingale measure whose *density* has a specific form. We show that an extended stochastic representation result for the density *process* induces in a natural way the definition of a valuation process which has a nice economic interpretation, see

Remark 1.4.7. In subsequent chapters, the characterization is used as a verification theorem, and the valuation process turns out to be the central mathematical object for the description and construction of explicit solutions in more specific financial models.

1.1 General semimartingale framework

Our mathematical framework is given by a probability space $(\Omega, \mathcal{F}, P)$, a finite time horizon T and a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions of right-continuity and completeness. For simplicity assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. On occasion, we will also introduce further filtrations. All semimartingales with respect to a filtration that satisfies the usual conditions are taken to have right continuous paths with left limits. Expectations are taken with respect to P unless specified otherwise.
Let $S = (S^i)_{i=1,\dots,d}$ be an $\mathbb{R}^d$-valued $(P, \mathbb{F})$-semimartingale. We consider S as the discounted prices of the risky assets in a financial market which contains a risk-less asset with discounted price constant at 1, and always assume that

$$S \text{ is } \mathbb{F}\text{-locally bounded.} \tag{1.1}$$

The sets of absolutely continuous and equivalent (local) martingale measures for S with respect to $\mathbb{F}$, and those with finite relative entropy are defined as

$$\begin{aligned}
\mathbb{P}_a &:= \{\, Q \ll P \,|\, S \text{ is a local } (Q,\mathbb{F})\text{-martingale}\} \\
\mathbb{P}_e &:= \{\, Q \sim P \,|\, S \text{ is a local } (Q,\mathbb{F})\text{-martingale}\} \\
\mathbb{P}_f &:= \mathbb{P}_f(P) := \{\, Q \in \mathbb{P}_a \,|\, H(Q|P) < \infty\}\,,
\end{aligned}$$

where $H(Q|P)$ denotes the relative entropy of Q with respect to P. We assume throughout that our financial model is free of arbitrage in the sense that

$$\mathbb{P}_e \cap \mathbb{P}_f \neq \emptyset\,, \tag{1.2}$$

and denote by Θ some set of "permitted" trading strategies. Technically, Θ is a cone in the space $L(S)$ of predictable and S-integrable processes. We refrain from a concrete specification of Θ because Delbaen et al. [DGRS^{3}00] and Schachermayer [Scha00a] emphasize that there are many possible choices for Θ which lead to the same maximal expected utility function u. Mathematically, this is due to the fact that for several choices of Θ the martingale approach to the (primal) utility maximization problem leads to the same dual problem, and thereby to the same value function u. Since both the utility indifference price and the corresponding hedging strategy will be defined on the basis of u, they are the same for all those choices of Θ. To obtain more universal results, it is reasonable to rely on the robust dual side of the optimization problem (see below) but not on a specific choice of Θ.

1.2 Duality results on exponential utility optimization

We need some preparations for the formulation of this robust duality. To this end, let B denote some random variable and fix some $\alpha \in (0,\infty)$. If $\exp(\alpha B)$ is P-integrable we can define a measure P_B via

$$dP_B := \frac{1}{E[e^{\alpha B}]} e^{\alpha B}\,. \tag{1.3}$$

In the sequel, it will often be convenient to work under P_B instead of P. Since $P_B \sim P$ we could obviously replace P by P_B in the definitions of $\mathbb{P}_a$ and $\mathbb{P}_e$ without altering these sets of measures. Concerning $\mathbb{P}_f$, the situation is less trivial, and we need some further assumptions to obtain

$$\mathbb{P}_f(P) = \mathbb{P}_f(P_B)\,. \tag{1.4}$$

Delbaen et al. [DGRS^{3}00] considered the case where B is bounded from below. For random variables B that are unbounded from below, the next Lemma will show that

$$E\left[e^{(\alpha+\varepsilon)B}\right] < \infty \quad \text{and} \quad E\left[e^{-\varepsilon B}\right] < \infty \quad \text{for some } \varepsilon > 0 \tag{1.5}$$

implies (1.4). We can then simply write $\mathbb{P}_f$ without any ambiguity with respect to P or P_B.

Lemma 1.2.1 *Assume (1.5). Then (1.4) holds and for all $Q \in \mathbb{P}_f$ we have that $B \in L^1(Q)$ and*

$$H(Q|P) = E_Q\left[\log\frac{dQ}{dP_B} + \log\frac{1}{E[e^{\alpha B}]} + \alpha B\right] = H(Q|P_B) + \log\frac{1}{E[e^{\alpha B}]} + E_Q[\alpha B]\,. \tag{1.6}$$

Proof: First, note that (1.5) implies $\exp(\varepsilon|B|) \in L^1(P)$. By Lemma 8 in [DGRS^{3}00] we have

$$E_Q[\varepsilon|B|] \leq H(Q|P) + \frac{1}{e} E_P\left[e^{\varepsilon|B|}\right] \qquad \text{for } Q \ll P. \tag{1.7}$$

Using (1.5), we obtain that $B \in L^1(Q)$ for $Q \in \mathbb{P}_f(P)$. This establishes the second equality in (1.6) for $Q \in \mathbb{P}_f(P)$ while the first equality holds by the definition of P_B. We conclude that $\mathbb{P}_f(P) \subseteq \mathbb{P}_f(P_B)$.

For the converse inclusion, Lemma 8 in [DGRS^{3}00] yields that (1.7) also holds with P_B instead of P. Thus, we obtain that $E_Q[\varepsilon|B|] < \infty$ for $Q \in \mathbb{P}_f(P_B)$ since

$$E_{P_B}\left[e^{\varepsilon|B|}\right] = \text{const}\, E_P\left[e^{\alpha B+\varepsilon|B|}\right] \leq \text{const} E_P\left[e^{(\alpha+\varepsilon)B^+ + \varepsilon B^-}\right]$$

and the right hand side is finite by hypothesis (1.5). We conclude that (1.6) also holds for $Q \in \mathbb{P}_f(P_B)$ and this implies the converse inclusion $\mathbb{P}_f(P_B) \subseteq \mathbb{P}_f(P)$. □

We will now specify conditions on Θ that are used in the sequel. We will mostly suppose that Θ satisfies the robust duality, i.e. we will suppose that the equality

$$\sup_\Theta E\left[-\exp\left(-\alpha(\int_0^T \vartheta dS - B)\right)\right] = -\exp\left(\sup_{Q\in\mathbb{P}_a}\{E_Q[\alpha B] - H(Q|P)\}\right) . \tag{1.8}$$

holds for $\alpha > 0$ and B satisfying (1.5).

Occasionally, we will also require that

$$\Theta \quad \text{contains} \quad \Theta_{\mathcal{M}} := \left\{\vartheta \in L(S) \,\middle|\, \vartheta \cdot S \text{ is a } (Q,\mathbb{F})\text{-martingale for all } Q \in \mathbb{P}_f\right\} . \tag{1.9}$$

For appropriate spaces Θ, both properties constitute key duality results from [DGRS^{3}00, KS00b, Scha00a, Scha01]: (1.8) states that the value of the utility optimization problem over Θ is given by value of the corresponding dual problem over the set $\mathbb{P}_a$ and (1.9) ensures that the optimal investment strategy is attained in Θ.

Example 1.2.2 *Consider the following spaces of strategies*

$$\begin{aligned}
\Theta_1 &= \left\{\vartheta \in L(S) \,\middle|\, \vartheta \cdot S \ \textit{ is a } Q^B\textit{-martingale for} \right. \\
&\qquad\qquad\qquad \left. Q^B \ \textit{given by Proposition 1.2.3}\right\} \\
\Theta_2 &:= \Theta_{\mathcal{M}} \\
\Theta_3 &:= \left\{\vartheta \in L(S) \,\middle|\, \vartheta \cdot S \quad \textit{is uniformly bounded}\right\} , \\
\Theta_4 &:= \left\{\vartheta \in L(S) \,\middle|\, \vartheta \cdot S \quad \textit{is uniformly bounded from below}\right\} .
\end{aligned}$$

Then (1.8) is satisfied for Θ_1,Θ_2,Θ_3 and Θ_4 (see [DGRS^{3}00, Scha00a]). (1.9) is satisfied for Θ_2 (by definition) and thereby also for Θ_1 that contains Θ_2.

In the next proposition, we recall central duality results from Delbaen et al. and Kabanov and Stricker (see Theorem 2 in [DGRS^{3}00] and Theorem 2.1 in [KS00b]) on the exponential utility maximization problem

$$u(x-B) := u(x-B;\alpha) := \sup_{\vartheta\in\Theta} E\left[-\exp\left(-\alpha(x + \int_0^T \vartheta dS - B)\right)\right] \tag{1.10}$$

and the corresponding dual problem

$$\sup_{Q\in\mathbb{P}_f} \{E_Q[\alpha B] - H(Q|P)\} . \tag{1.11}$$

Proposition 1.2.3 *Assume (1.1), (1.2), (1.5) and (1.8). Then*

1. *there exists a unique $Q^B \in \mathbb{P}_f \cap \mathbb{P}_e$ which maximizes (1.11).*

2. *The density of Q^B takes the form*

$$\exp\left(-\alpha(c^B + \int_0^T \vartheta^B dS - B)\right) = \frac{dQ^B}{dP}$$

for some ϑ^B such that $\vartheta^B \cdot S$ is a Q^B-martingale and $c^B \in \mathbb{R}$. Moreover, ϑ^B is in $\Theta_{\mathcal{M}}$.

3. *The maximal expected utility in (1.10) is given by*

$$u(x-B;\alpha) = -e^{-\alpha x} \exp\left(\sup_{Q\in\mathbb{P}_f} \{\alpha E_Q[B] - H(Q|P)\}\right)$$

4. *and*

$$u(x-B;\alpha) = E\left[-\exp\left(-\alpha(x + \int_0^T \vartheta^B dS - B)\right)\right] = -\exp(-\alpha(x - c^B))\,.$$

Notation: Actually, the notation should be $Q^{B,\alpha}$, $c^{B,\alpha}$, and $\vartheta^{B,\alpha}$ to express the dependence on the risk aversion parameter α (except for Q^0). For ease of notation, we omit α when there is no risk of ambiguity about the risk aversion parameter. The same convention applies for $u(x-B;\alpha)$.

The measure Q^0 minimizes $H(Q|P)$ over $Q \in \mathbb{P}_a$ and is therefore called *minimal entropy martingale measure.*
There are minor modifications in the assumptions on B and on the space of strategies in comparison to Theorem 2 in Delbaen et al. [DGRS^{3}00]. But it is straightforward to verify that their arguments are still valid with these modifications. However, we provide the details for the reader's convenience.

Proof: (of Proposition 1.2.3)
The exponential utility maximization problem (1.10) with terminal liability B reduces to the ordinary utility maximization problem

$$E[e^{\alpha B}] \sup_{\vartheta\in\Theta} E_{P_B}\left[-\exp\left(-\alpha(x + \int_0^T \vartheta dS)\right)\right] \tag{1.12}$$

without terminal liability but with P_B instead of P by a measure transformation, namely

$$-\exp\left(-\alpha(x + \int_0^T \vartheta dS - B)\right) = -\exp\left(-\alpha(x + \int_0^T \vartheta dS)\right)\left(E[e^{\alpha B}]\frac{dP_B}{dP}\right). \tag{1.13}$$

Also, (1.6) shows that

$$\sup_{Q\in\mathbb{P}_f} \{E_Q[\alpha B] - H(Q|P)\} + \log\frac{1}{E\left[e^{\alpha B}\right]} = \sup_{Q\in\mathbb{P}_f} \{-H(Q|P_B)\} \tag{1.14}$$

so that both suprema are attained by the same Q^B if the suprema are maxima. Recall that under (1.1) and $\mathbb{P}_f(P_B) \cap \mathbb{P}_e \neq \emptyset$ the optimum Q^B exists and its density has the form

$$\frac{dQ^B}{dP_B} = \exp\left(-\alpha(\text{const} + (\vartheta^B \cdot S)_T)\right)$$

for some $\vartheta^B \in L(S)$ such that $\vartheta^B \cdot S$ is a Q^B-martingale (see Corollary 2.1 of Frittelli [Ft00] and Proposition 3.2 and the proof of Theorem 4.3 of Grandits and Rheinländer [GraR99]). This gives the first half of part 2. Now, we can apply Theorem 2 in [DGRS^{3}00] (in the extended form due to Kabanov and Stricker [KS00b]) to the optimization problem (1.12) with Θ given by

$$\tilde{\Theta}_2 := \Theta_{\mathcal{M}} \cap \left\{ \vartheta \in L(S) \,\middle|\, e^{-\alpha(\vartheta \cdot S)_T} \in L^1(P_B) \right\}.$$

Rewriting the results with respect to P instead of P_B then yields all claims of Proposition 1.2.3 with $\Theta = \tilde{\Theta}_2$. Note that $\Theta_{\mathcal{M}}$ contains $\tilde{\Theta}_2$ and in addition only suboptimal strategies which yield expected utility $-\infty$. Thus, the results also hold for $\Theta = \Theta_{\mathcal{M}}$ and therefore for any Θ which satisfies (1.8). Note that the second equality in part 4 follows by part 2. □

In the sequel, we will frequently use a result by Csiszár ([Cs75], Theorem 2.2) on entropy minimizing probability measures. For ease of future reference, let us state his result in our framework:

Lemma 1.2.4 *A measure $R \in \mathbb{P}_f$ minimizes $H(Q|P)$ over $Q \in \mathbb{P}_a$ (or $\mathbb{P}_f$) if and only if*

$$H(Q|P) \geq H(Q|R) + H(R|P) \quad \textit{for every} \quad Q \in \mathbb{P}_a\,.$$

As an immediate consequence, the entropy minimizing martingale measure Q^0 from Proposition 1.2.3 satisfies $H(Q|P) \geq H(Q|Q^0) + H(Q^0|P)$ for every $Q \in \mathbb{P}_a$.

1.3 Utility indifference pricing

We now introduce one of our main objects of interest:

Definition 1.3.1 *If there is a unique solution $\pi(B) = \pi(B;\alpha)$ to the equation*

$$u(x;\alpha) = u(x + \pi - B;\alpha)\,, \tag{1.15}$$

we call this solution the utility indifference (selling) price *for B.*

In the absence of dynamical trading opportunities – this corresponds to letting $S \equiv S_0$ or $\Theta \equiv \{0\}$ – the indifference price π is defined as the solution of the equation

$$U(x) = E[U(x + \pi - B)] \tag{1.16}$$

with a utility function U which is exponential in our framework, i.e. $U(x) = -\exp(-\alpha x)$. The solution π to eq. (1.16) yields a valuation method which has been known for a long time. In fact, the origins of the idea can be traced back to the 18th century when Daniell Bernoulli [B1738] suggested that an investor, say gambler, will rank risky ventures, e.g. lotteries, by their expected utilities. In actuarial mathematics, the valuation method is called the "premium principle of equivalent utility" and this principle is known to have certain desirable properties if and only if the utility function U is exponential (see Gerber [Ge79], ch. 5, for details).

In the presence of a financial market, investors can maximize their expected utility and reduce risk from a terminal liability B by dynamic trading. Taking this into account, we are led to the Definition 1.3.1 for the utility indifference price. This adaptation of the classical idea from the static situation to the dynamic financial market case appeared first in Hodges and Neuberger [HodN89]. $\pi(B)$ can be interpreted as the adjustment to the initial capital that compensates an investor for the additional terminal liability B in terms of maximal expected (exponential) utility. In this sense, $\pi(B)$ is a subjective fair valuation of the liability B from the perspective of a risk averse investor. We emphasize that $\pi(B)$ should not be considered as market price for which B can be bought and sold in the financial market.

Some authors use the term *reservation (selling) price* instead of utility indifference price.

Remark 1.3.2 (Possible extensions)

In analogy to the previous definition, one could define a corresponding *utility indifference buying price* as the unique solution $\pi^b(B)$ to the equation

$$u(x) = u(x - \pi^b + B)\,.$$

It is easy to see that the solution is given by $\pi^b(B) = -\pi(-B)$ if the latter is defined. Furthermore, one might be interested in the utility indifference (selling) price of an additional claim B for an issuer who already has a liability, say, C. Again, it is easy to verify that the unique solution $\pi(B|C)$ to the equation

$$u(x - C) = u(x + \pi(B|C) - (B + C))$$

is given by $\pi(B|C) = \pi(B + C) - \pi(C)$ if the terms on the right-hand side are defined. To see this, note that the exponential utility implies that π does not depend on the initial capital x. Hence $u(x - C) = u(x - \pi(C)) = u(x - \pi(C) + \pi(B + C) - (B + C))$, which implies the claim. $\diamond$

Under the assumptions of Proposition 1.2.3, one obtains (see eq. (4.6) in [DGRS^{3}00])

$$\pi(B;\alpha) = \sup_{Q\in\mathbb{P}_f} \left\{ E_Q[B] - \frac{1}{\alpha}\Big(H(Q|P) - H(Q^0|P)\Big) \right\}, \tag{1.17}$$

and thereby the following properties (1.18)-(1.24) of the indifference price $\pi(B;\alpha)$, cf. Frittelli [Ft00a] and Rouge and El Karoui [RouE00]. First, we observe that

(1.18) (Independence on initial capital) $\pi(B;\alpha)$ does not depend on x.

Since $H(Q|P) \geq H(Q^0|P)$ for any $Q \in \mathbb{P}_f$ by definition of Q^0, we have

(1.19) (Monotonicity in α) $\alpha \mapsto \pi(B;\alpha)$ is increasing in α,

and an elementary algebraic transformation yields

(1.20) (Volume-scaling) $\pi(\beta B;\alpha) = \beta\pi(B;\beta\alpha)$ for $\beta \in (0,1]$.

If B is bounded, the previous equation holds for $\beta \in (0,\infty)$.
For bounded random variables B_1 and B_2, formula (1.17) yields

(1.21) (Convexity) $\pi(\lambda B_1 + (1-\lambda)B_2;\alpha) \leq \lambda\pi(B_1;\alpha) + (1-\lambda)\pi(B_2;\alpha)$ for $\lambda \in [0,1]$,

and (1.17) moreover implies

(1.22) (Monotonicity) $\pi(B_1;\alpha) \leq \pi(B_2;\alpha)$ if $B_1 \leq B_2$, and

(1.23) (Translation invariance) $\pi(B+c;\alpha) = \pi(B;\alpha) + c$ for $c \in \mathbb{R}$.

The last three properties lead to an interesting alternative interpretation of the utility indifference price as a measure of risk. Although we will not stress this point in the sequel, the reader may find it helpful for understanding and interpretation of some subsequent results, e.g. Corollary 1.3.5.

Remark 1.3.3 (Interpretation as a convex measure of risk)
By (1.21)-(1.23), the mapping $\varrho : X \mapsto \pi(-X;\alpha)$ satisfies the properties which define a *convex measure of risk* on the set of bounded random variables X. For the notion and a discussion of such risk measures, we refer to Föllmer and Schied [FSchi01]. Let us just note here that ϱ assigns a quantitative measure of risk to a financial position X. Since our indifference value π is specified with respect to liabilities, it is applied to the liability $B = -X$ that corresponds to the financial position X. $\diamond$

For any $B \in L^\infty(P)$ that is *attainable* in the sense that $B = b + \int_0^T \vartheta\, dS$ for $\vartheta \in L(S)$ with $\int \vartheta\, dS$ bounded (uniformly in t) we have $E_Q[B] = b$ for all $Q \in \mathbb{P}_a$. By (1.17), this yields

(1.24) (Elementary No-Arbitrage consistency)

$$\pi\left(b + \int_0^T \vartheta\, dS;\alpha\right) = b \qquad \text{for } \vartheta \in L(S) \text{ with } \int \vartheta\, dS \text{ bounded.}$$

Since Q^0 is in $\mathbb{P}_e$ the following Proposition 1.3.4 implies in particular that the utility indifference price of any bounded claim lies within the interval of possible arbitrage free valuations, that is

$$\inf_{Q\in\mathbb{P}_e} E_Q[B] \le \pi(B;\alpha) \le \sup_{Q\in\mathbb{P}_e} E_Q[B] \quad \text{for } B \in L^\infty(P).$$

In this sense, we have consistency with the No-Arbitrage principle also for non attainable claims.

Proposition 1.3.4 *(risk-aversion asymptotics)*
Assume (1.1), (1.2), (1.8) and $B \in L^\infty(P)$. Then $\alpha \mapsto \pi(B;\alpha)$, $\alpha \in (0,\infty)$, is non-decreasing and

$$\lim_{\alpha\uparrow\infty} \pi(B;\alpha) = \sup_{Q\in\mathbb{P}_e} E_Q[B]\,, \tag{1.25}$$

$$\lim_{\alpha\downarrow 0} \pi(B;\alpha) = E_{Q^0}[B]\,. \tag{1.26}$$

Rouge and El Karoui [RouE00] proved these asymptotics in a Brownian setting using backward stochastic differential equations. In the present semimartingale framework, (1.25) was proven by Delbaen et al. [DGRS^{3}00] and it remains to show (1.26) here.

Proof: It is clear from (1.17) and (1.19) that $\pi(B;\alpha) \ge E_{Q^0}[B]$ for all α and $\pi(B;\alpha)$ decreases when α decreases. Hence,

$$E_{Q^0}[B] \le \lim_{\alpha\downarrow 0} \pi(B;\alpha) = \inf_{\alpha>0} \sup_{Q\in\mathbb{P}_f} \left\{ E_Q[B] - \frac{1}{\alpha}\Big(H(Q|P) - H(Q^0|P)\Big)\right\}.$$

Suppose $E_{Q^0}[B] + \varepsilon \le \lim_{\alpha\downarrow 0} \pi(B;\alpha)$ for some $\varepsilon > 0$. Then, we can find for any $\alpha > 0$ some $Q_\alpha \in \mathbb{P}_f$ such that

$$\Big(E_{Q_\alpha}[B] - E_{Q^0}[B]\Big) - \frac{1}{\alpha}\Big(H(Q_\alpha|P) - H(Q^0|P)\Big) > \frac{\varepsilon}{2}\,. \tag{1.27}$$

The term in the first bracket is bounded uniformly in α since $B \in L^\infty$, and the term in the second bracket nonnegative for all α. By (1.27) this yields $\lim_{\alpha\downarrow 0} H(Q_\alpha|P) = H(Q^0|P)$ and we obtain

$$\lim_{\alpha\downarrow 0} H(Q_\alpha|Q^0) = 0$$

since $H(Q_\alpha|P) \ge H(Q_\alpha|Q^0) + H(Q^0|P)$ by Lemma 1.2.4. We conclude that Q_α converges to Q^0 in entropy and hence also in total variation. This implies $E_{Q_\alpha}[B] \to E_{Q^0}[B]$ for $\alpha \downarrow 0$ and so the limes inferior for $\alpha \downarrow 0$ of the left-hand side in (1.27) is less than or equal to 0. Clearly, this is a contradiction. □

In view of the volume scaling property (cf. (1.20) and the subsequent remark there), the limits in Proposition 1.3.4 can be rewritten as volume-asymptotics of the utility indifference price. This yields the subsequent corollary.

Corollary 1.3.5 *(volume asymptotics)*
Assume (1.1), (1.2), (1.8) and $B \in L^\infty(P)$. *For any* $\alpha \in (0,\infty)$, *the mapping* $\beta \mapsto \frac{1}{\beta}\pi(\beta B;\alpha)$, $\beta \in (0,\infty)$, *is non-decreasing and*

$$\lim_{\beta\uparrow\infty} \frac{1}{\beta}\pi(\beta B;\alpha) = \sup_{Q\in\mathbb{P}_e} E_Q[B]\,, \tag{1.28}$$

$$\lim_{\beta\downarrow 0} \frac{1}{\beta}\pi(\beta B;\alpha) = E_{Q^0}[B]\,. \tag{1.29}$$

The appearing quantity $\frac{1}{\beta}\pi(\beta B;\alpha)$ can be interpreted as price per unit for a given amount (volume) β of claims B. It is increasing in β since π is defined as a risk-averse and subjective fair valuation from the issuer's perspective. For concreteness, consider an insurance company that insures skyscrapers against earthquakes and has already many clients in San Francisco. Adding a further client there, would increase the volume of the insured risk that is exposed to the next big earthquake in northern California. Provided the insurer has to keep its risks and cannot resell them, he would prefer an otherwise comparable risk somewhere else. That is, his risk-averse valuation of an additional earthquake-related policy is higher if it increases his too large exposure to the next big earthquake in California.

Remark 1.3.6 (Relation to Davis' "fair price")
Suppose B is bounded, α equals 1 and $\Theta_{\mathcal{M}}$ is a subset of Θ. By Parts 2 and 4 of Proposition 1.2.3, we have

$$E_{Q^0}[B] = E_P\left[e^{-c^0-\int_0^T \vartheta^0\,dS}\,B\right] = \frac{E_P\left[U'(x+\int_0^T \vartheta^0\,dS)\,B\right]}{\frac{\partial}{\partial x}\left(-e^{-(x-c^0)}\right)} \tag{1.30}$$

with $U(x) := -e^{-x}$. Thanks to formula (3) in [D97], this implies that $E_{Q^0}[B]$ equals Davis' "fair price" $\widehat{p}$ of the claim B with respect to the exponential utility function U. The definition of this price is based on the idea of "zero marginal rate of substitution" which is very common in economics and was applied in [D97] to the pricing of options. Intuitively, $\widehat{p}$ is defined such that an investor cannot increase his expected (exponential) utility by diverting an infinitesimal amount of his capital into the option contract.
By (1.30), the results (1.26) (or (1.29)) show that Davis' "fair price" $\widehat{p}$ can be seen as the limit of the utility indifference valuations for B for vanishing risk-aversion (or infinitesimal contract volume, respectively). Moreover, $\widehat{p}$ provides a lower bound for the utility indifference prices $\pi(B;\alpha)$, $\alpha \in (0,\infty)$. ◇

1.4 Utility indifference hedging

We now define the hedging strategy $\psi(B;\alpha)$ as the adjustment of the optimal strategy without liability that is necessary to obtain a strategy that is optimal under the terminal liability B.

More vaguely, it is the part of the optimal strategy ϑ^B that stems from the additional liability B.

Definition 1.4.1 *Under the assumptions of Proposition 1.2.3, we define the* utility indifference hedging strategy ψ *for* B *by*

$$\psi(B) = \psi(B;\alpha) := \vartheta^{B,\alpha} - \vartheta^{0,\alpha}\,. \tag{1.31}$$

Part 2 of Proposition 1.2.3 implies that ϑ^B and ϑ^0 are unique in the sense that the processes $\int \vartheta^B\, dS$ and $\int \vartheta^0\, dS$, respectively, are unique. Hence, the hedging strategy is unique in the sense that $\psi \cdot S$ is unique. More formally, ψ is defined in the quotient space of $L(S)$ with respect to the equivalence relation $\vartheta \sim \xi :\Leftrightarrow \int \vartheta dS = \int \xi dS$.
The main goal of this section is to characterize the utility indifference hedging strategy and the indifference price by necessary and sufficient criteria. We introduce a utility indifference price process to formulate a martingale criterion, which is sufficient to identify the indifference hedging strategy and the indifference price. Moreover, this price process provides a useful ansatz to solve the pricing and hedging problem in concrete models as later chapters will show.
Assume P_B is well defined. Grandits and Rheinländer [GraR99] showed that a measure $\bar{Q} \in \mathbb{P}_a$ minimizes $H(Q|P_B)$ over $Q \in \mathbb{P}_a$ if its density has the form

$$\frac{d\bar{Q}}{dP_B} = \exp\left(-\alpha(c + \int_0^T \bar{\vartheta}\, dS)\right) \quad \text{for some } c \in \mathbb{R} \text{ and } \bar{\vartheta} \in L(S) \tag{1.32}$$

and one of the following two conditions holds:

(i) $\bar{\vartheta} \cdot S$ is a $\bar{Q}$-$\mathcal{BMO}$-martingale and $e^{\varepsilon\alpha \int_0^T \bar{\vartheta}\, dS} \in L^1(P_B)$ for some $\varepsilon > 0$, or

(ii) $\bar{Q} \in \mathbb{P}_f$ and $\bar{\vartheta} \in \Theta_{\mathcal{M}}$.

Note that (1.32) in combination with (i) or (ii) already implies that $\bar{Q} \in \mathbb{P}_f \cap \mathbb{P}_e$ since $d\bar{Q}/dP_B > 0$ and $H(\bar{Q}|P_B) = -\alpha c < \infty$. Variant (i) is just a redraft of Proposition 3.4 in [GraR99] and variant (ii) readily follows from Proposition 3.2 in [GraR99]. Rewriting these results from Grandits and Rheinländer [GraR99] with respect to P instead of P_B then yields the following formulation:

Proposition 1.4.2 *Assume (1.1) and (1.5). Suppose the density of* $\bar{Q} \in \mathbb{P}_a$ *takes the form*

$$\frac{d\bar{Q}}{dP} = \exp\left(-\alpha(\bar{c} + \int_0^T \bar{\vartheta}\, dS - B)\right) \quad \textit{for some } \bar{c} \in \mathbb{R} \textit{ and } \bar{\vartheta} \in L(S)$$

and one of the following two conditions holds:

(i) $\bar{\vartheta} \cdot S$ *is a* $\bar{Q}$-$\mathcal{BMO}$*-martingale and* $e^{\alpha(B + \varepsilon \int_0^T \bar{\vartheta}\, dS)} \in L^1(P)$ *for some* $\varepsilon > 0$*, or*

(ii) $\bar{Q} \in \mathbb{P}_f$ *and* $\bar{\vartheta} \in \Theta_{\mathcal{M}}$.

Then $\bar{Q}$ is in $\mathbb{P}_e \cap \mathbb{P}_f$ and gives the solution to the dual problem (see Proposition 1.2.3), i.e. $\bar{Q} = Q^B$, $\bar{\vartheta} = \vartheta^B$ and $\bar{c} = c^B$.

As a consequence from variant (ii), we can characterize the indifference price π and the hedging strategy ψ by a condition that is both necessary and sufficient:

Corollary 1.4.3 *Suppose the assumptions of Proposition 1.2.3 hold.*

1. *The hedging strategy $\psi(B)$ is in $\Theta_{\mathcal{M}}$ and satisfies*

$$\exp\left(-\alpha\Big(\pi(B) + \int_0^T \psi(B)\, dS - B\Big)\right) = \frac{dQ^B}{dQ^0}\,.$$

2. *Conversely: Let $\bar{\psi} \in \Theta_{\mathcal{M}}$, $\bar{\pi} \in \mathbb{R}$ and $\bar{Q} \in \mathbb{P}_f$ such that*

$$\exp\left(-\alpha\Big(\bar{\pi} + \int_0^T \bar{\psi}\, dS - B\Big)\right) = \frac{d\bar{Q}}{dQ^0} \tag{1.33}$$

holds. Then we have $\psi(B) = \bar{\psi}$, $\pi(B) = \bar{\pi}$ and $Q^B = \bar{Q}$.

Remark 1.4.4 Part 2 of Corollary 1.4.3 yields that the replicating strategy for a bounded attainable claim B is the hedging strategy. To see this, let $B = b + \int_0^T \bar{\psi}\, dS$ with $b \in \mathbb{R}$ and $\int \bar{\psi}\, dS$ bounded (uniformly in t). Taking $\bar{Q} := Q^0$ and $\bar{\pi} := b$, we obtain from part 2 of Corollary 1.4.3 that $\pi(B) = \bar{\pi}$ and $\psi(B) = \bar{\psi}$; that is, the indifference price equals the initial capital that is needed for the replication of B and the hedging strategy is the replicating strategy, as it should be. ◇

Proof: (of Corollary 1.4.3)
1. By Proposition 1.2.3 we have

$$u(c^B - B) = E_P\left[-\frac{dQ^B}{dP}\right] = -1 = E_P\left[-\frac{dQ^0}{dP}\right] = u(c^0 - 0)$$

and conclude that $\pi(B) = c^B - c^0$. Moreover, the hedging strategy $\psi(B) = \vartheta^B - \vartheta^0$ is in the space $\Theta_{\mathcal{M}}$ and

$$\frac{dQ^B}{dQ^0} = \frac{dQ^B}{dP}\left(\frac{dQ^0}{dP}\right)^{-1} = \exp\left(-\alpha\Big((c^B - c^0) + \int_0^T \left(\vartheta^B - \vartheta^0\right) dS - B\Big)\right).$$

2. For the "sufficiency"-part, consider

$$\frac{d\bar{Q}}{dP} = \left(\frac{dQ^0}{dP}\right)\frac{d\bar{Q}}{dQ^0} = \exp\left(-\alpha\left((c^0 + \bar{\pi}) + \int_0^T (\vartheta^0 + \bar{\psi})dS - B\right)\right). \tag{1.34}$$

By variant (ii) of Proposition 1.4.2 we obtain $Q^B = \bar{Q}$ and then $\vartheta^B = \vartheta^0 + \bar{\psi}$ and $c^B = c^0 + \bar{\pi}$ via Proposition 1.2.3. By the same arguments as in the first part of this proof, we then obtain $\pi(B) = c^B - c^0 = \bar{\pi}$ and by the definition of ψ we have $\psi(B) = \vartheta^B - \vartheta^0 = \bar{\psi}$. □

We obtain from part 1 of Corollary 1.4.3 that the density process of Q^B with respect to Q^0 takes the form

$$\frac{dQ^B}{dQ^0}\bigg|_{\mathcal{F}_t} = \exp\left(-\alpha\Big(\pi(B) + \int_0^t \psi(B)\,dS - \pi_t\Big)\right), \quad t \in [0,T], \tag{1.35}$$

with $(\pi_t)_{t\in[0,T]}$ given by

$$\pi_t = \frac{1}{\alpha}\log E_{Q^0}\left[\exp(\alpha(B - \int_t^T \psi(B)dS)\Big|\mathcal{F}_t\right], \quad t \in [0,T]. \tag{1.36}$$

Note that $(\pi_t)_{t\in[0,T]}$ is a semimartingale and satisfies $\pi_T = B$ and $\pi_0 = \pi(B)$.

Definition 1.4.5 *Suppose the assumptions of Proposition 1.2.3 hold. We call the process $(\pi_t)_{t\in[0,T]}$ in the representation (1.35) the* utility indifference price process *of B.*

In the following chapters, the price process (π_t) will play a central role for the solution to our indifference pricing and hedging problem. This is best seen in Chapter 3. There, we first obtain results on the process (π_t) and then derive the utility indifference price $\pi(B)$ and the hedging strategy $\psi(B)$ from (π_t). Note that the order of these derivations is opposite to the one in formula (1.36) where π_t is obtained from ψ.
For these later purposes, we formulate a density process version of Corollary 1.4.3:

Corollary 1.4.6 *Suppose the assumptions of Proposition 1.2.3 hold and*

$$\frac{d\bar{Q}}{dQ^0}\bigg|_{\mathcal{F}_t} = \exp\left(-\alpha\Big(\bar{\pi}_0 + \int_0^t \bar{\psi}\,dS - \bar{\pi}_t\Big)\right), \quad t \in [0,T], \tag{1.37}$$

holds for some $\bar{Q} \in \mathbb{P}_f$, $\bar{\psi} \in \Theta_{\mathcal{M}}$ and a semimartingale $(\bar{\pi}_t)_{t\in[0,T]}$ with $\bar{\pi}_T = B$. Then $\psi(B) = \bar{\psi}$, $\pi(B) = \bar{\pi}_0$, $Q^B = \bar{Q}$ and $(\bar{\pi}_t)$ is the indifference price process.

Proof: Taking $t = T$, we obtain by part 2 of Corollary 1.4.3 that $\psi(B) = \bar{\psi}$, $\pi(B) = \bar{\pi}_0$ and $Q^B = \bar{Q}$. Comparing (1.35) and (1.37) then implies that $(\bar{\pi}_t)$ is the utility indifference price process. □

Remark 1.4.7 An analogous reasoning leads to a density process formulation of the sufficient criteria from Proposition 1.4.2. We are going to show that this offers an economic interpretation for a process which appears in Chapter 3 (cf. Theorem 3.4.3 and the subsequent remarks) and also the indifference price process $(\pi_t)_{t\in[0,T]}$.
Suppose that (1.1) and (1.5) hold and that we are given some $\bar{Q} \in \mathbb{P}_a$, $\bar{\vartheta} \in L(S)$ and a semimartingale $(c_t^B)_{t\in[0,T]}$ such that $c_T^B = B$, $\bar{\vartheta}\cdot S$ is a $\bar{Q}$-$\mathcal{BMO}$-martingale,

$$\frac{d\bar{Q}}{dP}\bigg|_{\mathcal{F}_t} = \exp\left(-\alpha(c_0^B + \int_0^t \bar{\vartheta}\,dS - c_t^B)\right), \quad t \in [0,T], \tag{1.38}$$

holds, and $e^{\alpha(B+\varepsilon\int_0^T \bar\vartheta\,dS)}$ is in $L^1(P)$ for some $\varepsilon > 0$. Via Proposition 1.4.2, this yields $\bar Q = Q^B$ and $\bar\vartheta = \vartheta^B$ and $c_0^B = c^B$, that is, we have the solution to the dual problem in Proposition 1.2.3. Since the terminal density determines the density process, we find that the process $(c_t^B)_{t\in[0,T]}$ is determined by ϑ^B and B. This dependence is analogous to the one in eq. (1.36) and given by the formula

$$c_t^B = \frac{1}{\alpha}\log E_P\left[\exp\left(\alpha(B - \int_t^T \vartheta^B\,dS)\right)\Big|\mathcal{F}_t\right], \quad t\in[0,T]. \tag{1.39}$$

Note that the expectation here is under the objective probability measure P and not under Q^0 as in eq. (1.36).

This expression provides some interpretation to the processes (c_t^B) and thereby also for (π_t). The investor who maximizes his P-expected utility under terminal liability B follows his optimal trading strategy ϑ^B. At time $t\in[0,T]$, he then faces the effective liability $B-\int_t^T \vartheta^B\,dS$ which is the difference between B and the gains from trade that he is going to realize in the remaining time $(t,T]$. Note that the effective liability can be positive as well as negative. In the latter case, the investor enjoys an effective gain from present time t until T.

By eq. (1.39), we can interpret c_t^B as the current (exponential) P-*certainty equivalent* of this remaining effective liability given the information at time t: At time t, c_t^B is the "time t-certain" liability that an investor would rate as good as the remaining effective liability in terms of expected exponential utility. For the investor with no liability, we have

$$c_t^0 = \frac{1}{\alpha}\log E_P\left[\exp\left(\alpha(- \int_t^T \vartheta^0\,dS)\right)\Big|\mathcal{F}_t\right], \quad t\in[0,T]. \tag{1.40}$$

We can calculate the density process $\frac{dQ^B}{dQ^0}\Big|_{\mathcal{F}_t} = \frac{dQ^B}{dP}\Big|_{\mathcal{F}_t}\left(\frac{dQ^0}{dP}\Big|_{\mathcal{F}_t}\right)^{-1}$, $t\in[0,T]$, using formula (1.38) (with $\bar Q = Q^B$ and $\bar\vartheta = \vartheta^B$). Substituting $\psi(B) = \vartheta^B - \vartheta^0$ and comparing the result with (1.37) then yields

$$\pi_t = c_t^B - c_t^0, \quad t\in[0,T].$$

Thus, the current utility indifference price process is the difference of the current certainty equivalents of the effective liabilities between the investor with terminal liability B and the ordinary investor who has no terminal liability. ◇

Chapter 2

Results on rational pricing and hedging in semi-complete product models

In the present chapter, we aim for more constructive results and further properties of our utility based pricing and hedging approach. To this end, we impose certain structural assumptions on our financial market model. More precisely, the framework for this chapter will be a so-called *semi-complete product model* which consists of a *complete* financial sub-market and additional *independent* sources of uncertainty and risk. The model is formally introduced and specified in Section 2.1 Thanks to the structural properties of the semi-complete product model, the set of equivalent martingale measures has a very convenient structure which is explored in Section 2.2. In Section 2.3, we show that the utility indifference price and the corresponding hedging strategy for an aggregated amount of contingent claims are given by the sums of the prices and hedging strategies, respectively, for the single claims whenever these claims are "sufficiently independent". More precisely, this additivity holds if the single claims are conditionally independent given the information of the complete sub-market. Under similar conditional independence assumptions, we then show that diversification effects among a large number of individual risks can asymptotically lead to a utility based valuation for the aggregate which is risk neutral with respect to untradable factors of risk under the objective probability measure P.
Section 2.4 provides explicit upper and lower bounds for the utility indifference price. If the additional information is piecewise constant, the utility indifference price can be obtained by an explicit backward computation scheme. This scheme resembles the well known backward computation method in the binomial tree model in the sense that it only involves the calculation of conditional expectations for each period.
In the following section, we specify the semi-complete model and the technical assumptions which are supposed to hold throughout this chapter.

2.1 Chapter Assumption: Semi-complete product model

Let $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t\in[0,T]}$ and $\mathbb{I}^i = (\mathcal{I}_t^i)_{t\in[0,T]}$, $i \in \mathbb{N}$, be filtrations which satisfy the usual conditions and have trivial σ-fields at $t = 0$. We suppose that S is adapted and locally bounded with respect to $\mathbb{F}^0$ and

$$\mathcal{F}_T^0,\ \mathcal{I}_T^1,\ \mathcal{I}_T^2, \dots \quad \text{are independent under } P. \tag{2.1}$$

Let $\mathbb{I} = (\mathcal{I}_t)$ be given by $\mathcal{I}_t := \bigvee_{i=1}^{\infty} \mathcal{I}_t^i$ and define $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ via

$$\mathcal{F}_t := \mathcal{F}_t^0 \vee \mathcal{I}_t = \mathcal{F}_t^0 \vee \bigvee_{i=1}^{\infty} \mathcal{I}_t^i\,, \qquad t \in [0,T]\,. \tag{2.2}$$

We assume that there is a unique Q^* which is in $\mathbb{P}_e \cap \mathbb{P}_f$ and has an $\mathcal{F}_T^0$-measurable density, i.e.

$$\left\{ Q \in \mathbb{P}_e \,\middle|\, Q \in \mathbb{P}_f,\ \frac{dQ}{dP} \text{ is } \mathcal{F}_T^0\text{-measurable} \right\} = \{Q^*\}\,. \tag{2.3}$$

Hence, S has the strong predictable representation property with respect to $(\mathbb{F}^0, Q^*)$ (see Jacod [Ja79], Corollaire 11.4) and the conditions (1.1) and (1.2) hold. By (2.3), condition (2.1) is equivalent to assuming that

$$\mathcal{F}_T^0,\ \mathcal{I}_T^1,\ \mathcal{I}_T^2, \dots \quad \text{are independent under } Q^*. \tag{2.4}$$

As the next result shows, both $\mathbb{F}$ and $\mathbb{I}$ inherit the usual conditions from their sub-filtrations thanks to the independence assumption (2.1).

Lemma 2.1.1 *$\mathbb{F}$ and $\mathbb{I}$ satisfy the usual conditions.*

Proof: It suffices to show right-continuity. We just show the argument for $\mathbb{I}$: Let $A_i \in \mathcal{I}_T^i$, $i = 1, \dots, n$, for some $n \in \mathbb{N}$. By (2.4), we have for any $s \in [0,T]$

$$E\left[\prod_{i=1}^n 1_{A_i} \,|\, \mathcal{I}_s\right] = \prod_{i=1}^n E[1_{A_i}|\mathcal{I}_s^i]\,.$$

Letting $s \downarrow t$ for some $t \in [0,T]$ and using (2.4) and the usual conditions for the $\mathbb{I}^i$, $i = 1. \dots, n$, yields

$$E\left[\prod_{i=1}^n 1_{A_i} \,|\, \mathcal{I}_{t+}\right] = \prod_{i=1}^n E[1_{A_i}|\mathcal{I}_{t+}^i] = \prod_{i=1}^n E[1_{A_i}|\mathcal{I}_t^i] = E\left[\prod_{i=1}^n 1_{A_i} \,|\, \mathcal{I}_t\right]\,.$$

By monotone class arguments, this implies $E[f|\mathcal{I}_{t+}] = E[f|\mathcal{I}_t]$ for all bounded $\mathcal{I}_T$-measurable functions f and therefore $\mathcal{I}_{t+} = \mathcal{I}_t$. □

Remark 2.1.2 Assumptions (2.3) and (2.1) mean that our model is composed of a complete financial market $(S, \mathbb{F}^0)$ and additional independent sources $\mathbb{I}^1, \mathbb{I}^2, \dots$ of uncertainty and risk which are independent from the complete (sub-)market under the objective probability measure P. ◇

2.2 The structure of the equivalent martingale measures

To prepare for our main results, we start with some structural results on the optimal measure Q^B for the dual problem and on the set $\mathbb{P}_e$ of equivalent martingale measures in the semi-complete setting.
Since $dQ^* = \frac{dQ}{dP}\Big|_{\mathcal{F}^0_T} dP$ holds for any $Q \in \mathbb{P}_f$ we have $H(Q^*|P) \leq H(Q|P)$ for $Q \in \mathbb{P}_f$, i.e. Q^* is the entropy minimizing martingale measure:

$$Q^* = Q^0 \,. \tag{2.5}$$

A similar reasoning yields

Lemma 2.2.1 *Suppose B is $\mathcal{F}^0_T \vee \mathcal{I}^k_T$-measurable for some k and satisfies (1.5). Then*

$$\frac{dQ^B}{dQ^*} \quad \text{is } \mathcal{F}^0_T \vee \mathcal{I}^k_T \text{ -measurable}\,.$$

Proof: We can assume that S is bounded; otherwise one can $\mathbb{F}^0$-localize so that it is. By symmetry, it suffices to consider the case $k = 1$. Define $dQ' := \frac{dQ^B}{dP_B}\Big|_{\mathcal{F}^0_T \vee \mathcal{I}^1_T} dP_B$ with P_B given by (1.3). Then S is a martingale with respect to $(Q', (\mathcal{F}^0_t \vee \mathcal{I}^1_t)_{t\in[0,T]})$ and

$$\frac{dQ'}{dQ^*} = \left(\frac{dQ'}{dP_B}\right)\left(\frac{dP_B}{dP}\right)\left(\frac{dP}{dQ^*}\right) \quad \text{is } \mathcal{F}^0_T \vee \mathcal{I}^1_T\text{-measurable.}$$

In combination with (2.4), this implies that $\mathcal{F}^0_T \vee \mathcal{I}^1_T$ is Q'-independent from $\mathcal{I}^2_T \vee \mathcal{I}^3_T \vee \ldots$ and so S is also a $(Q', \mathbb{F})$-martingale. By the construction of Q' we have $H(Q'|P_B) \leq H(Q^B|P_B)$ and $Q' \sim P$. Since Q^B is unique, this implies $Q' = Q^B$ and thereby yields the claim. □

In the present semi-complete model, the set $\mathbb{P}_e$ of equivalent martingale measures has a rather simple structure which is analyzed in the next result.

Lemma 2.2.2 *Let Q be some measure equivalent to P and let Z denote the $\mathbb{F}$-density process of Q with respect to Q^*. Then Q is an element of $\mathbb{P}_e$ if and only if*

$$E^*[Z_t|\mathcal{F}^0_t \vee \mathcal{I}_s] = Z_s \qquad \textit{for all} \quad 0 \leq s \leq t \leq T\,. \tag{2.6}$$

In particular, taking $s = 0$ yields $E^*[Z_t|\mathcal{F}^0_t] = 1$ for $t \in [0, T]$.

Proof: Again, we can assume that S is bounded. Otherwise we can $\mathbb{F}^0$-localize appropriately.

1. Let us first show necessity: Fix $s \leq T$ and define

$$\widetilde{Z}_t := E^*[Z_T|\mathcal{F}^0_t \vee \mathcal{I}_s] = E^*[Z_t|\mathcal{F}^0_t \vee \mathcal{I}_s]\,, \quad t \in [s, T]\,,$$

where the equality follows via conditioning on $\mathcal{F}_t$. To show (2.6), it suffices to prove

$$\widetilde{Z}_t = \widetilde{Z}_s \quad \text{for} \quad t \in [s,T]\,. \tag{2.7}$$

To this end, define $(L_t)_{t\in[0,T]}$ by

$$L_t := \begin{cases} 1 & : \; t < s\,, \\ \frac{\widetilde{Z}_t}{\widetilde{Z}_s} & : \; t \geq s\,. \end{cases}$$

Then

(2.8) S and L are martingales with respect to $(Q^*, (\mathcal{F}^0_t \vee \mathcal{I}_s)_{t\in[0,T]})$

by the independence hypothesis (2.4) and the construction of L, respectively. A little calculation shows that also LS is a $(Q^*, (\mathcal{F}^0_t \vee \mathcal{I}_s)_{t\in[0,T]})$-martingale. In fact,

$$Z_u S_u = E^*[Z_T S_t | \mathcal{F}_u] = E^*\left[E^*[Z_T|\mathcal{F}^0_t \vee \mathcal{I}_u] S_t \middle| \mathcal{F}_u\right], \qquad s \leq u \leq t \leq T,$$

implies – by taking $E^*[\,\cdot\,|\mathcal{F}^0_u \vee \mathcal{I}_s]$ on both sides – that

$$E^*[Z_u|\mathcal{F}^0_u \vee \mathcal{I}_s] S_u = E^*\left[E^*[Z_T|\mathcal{F}^0_t \vee \mathcal{I}_u] S_t \middle| \mathcal{F}^0_u \vee \mathcal{I}_s\right], \qquad s \leq u \leq t \leq T. \tag{2.9}$$

Via conditioning an $\mathcal{F}^0_t \vee \mathcal{I}_s$ we see that the right hand side of this equation is equal to $E^*\left[E^*[Z_T|\mathcal{F}^0_t \vee \mathcal{I}_s] S_t \middle| \mathcal{F}^0_u \vee \mathcal{I}_s\right]$. Substituting the definitions of $\widetilde{Z}$ and L, we can then rewrite eq. (2.9) as

$$L_u S_u = E^*[L_t S_t|\mathcal{F}^0_u \vee \mathcal{I}_s] \quad \text{for} \quad s \leq u \leq t \leq T\,. \tag{2.10}$$

In combination with the definition of L and (2.8), this establishes that LS is a martingale with respect to $(Q^*, (\mathcal{F}^0_t \vee \mathcal{I}_s)_{t\in[0,T]})$. From (2.4) and (2.3) follows by Theorem 5.2.3 that S has the strong predictable representation property with respect to $(Q^*, (\mathcal{F}^0_t \vee \mathcal{I}_s))_{t\in[0,T]})$. This implies that any local $((\mathcal{F}^0_t \vee \mathcal{I}_s)_{t\in[0,T]})$-martingale that is orthogonal to S must be constant (see Corollary 13.6 in [HWY92]). Hence, L is constant at $L_0 = 1$ and this establishes (2.7).

2. To show sufficiency, note that (2.6) and (2.4) imply

$$E^*\left[Z_t S_t|\mathcal{F}^0_s \vee \mathcal{I}_s\right] = E^*\left[S_t E^*[Z_t|\mathcal{F}^0_t \vee \mathcal{I}_s] \middle| \mathcal{F}^0_s \vee \mathcal{I}_s\right] = E^*[S_t Z_s|\mathcal{F}^0_s \vee \mathcal{I}_s] = Z_s S_s\,.$$

□

2.3 Additivity and diversification

We already know that the mapping $B \mapsto \pi(B;\alpha)$ is linear on the space of bounded attainable claims in general (see (1.24)).
In the static situation – which corresponds to setting $S \equiv 0$ – the utility indifference price $\pi(B;\alpha) = 1/\alpha \log E_P[\exp(\alpha B)]$ is known to be additive with respect to independent claims, i.e., $\pi(B_1+B_2) = \pi(B_1)+\pi(B_2)$ holds for P-independent variables $B_{1,2}$ with sufficient integrability (see Gerber [Ge79], ch. 5, on the “exponential premium principle”). For dependent claims π need not be additive (or sub-additive) even in the static situation.
In the present section, we are going to obtain similar results on additivity for the dynamic case. More precisely, we will show that the utility indifference price and the hedging strategy are additive when applied to a sum of claims which are conditionally P-independent given the information of the complete financial market. Clearly, this can considerably facilitate the computations for an aggregated amount of claims since the problem is reduced to the level of individual claims. For an example, we refer to Section 4.3. Using our additivity result, we prove that diversification leads asymptotically to a risk-neutral valuation with respect to a large number of independent additional sources of risk.

Theorem 2.3.1 *Suppose (1.8) holds. Let $B_1, \dots, B_n$ be such that each B_i is $\mathcal{F}_T^0 \vee \mathcal{I}_T^i$-measurable and satisfies $\exp(\gamma|B_i|) \in L^1(P)$ for all $\gamma \in \mathbb{R}$. Then*

$$\pi\left(\sum_{i=1}^n B_i;\alpha\right) = \sum_{i=1}^n \pi(B_i;\alpha)\,, \tag{2.11}$$

$$\psi\left(\sum_{i=1}^n B_i;\alpha\right) = \sum_{i=1}^n \psi(B_i;\alpha)\,. \tag{2.12}$$

Proof: It suffices to consider the case $n=2$ since the argument can be iterated. Let $Q^i := Q^{B^i}$, $i=1,2$, and denote by Z^i the $\mathbb{F}$-density process of Q^i with respect to Q^*. We first show that $d\bar{Q} := Z_T^1 Z_T^2 dQ^*$ defines an element of $\mathbb{P}_e$. Z_T^i is $\mathcal{F}_T^0 \vee \mathcal{I}_T^i$ measurable by Lemma 2.2.1. Hence, Z_T^1 and Z_T^2 are Q^*-independent given $\mathcal{F}_T^0$ and we obtain

$$E^*[Z_T^1 Z_T^2|\mathcal{F}_T^0] = E^*[Z_T^1|\mathcal{F}_T^0]E^*[Z_T^2|\mathcal{F}_T^0] = 1\cdot 1 = 1 \tag{2.13}$$

using Lemma 2.2.2. Thus, $\bar{Q}$ is a probability measure and $\bar{Q} \sim P$ since $Z_T^i > 0$. Again by (2.4) and Lemma 2.2.2, we have

$$E^*[Z_T^j|\mathcal{F}_T^0 \vee \mathcal{I}_T^i \vee \mathcal{I}_t^j] = E^*[Z_T^j|\mathcal{F}_T^0 \vee \mathcal{I}_t^j] = Z_t^j \quad \text{for } i,j \in \{1,2\} \text{ with } i \neq j$$

and conclude that

$$\begin{aligned}
E^*[Z_T^1 Z_T^2 | \mathcal{F}_t] &= E^*[Z_T^1 Z_T^2 | \mathcal{F}_t^0 \vee \mathcal{I}_t^1 \vee \mathcal{I}_t^2] \\
&= E^*\left[Z_T^1 E^*[Z_T^2 | \mathcal{F}_T^0 \vee \mathcal{I}_T^1 \vee \mathcal{I}_t^2] \middle| \mathcal{F}_t^0 \vee \mathcal{I}_t^1 \vee \mathcal{I}_t^2 \right] \\
&= E^*\left[Z_T^1 Z_t^2 | \mathcal{F}_t^0 \vee \mathcal{I}_t^1 \vee \mathcal{I}_t^2 \right] \\
&= E^*\left[E^*[Z_T^1 | \mathcal{F}_T^0 \vee \mathcal{I}_t^1 \vee \mathcal{I}_T^2] \, Z_t^2 \middle| \mathcal{F}_t^0 \vee \mathcal{I}_t^1 \vee \mathcal{I}_t^2 \right] \\
&= Z_t^1 Z_t^2 \qquad \text{for } t \leq T.
\end{aligned}$$

Hence, $Z^1 Z^2$ is the density process of $\bar{Q}$ with respect to Q^*. Using (2.4) and Lemma 2.2.2 we obtain that

$$E^*[Z_t^1 Z_t^2 | \mathcal{F}_t^0 \vee \mathcal{I}_s] = E^*[Z_t^1 | \mathcal{F}_t^0 \vee \mathcal{I}_s] E^*[Z_t^2 | \mathcal{F}_t^0 \vee \mathcal{I}_s] = Z_s^1 Z_s^2 \,, \quad s \leq t \leq T \,,$$

and conclude via Lemma 2.2.2 that $\bar{Q}$ is in $\mathbb{P}_e$.
We next show that $H(\bar{Q}|P)$ is finite. To this end, let $Z_T^0 := dQ^*/dP$ and consider

$$\begin{aligned}
H(\bar{Q}|P) &= E^*[Z_T^1 Z_T^2 \log(Z_T^0 Z_T^1 Z_T^2)] \\
&= E^*[Z_T^1 Z_T^2 \log Z_T^0] + E^*[Z_T^1 Z_T^2 \log Z_T^1] + E^*[Z_T^1 Z_T^2 \log Z_T^2] \,.
\end{aligned}$$

We will prove that the three summands are all finite and thereby establish the last equality. By (2.13) the first summand equals $E^*[\log Z_T^0] = H(Q^*|P) < \infty$. Next $E^*[Z_T^2 | \mathcal{F}_T^0 \vee \mathcal{I}_T^1] = E^*[Z_T^2 | \mathcal{F}_T^0] = 1$ implies via conditioning on $\mathcal{F}_T^0 \vee \mathcal{I}_T^1$ that the second summand equals $E^*[Z_T^1 \log Z_T^1] = H(Q^1|Q^*)$. From $H(Q^1|P) \geq H(Q^1|Q^*) + H(Q^*|P)$ (see Lemma 1.2.4) and $Q^1 \in \mathbb{P}_f$ follows $H(Q^1|Q^*) < \infty$. Analogously, the third summand is equal to $H(Q^2|Q^*)$ and also finite.
Hence, $\bar{Q}$ is in $\mathbb{P}_f \cap \mathbb{P}_e$. Hölder's inequality yields $\exp(\gamma |B_1 + B_2|) \in L^1(P)$ for any $\gamma \in \mathbb{R}$ since $\exp(2\gamma |B_i|) \in L^1(P)$ by hypothesis. Recalling the equality $Q^* = Q^0$, we have by part 1 of Corollary 1.4.3

$$\frac{d\bar{Q}}{dQ^0} = \frac{dQ^1}{dQ^0} \frac{dQ^2}{dQ^0} = \exp\left(-\alpha((\pi(B_1) + \pi(B_2)) + \int_0^T (\psi(B_1) + \psi(B_2)) \, dS - (B_1 + B_2) \right)$$

and part 2 of Corollary 1.4.3 then yields the claim. □

As an application of the previous theorem, we are going to show that diversification leads asymptotically to a risk-neutral valuation with respect to a large number of independent sources $\mathbb{I}^k$ of risk. This is intuitively very plausible: The further (untradable) sources of uncertainty in the claim are not hedged by dynamical trading but in a way by the law of large numbers. To explain the contribution of the next result let us first introduce some notations. Let $(N_t^i)_{t \in [0,T]}$, $i = 1, 2, \ldots$, be cadlag processes which are i.i.d. and independent of $\mathcal{F}_T^0$ under P.

(2.14) Let $\mathbb{I}^i$ be the usual P-augmentation of the filtration generated by N^i.

Each (S, N^i) has paths in the Skohorod space. Let f be a fixed real-valued measurable function on this space and let the claims be given by

$$B_i := f(S, N^i) \quad , i \in \mathbb{N} . \tag{2.15}$$

To explain the crucial difference between the intuitive law of large numbers argument from above and the subsequent corollary, suppose for the moment that f is bounded. The intuitive reasoning from above then argues that $\frac{1}{n}\sum_{i=1}^n B_i$ converges a.s. to $\bar{B} := E^*[f(S, N^1)|\mathcal{F}_T^0]$ by a conditional version of the law of large numbers. This limit can be interpreted as follows: By diversifying his portfolio to more but smaller claims, the issuer can eliminate the risk associated with the uncertainty about the outcomes of the N^i's. But the risk which stems from the uncertainty about the future evolution of the asset prices S still remains with the issuer since all claims B_i are linked to the same S.
By the $(Q^*, \mathbb{F}^0)$-representation property of S and (1.24) we obtain $\pi(\bar{B}; \alpha) = E^*[\bar{B}]$. Summing up, we have

$$\pi\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n B_i; \alpha\right) = E^*[\bar{B}] .$$

Hence, the indifference price of the aggregated portfolio of claims $\bar{B}$ that represents the case of "infinite diversification" is $E^*[\bar{B}]$. In comparison to this, the subsequent corollary proves that the utility indifference price of the "finitely diversified" portfolio $\frac{1}{n}\sum_{i=1}^n B_i$ itself tends to $E^*[\bar{B}]$ if the diversification increases; i.e.,

$$\lim_{n\to\infty} \pi\left(\frac{1}{n}\sum_{i=1}^n B_i; \alpha\right) = E^*[\bar{B}] .$$

In reality, any investor can diversify his risks only by holding a large but finite portfolio. Therefore, the latter limit is the one that is relevant for answering the following economic question: How does increasing diversification asymptotically affect the indifference price of the overall portfolio of claims? It turns out that both limits coincide in the present setting. In this sense, the following result justifies the conclusion of the intuitive reasoning outlined above.

Corollary 2.3.2 *(Diversification)*
Suppose (2.14) and (1.8) hold. Let $B_1, B_2, \ldots$ *be given by (2.15) and suppose that* $\exp(\gamma|B_1|) \in L^1(P)$ *for all* $\gamma \in \mathbb{R}$*. Then*

$$\pi\left(\frac{1}{n}\sum_{i=1}^n B_i\,;\,\alpha\right) \stackrel{n\to\infty}{\longrightarrow} E^*[B_1] . \tag{2.16}$$

By (2.4), the limit in (2.16) takes the form $E^*[h(S)]E_P[g(N^1)]$ when $B_1 = f(S, N^1) = h(S)g(N^1)$ for measurable functions g and h.

Proof:
By symmetry, we have $\pi(f(S,N^i)/n;\alpha) = \pi(f(S,N^1)/n;\alpha)$ for all i. By Theorem 2.3.1, (1.20) and Proposition 1.3.4 we then obtain

$$\pi\left(\frac{1}{n}\sum\nolimits_{i=1}^{n} B_i;\alpha\right) = n\,\pi\left(\frac{f(S,N^1)}{n};\alpha\right) = \pi\left(f(S,N^1);\frac{\alpha}{n}\right) \overset{n\to\infty}{\longrightarrow} E_{Q^0}\left[f(S,N^1)\right]\,.$$

Since $Q^0 = Q^*$, this yields the claim. □

To illustrate the result, we now give a simple example. Section 4.3 contains a similar application of our diversification result to index-linked life insurances in a much more advanced modelling framework.

Example 2.3.3 *(Equity-linked life insurance)*
We consider the (discounted) standard Black-Scholes model: Let $dS = S(\gamma dt + \sigma dW)$ be a geometric Brownian motion with constant coefficients $\gamma \in \mathbb{R}$ and $\sigma > 0$. Let $N^1, N^2, \dots$ be i.i.d. Poisson processes with deterministic intensities $\lambda(t)$, independent of W, and suppose that the information flow $\mathbb{F}$ is generated by $W, N^1, N^2, \dots$.
N^i models the survival process of policy holder i who is thought to be alive until the first jump of N^i occurs. We consider claims of the form $B_i = h(S)I_{\{N^i_T=0\}}$ with h measurable and bounded, e.g. a bull spread $h(S) = K_1 + (S_T - K_1)^+ \wedge K_2$ with constants $K_1 < K_2$. Corollary 2.3.2 yields

$$\lim_{n\to\infty} \pi\left(\frac{1}{n}\sum_{i=1}^{n} h(S)I_{\{N^i_T=0\}}\right) = E^*[h(S)I_{\{N^1_T=0\}}] = E^*[h(S)]P[N^1_T = 0]\,.$$

This shows that for a large number of policy holders with small individual contracts the utility indifference price of the aggregated portfolio of insurance claims tends to a valuation which is risk-neutral with respect to mortality risk. This is in accordance with the commonly used practice (cf. [RolSST98], ch. 13.3.2) and provides a utility based theoretical justification for it.

2.4 A computation scheme and explicit bounds

We now derive an explicit backward computation scheme for the utility indifference price process in the case that the additional information flow $\mathbb{I} = (\mathcal{I}_t)_{t\in[0,T]}$ is piecewise constant, i.e.,

$$\mathcal{I}_t = \mathcal{I}_{t_k} \quad \text{for all } t \in [t_k, t_{k+1}) \tag{2.17}$$

with deterministic times $0 = t_0 < t_1 < \cdots < t_n = T$. As an application, we obtain simple formulae which give upper and lower bounds for the utility indifference price. These bounds are valid even if condition (2.17) does not hold.

Theorem 2.4.1 *Assume (1.5), (1.8), and (2.17). Then the utility indifference price process* (π_t) *is recursively determined by*

$$\begin{aligned} \pi_{t_n} &= B\,, \\ \pi_t &= E^*\left[\frac{1}{\alpha}\log E_P\left[e^{\alpha\pi_{t_{k+1}}}\middle|\mathcal{F}^0_{t_{k+1}}\vee\mathcal{I}_{t_k}\right]\middle|\mathcal{F}^0_t\vee\mathcal{I}_{t_k}\right] \quad ,t\in[t_k,t_{k+1})\,, \end{aligned} \tag{2.18}$$

and the utility indifference price is $\pi(B;\alpha)=\pi_0$.

Having this result, a second reading of the proof reveals (via eq. (2.20)) that

$$\pi_t = \pi_{t_k} + \int_{t_k}^t \psi(B)dS\,, \qquad t\in[t_k,t_{k+1})\,.$$

This means that the indifference price process (π_t) is piecewise self-financing.

In a sense, the previous computation scheme for the utility indifference price resembles the common backwards computation method of the replication price in the binomial tree model: The price at time t_k is computed as a conditional expectation of a functional which involves the price at t_{k+1}. However, since our price is based on exponential utility preferences of the investor, the backwards calculation step demands the calculation of two nested conditional expectations which involve exponential and logarithmic functions.
It is remarkably that the utility indifference price can be calculated period by period and in this sense "locally" (in time), although its definition is based on the optimization problem to maximize exponential utility until terminal time T – a problem which does not reduce to a local one in general.

Remark 2.4.2 Before proving Theorem 2.4.1, let us outline a heuristic explanation of the recursion formula (2.18). At time t_{k+1} the utility indifference price of the claim B for an investor with exponential utility is $\pi_{t_{k+1}}$ and the investor possesses the information $\mathcal{F}^0_{t_{k+1}}\vee\mathcal{I}_{t_{k+1}}$. Before he comes to know the new additional information, he has information $\mathcal{F}^0_{t_{k+1}}\vee\mathcal{I}_{t_k}$ and would assign the certainty equivalent $\tilde{\pi}_{t_{k+1}} := \frac{1}{\alpha}\log E_P\left[e^{\alpha\pi_{t_{k+1}}}\middle|\mathcal{F}^0_{t_{k+1}}\vee\mathcal{I}_{t_k}\right]$ to $\pi_{t_{k+1}}$. By no-arbitrage considerations, the only reasonable price for $\tilde{\pi}_{t_{k+1}}$ at time $t\in[t_k,t_{k+1})$ is $\pi_t = E^*[\tilde{\pi}_{t_{k+1}}|\mathcal{F}_t]$ since $\tilde{\pi}_{t_{k+1}}$ can be replicated from π_t by a self-financing trading strategy. This replication is possible because the model is "piecewise complete". More precisely, it could be shown by Theorem 5.2.3 in [ABS00] that $(S_t)_{t\in[t_k,t_{k+1}]}$ has the strong predictable representation property with respect to $(Q^*,(\mathcal{F}^0_t\vee\mathcal{I}_{t_k})_{t\in[t_k,t_{k+1}]})$. ◇

Proof: (of Theorem 2.4.1)
Proposition 1.4.2 and the subsequent remarks show that the $\mathbb{F}$-density process of Q^B with respect to Q^0 has the form

$$Z_t = \exp\left(-\alpha\left(\pi_0 + \int_0^t \psi(B)dS - \pi_t\right)\right)\,, \quad t\in[0,T]\,,$$

with $\pi_T = B$ and $\psi(B) \in \Theta_{\mathcal{M}}$. For $t \in [t_k, t_{k+1})$ we obtain

$$\begin{aligned}(2.19) \qquad 1 &= E^* \left[\frac{Z_{t_{k+1}}}{Z_t} \,\middle|\, \mathcal{F}^0_{t_{k+1}} \vee \mathcal{I}_t \right] \\ &= \exp\left(-\alpha \left(\int_{(t,t_{k+1}]} \psi(B) dS + \pi_t \right) \right) E^* \left[e^{\alpha \pi_{t_{k+1}}} \,\middle|\, \mathcal{F}^0_{t_{k+1}} \vee \mathcal{I}_{t_k} \right]\end{aligned}$$

using Lemma 2.2.2, assumption (2.17) and the fact that

$$\int_{(t,t_{k+1}]} \psi(B) dS$$

is $\mathcal{F}^0_{t_{k+1}} \vee \mathcal{I}_{t_k}$-measurable due to (2.17) and the predictability of ψ. This measurability is readily verified when the integrand is an elementary predictable process. By standard approximation arguments, the measurability then also holds for left continuous processes with right limits, for bounded predictable processes (see Protter [Pr90], II.4 and IV.2 for details) and, hence, also for integrable predictable processes (see Mémin [Mem80], Lemme V.3).
Solving for π_t in eq. (2.19) leads to

$$(2.20) \qquad \pi_t = -\int_{(t,t_{k+1}]} \psi(B)\, dS + \frac{1}{\alpha} \log E^* \left[e^{\alpha \pi_{t_{k+1}}} \,\middle|\, \mathcal{F}^0_{t_{k+1}} \vee \mathcal{I}_t \right], \quad t \in [t_k, t_{k+1}),$$

and taking Q^*-expectations conditional on $\mathcal{F}^0_t \vee \mathcal{I}_t = \mathcal{F}^0_t \vee \mathcal{I}_{t_k}$ yields (2.18). To see this, note that $\psi(B) \cdot S$ is a Q^*-martingale and the independence hypothesis (2.4) implies

$$E^* \left[e^{\alpha \pi_{t_{k+1}}} \,\middle|\, \mathcal{F}^0_{t_{k+1}} \vee \mathcal{I}_{t_k} \right] = E_P \left[e^{\alpha \pi_{t_{k+1}}} \,\middle|\, \mathcal{F}^0_{t_{k+1}} \vee \mathcal{I}_{t_k} \right].$$

□

The next result provides bounds for the utility indifference price. For an interpretation of these bounds, let us consider the following extreme cases of additional information. First, there might be no additional information before terminal time T, i.e. $\mathcal{I}_t$ is trivial for $t \in [0, T)$. The opposite extreme is the case where all additional information is available from the beginning, i.e. $\mathcal{I}_0 = \mathcal{I}_T$. The proof of Corollary 2.4.3 will show that the utility indifference prices which correspond to these extreme situations provide bounds for the utility indifference price in an intermediate information stage. Due to Theorem 2.4.1, these bounds can be given in explicit form. To compute them, one just has to calculate two expectations – one of them conditional – under the measures P and Q^*. We emphasize that the measures P and Q^* are typically known – in contrast to Q^B.

Corollary 2.4.3 *Assume (1.5) and (1.8). Then*

$$\begin{aligned}(2.21) \qquad \pi(B;\alpha) &\geq \frac{1}{\alpha} \log E_P \left[\exp\left(\alpha E^* \left[B \,|\, \mathcal{I}_T \right] \right) \right], \\ (2.22) \qquad \pi(B;\alpha) &\leq \frac{1}{\alpha} E^* \left[\log E_P \left[\exp(\alpha B) | \mathcal{F}^0_T \right] \right].\end{aligned}$$

Note that the previous result does not require assumption (2.17).

Proof: We define filtrations $\mathbb{F}^i = (\mathcal{F}_t^i)_{t\in[0,T]}$, $i = 1, 2$, via $\mathcal{F}_t^1 := \mathcal{F}_t^0$ for $t \in [0, T)$, $\mathcal{F}_T^1 := \mathcal{F}_T$ and $\mathcal{F}_t^2 := \mathcal{F}_t^0 \vee \mathcal{I}_T$ for $t \in [0, T]$. Then

$$\mathcal{F}_t^1 \subseteq \mathcal{F}_t \subseteq \mathcal{F}_t^2 \quad \text{for } t \in [0, T]. \tag{2.23}$$

The basic idea for the proof is to establish the relations

$$\pi(B; \mathbb{F}^1) \geq \pi(B; \mathbb{F}) \geq \pi(B; \mathbb{F}^2) \tag{2.24}$$

between the utility indifference prices for investors with information flow $\mathbb{H} \in \{\mathbb{F}^1, \mathbb{F}, \mathbb{F}^2\}$, and to write these bounds for $\pi(B; \mathbb{F})$ via Theorem 2.4.1 in explicit form.

Beforehand, we have to overcome a technical problem: $\mathcal{F}_0^2$ is not trivial so that the filtration $\mathbb{F}^2$ a priori does not fit in our general framework which assumes – basically for simplicity in the exposition – that the initial filtration is trivial. But this problem can be overcome as follows: We consider the time interval $[-1, T]$ (instead of $[0, T]$), letting $S_t := S_0$ for $t \in [-1, 0)$ and all filtrations be trivial on $[-1, 0)$. By this trick, we have placed ourself in the general framework for all three filtrations $\mathbb{F}^1$, $\mathbb{F}$ and $\mathbb{F}^2$.

We let the strategy space for the investor with information flow $\mathbb{H} \in \{\mathbb{F}^1, \mathbb{F}, \mathbb{F}^2\}$ be given by $\Theta(\mathbb{H}) := \Theta_3(\mathbb{H})$. This causes no loss of generality since $\Theta_3(\mathbb{F})$ satisfies (1.8) and leads to the same utility indifference value $\pi(B, \mathbb{F})$ (given by (1.17)) as any other Θ for which (1.8) holds. The sets $\mathbb{P}_f$ of equivalent local martingale measures corresponding to the different filtrations satisfy the inclusions

$$\mathbb{P}_f(\mathbb{F}^1) \supseteq \mathbb{P}_f(\mathbb{F}) \supseteq \mathbb{P}_f(\mathbb{F}^2)\,. \tag{2.25}$$

To see this, note that S is locally bounded and adapted with respect to $\mathbb{F}^0$, $\mathbb{F}^0$ is a sub-filtration of $\mathbb{F}^1$ and (2.23) holds. The independence of $\mathcal{F}_T^0$ and $\mathcal{I}_T$ under Q^* implies that Q^* is in $\mathbb{P}_f(\mathbb{F}^2)$ and this yields

$$Q^* \in \mathbb{P}_f(\mathbb{H}) \quad \text{for} \quad \mathbb{H} \in \{\mathbb{F}^1, \mathbb{F}, \mathbb{F}^2\}\,.$$

We are going to show now, that Q^* minimizes $H(Q|P)$ over $\mathbb{P}_f(\mathbb{H})$ for any $\mathbb{H} \in \{\mathbb{F}^1, \mathbb{F}, \mathbb{F}^2\}$, i.e.,

$$Q^0(\mathbb{H}) = Q^* \quad \text{for} \quad \mathbb{H} \in \{\mathbb{F}^1, \mathbb{F}, \mathbb{F}^2\}\,. \tag{2.26}$$

For $\mathbb{H} = \mathbb{F}^1$, we have $H(Q^*|P) \leq H(Q|P)$ for any $Q \in \mathbb{P}_f(\mathbb{F}^1)$ since $dQ^* = \frac{dQ}{dP}\Big|_{\mathcal{F}_T^0} dP$. By (2.25), this yields (2.26).

From (2.25), (2.26) and (1.17) we finally conclude that the relations

$$\pi(B; \mathbb{F}^1) \geq \pi(B; \mathbb{F}) \geq \pi(B; \mathbb{F}^2)$$

hold. Since the filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$ are piecewise constant, we can compute these bounds via Theorem 2.4.1. This yields the claim. $\square$

Chapter 3

PDE-solutions for rational pricing and hedging in incomplete SDE-models

In this chapter, we derive explicit and constructive results on the utility indifference price and the corresponding hedging strategy. We consider a financial market model that is given by a system of stochastic differential equations. The price process S of the tradable risky assets is modeled by an Itô process. Our model is incomplete as there are additional untradable factors of uncertainty and risk which are represented by a finite-state process η. It permits for various dependencies between S and η – in particular it is not assumed that S and η are independent. More precisely, on the one hand both S and η enter the coefficients of the stochastic differential equation that describes the dynamics of S. On the other hand, η jumps to other states with intensities that depend on the current value of S. This model for the present chapter is formally introduced in Section 3.1 and discussed in Sections 3.2 and 3.3
For a general class of contingent claims, we are going to describe the solution to the utility indifference pricing and hedging problem by the unique solution to a system of interacting partial differential equations. In the literature, such systems are also called reaction diffusion equations. Beforehand, we prove a similar solution for the utility maximization problem with an additional liability.
The subsequent Chapter 4 will illustrate the flexibility of our financial market model and exemplify the results of the present chapter by applications to various financial products like index-linked life insurance, default and credit-risk derivatives and weather derivatives.

3.1 Chapter Assumption: Incomplete SDE-model

Let D be a domain in $\mathbb{R}^d$. We assume that there exists a sequence $(D_n)_{n\in\mathbb{N}}$ of bounded domains with closure $\bar{D}_n \subseteq D$ such that $D = \cup_{n=1}^{\infty} D_n$ and each D_n has a C^2-boundary.

Typical examples are $D = \mathbb{R}^d$ and $D = (0,\infty)^d$. Let (S,η) be a solution of the following system of stochastic differential equations with paths in $D \times \{1,\ldots,m\}$ $(m \in \mathbb{N})$.

$$dS_t = \Gamma(t,S_t,\eta_{t-})\,dt + \Sigma(t,S_t,\eta_{t-})\,dW_t\,, \qquad S_0 \in D\,, \tag{3.1}$$

$$d\eta_t = \sum_{k,j=1}^{m} (j-k) I_k(\eta_{t-})\,dN_t^{kj}\,, \qquad \eta_0 \in \{1,\ldots,m\}\,, \tag{3.2}$$

where $\Gamma : [0,T] \times D \times \{1,\ldots,m\} \to \mathbb{R}^d$ and $\Sigma : [0,T] \times D \times \{1,\ldots,m\} \to \mathbb{R}^{d\times d}$ are functions which are of class C^1 with respect to $(t,x) \in [0,T]\times D$; $I_k := I_{\{k\}}$ denotes the indicator function on $\{k\}$; $W = (W^i)_{i=1,\ldots,d}$ is an $\mathbb{R}^d$-valued $(P,\mathbb{F})$-Brownian motion and $N = (N^{kj})_{k,j\in\{1,\ldots,m\}}$ is a multivariate $\mathbb{F}$-adapted point process such that

$$\left(N_t^{kj}\right) \quad \text{has } (P,\mathbb{F})\text{-intensity} \quad \lambda^{kj}(t,S_t) \quad \text{for} \quad k,j \in \{1,\ldots,m\} \tag{3.3}$$

where $\lambda^{kj} : [0,T] \times D \to [0,\infty)$ are bounded functions of class C^1.

In the sequel (see (3.14)) we will specify conditions which ensure that the set $\mathbb{P}_e$ of equivalent martingale measures is non-empty. It is easy to see that the set $\mathbb{P}_e$ – provided it is non-empty – is not a singleton in general. In this sense, the model is incomplete.

The process (S,η) constitutes a Markov-type model in the sense that the coefficients of the SDEs (3.1) and (3.2) and also the intensities (3.3) of N are functions of (t,S_t,η_{t-}). We refer to Remark 3.1.2 for further comments on the Markov property of the model.

In the case $D \subseteq (0,\infty)^d$ we can rewrite the SDE (3.1) as a generalized Black-Scholes model. With the notation $\frac{dS}{S} = \left(\frac{dS^1}{S^1},\ldots,\frac{dS^d}{S^d}\right)^{\mathrm{tr}}$ we have

$$\frac{dS_t}{S_t} = \gamma(t,S_t,\eta_{t-})dt + \sigma(t,S_t,\eta_{t-})\,dW_t\,, \tag{3.4}$$

for $\gamma(t,x,k) = \operatorname{diag}\left(\frac{1}{x^i}\right)_{i=1,\ldots,d} \Gamma(t,x,k)$ and $\sigma(t,x,k) = \operatorname{diag}\left(\frac{1}{x^i}\right)_{i=1,\ldots,d} \Sigma(t,x,k)$.

Remark 3.1.1 (On the construction of the model)

Note that the process η enters the coefficients in the stochastic differential equation (3.1) that describes the dynamics of S. In turn, the intensities (3.3) of the multivariate point process N driving η depend on the process S. At first sight, it seems difficult to construct models with such mutual dependencies since we are faced with a non-standard SDE-system. The solution (S,η) does not only enter the coefficients of the SDE-system (3.1)-(3.2) but also the intensities (3.3) of the driving process N depend on the solution. However, we will see in the next section that this problem can be reduced by a change of measure to the case where N is a standard Poisson process with intensities constant at 1. This situation is much simpler, since η is then an autonomous process and well-defined by (3.2). ◇

Remark 3.1.2 (On the Markov property of (S, η))
It is plausible from the SDE-system (3.1)-(3.3) that (S, η) should be a (strong) Markov process with respect to $(P, \mathbb{F})$. The basic idea to prove this is to show uniqueness of the corresponding martingale problem: To this end, one has to verify the uniqueness of the one-dimensional distributions for the solutions of the martingale problem (see condition (4.9) in [EtK86], Theorem 3.4.2) by using the uniqueness results on PDEs from Chapter 7. This would yield the (strong) $(P, \mathbb{F})$-Markov property of (t, S_t, η_t) and also uniqueness in law. Since we do not need the Markov property explicitly in the sequel, we do not carry out this program. Let us just motivate the Markov property by a rather special example here. Suppose N^{kj}, $k, j \in \{1, \dots, m\}$, are independent standard Poisson processes and the assumptions of the subsequent Example 3.3.1 hold. Then there is an unique solution to the SDE-system (3.1)-(3.2) and this solution is a Markov process (see Protter [Pr90], V.6, Theorem 32). ◇

In the sequel, we are going to apply the results from Chapter 7 on interacting systems of partial differential equations. To do so, we have to impose further regularity assumptions on the coefficients of the SDE (3.1). By assumption, $(t, x) \mapsto \Sigma(t, x, k)$ is of class C^1 on $[0, T] \times D$ and therefore Lipschitz continuous on $[0, T] \times K$ for any compact set $K \subset D$ and any k. This implies that $x \mapsto \Sigma(t, x, k)$ locally Lipschitz continuous in x, uniformly in t and k. These local Lipschitz condition implies (see [Pr90], Th. V.38, or [Ku84], Th. II.5.2) that for any $(t, x, k) \in [0, T] \times D \times \{1, \dots, m\}$ there is a unique strong solution $X^{t,x,k}$ to the SDE

$$X_t^{t,x,k} = x \in D\,, \qquad dX_s^{t,x,k} = \Sigma(s, X_s^{t,x,k}, k)\, dW_s^j \quad \text{for } s \in [t, T] \tag{3.5}$$

up to a possibly finite explosion time. We assume furthermore that $X^{t,x,k}$ does not explode or leave D during $[0, T]$, i.e., for any t, x and k we have

$$P\left[\sup_{s \in [t,T]} \left| X_s^{t,x,k} \right| < \infty \right] = 1 \quad \text{and} \quad P\left[X_s^{t,x,k} \in D \ \text{ for all } s \in [t, T]\right] = 1\,. \tag{3.6}$$

In the following Section 3.3, we shall provide examples where a solution (S, η) to the SDE-system (3.1) - (3.3) exists and (3.6) can be verified. Let us note that the SDE (3.5) coincides with the SDE (7.1) in Chapter 7 (for $b^k(t, x) := 0$ and $\Sigma_k(t, x) := \Sigma(t, x, k)$ there) and (3.6) corresponds to the condition (7.3).

Remark 3.1.3 It is natural to ask how the SDE (3.5) is related to the SDE that describes S. Under some conditions (see (3.14) and part 4 of the proof for Theorem 3.4.2) the minimal martingale measure $\widehat{Q} \in \mathbb{P}_e$ exists and one can show that the dynamics of (S, η) under $\widehat{Q}$ are described by the SDE-system

$$\begin{aligned} dS_t &= \Sigma(t, S_t, \eta_{t-})\, d\widehat{W}_t\,, && S_0 \in D\,, \\ d\eta_t &= \sum_{k,j \in \{1, \dots, m\}} (j - k) I_k(\eta_{t-})\, dN_t^{kj}\,, && \eta_0 \in \{1, \dots, m\}\,, \end{aligned} \tag{3.7}$$

where $\widehat{W}$ is a $\widehat{Q}$-Brownian motion and the point process N has the same intensities (3.3) under $\widehat{Q}$ as under P. Since the solution to (3.5) is strong, we could replace (P, W) by $(\hat{Q}, \hat{W})$ and obtain that the corresponding solution satisfies (3.6) under $\hat{Q}$ then. From this perspective, one might view condition (3.6) on the SDE (3.5) as an additional regularity condition on the SDE (3.1) whose solution S is also supposed to stay in the domain D. ◇

3.2 Construction of S-dependent intensities by a change of measure

The key idea to construct a model with (3.1), (3.2) and (3.3) is to reduce the problem by constructing at first just a solution (S, η) to the SDE-system (3.1) and (3.2) in the case that $N = (N^{kj})$ is a standard multivariate Poisson process. Then, the intensities of this process are constant at 1 and in particular do not depend on S. Having a solution (S, η) in that reduced case, we can then construct the desired (t, S)-dependent intensities for N by a suitable change of measure.

More precisely, we start with a filtered probability space $(\Omega, \mathcal{F}', \mathbb{F}', P')$ carrying a d-dimensional $(P'\mathbb{F}')$-Brownian motion $W = (W^i)_{i\in\{1,\dots,d\}}$ and a multivariate adapted point process $N = (N^{kj})_{k,j\in\{1,\dots,m\}}$ such that N^{kj} has constant $(P'\mathbb{F}')$-intensity 1 for any k and j, i.e.,

$$\text{(3.8)} \qquad N^{kj},\ k, j \in \{1, \dots, m\}, \text{ are independent standard Poisson processes.}$$

We assume that $\mathcal{F}'_0$ is trivial, $\mathcal{F}'_T = \mathcal{F}'$ and $\mathbb{F}'$ satisfies the usual conditions. On this basis, eq. (3.2) defines a unique process η. Given this autonomous process there is a solution S to (3.1) under suitable assumptions. Examples are given in the subsequent Section 3.3. Then,

$$\text{(3.9)} \qquad dP := \mathcal{E}\left(\sum_{k,j\in\{1,\dots,m\}} \int (\lambda^{kj}(t, S_t) - 1)\,(dN_t^{kj} - dt) \right)_T dP'$$

defines a probability measure $P \ll P'$ such that N is a multivariate point process having the intensities from (3.3) under $(P, \mathbb{F}')$ (see Brémaud [Br81], VI.2.,Theorems 3 and 4). By Girsanov's theorem, W is a (local) $(P, \mathbb{F}')$-martingale. The covariance process $\langle W \rangle$ is the same under P' and P since it can be computed pathwise and therefore W is also a Brownian motion under $(P, \mathbb{F}')$ by Lévy's characterization.

Finally, let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be the standard P-completion of $(\Omega, \mathcal{F}', \mathbb{F}', P)$. Then $\mathbb{F}$ satisfies the usual conditions, W is a Brownian motion and N a multivariate point process with the desired intensities (3.3) under $(P, \mathbb{F})$. Moreover, (S, η) solves the SDE-system (3.1)-(3.2) under $(P, \mathbb{F})$ (see Protter [Pr90], II.5, Theorems 14 and 16).

3.3 Some case studies

We provide some examples where a solution (S, η) to the system (3.1)-(3.2) exists and also condition (3.6) can be verified. We can restrict ourself to the simpler case (3.8) where N is a standard multidimensional Poisson process. Having a solution (S, η) in this situation, a model with intensities of the form (3.3) is easily constructed by the change of measure technique described previously.

Example 3.3.1 (SDE-coefficients are Lipschitz continuous in (x, k))
Suppose Γ and Σ are globally Lipschitz in $(x, k) \in D \times \{1, \dots, m\}$, uniformly in $t \in [0, T]$ and (3.8) holds. Then there exists (see Protter [Pr90], V.3, Th.7) an unique solution $X^{t,x,k}$ in $D = \mathbb{R}^d$ to (3.5) for any t, x, k and also a unique solution (S, η) to SDE-system (3.1)-(3.2).

Example 3.3.2 (Explicit solution by stochastic exponentials)
Let $D = (0, \infty)^d$ and suppose that $\gamma = \gamma(t, k)$ and $\sigma = \sigma(t, k)$ depend only on (t, k) but not on x and are continuous on $[0, T]$ for any $k \in \{1, \dots, k\}$. Then the unique solution $X^{t,x,k}$ to (3.5) is given by

$$(X_s^{t,x,k})^i = x^i \frac{\mathcal{E}\left(\left(\int \sigma(u,k)\, dW_u\right)^i\right)_s}{\mathcal{E}\left(\left(\int \sigma(u,k)\, dW_u\right)^i\right)_t}, \quad s \in [t, T]\,, i = 1, \dots, d\,,$$

and stays in $D = (0, \infty)^d$. When (3.8) holds, η is defined by (3.2) and the unique solution S to (3.1) is given by

$$S^i = S_0^i\, \mathcal{E}\left(\int \gamma^i(u, \eta_{u-})\, du + \left(\int \sigma(u, \eta_{u-})\, dW_u\right)^i\right).$$

Example 3.3.3 (SDE-coefficients for S are Lipschitz continuous in x for all k)
Let $D = \mathbb{R}^d$ and suppose that for each $k \in \{1, \dots, m\}$, the functions $x \mapsto \Gamma(t, x, k)$ and $x \mapsto \Sigma(t, x, k)$ are globally Lipschitz continuous in x, uniformly in $t \in [0, T]$. Then (3.6) holds (see Protter [Pr90], V.3, Th. 7). If (3.8) holds, η is defined by (3.2) and an induction argument yields the existence of a strong solution S to (3.1): To that end, let $\tau^0 := 0$ and define the sequence $\tau^{n+1} := \inf\{t > \tau^n | \Delta\eta_t \neq 0\} \wedge T$ of jump times of η. We have that $P[\tau^n \geq T] \to 1$ for $n \to \infty$ since N is non-exploding. So, it suffices to show that there is a unique solution on $[\![0, \tau^n]\!]$ to the SDE (3.1) for all n. Suppose that

(3.10) $\qquad X^n$ is the unique solution to the SDE (3.1) on $[\![0, \tau^n]\!]$.

By Theorem V.3.7 in Protter [Pr90], the following system of SDEs

$$X_t^{n+1} = X_{t\wedge\tau^n}^n + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Gamma(s, X_s^{n+1}, \eta_{\tau^n})\, ds + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Sigma(s, X_s^{n+1}, \eta_{\tau^n})\, dW_s \tag{3.11}$$

has a unique solution $(X_t^{n+1})_{t\in[0,T]}$. By construction, we have $X^{n+1} = X^n$ on $[\![0,\tau^n]\!]$ and $\eta_{s-} = \eta_{\tau^n}$ for $(\omega,s) \in]\!]\tau^n,\tau^{n+1}]\!]$. (3.10) and (3.11) yield

$$\begin{aligned} X_t^{n+1} &= X_{t\wedge\tau^n}^n + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Gamma(s,X_s^{n+1},\eta_{s-})\,ds + \int_0^t I_{]\!]\tau^n,\tau^{n+1}]\!]}\Sigma(s,X_s^{n+1},\eta_{s-})\,dW_s \\ &= S_0 + \int_0^t \Gamma(s,X_s^{n+1},\eta_{s-})\,ds + \int_0^t \Sigma(s,X_s^{n+1},\eta_{s-})\,dW_s\,, \quad t\in[0,T]. \end{aligned}$$

Hence, X^{n+1} is a solution to the SDE (3.1) on $[\![0,\tau^{n+1}]\!]$ and this solution is unique on $[\![0,\tau^{n+1}]\!]$. To see the latter, recall that the solution is unique on $[\![0,\tau^n]\!]$ by hypothesis. This implies that any solution S to the SDE (3.1) has to satisfy the SDE (3.11) on $[\![0,\tau^{n+1}]\!]$ and therefore coincides with X^{n+1} on $[\![0,\tau^{n+1}]\!]$.

3.4 Solutions to the utility maximization problem

In this section we will show that the solution to the dual and thereby also the primal utility maximization problem is given via the unique solution to a system of interacting semi-linear partial differential equations. To obtain our result we basically have to obtain the optimal measure Q^B (and Q^0) for the dual problem (see Proposition 1.2.3).

We consider claims B of the form

$$B = h(S_T,\eta_T) + \int_0^T f(t,S_t,\eta_t)\,dt + \sum_{t\le T}\sum_{\substack{k,j=1\\k\ne j}}^m 1_{\{\eta_{t-}=k,\eta_t=j\}}f^{kj}(t,S_t) \tag{3.12}$$

with continuous bounded functions $h : D\times\{1,\dots,m\}\to\mathbb{R}$, $f : [0,T]\times D\times\{1,\dots,m\}\to\mathbb{R}$ and $f^{kj} : [0,T]\times D\to\mathbb{R}$, $k,j\in\{1,\dots,m\}$. Furthermore, $f(\cdot,\cdot,k)$ and $f^{kj}(\cdot,\cdot)$ are supposed to be of class C^1 on $[0,T]\times D$ for all $k,j\in\{1,\dots,m\}$. Note that $\sum_{t\le T}\cdots$ is a finite sum since η has only a finite number of jumps in finite time by construction.
The claim B is the sum of three components with the following interpretations. The first term describes a terminal payoff that depends on the final state (S_T,η_T). The second term models payments which are made continuously at rate $f(t,S_t,\eta_t)$. The third term specifies further payments for each time when η jumps from one state to another.

Lemma 3.4.1 *When B is of the form (3.12) we have* $\exp(\beta B)\in L^1(P)$ *for all* $\beta\in\mathbb{R}$. *In particular, B satisfies condition (1.5).*

Proof: It suffices to prove that

$$E[\exp(\beta N'_T)] < \infty \quad \text{for any } \beta<\infty \tag{3.13}$$

with $N' := \sum_{k,j} N^{kj}$. To see that this is enough, note that the first two terms in (3.12) are bounded by hypothesis and the third term satisfies

$$\left| \sum_{t \le T} \sum_{\substack{k,j=1 \\ k \neq j}}^{m} 1_{\{\eta_{t-}=k, \eta_t = j\}} f^{kj}(t, S_t) \right| \le \sup_{k,j \in \{1,\dots,m\}} \left\| f^{kj} \right\|_\infty N'_T \,.$$

By construction, N' is a point process with the bounded $(P, \mathbb{F})$-intensity $\lambda'_t := \sum_{k,j} \lambda^{kj}(t, S_t)$, $t \in [0,T]$. We can assume that $N' = (N'_t)$ is defined on the infinite time interval $[0, \infty)$ with intensity $\lambda'_t = 1$ for $t > T$ (this can always be constructed on an enlargement of the probability space). Let $c < \infty$ denote an upper bound for the intensity λ'. A random change of time argument (see Brémaud [Br81], II, Theorem 16) shows that N'_τ with τ defined via $\int_0^\tau \lambda'_s \, ds = cT$ is Poisson distributed with parameter cT. In particular, $\exp(\beta N'_\tau)$ is in $L^p(P)$ for any $p \in [1, \infty)$. This implies (3.13) since $\tau \ge T$ by construction. In fact, $cT = \int_0^\tau \lambda'_s \, ds \le c\tau$.
□

For the next result we will need the assumption that

(3.14) $\qquad \Sigma$ is invertible and $\Sigma^{-1}\Gamma$ is bounded on $[0,T] \times D \times \{1, \dots, m\}$.

This condition implies in particular that

$$\mathbb{P}_e \cap \mathbb{P}_f \neq \emptyset$$

since $d\widehat{Q} := \mathcal{E}(-\int \Sigma^{-1}\Gamma(t, S_t, \eta_{t-}) \, dW_t)_T \; dP$ defines an element of $\mathbb{P}_e \cap \mathbb{P}_f$ if (3.14) holds. Moreover, the existence of Σ^{-1} and the continuity and differentiability properties of Σ yield that

(3.15) $\qquad (t,x) \mapsto \Sigma^{-1}(t,x,k) \quad$ is continuous and of class C^1 on $[0,T] \times D$

for any $k \in \{1, \dots, m\}$. For claims B of the form (3.12), the subsequent theorem will show that the density process of Q^B is described via the unique bounded classical solution $v : [0,T] \times D \times \{1, \dots, m\} \to \mathbb{R}$ to the following system of m partial differential equations with terminal condition at T and $k = 1, \dots, m$ (subscripts are denoting partial derivatives):

$$\begin{aligned} & v_t(t,x,k) + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t,x,k) \, v_{x^i x^j}(t,x,k) \\ & - \frac{1}{2\alpha} ||\Sigma^{-1}(t,x,k)\Gamma(t,x,k)||^2 + f(t,x,k) \\ & + \sum_{\substack{j=1 \\ j \neq k}}^{m} \lambda^{kj}(t,x) \frac{1}{\alpha} \left(e^{\alpha(v(t,x,j) - v(t,x,k) + f^{kj}(t,x))} - 1 \right) \quad = \quad 0 \,, \quad (t,x) \in [0,T) \times D \,, \\ & \qquad\qquad \text{and} \qquad v(T,x,k) \quad = \quad h(x,k) \,, \quad x \in D \,, \end{aligned} \tag{3.16}$$

with $a = (a^{ij})_{i,j\in\{1,\dots,d\}} := \Sigma\,\Sigma^{\mathrm{tr}}$ and $||\Sigma^{-1}\Gamma||^2 := (\Sigma^{-1}\Gamma)^{\mathrm{tr}}(\Sigma^{-1}\Gamma) = \Gamma^{\mathrm{tr}}a^{-1}\Gamma$. Let

$$dM^{kj} := dN^{kj} - \lambda^{kj}(t,S_t)\,dt\,, \quad k,j \in \{1,\dots,m\}\,,$$

denote the compensated $(\mathbb{F},P)$-point processes. M^{kj} is in $\mathcal{M}^2(P,\mathbb{F})$ since

$$E\left[[M^{kj}]_T\right] = E\left[N_T^{kj}\right] = E\left[\int_0^T \lambda^{kj}(u,S_u)\,du\right]$$

is finite due to the boundedness of λ^{kj}.

Notation: $v_x := (v_{x^1},\dots,v_{x^d})^{\mathrm{tr}}$ denotes the gradient vector of $v(t,x,k)$ with respect to $x = (x^i)_{i=1,\dots,d}$. $C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},\mathbb{R})$ denotes the space of continuous bounded functions $v : [0,T]\times D\times\{1,\dots,m\} \to \mathbb{R}$ with $(t,x)\mapsto v(t,x,k)$ being of class $C^{1,2}$ on $[0,T)\times D$ for any $k\in\{1,\dots,m\}$. Note that the $C^{1,2}$-condition is imposed only on $[0,T)\times D$ while continuity is required on $[0,T]\times D$ for every k.

Now we are in position to formulate our main result in this Chapter.

Theorem 3.4.2 *Suppose (3.14) holds and B is of the form (3.12). There is a unique function $v^B \in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},\mathbb{R})$ which solves the PDE-system (3.16). The optimal measure $Q^B \in \mathbb{P}_e\cap\mathbb{P}_f$ for the dual problem (1.11) is given by*

$$\begin{aligned}\frac{dQ^B}{dP} &= \exp\left(-\alpha\Big(v^B(0,S_0,\eta_0) + \int_0^T \vartheta_u^B\,dS_u - B\Big)\right) \quad \textit{with}\\ \vartheta_t^B &= v_x^B(t,S_t,\eta_{t-}) + \frac{1}{\alpha}\left(\Sigma(t,S_t,\eta_{t-})\,\Sigma^{\mathrm{tr}}(t,S_t,\eta_{t-})\right)^{-1}\Gamma\,(t,S_t,\eta_{t-})\,, \quad t\in[0,T)\,,\end{aligned}$$

and $\vartheta_T^B = 0$. Moreover, ϑ^B is in $\Theta_{\mathcal{M}}$.

It is rather arbitrary to set $\vartheta_T^B = 0$ since the value of ϑ_T^B does not affect the stochastic integral $\int \vartheta^B\,dS$ – provided that $(\vartheta^B)_{t\in[0,T]}$ is in $L(S)$. However, we cannot extend the equation for ϑ_t^B in the theorem from $[0,T)$ to $[0,T]$ since the gradient v_x may not exist at the terminal time T.

Proof: We proceed in several steps.

Step 1:
By the assumptions in place, we obtain from Theorem 7.3.3 that there exists a unique function $v^B \in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},\mathbb{R})$ which solves the PDE-system (3.16).
To be more precise, Theorem 7.3.3 yields the existence of a unique solution $(v^k(t,x))_{k=1,\dots,m}$ in $C_b^{1,2}([0,T)\times D,\mathbb{R}^m)$ to the PDE-system (7.8) with $b_k := 0$, $c^k := 0$, $\Sigma_k := \Sigma(\cdot,\cdot,k)$, $f^k(t,x) := f(t,x,k) - \frac{1}{2\alpha}||\Sigma^{-1}\Gamma(t,x,k)||^2$, $g^k(t,x,v) = \sum\limits_{j:j\neq k} \frac{\lambda^{kj}(t,x)}{\alpha}\left(e^{\alpha(v^j - v^k + f^{kj}(t,x))} - 1\right)$

and $h^k(x) := h(x,k)$. By identifying $C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},\mathbb{R})$ and $C_b^{1,2}([0,T)\times D,\mathbb{R}^m)$ we obtain that $v^B(t,x,k) := v^k(t,x)$ is the unique solution in $C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},\mathbb{R})$ to the PDE-system (3.16).

Step 2:
For ease of notation, let $v := v^B$ and define the short-hand notations

$$\begin{aligned} dN_t^{\eta j} &:= \sum_{k=1}^m I_k(\eta_{t-})\,dN_t^{kj}\,, & dM_t^{\eta j} &:= \sum_{k=1}^m I_k(\eta_{t-})\,dM_t^{kj}\,, \\ \lambda^{\eta j}(t,x) &:= \sum_{k=1}^m I_k(\eta_{t-})\,\lambda^{kj}(t,x)\,, & f^{\eta j}(t,x) &:= \sum_{k=1}^m I_k(\eta_{t-}) f^{kj}(t,x)\,. \end{aligned} \tag{3.17}$$

Note that $dN_t^{\eta j} = dM_t^{\eta j} + \lambda^{\eta j}(t,S_t)\,dt$. We are going to show that

$$Z := \mathcal{E}\left(-\int \Sigma^{-1}(t,S_t,\eta_{t-})\Gamma(t,S_t,\eta_{t-})\,dW_t + \sum_{j=1}^m \int \xi^j(t,S_t,\eta_{t-})\,dM_t^{\eta j}\right) \tag{3.18}$$

with

$$\xi^j(t,S_t,\eta_{t-}) := 1_{\{\eta_{t-}\neq j\}}\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))} - 1\right)\,, \quad j=1,\dots,m\,, \tag{3.19}$$

is the density process of the optimal measure Q^B for the dual problem. The process Z is strictly positive since $\xi^j > -1$ holds for all j. To reduce notation, we make the standing convention that functions in stochastic integrals are evaluated at (t,S_t,η_{t-}) unless specified otherwise. Then we have by Yor's formula

$$Z = \mathcal{E}\left(-\int \Sigma^{-1}\Gamma\,dW\right)\,\mathcal{E}\left(\sum_{j=1}^m \int \xi^j\,dM^{\eta j}\right)\,. \tag{3.20}$$

By Novikov's criterion, the boundedness of $\Sigma^{-1}\Gamma$ implies that

$$\mathcal{E}\left(-\int \Sigma^{-1}\Gamma\,dW\right) \quad \text{is in the martingale space } \mathcal{H}^p(P) \text{ for any } p\in[1,\infty)\,. \tag{3.21}$$

ξ^j is bounded (uniformly in t) since v and the f^{kj}'s are bounded. So, there is a constant $c<\infty$ such that $|\xi^j(t,x,k)|\le c$ for all t,x,k,j and

$$\begin{aligned} \mathcal{E}\left(\sum_{j=1}^m \int \xi^j\,dM^{\eta j}\right)_t &= \prod_{j=1}^m \mathcal{E}\left(\int \xi^j\,dM^{\eta j}\right)_t \\ &= \prod_{j=1}^m \left(e^{-\int_0^t \xi_s^j \lambda^{\eta j}(s,S_s)\,ds} \prod_{0<s\le t} (1+\xi_{s-}^j\,\Delta N_s^{\eta j})\right) \\ &\le e^{mTc\|\lambda\|}(1+c)^{\sum_{k,j=1}^m N_T^{kj}}\,, \qquad t\in[0,T]\,, \end{aligned} \tag{3.22}$$

with $||\lambda|| := \sup_{k,j\in\{1,\dots,m\}} ||\lambda^{kj}||_\infty < \infty$. Therefore, the exponential on the left hand side in (3.22) is bounded (uniformly in t) by the random variable on the right hand side. We are going to show that this bound is in $L^p(P)$ for any $p \in [1,\infty)$ and thereby obtain that

$$\mathcal{E}\left(\sum_{j=1}^m \int \xi^j \, dM^{\eta j}\right) \quad \text{is in the martingale space } \mathcal{H}^p(P) \text{ for any } p \in [1,\infty)\,. \tag{3.23}$$

For the integrability in question, it suffices to verify that $E[\exp(\beta N'_T)] < \infty$ holds for any $\beta < \infty$ with $N' := \sum_{k,j} N^{kj}$. This has been done in the proof of Lemma 3.4.1 (see (3.13)). By (3.20), (3.21), (3.23) and Hölder's inequality we obtain that

$$Z \text{ is in the martingale space } \mathcal{H}^p(P) \text{ for any } p \in [1,\infty). \tag{3.24}$$

In particular, Z is a P-martingale.

Step 3:
So, $dQ := Z_T\, dP$ defines a probability measure which is equivalent to P since $Z > 0$. We are going to show that Q is in $\mathbb{P}_e$. By Girsanov's theorem and Lévy's characterization, (3.18) yields that $\widetilde{W} := W + \int \Sigma^{-1}\Gamma\, dt$ is a Q-Brownian motion. Hence,

$$dS_t = \Sigma(t, S_t, \eta_{t-})\, d\widetilde{W}_t \tag{3.25}$$

and Q is an equivalent local martingale measure for S.

Step 4:
Our aim is to show that Q coincides with the optimal measure Q^B from the dual problem. To this end, we are going to verify the list of sufficient conditions from Proposition 1.4.2, variant (i). Let

$$\bar\vartheta_t := v_x(t,S_t,\eta_{t-}) + \frac{1}{\alpha}\left(\Sigma(t,S_t,\eta_{t-})\,\Sigma^{\mathrm{tr}}(t,S_t,\eta_{t-})\right)^{-1}\Gamma\,(t,S_t,\eta_{t-})\,,\quad t\in[0,T)\,,$$

and $\bar\vartheta_T := 0$. In the present part of the proof, we now show that

$$\bar\vartheta \text{ is in } L(S) \text{ and } \left(\int_0^t \bar\vartheta\, dS\right)_{t\in[0,T]} \text{ is a } Q\text{-}\mathcal{BMO}\text{-martingale}\,. \tag{3.26}$$

From the definition of $\bar\vartheta$, we have that $\bar\vartheta$ is left-continuous with right limits on $[0,T)$. Thus, $(1_{[0,t]}(u)\bar\vartheta_u)_{u\in[0,T]}$ is integrable with respect to S in the sense of local Q-martingales and $\int_0^t \bar\vartheta\, dS$ is well defined for $t\in[0,T)$. To obtain (3.26), it suffices to prove that

$$\left(\int_0^t \bar\vartheta\, dS\right)_{t\in[0,T)} \text{ is a } Q\text{-}\mathcal{BMO}\text{-martingale} \tag{3.27}$$

and thereby in the martingale space $\mathcal{H}^2(Q)$. To see the latter, note that $d[S]$ is absolutely continuous with respect to Lebesgue measure so that (3.27) implies

$$E_Q\left[\int_0^T \bar\vartheta^{\mathrm{tr}} d[S]\ \bar\vartheta\right] = \sup_{t\in[0,T)} E_Q\left[\int_0^t \bar\vartheta^{\mathrm{tr}}\, d[S]\ \bar\vartheta\right] = \left\|\left(\int_0^t \bar\vartheta\, dS\right)_{t\in[0,T)}\right\|^2_{\mathcal{H}^2(Q)} < \infty\,,$$

i.e. $(\bar\vartheta_u)_{u\in[0,T]} \in L^2(S,Q) \subset L(S)$. Hence, $\left(\int_0^t \bar\vartheta\, dS\right)_{t\in[0,T]} \in \mathcal{H}^2(Q)$ and by the continuity of $\int \bar\vartheta\, dS$ and the definition of the $\mathcal{BMO}(Q)$-norm (see [HWY92], 10.6), we conclude that the extended process $\left(\int_0^t \bar\vartheta\, dS\right)_{t\in[0,T]}$ is also a Q-$\mathcal{BMO}$-martingale if (3.27) holds.
The definition of $\bar\vartheta$ and (3.25) imply

$$\int_0^t \bar\vartheta\, dS = \int_0^t v_x\, dS + \frac{1}{\alpha}\int_0^t \Sigma^{-1}\Gamma\, d\widetilde{W}\,, \quad t\in[0,T).$$

To establish (3.27) it suffices to show that $(\int_0^t v_x\, dS)_{t\in[0,T)}$ is a $Q-\mathcal{BMO}$-martingale since $\Sigma^{-1}\Gamma$ is bounded by hypothesis. By (3.21)

$$d\widehat{Q} := \mathcal{E}\left(-\int \Sigma^{-1}\Gamma\, dW\right)_T dP$$

defines a probability measure which is in $\mathbb{P}_e$ (by Girsanov's theorem) and is called minimal martingale measure. Again by Girsanov's theorem, the processes M^{kj} are also $\widehat{Q}$-martingales and the multivariate point process (N^{kj}) has the same bounded intensities $\lambda^{kj}(t,S_t)$ under $\widehat{Q}$ as under P. It follows from (3.20) and the definition of $\widehat{Q}$ that the density process of Q with respect to $\widehat{Q}$ is given by the stochastic exponential

$$\mathcal{E}\left(\sum_{j=1}^m \int \xi_t^j\, dM_t^{\eta j}\right) = \mathcal{E}\left(\sum_{\substack{k,j=1\\k\neq j}}^m \int I_k(\eta_{t-})\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,k)+f^{kj}(t,S_t))}-1\right) dM_t^{kj}\right).$$

By Theorem VI.3 in Brémaud [Br81] this implies that N^{kj}, $k,j\in\{1,\dots,m\}$, has $(Q,\mathbb{F})$-intensity

$$\tilde\lambda_t^{kj} := \lambda^{kj}(t,S_t)\left(1+I_k(\eta_{t-})I_{\{k\neq j\}}\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,k)+f^{kj}(t,S_t))}-1\right)\right). \tag{3.28}$$

Let $\widetilde{M}^{kj} := N^{kj} - \int \tilde\lambda_t^{kj}\, dt$ denote the compensated $(Q,\mathbb{F})$-point processes and note that each $\tilde\lambda^{kj}$ is bounded since v, λ^{kj} and f^{kj} are bounded. Due to the boundedness of the intensities, $\widetilde{M} = (\widetilde{M}^{kj})_{k,j\in\{1,\dots,m\}}$ is in $\mathcal{M}^2(Q,\mathbb{F})$.
By Itô's formula, we obtain – recalling our convention that functions in integrands are evaluated at (t,S_t,η_{t-}) unless specified otherwise –

$$v_x^{\mathrm{tr}}\, dS_t \;=\; dv(t,S_t,\eta_t) - \Delta v(t,S_t,\eta_t) - \left(v_t + \frac{1}{2}\sum_{i,j=1}^d a^{ij}\, v_{x^i x^j}\right) dt\,,\quad t\in[0,T)\,,$$

with $\Delta v(t,S_t,\eta_t) := v(t,S_t,\eta_t) - v(t-,S_{t-},\eta_{t-}) = v(t,S_t,\eta_t) - v(t,S_t,\eta_{t-})$. Substituting

$$\Delta v(t,S_t,\eta_t) = \sum_{\substack{k,j=1\\k\neq j}}^{m} I_k(\eta_{t-})(v(t,S_t,j) - v(t,S_t,k))dN_t^{kj}$$

and $dN_t^{kj} = d\widetilde{M}_t^{kj} + \tilde{\lambda}_t^{kj}dt$ and using the PDE for v yields

$$v_x^{\mathrm{tr}}\, dS_t \;=\; dv(t,S_t,\eta_t) + L_t\, dt + \sum_{k,j=1}^{m} H_t^{kj}\, d\widetilde{M}^{kj}\,, \quad t\in[0,T)\,, \tag{3.29}$$

with L and H^{kj}, $k,j=1,\dots,m$, given by

$$\begin{aligned} L_t \;:=&\; f(t,S_t,\eta_{t-}) - \frac{1}{2\alpha}||\Sigma^{-1}(t,S_t,\eta_{t-})\Gamma(t,S_t,\eta_{t-})||^2 \\ &+ \sum_{\substack{j=1\\j\neq\eta_{t-}}}^{m} \tfrac{\lambda^{\eta j}(t,S_t)}{\alpha}\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))}-1\right) - \tilde{\lambda}_t^{\eta j}\left(v(t,S_t,j)-v(t,S_t,\eta_{t-})\right), \\ H_t^{kj} \;:=&\; -I_k(\eta_{t-})I_{\{k\neq j\}}(v(t,S_t,j)-v(t,S_t,k))\,. \end{aligned}$$

Note that L and H^{kj} are bounded. Hence, the stochastic integral with respect to $\widetilde{M}$ on the right hand side of eq. (3.29) is a square integrable Q-martingale. Using also the boundedness of $\tilde{\lambda}^{kj}$, we conclude that there is some constant $c<\infty$ such that

$$\begin{aligned} & E_Q\left[\left[\int H^{kj}d\widetilde{M}^{kj}\right]_T - \left[\int H^{kj}d\widetilde{M}^{kj}\right]_t \Big| \mathcal{F}_t\right] \\ =\; & E_Q\left[\int_t^T (H^{kj})^2 d[\widetilde{M}^{kj}]\Big|\mathcal{F}_t\right] = E_Q\left[\int_t^T (H_u^{kj})^2\tilde{\lambda}_u^{kj}\,du\Big|\mathcal{F}_t\right] \le (T-t)c \end{aligned} \tag{3.30}$$

for all k,j and $t\in[0,T]$. This implies that the $d\widetilde{M}$-term in (3.29) is even a $Q-\mathcal{BMO}$-martingale (see Theorem X.10.9 in [HWY92]). Hence, (3.29) shows that $(\int v_x\,dS)_{t\in[0,T)}$ equals the sum of a bounded process (the first two terms on the right) and a $Q-\mathcal{BMO}$-martingale (the third term on the right). This implies that $(\int v_x\,dS)_{t\in[0,T)}$ is a square integrable Q-martingale and is moreover itself a $Q-\mathcal{BMO}$-martingale. The latter assertion follows from the observation that $\mathcal{BMO}$-martingales constitute a linear space that contains all bounded martingales (see [HWY92], 10.6 and 10.11). This establishes (3.27) and thereby (3.26).

Step 5:
Recall that Z is the density process of Q with respect to P and that our aim is to show $Q=Q^B$. In the present part, we prove $Z=\widetilde{Z}$ with $\widetilde{Z}=(\widetilde{Z}_t)_{t\in[0,T]}$ defined by

$$\widetilde{Z}_t \;:=\; \exp\left(-\alpha\Big\{v^B(0,S_0,\eta_0) + \int_0^t \vartheta_u^B\,dS_u - v^B(t,S_t,\eta_t) - \int_0^t f(u,S_u,\eta_u)du - \sum_{u\le t}\sum_{\substack{k,j=1\\k\neq j}}^{m} 1_{\{\eta_{u-}=k,\eta_u=j\}}f^{kj}(u,S_u)\Big\}\right). \tag{3.31}$$

In the sequel, we will use that that $\int f(u,S_u,\eta_u)du$ is indistinguable from $\int f(u,S_u,\eta_{u-})du$. By Itô's formula one obtains – with the standing convention that all functions in integrands are evaluated at (t,S_t,η_{t-}) unless specified otherwise –

$$\begin{aligned}\frac{d\widetilde{Z}_t}{\widetilde{Z}_{t-}} &= \alpha\left(v_t+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}v_{x^ix^j}+(v_x-\bar{\vartheta})^{\mathrm{tr}}\,\Gamma+\frac{\alpha}{2}||\Sigma^{\mathrm{tr}}(v_x-\bar{\vartheta})||^2+f\right)dt\\ &\quad+\alpha(v_x-\bar{\vartheta})^{\mathrm{tr}}\Sigma\,dW_t\;+\frac{\Delta\widetilde{Z}_t}{\widetilde{Z}_{t-}},\quad t\in[0,T)\,,\end{aligned}$$

and this yields

$$\begin{aligned}\frac{d\widetilde{Z}_t}{\widetilde{Z}_{t-}} &= \alpha\left(v_t+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}v_{x^ix^j}-\frac{1}{2\alpha}||\Sigma^{-1}\Gamma||^2+f\right)dt\\ &\quad-\left(\Sigma^{-1}\Gamma\right)^{\mathrm{tr}}\,dW_t\;+\frac{\Delta\widetilde{Z}_t}{\widetilde{Z}_{t-}},\quad t\in[0,T)\end{aligned}\tag{3.32}$$

by putting in the definition of $\bar{\vartheta}$. Recalling the short-hand notations $N^{\eta j}$, $M^{\eta j}$, $\lambda^{\eta j}$ and $f^{\eta j}$ from (3.17), we have

$$\begin{aligned}\frac{\Delta\widetilde{Z}_t}{\widetilde{Z}_{t-}} &= \frac{\widetilde{Z}_t}{\widetilde{Z}_{t-}}-1\\ &= \exp\left(\alpha\Big(v(t,S_t,\eta_t)-v(t,S_t,\eta_{t-})+\sum_{\substack{k,j=1\\k\neq j}}^{m}f^{kj}(t,S_t)1_{\{\eta_{t-}=k,\eta_t=j\}}\Big)\right)-1\\ &= \sum_{j=1}^{m}I_{\{\eta_{t-}\neq j\}}\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))}-1\right)\Delta N_t^{\eta j}\,,\quad t\in[0,T)\,,\end{aligned}$$

and $\Delta N_t^{\eta j}=dN_t^{\eta j}=dM_t^{\eta j}+\lambda^{\eta j}(t,S_t)\,dt$. Substituting these terms into eq. (3.32) yields

$$\begin{aligned}\frac{d\widetilde{Z}}{\widetilde{Z}_-} &= \alpha\left(v_t+\frac{1}{2}\sum_{i,j=1}^{d}a^{ij}v_{x^ix^j}-\frac{1}{2\alpha}||\Sigma^{-1}\Gamma||^2+f\right.\\ &\qquad\left.+\frac{1}{\alpha}\sum_{j=1}^{m}I_{\{\eta_{t-}\neq j\}}\lambda^{\eta j}(t,S_t)\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))}-1\right)\right)dt\\ &\quad-\left(\Sigma^{-1}\Gamma\right)^{\mathrm{tr}}\,dW_t\\ &\quad+\sum_{j=1}^{m}I_{\{\eta_{t-}\neq j\}}\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))}-1\right)dM_t^{\eta j}\,,\quad t\in[0,T)\,;\end{aligned}$$

and thanks to the PDE for v this simplifies to

$$\frac{d\widetilde{Z}_t}{\widetilde{Z}_{t-}}=-(\Sigma^{-1}\Gamma)^{\mathrm{tr}}\,dW_t+\sum_{j=1}^{m}\xi^j\,dM_t^{\eta j}\,,\quad t\in[0,T)\,,\tag{3.33}$$

with ξ^j given by (3.19). By the definition (3.18) of Z and the SDE (3.33) for $\widetilde{Z}$ we obtain

$$Z_t = \widetilde{Z}_t\,, \quad t \in [0,T). \tag{3.34}$$

Since $N_T = N_{T-}$ (P-a.s.) we have $\eta_T = \eta_{T-}$, $M_T = M_{T-}$, $\widetilde{Z}_T = \widetilde{Z}_{T-}$ and $Z_T = Z_{T-}$ by the definitions of η, $M = (M^{kj})_{k,j\in\{1,\dots,m\}}$, Z and $\widetilde{Z}$, respectively. Hence, (3.34) also holds on the closed interval $[0,T]$. This establishes the desired identity $Z = \widetilde{Z}$.

Step 6:
Our goal is to show that Q coincides with the optimal measure Q^B from the dual problem. By the preceding steps of the proof and the terminal conditions on v we already know that $\int \bar{\vartheta}\, dS$ is a $Q - \mathcal{BMO}$-martingale and

$$\frac{dQ}{dP} = Z_T = \exp\left(-\alpha\left(v(0,S_0,\eta_0) + \int_0^T \bar{\vartheta}\, dS - B\right)\right). \tag{3.35}$$

In order to obtain our claim via variant (i) of Proposition 1.4.2, it remains to show that

$$\text{there exists } \varepsilon > 0 \text{ such that} \quad \exp\left(\alpha\left(B + \varepsilon\int_0^T \bar{\vartheta}_t\, dS_t\right)\right) \in L^1(P)\,. \tag{3.36}$$

By Itô's formula and the definition of $\bar{\vartheta}$ we have – recalling our convention that all functions in integrands are evaluated at (t, S_t, η_{t-}) unless specified otherwise –

$$\begin{aligned}\bar{\vartheta}_t^{\mathrm{tr}}\, dS_t &= dv(t,S_t,\eta_t) - \left(v_t + \frac{1}{2}\sum_{i,j=1}^d a^{ij} v_{x^i x^j} - \frac{1}{\alpha}||\Sigma^{-1}\Gamma||^2\right) dt \\ &\quad + \frac{1}{\alpha}(\Sigma^{-1}\Gamma)^{\mathrm{tr}}\, dW_t \;- \Delta v(t,S_t,\eta_t)\,, \quad t \in [0,T)\,.\end{aligned}$$

Substituting

$$\Delta v(t,S_t,\eta_t) = \sum_{j=1}^m \left(v(t,S_t,j) - v(t,S_t,\eta_{t-})\right)\left(dM_t^{\eta j} + \lambda^{\eta j}(t,S_t)\right)$$

and using the PDE for v leads to

$$\begin{aligned}\bar{\vartheta}_t^{\mathrm{tr}}\, dS_t &= dv(t,S_t,\eta_t) + \Bigg(\frac{1}{2\alpha}||\Sigma^{-1}\Gamma||^2 + f + \sum_{j=1}^m I_{\{\eta_{t-}\neq j\}}\lambda^{\eta j}(t,S_t)\Bigg\{ \\ &\quad \frac{1}{\alpha}\left(e^{\alpha(v(t,S_t,j)-v(t,S_t,\eta_{t-})+f^{\eta j}(t,S_t))} - 1\right) - \left(v(t,S_t,j) - v(t,S_t,\eta_{t-})\right)\Bigg\}\Bigg) dt \\ &\quad + \frac{1}{\alpha}\left(\Sigma^{-1}\Gamma\right)^{\mathrm{tr}} dW_t \;- \sum_{j=1}^m \left(v(t,S_t,j) - v(t,S_t,\eta_{t-})\right) dM_t^{\eta j}\,, \qquad t \in [0,T]\,.\end{aligned} \tag{3.37}$$

More precisely, we obtain the previous equality at first only for $t \in [0,T)$. But both sides of the previous equation describe continuous processes on $[0,T]$ since

$$v(t,S_t,\eta_t) - v(t,S_t,\eta_{t-}) = \sum_{j=1}^m (v(t,S_t,j) - v(t,S_t,\eta_{t-}))\Delta M_t^{\eta j}\,, \quad t \in [0,T]\,,$$

and therefore the equation also holds for $t \in [0,T]$.
The dt-integrand and the process v on the right hand side of (3.37) are bounded and the stochastic integrals with respect to W and M are P-$\mathcal{BMO}$-martingales. To obtain the latter assertion, we can apply similar arguments as in (3.30) since the integrands are again bounded. Hence, $(\int_0^t \bar\vartheta\, dS)_{t\in[0,T]}$ equals a bounded process plus a P-$\mathcal{BMO}$-martingale and by the John-Nirenberg inequality (see [HWY92], 10.43) there is some $\bar\varepsilon > 0$ such that

$$\exp\left(\alpha\bar\varepsilon \int_0^T \bar\vartheta dS\right) \in L^1(P)\,.$$

In combination with Lemma 3.4.1 and Hölder's inequality this establishes (3.36).

Step 7:
Summing up, we have verified all the conditions required for variant (i) of Proposition 1.4.2 and conclude that Q is the optimal martingale measure and $\bar\vartheta$ is the corresponding strategy for the dual problem, i.e. $Q^B = Q$ and $\vartheta^B = \bar\vartheta$. Via Proposition 1.2.3 this implies that ϑ^B is in $\Theta_{\mathcal{M}}$.

□

In the previous proof we have derived an explicit description of the $\mathbb{F}$-density process of the optimal dual measure Q^B in an exponential form (see (3.31)) and also in the form of a stochastic exponential (see (3.18)). We state this a separate result:

Theorem 3.4.3 *Suppose the assumptions of Theorem 3.4.2 hold. Then the density process of Q^B with respect to P is given by*

$$\begin{aligned}\left.\frac{dQ^B}{dP}\right|_{\mathcal{F}_t} &= \mathcal{E}\Bigg(-\int \left(\Sigma^{-1}(u,S_u,\eta_{u-})\Gamma(u,S_u,\eta_{u-})\right)^{\mathrm{tr}} dW_u \\ &\qquad + \sum_{\substack{k,j=1\\k\neq j}}^m \int I_k(\eta_{u-})\lambda^{kj}(u,S_u)\left(e^{\alpha(v^B(u,S_u,j)-v^B(u,S_u,k)+f^{kj}(u,S_u))}-1\right) dM_u^{kj}\Bigg)_t \\ &= \exp\Bigg(-\alpha\Big\{v^B(0,S_0,\eta_0)+\int_0^t \vartheta_u^B\,dS_u - v^B(t,S_t,\eta_t) \\ &\qquad -\int_0^t f(u,S_u,\eta_u)du - \sum_{u\le t}\sum_{\substack{k,j=1\\k\neq j}}^m 1_{\{\eta_{u-}=k,\eta_u=j\}} f^{kj}(u,S_u)\Big\}\Bigg)\,, \end{aligned} \tag{3.38}$$

$t \in [0,T]$, with v^B and ϑ^B provided by Theorem 3.4.2.

The term $c_t^B := v^B(t,S_t,\eta_t) + \int_0^t f(u,S_u,\eta_u)du + \sum_{u\le t}\sum_{k,j:k\neq j} f^{kj}(u,S_u)$ that appears in (3.38) can be interpreted as the (exponential) certainty equivalent of the remaining effective liability at time t. We refer to Remark 1.4.7 for details.

For $B = 0$, the previous results provide explicit formulas for the density of the entropy minimizing martingale measure Q^0. We refer to Grandits and Rheinländer [GraR99] and Miyahara [Mi96, Mi00] for related results in different models. Our major contribution is that we have developed an incomplete multidimensional model which allows for various mutual dependencies between the financial assets S and further (untraded) factors of risk η, but nevertheless still permits for a fairly explicit and interpretable solution.
Moreover, we obtained the solution Q^B to the dual side of the utility maximization problem also *with an additional terminal liability* B. From the description of this dual solution, we immediately obtain PDE-type results on the optimal investment problem with additional terminal liability B (via Proposition 1.2.3). In the following section, we will see that the solution to our utility indifference pricing and hedging problem is also given by the solution to a system of interacting semi-linear PDEs.

Remark 3.4.4 (A comparison with related work in the literature)
Hamilton-Jacobi-Bellman and PDE-type solutions for expected utility maximization in the presence of a terminal liability have been studied by several authors (see [D00, DPZ93, HenH00, Hen01, Z01], Section 4.3 in [DGRS^{3}00] and references therein). Throughout, these authors consider classical incomplete models, i.e., models with a Brownian filtration where the asset prices S are given by a diffusion process driven by Brownian motions and the incompleteness comes from the fact that there are more Brownian motions (sources of risk) than assets. This modeling differentiates their work from ours.
Davis [D00] investigates the pricing and hedging of a claim that is written on a non-traded asset when there is a 'closely related' tradable asset. The price processes of both assets are given by correlated geometric Brownian motions and the investor's objective is to maximize his expected exponential utility. Davis obtains a semi-linear PDE which describes the optimal solution to the combined investment and hedging problem. This PDE contains squares of derivatives. In comparison, the solution to the investment and hedging problem in our model is given by an interacting system of semi-linear PDEs which are linear in all derivatives.
Henderson and Hobson [HenH00, Hen01] studied the same problem as Davis [D00] when the utility function is of power-type instead of exponential. Zariphopoulou [Z01] considers the optimal investment problem without terminal liability for a power-type utility function in an incomplete diffusion model with one traded risky asset and a further (untraded) stochastic factor of risk. She shows that the value function is a transformation of the viscosity solution to a linear parabolic PDE; the optimal investment strategy is then derived from this value function (see [Z01], Theorems 3.2 and 3.3). An Example ([Z01], Section 4, C) provides an upper bound for the utility indifference price for a claim that is a function of the traded asset. So far, known results for the optimal investment problem under terminal liability and the

dynamics of the utility based price process are much less explicit if the market contains several tradable assets. For example, the merely formal Hamilton-Jacobi-Bellman equation in Sect. 4.3 in [DGRS^{3}00] for a general diffusion model is too involved to yield a PDE for the value function v^B of the optimal investment problem under terminal liability; the reason is that it seems not possible to obtain the minimizer for the HJB-equation – i.e. the optimal strategy ϑ^B – explicitly in terms of the value function. Solutions are only known for special cases in dimension one (see Section 4.3 in [DGRS^{3}00] for references).

In a Brownian model which is non-Markovian, Rouge and El Karoui [RouE00] describe the dynamics of a utility based option price by a backward stochastic differential equation and also provide a formula for the optimal strategy. However, both equations involve a process z which is not explicitly given in terms of the parameters of the model but arises from a pure existence result.

In comparison to this, let us emphasize a major contribution of our results. For our multidimensional incomplete model, Sections 3.4 and 3.5 provide PDE-solutions to the optimal investment and hedging problem where all appearing coefficients are explicitly given in terms of parameters of the financial model. Finally, we note a difference in the techniques of the proofs. Our formulation relies solely on martingale arguments while other authors (at least partly) use the HJB dynamic programming approach [D00, DGRS^{3}00, HenH00, Hen01, Z01] or the theory of backward stochastic equations [RouE00]. $\diamond$

From Theorem 3.4.2 we immediately obtain the solution to the primal exponential utility maximization problem via the duality results (see Proposition 1.2.3):

Corollary 3.4.5 *Suppose the assumptions of Theorem 3.4.2 and (1.8) hold. Then*

$$E\left[-\exp\left(-\alpha(x+\int_0^T \vartheta^B dS - B)\right)\right] = \sup_{\vartheta\in\Theta} E\left[-\exp\left(-\alpha(x+\int_0^T \vartheta dS - B)\right)\right], \quad x\in\mathbb{R},$$

with $\vartheta^B \in \Theta_{\mathcal{M}}$ given by Theorem 3.4.2. In particular, the supremum is attained by ϑ^B if ϑ^B is in Θ.

3.5 Solutions to the rational pricing and hedging problem

We are going to describe the structure of the solution for the indifference price process and the corresponding hedging strategy by means of an interacting system of non-linear partial differential equations. We rely on the previous section and results from Section 1.4.

If the assumptions of Theorems 3.4.2 hold (with B), they are also satisfied for the claim $B = 0$ and so the density process of the entropy minimizing martingale measure $Q^0 \in \mathbb{P}_f \cap \mathbb{P}_e$ is described by the unique solution $v^0 \in C_b^{1,2}([0,T)\times D\times\{1,\dots,m\},\mathbb{R})$ to the following system of PDEs with $k = 1, \dots, m$:

$$\begin{aligned}
& v^0_t(t,x,k) + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,k)\, v^0_{x^i x^j}(t,x,k) \\
& - \frac{1}{2\alpha}||\Sigma^{-1}(t,x,k)\Gamma(t,x,k)||^2 \\
& + \sum_{\substack{j=1\\k\neq j}}^{m} \lambda^{kj}(t,x)\frac{1}{\alpha}\left(e^{\alpha(v^0(t,x,j)-v^0(t,x,k))}-1\right) = 0\,, \qquad (t,x)\in[0,T)\times D\,, \\
& \text{and} \quad v^0(T,x,k) = 0\,, \qquad x\in D\,.
\end{aligned} \tag{3.39}$$

Since v^0 solves (3.39) and v^B solves (3.16), the difference $v^\pi = v^B - v^0$ solves the following PDE-system

$$\begin{aligned}
& v^\pi_t(t,x,k) + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,k)\, v^\pi_{x^i x^j}(t,x,k) \\
& + f(t,x,k) \\
& + \sum_{\substack{j=1\\k\neq j}}^{m} \Lambda^{kj}(t,x)\frac{1}{\alpha}\left(e^{\alpha(v^\pi(t,x,j)-v^\pi(t,x,k)+f^{kj}(t,x))}-1\right) = 0\,, \quad (t,x)\in[0,T)\times D\,, \\
& \text{and} \quad v^\pi(T,x,k) = h(x,k)\,, \quad x\in D\,,
\end{aligned} \tag{3.40}$$

with

$$\Lambda^{kj}(t,x) := \lambda^{kj}(t,x)e^{\alpha(v^0(t,x,j)-v^0(t,x,k))}\,, \quad (t,x)\in[0,T]\times D\,. \tag{3.41}$$

To see this, note that the PDEs which describe v^B and v^0 are linear in all terms but one. Calculating the difference of these non-linear terms, using the equality

$$\begin{aligned}
& e^{\alpha(v^B(t,x,j)-v^B(t,x,k)+f^{kj}(t,x))} - e^{\alpha(v^0(t,x,j)-v^0(t,x,k))} \\
& \quad = e^{\alpha(v^0(t,x,j)-v^0(t,x,k))}\left(e^{\alpha(v^\pi(t,x,j)-v^\pi(t,x,k)+f^{kj}(t,x))}-1\right)
\end{aligned}$$

and the definition of (3.41) then lead to (3.40).

Remark 3.5.1 (Interpretation of the functions Λ^{kj})
Let us provide some interpretation for the functions Λ^{kj}, $k \neq j$, which appear in the PDE-system (3.40) for v^π. By eq. (3.28) in the proof of Theorem 3.4.2 – applied for $B = 0$ – and the definition (3.2) of η, we have that $I_k(\eta_{t-})\Lambda^{kj}(t, S_t)$ is the $(Q^0, \mathbb{F})$-intensity of the point process $\left(\sum_{s\le t} I_{\{\eta_{s-}=k,\eta_s=j\}}\right)_{t\in[0,T]}$, $k \neq j$, which counts the jumps of η from k to j. In this sense, the functions Λ^{kj}, $k \neq j$, describe the intensities of the process η to jump from one state to another with respect to the entropy minimizing martingale measure Q^0. In comparison, the corresponding functions with respect to the objective measure P are given by λ^{kj}, $k \neq j$. ◇

We will show that there is a unique solution v^π to the PDE-system (3.40) and this solution describes both the utility indifference price process for B and the corresponding hedging strategy. Let us recall the notation $C_b^{1,2}([0,T) \times D \times \{1,\dots,m\})$ from p. 48.

Theorem 3.5.2 *Suppose the assumptions of Theorem 3.4.2 and (1.8) hold. Then there is a unique function $v^\pi \in C_b^{1,2}([0,T) \times D \times \{1,\dots,m\})$ which solves the PDE-system (3.40), and the utility indifference price process $(\pi_t)_{t\in[0,T]}$ and the hedging strategy $\psi \in \Theta_{\mathcal{M}}$ for B are given by*

$$\begin{aligned} \pi_t(B) &= v^\pi(t, S_t, \eta_t) + \int_0^t f(u, S_u, \eta_u)\, du + \sum_{u\le t} \sum_{\substack{k,j=1 \\ k\neq j}}^{m} 1_{\{\eta_{u-}=k, \eta_u = j\}} f^{kj}(u, S_u) \quad \textit{and} \\ \psi_t(B) &= v_x(t, S_t, \eta_{t-})\,, \quad t \in [0,T)\,, \end{aligned}$$

and $\psi_T(B) = 0$. In particular, $\pi(B;\alpha) = \pi_0(B) = v^\pi(0, S_0, \eta_0)$ is the utility indifference price.

Again (cf. the remark following Theorem 3.4.2) it is rather arbitrary to set $\psi_T(B) = 0$ since the value of $\psi(B)$ at T does not affect the stochastic integral $\int \psi(B)\, dS$, i.e. the equivalence class of $\psi(B)$ in $L(S)$. In other words, the equivalence class of $\psi \in \Theta_{\mathcal{M}}$ is already determined by the eq. for $(\psi_t)_{t\in[0,T)}$.

Proof: Under the given assumptions, Theorem 3.4.2 applies to both claims B and 0. In particular, there exist unique solutions v^0 and v^B in $C_b^{1,2}([0,T) \times D \times \{1,\dots,m\})$ to the PDE-systems (3.39) and (3.16), respectively. By the foregoing considerations, we know that $v^B - v^0$ solves (3.40). Moreover, this solution is unique. To see this, let $v^\pi \in C_b^{1,2}([0,T) \times D \times \{1,\dots,m\})$ be a solution to the PDE-system (3.40). It is straightforward to verify that $v^0 + v^\pi$ satisfies the same PDE-system (3.16) as v^B. Hence, $v^\pi = v^B - v^0$ by the uniqueness of v^B. This implies

$$v_x^\pi(t,x,k) = v_x^B(t,x,k) - v_x^0(t,x,k)\,, \quad (t,x,k) \in [0,T) \times D \times \{1,\dots,m\}\,,$$

and

$$v^\pi(T, S_T, \eta_T) = v^B(T, S_T, \eta_T) = h(S_T, \eta_T)\,. \tag{3.42}$$

Substituting the formulae for ϑ^B and ϑ^0 from Theorem 3.4.2 in the definition of $\psi(B) := \vartheta^B - \vartheta^0$ yields

$$\psi_t(B) = \vartheta_t^B - \vartheta_t^0 = v_x^B(t, S_t, \eta_{t-}) - v_x^0(t, S_t, \eta_{t-}) = v_x^\pi(t, S_t, \eta_{t-})\,, \quad t \in [0,T).$$

Moreover, ψ is in $\Theta_{\mathcal{M}}$ since ϑ^B and ϑ^0 are in this linear space. Using the formulae from Theorem 3.4.3 for the densities of Q^B and Q^0 we can calculate the density process of Q^B with respect to Q^0 as

$$\left.\frac{dQ^B}{dQ^0}\right|_{\mathcal{F}_t} = \left.\frac{dQ^B}{dP}\right|_{\mathcal{F}_t} \left(\left.\frac{dQ^0}{dP}\right|_{\mathcal{F}_t} \right)^{-1} = \exp\left(-\alpha\Big(\bar\pi_0 + \int_0^t \psi_t(B)\, dS_u - \bar\pi_t \Big) \right)\,, \quad t \in [0,T],$$

with

$$\bar{\pi}_t := v^\pi(t, S_t, \eta_t) + \int_0^t f(u, S_u, \eta_u)du + \sum_{u \le t} \sum_{\substack{k,j=1 \\ k \neq j}}^{m} 1_{\{\eta_{u-}=k, \eta_u=j\}} f^{kj}(u, S_u) .$$

By (3.42) we have $\bar{\pi}_T = B$ and Corollary 1.4.6 then yields $\pi_t(B) = \bar{\pi}_t$, $t \in [0, T]$. □

Remark 3.5.3 The theorem shows that the solution to our indifference pricing and hedging problem can be determined from the unique solution to the PDE-system (3.40). Note that this system involves v^0 by eq. (3.41). So one needs first the unique solution v^0 to the system (3.39) – which involves only the coefficients of the underlying financial model – in order to determine (π_t) from (3.40). In some situations this dependence on v^0 disappears and

$$\Lambda^{kj}(t, x) = \lambda^{kj}(t, x) , \quad (t, x) \in [0, T] \times D ,$$

holds for all $k \neq j$. Then, eq. (3.40) alone determines the solution to our problem.
Suppose for instance that the coefficients Σ and Γ in the SDE (3.1) that drives S depend only on (t, x) but do not vary in k, i.e. $\Gamma(t, x, k) = \Gamma(t, x, j)$ and $\Sigma(t, x, j) = \Sigma(t, x, k)$ for any $k, j = 1, \dots, m$ and $(t, x) \in [0, T] \times D$. This means that the untradable stochastic risk factor (η_t) does not affect the dynamics of the asset prices S but only the asset prices can affect the intensities of η. The liability B, however, might depend on the common evolution of η and S. In this situation the solution v^0 to the PDE (3.39) does not vary in k and therefore

$$e^{\alpha(v^0(t,x,j) - v^0(t,x,k))} = 1 \quad \text{for} \quad (t, x) \in [0, T] \times D, \ k, j \in \{1, \dots, m\} .$$

Hence, Λ^{kj} equals λ^{kj} for all k, j and the PDE-system (3.40) for $v^\pi(t, x, k)$ involves only coefficients which are directly given by the underlying financial model. ◇

Chapter 4

Applications

In the present chapter, we illustrate our previous results and the range of possible applications for the model from Chapter 3 in particular. Section 4.3 moreover exemplifies the additivity and diversification results from Chapter 2.
To this end, we consider a variety of concrete financial products like equity linked life insurances, default-risk contracts and weather derivatives. Furthermore, we will discuss our results in the context of stochastic volatility models. To put our modeling framework and our contributions into perspective, we thereby also point out similarities and crucial differences to related work in the literature. Our references however are selective and we do not claim to give an exhaustive account of the financial contracts mentioned.

4.1 Chapter Assumption: The financial market model

We consider the framework of Section 3. For a formal description of the model, the reader is referred to Section 3.1. Throughout, it is supposed that all assumptions of Section 3.1 hold. For our purposes, it is convenient to assume furthermore $D \subseteq (0,\infty)^d$. With the re-parameterizations (cf. (3.4))

$$\Gamma(t,x,k) = \operatorname{diag}(x^i)_{i=1,\dots,d}\,\gamma(t,x,k) \quad \text{and} \quad \Sigma(t,x,k) = \operatorname{diag}(x^i)_{i=1,\dots,d}\,\sigma(t,x,k) \tag{4.1}$$

we can then rewrite the stochastic differential equation for the (discounted) prices S of the tradable financial assets as a generalized Black-Scholes SDE

$$dS_t = \operatorname{diag}(S_t^i)_{i=1,\dots,d}\Big(\gamma(t,S_t,\eta_{t-})dt + \sigma(t,S_t,\eta_{t-})\,dW_t\Big), \quad S_0 \in D. \tag{4.2}$$

Considering discounted prices does not exclude random interest rates from our analysis if we interpret S as *forward price process*, i.e. if we choose the zero coupon bond with maturity T as the numeraire (see [GER95] or [Bj97], Ch. 5).
The coefficients in the SDE (3.1) for S depend on an additional process η which represents some untradable factors of risk. Let us recall from Section 3.1 that η is an adapted finite state

process with values in $\{1, \dots, m\}$ and (cf. (3.2))

$$\text{if } \eta_{t-} = k \text{ then } \eta \text{ jumps at time } t \text{ with } (P, \mathbb{F})\text{-intensity } \lambda^{kj}(t, S_t) \text{ from } k \text{ to } j. \tag{4.3}$$

We assume that

$$\sigma \text{ is invertible and } \sigma^{-1}\gamma \text{ is bounded on } [0,T] \times D \times \{1, \dots, m\}. \tag{4.4}$$

Due to (4.1), this condition is equivalent to (3.14) in the present framework. As in the previous section, we consider contingent claims of the general type

$$B = h(S_T, \eta_T) + \int_0^T f(t, S_t, \eta_t)\, dt + \sum_{t \le T} \sum_{\substack{k,j=1 \\ k \neq j}}^{m} 1_{\{\eta_{t-}=k, \eta_t = j\}} f^{kj}(t, S_t) \tag{4.5}$$

with continuous bounded functions $h : D \times \{1, \dots, m\} \to \mathbb{R}$, $f : [0,T] \times D \times \{1, \dots, m\} \to \mathbb{R}$ and $f^{kj} : [0,T] \times D \to \mathbb{R}$. It is supposed that $f(\cdot, \cdot, k)$ and $f^{kj}(\cdot, \cdot)$ are of class C^1 with respect to (t, x) on $[0,T] \times D$ for all k, j.
Thus, we consider financial contracts which specify a terminal payoff $h(S_T, \eta_T)$, payments according to a continuously payable rate $f(t, S_t, \eta_t)$, payments which are triggered by jumps of the process η or a combination of all of these. Typical examples for contingent claims whose payoffs meet one of these three forms are (in the above order) European options, Asian options and so-called general life insurances (cf. [Hoe69] or [No92]), respectively.

4.2 General solution by reaction-diffusion equations

Since our model is incomplete, a claim like B – specified in (4.5) – cannot be replicated by dynamic trading in the financial market S in general. Therefore, an investor who issues B is faced with some inevitable risk, and his valuation and the criteria for his "best" possible hedging strategy have to depend on his preferences and his attitude towards risk. Our definitions of the *utility indifference price* and the corresponding *hedging strategy*, respectively, are based on the assumption that the investor is rational and has constant absolute risk aversion; i.e., his objective is to maximize his expected exponential utility according to some level of risk aversion $\alpha \in (0, \infty)$.
Let us recall the general solution as it is provided by Theorem 3.5.2. Under the given assumptions, both the utility indifference price process and the corresponding hedging strategy are described via the unique solution v^π to a system of m interacting semi-linear partial differential equations with terminal conditions – a so-called *reaction-diffusion equation*. In fact, the solution involves two such PDE-systems since the coefficients in the PDE-system for v^π involve the solution v^0 to another similar PDE-system.
The function v^π basically describes the valuation of the claim (cf. (4.11) for details). Each single PDE in the system corresponds to one possible state of the process η which models some

untradable additional factors of risk. The uncertainty about the evolution of these untradable factors of risk has an impact on the valuation which is reflected by the interaction between the single PDEs. Similarly to the Black-Scholes model, the utility based hedging strategy $\psi(B)$ is given by the gradient of the valuation function v^π with respect to the prices of the tradable assets.
By (4.1), the matrix $\tilde{a} := \sigma\sigma^{tr}$ is related to $a := \Sigma\Sigma^{tr}$ by the equation

$$(a^{ij}(t,x,k))_{i,j\in\{1,\dots,d\}} = \Sigma(t,x,k)\Sigma^{\mathrm{tr}}(t,x,k) = (x^i x^j \tilde{a}^{ij}(t,x,k))_{i,j\in\{1,\dots,d\}} \tag{4.6}$$

which holds for all $(t,x,k) \in [0,T] \times D \times \{1,\dots,m\}$. Using γ, σ and $\tilde{a}$ instead of Γ, Σ and a we can now rewrite the PDE-systems (3.39) and (3.40) for v^0 and v^π in a form which is more reminiscent of the classical Black-Scholes situation: The functions $v^0(\cdot,\cdot,k) : [0,T] \times D \to \mathbb{R}$, $k = 1,\dots,m$, are described by

$$\begin{aligned}
&v^0_t(t,x,k) + \frac{1}{2}\sum_{i,j=1}^{d} x^i x^j \tilde{a}^{ij}(t,x,k)\, v^0_{x^i x^j}(t,x,k) \\
&- \frac{1}{2\alpha}||\sigma^{-1}(t,x,k)\gamma(t,x,k)||^2 \\
&+ \sum_{\substack{j=1\\ j\neq k}}^{m} \lambda^{kj}(t,x)\frac{1}{\alpha}\left(e^{\alpha(v^0(t,x,j)-v^0(t,x,k))} - 1\right) \quad = \quad 0\,, \qquad (t,x) \in [0,T) \times D\,, \\
&\qquad\qquad \text{and} \qquad v^0(T,x,k) \quad = \quad 0\,, \qquad x \in D\,;
\end{aligned} \tag{4.7}$$

and the functions $v^\pi(\cdot,\cdot,k) : [0,T] \times D \to \mathbb{R}$, $k = 1,\dots,m$, are given by

$$\begin{aligned}
&v^\pi_t(t,x,k) + \frac{1}{2}\sum_{i,j=1}^{d} x^i x^j \tilde{a}^{ij}(t,x,k)\, v^\pi_{x^i x^j}(t,x,k) \\
&+ f(t,x,k) \\
&+ \sum_{\substack{j=1\\ j\neq k}}^{m} \Lambda^{kj}(t,x)\frac{1}{\alpha}\left(e^{\alpha(v^\pi(t,x,j)-v^\pi(t,x,k)+f^{kj}(t,x))} - 1\right) \quad = \quad 0\,, \quad (t,x) \in [0,T) \times D\,, \\
&\qquad\qquad \text{and} \qquad v^\pi(T,x,k) \quad = \quad h(x,k)\,, \quad x \in D\,,
\end{aligned} \tag{4.8}$$

with

$$\Lambda^{kj}(t,x) := \lambda^{kj}(t,x)e^{\alpha(v^0(t,x,j)-v^0(t,x,k))}\,, \quad (t,x) \in [0,T] \times D\,, k,j \in \{1,\dots,m\}\,. \tag{4.9}$$

Recall from Remark 3.5.3 that the functions Λ^{kj} and λ^{kj} coincide for all k,j if γ and σ do not vary in the η-argument, i.e. if

$$\gamma(t,x,1) = \gamma(t,x,k) \text{ and } \sigma(t,x,1) = \sigma(t,x,k),\ (t,x,k) \in [0,T] \times D \times \{1,\dots,m\}\,. \tag{4.10}$$

By Theorem 3.5.2, there exist unique solutions v^0 and v^π in $C_b^{1,2}([0,T) \times D \times \{1,\dots,m\},\mathbb{R})$ to the PDE-systems above, and the utility indifference price process (π_t) and the hedging strategy $\psi \in \Theta_{\mathcal{M}}$ for the claim B are given by

$$(4.11)\quad \pi_t(B) = v^\pi(t,S_t,\eta_t) + \int_0^t f(u,S_u,\eta_u)\,du + \sum_{u\le t}\sum_{\substack{k,j=1\\k\neq j}}^{m} 1_{\{\eta_{u-}=k,\eta_u=j\}} f^{kj}(u,S_u)\,,$$

$$(4.12)\quad \psi_t(B) = v_x^\pi(t,S_t,\eta_{t-})\,,\quad t\in[0,T)\,.$$

In particular, $\pi(B;\alpha) = \pi_0 = v^\pi(0,S_0,\eta_0)$ is the utility indifference price of B. Recall that v_x^π denotes the gradient of v^π with respect to x.

We emphasize that the prices of the tradable assets are given by S and that there are no further trading opportunities for the issuer of the claim in the model. Changing the model by enlarging or restricting the set of tradable assets can alter the utility indifference price and the corresponding hedging strategy, since such changes in general affect the possibilities of the issuer to (partially) hedge risk. For practical implementations, the choice of the set of tradable assets would be a crucial task of the modeling. Those tradable assets which are related to the claim should be incorporated in the model and it has to be ensured that the implementation of the model can be sufficiently fitted to real market prices of traded securities. Recall that (bounded) attainable claims are replicated by the hedging strategy and valued by the initial capital that is necessary for the replication (cf. Remark 1.4.4). Non-attainable claims which can be hedged only partially leave the investor with some inevitable risk. In our model, the investor's approach to the valuation and hedging of such risks is based on the objective to maximize his expected exponential utility under P. Therefore, the way we model the untradable factors of risk η and the tradable assets S under the objective probability measure P should be adequate to the real application problem. As a rule of thumb, the less attainable the claim B the more important is the role of P.

4.3 Equity-linked life insurance contracts

Let us consider a life insurance contract whose payoff is explicitly linked to an equity, e.g. a portfolio or some financial index. For simplicity, we restrict ourselves to the case $d = 1$ here. Let

$$\frac{dS_t}{S_t} = \gamma(t,S_t)\,dt + \sigma(t,S_t)\,dW_t\,,\quad S_0\in D\,,$$

and consider S as a stock market index. The survival of a policy holder is modeled by a Poisson process $\breve{N}$ with deterministic intensity $\breve{\lambda}(t)$ which is independent of W. The policy holder is thought to be alive until $\breve{N}$ jumps for the first time, i.e. he dies at $\tau := \inf\{t \ge 0 | \breve{N}_t > 0\}$. We take the mortality risk of the policy holder as an untradable factor of risk and consider a combination of an *endowment insurance* which pays $H(S_T)$ provided the policy holder is still alive at time T and a *term insurance* which pays an equity related amount $F(\tau, S_\tau)$ when the policy holder has died, i.e.

$$B = \begin{cases} H(S_T) & : \ \tau > T \\ F(\tau, S_\tau) & : \ \tau \le T \,. \end{cases} \tag{4.13}$$

This problem can be embedded in the present framework by choosing $m := 2$, $\eta_t := 1 + 1_{\{\breve{N}_t > 0\}}$, $\lambda^{kj}(t,x) := \breve{\lambda}(t) 1_{\{(k,j)=(1,2)\}}$, $f := 0$, $f^{kj}(t,x) := F(t,x) 1_{\{(k,j)=(1,2)\}}$, $h(x,1) := H(x)$ and $h(x,2) := 0$. Under our standing assumptions (cf. Section 4.1), the indifference price process $(\pi_t(B))$ and the corresponding hedging strategy ψ for the claim are given by

$$\pi_t(B) = \begin{cases} v(t, S_t) & : \ t < \tau \\ F(\tau, S_\tau) & : \ t \ge \tau \end{cases} \qquad \text{and} \qquad \psi_t(B) = \begin{cases} v_x(t, S_t) & : \ t \le \tau \\ 0 & : \ t > \tau \end{cases}$$

where v is the unique solution in $C_b^{1,2}([0,T) \times D, \mathbb{R})$ to the non-linear PDE

$$v_t(t,x) + \frac{1}{2} x^2 \sigma^2(t,x) v_{xx}(t,x) \quad = \quad -\breve{\lambda}(t) \frac{1}{\alpha} \left(e^{\alpha(0 - v(t,x) + F(t,x))} - 1 \right)$$

with terminal condition $v(T,x) = H(x)$. To see that this is in fact a simplified presentation of the general solution for $\pi_t(B)$ and $\psi_t(B)$ (cf. (4.11) and (4.12)) for the present problem, observe that condition (4.10) holds and the solution v^π to the PDE (4.8) is related to v via $v^\pi(\cdot,\cdot,1) = v$ and $v^\pi(\cdot,\cdot,2) = 0$.

Note that the left-hand side of the PDE above coincides with the well known Black-Scholes PDE but we have an additional non-linear term on the right-hand side (RHS). Let us explain this additional term in the simplified case when $F = 0$. In this case, the function v describes the utility based valuation of the insurance contract if the policy holder is still alive, and the terminal condition at time T is therefore given by the endowment payoff function H. It is intuitively clear that the valuation of the contract is 0 when the policy holder is dead. The non-linear term on the RHS can be interpreted as an attraction of v towards 0. The strength of this attraction is given by the product of the current intensity $\breve{\lambda}$ that the policy holder dies and an exponentially weighted difference between the "policy holder has died"-value 0 and the current "still alive"-value v. Note that the absolute value of this exponentially weighted difference is increasing in the risk aversion parameter α.

As another example, let us consider an insurance which provides a life annuity payable continuously at rate $R(t, S_t)$ and a payment $F(\tau, S_\tau)$ at the time of death if this occurs before T, i.e.

$$B = \int_0^\tau R(t, S_t)\, dt + 1_{\{\tau \le T\}} F(\tau, S_\tau) \,. \tag{4.14}$$

This problem can be embedded in our framework by choosing m, λ^{kj} and η as above, $f(t,x,k) := R(t,x)1_{\{k=1\}}$, $f^{12} := F$, $f^{kj} := 0$ for $(k,j) \neq (1,2)$, $h(x,1) := H(x)$ and $h(x,2) := 0$. If B satisfies the standing assumptions (cf. Section 4.1), the indifference price process (π_t) and the corresponding hedging strategy ψ for this claim are then given by

$$\pi_t = \begin{cases} \int_0^t R(u,S_u)\,du + v(t,S_t) & : \ t < \tau \\ \int_0^\tau R(u,S_u)\,du + F(\tau,S_\tau) & : \ t \geq \tau \end{cases} \qquad \text{and} \qquad \psi_t = \begin{cases} v_x(t,S_t) & : \ t \leq \tau \\ 0 & : \ t > \tau \end{cases}$$

where v is the unique solution in $C_b^{1,2}([0,T) \times D, \mathbb{R})$ to the following non-linear PDE

$$v_t(t,x) + \frac{1}{2}x^2\sigma^2(t,x)v_{xx}(t,x) \quad = \quad -R(t,x) - \check{\lambda}(t)\frac{1}{\alpha}\left(e^{\alpha(0-v(t,x)+F(t,x))} - 1\right)$$

with terminal condition $v(T,x) = 0$. As in the foregoing example, this solution is a simplified presentation of our general solution for $\pi_t(B)$ and $\psi_t(B)$ for the present problem. Mathematically, the solution permits R to be negative. In that case, the insurer receives premiums from the policy holder.

We could easily consider more complex types of equity linked life insurance contracts within our general framework by taking a larger state space for η, e.g. $\eta_t \in \{\text{active, disabled, dead}\}$ with $m = 3$. Similar to contract (4.14), we might for instance consider an insurance where the policy holder pays premiums while he is active, receives life annuities while he is disabled and the insurance moreover pays a certain amount when the policy holder has died. Also, the model can deal with several tradable assets by taking $d > 1$.

Typically, an insurance company sells a large number of insurance policies. It is therefore natural to consider the valuation and the hedging possibilities for its aggregated portfolio of claims. Let us conclude this section with some comments on this problem.
In the literature on equity linked life insurance contracts, it is common to model the equity prices by a (generalized) Black-Scholes model and to consider the mortality risk as P-independent from S, and (cf. [AP94], [BrS79] or [RolSST98], Sect. 13.3.2) it is typically argued that equity linked payments which are conditional on the survival or death of a policy holder should be valued by the Black-Scholes value of the unconditional payoffs multiplied with the P-probabilities of survival or death, respectively. The intuitive but somewhat heuristic reasoning for this is basically that the mortality risk in the individual policies are eliminated by a law of large numbers and are therefore practically not present for the insurance company.
To make this more rigorous, we can analyze the utility based pricing and hedging problem for an aggregated portfolio of insurances and apply results from Chapter 2. Assuming that the individual policy holders die independently of each other, we obtain that the utility based valuation and the corresponding hedging strategy for an aggregated portfolio of insurances are given by the sum of the valuations and hedging strategies, respectively, for the individual poli-

cies. Moreover, diversification indeed causes the utility indifference price of a large aggregated portfolio to behave asymptotically in accordance with the intuitive reasoning outlined above. To be more precise, let us briefly explain how the model of the present section can be embedded into the framework of Section 2. Let $\breve{N}^i$, $i \in \mathbb{N}$, be a sequence of independent Poisson processes with intensities $\breve{\lambda}^i(t)$ which are furthermore independent of the Brownian motion driving S. Policy holder i is thought to be alive until time τ^i where $\breve{N}^i$ jumps for the first time. Suppose that the information flow $\mathbb{F}$ is generated by W and $\breve{N}^i$, $i \in \mathbb{N}$. Taking $\mathbb{F}^0 := \mathbb{F}^W$, $\mathbb{I}^i := \mathbb{F}^{\breve{N}^i}$ and $dQ^* := \mathcal{E}(-\int \sigma^{-1}\gamma \, dW)_T \, dP$, the model fits into the semi-complete framework of Section 2. Recall that $dQ^* := \mathcal{E}(-\int \sigma^{-1}\gamma \, dW)_T \, dP$ is the unique equivalent martingale measure on $\mathcal{F}_T^W$ and coincides with P on $\bigvee_i \mathcal{F}_T^{\breve{N}^i}$.
For each policy holder i, let us consider an equity linked life insurance B_i of the form (4.13) or (4.14), say, with τ^i instead of τ. By Theorem 2.3.1, the utility indifference price and the corresponding hedging strategy for the aggregated portfolio of claims, i.e. $B := \sum_{i=1}^n B_i$, are given by the sum of the utility indifference prices and hedging strategies, respectively, for the individual policies B_i.
Suppose moreover, that $\breve{N}_i$, $i \in \mathbb{N}$, are identically distributed under P and the individual policies are identical contracts written on different lives. For instance, let each B_i be given by (4.13) with τ^i instead of τ. By Corollary 2.3.2, the utility indifference value of a large aggregated portfolio of n policies is approximately given by $nE_{Q^*}[B_1]$ then. Since Q^* coincides with P on $\bigvee_i \mathcal{F}_T^{\breve{N}^i}$, this result can be interpreted as follows. Thanks to diversification effects, the large number of policies leads to a utility based valuation which is risk neutral under P with respect to the mortality risk – although this is not tradable in the model.

For indifference pricing and hedging approaches to equity linked life insurances which are based on quadratic criteria instead of exponential utility functions we refer the interested reader to Møller [Mø00, Mø01].

4.4 Default and credit risk

We now consider the pricing and hedging problem for a "default insurance" contract which pays 1 Euro at time T if a certain firm has defaulted in the time interval $[0, T]$. Suppose that the firm defaults or changes its rating class according to intensities which depend on the current values of some tradable assets S. For instance, S could model the stock price of the firm itself, stock prices of firms in related industries or some financial index.
To embed this problem in our present framework, let the dynamics of S and η be described by the general model in Section 4.1 (see (4.2) and (4.3)). The process η models the rating of the firm and the state space $\{1, \dots, m\}$ corresponds to $\{AAA, \dots, \text{default}\}$. We suppose that the default state m is absorbing, i.e. $\lambda^{mj} := 0$ for all j. Let $\tau := \inf\{t \geq 0 | \eta_t = m\}$ be the

random time of default.
Under our standing assumptions (cf. Section 4.1) there is a unique solution v^π to the PDE (4.8) with $f := 0$, $f^{kj} := 0$ for all k, j, and $h(x,k) := 1_{\{m\}}(k)$. By (4.11) and (4.12), this solution v^π determines both the indifference price process $(\pi_t(B))$ and the corresponding hedging strategy $\psi(B)$ for the "default insurance" contract

$$B = \begin{cases} 1 & : \quad \tau \leq T \\ 0 & : \quad \tau > T \end{cases} .$$

Note that $v^\pi(\cdot,\cdot,m)$ equals 1 since the constant function 1 solves the PDE for $v^\pi(\cdot,\cdot,m)$ in (4.8) and the solution is unique. This implies

$$\psi_t(B) = \begin{cases} v_x^\pi(t, S_t, \eta_{t-}) & : \quad t \leq \tau \\ 0 & : \quad t > \tau \end{cases} .$$

In particular, the hedging strategy is zero after default has happened.

By (4.9), the solution v^π in general also involves the second PDE (4.7) via (4.9). The function v^π is described solely by the PDE-system (4.8) if condition (4.10) holds, i.e. if γ and σ do not vary with η. Intuitively, condition (4.10) means that the rating process η depends on the asset prices S via its intensities while the current state of the rating does not affect the dynamics of S. This assumption could be appropriate if S models some broad financial index. However, the assumption is clearly inappropriate if S models the stock price of the firm itself since in our model the default event takes the investor by surprise and this should have an effect on the stock price.

As a further example for a contingent claim which is subject to default and credit risk, let us consider a claim which provides interest payable continuously at a rate $R(t, \eta_t)$ – depending on the rating class of the firm – until default happens, i.e.

$$B = \int_0^\tau R(t, \eta_t)\, dt\,.$$

Provided that B satisfies the assumptions from Section 4.1, there is a unique solution v^π to the PDE (4.8) with $h := 0$, $f(t,x,k) := 1_{\{k \neq m\}} R(t,k)$ and $f^{kj} := 0$ for all k, j. By the formulae (4.11) and (4.12), this function v^π determines the indifference price process and the corresponding hedging strategy for the claim B. Similar as in the previous example, no hedging activity takes place after default has happened since $v^\pi(\cdot,\cdot,m)$ is constant at 0.

Institutional investors are usually faced with several default risks. Therefore, the valuation and hedging problem for an aggregated amount of several correlated default risks is an interesting and relevant issue. Our model can deal with several defaults which might be stochastically dependent. To this end, take now $m = \text{card}\left(\{\text{AAA}, \ldots, \text{default}\}\right)^{\ell}$ with $\ell \in \mathbb{N}$ and consider $\eta = (\eta^i)_{i=1,\ldots,\ell}$ as the joint rating of ℓ companies whose default intensities depend on some tradable assets $S = (S^i)_{i=1,\ldots,d}$. For instance, the components of S might be the stock prices of the firms. The model permits for several types of dependencies among the defaults of individual firms. First of all, different firms might default at the same time. Even if the rating classes of different firms do not change simultaneously, the individual default events in the model can be dependent since their intensities might be functions of the same tradable assets or functions of different assets which are correlated.

To put the present section on credit risk into perspective, let us compare our approach to the pricing and hedging problem with some prominent approaches in the literature on default and credit risk.
In terms of the stochastic processes which appear in the model, there is a striking similarity to Lando's model [La98] for the pricing of credit risky securities. In fact, our modeling was inspired by Lando's model. However, the purpose of our work differs considerably from Lando's [La98], and therefore the interpretation of the processes in the stochastic model and the approach to the valuation problem are substantially different. Lando's work solely addresses the pricing of defaultable securities in the financial market. By no arbitrage considerations (cf. [HK79]), security prices in a financial market have to be martingales under a *pricing measure* Q which is equivalent to the objective probability measure P. Taking the perspective of the market, Lando postulates that the (market) prices for default-free and defaultable securities are given as expected payoffs under a pricing measure Q which is a priori given by his model. In fact, the model is specified right from the start under the pricing measure Q and, hence, directly constitutes a linear pricing rule. Under Q, the rating of a firm is supposed to change according to intensities which are functions of another given process X which represents the evolution of some economic state variables. Assuming some structural relations between the empirical intensities of the rating process and those under the pricing measure Q, this pricing rule has to be calibrated to observed market prices for tradable defaultable claims and the empirical intensities.
Our stochastic process (S, η) resembles the process (X, η) in [La98]. However, our model allows for a mutual dependence between S and η (cf. (4.2) and (4.3)). More importantly, we adopt the perspective of an individual investor to address *both* valuation and hedging and we are primarily concerned with claims which incorporate untradable risk. To this end, we suppose that our investor is risk-averse and that his objective is to maximize his P-expected exponential utility. As a consequence, our model must be specified under the objective probability measure P and incorporates the price process S of the tradable securities. Default and credit risk are linked to the evolution of the rating process η and not (directly) tradable. Nevertheless they might be partially hedgable thanks to the mutual dependencies between η and S. We then

derive the investor's valuation and moreover his (partial) *hedging strategy* which minimizes the inevitable risk – both in accordance to his risk-averse preferences.
The default in our model happens by surprise and is unpredictable for the investor. This is similar to the "intensity based"-models for credit risk which are also called *reduced form* models (see [La97] for a summary) and constitutes a crucial difference to the so-called *structural* approach which goes back to the seminal works of Black and Scholes [BlS73] and Merton [Mer74], cf. ch. 12 in [Mer90]. In their structural model, credit risky securities are regarded as contingent claims on the value of the issuing firm's assets and are valued according to the classical option pricing theory. To this end, the firm's value is considered as a tradable asset and modeled by a diffusion process. Default happens at the predictable stopping time when the firm's value first falls below the amount of liabilities. The main problem of this approach is that default could actually well arise before or after the market value of the firm falls below that of liabilities (cf. [DuSS96]). Therefore, the structural model relies on an abstract value of the firm which is typically neither observable nor tradable in practice (cf. [La97]).
On the other hand, our analysis embeds the default risk in a financial market with tradable assets. In this aspect, our model is more similar to the classical structural model in [BlS73, Mer74] and different from the reduced form models in [DuSS96, La98], since the latter are only concerned with pricing issues but do not address trading and hedging problems.

4.5 Stochastic volatility

In the literature on mathematical finance, stochastic volatility models provide a major part of the examples for incomplete financial market models. Not surprisingly, the incomplete model specified in Section 4.1 can be considered as a stochastic volatility model. Indeed, the process η may be interpreted as an economic state variable which affects both the drift and the volatility of the tradable risky assets S via the SDE (4.2). Our results provide explicit solutions for the utility based hedging and pricing of contingent claims B which are combinations of European and Asian options, i.e.

$$B = H(S_T) + \int_0^T F(t, S_t)\, dt\,. \tag{4.15}$$

Such claims can be incorporated into the framework from Section 4.1 by taking $h(x,k) = H(x)$, $f(t,x,k) = F(t,x)$ and $f^{kj} := 0$ for all $k, j \in \{1, \dots, m\}$. If B satisfies the standing assumptions, the solution for both the utility indifference price process $\pi(B)$ and the corresponding hedging strategy $\psi(B)$ are explicitly given in terms of the unique solution v^π to a certain system (4.8) of interacting semi-linear partial differential equations; see (4.11) and (4.12) for details.
Again, our main contribution is that we obtain an explicit solution in a wide class of incomplete models. To the best of our knowledge, there are no similar results in the literature on the

utility indifference approach to pricing and hedging in the context of stochastic volatility models so far. Therefore, let us compare our results with other ones that are based on a different approach. For this purpose, we find it instructive to single out the article of Di Masi, Kabanov and Runggaldier [DiMKR94] since their stochastic model for the financial market is similar, their approach to the hedging problem is substantially different, but – surprisingly – their solution is again similar to ours.

The stochastic model of the financial market in [DiMKR94] is an extension of the Black-Scholes model. The drift and the volatility are modulated by a (homogeneous) finite-state Markov process which is independent of the Brownian motion which drives the stock price. This incomplete model constitutes a special case of our general framework of Section 4.1 if we take $d = 1$, $D = (0, \infty)$, all λ^{kj} as constants, and suppose that η and W are independent under P and that both $\gamma(t, x, k)$ and $\sigma(t, x, k)$ do not vary in t and x. Di Masi et. al. [DiMKR94] restrict their analysis to contingent claims of European type. That is, F in (4.15) is taken to be 0 such that $B = H(S_T)$[1]. Their approach to the hedging and pricing problem is different from ours and relies on the concept of *local risk minimization* which is due to Föllmer and Schweizer [FSchw91]. Basically, the objective for this hedging approach is to minimize a remaining quadratic hedging error in expectation.

Interestingly, the hedging strategy and the corresponding valuation are – similarly to our solution – described by a system of m interacting PDE's (see [DiMKR94], Theorem 3.1), namely

$$v_t(t,x,k) + \frac{1}{2}x^2\sigma^2(k)v_{xx}(t,x,k) \quad = \quad -\sum_{\substack{j=1 \\ j\neq k}}^{m} \lambda^{kj}\left(v(t,x,j) - v(t,x,k)\right), \tag{4.16}$$

with $k = 1, \dots, m$ and terminal condition $v(T, x, k) = H(x)$.

For comparison, in the simpler framework of [DiMKR94] our PDE-system (4.8) for v^π takes the form

$$v^\pi_t(t,x,k) + \frac{1}{2}x^2\sigma^2(k)v^\pi_{xx}(t,x,k) \quad = \quad -\sum_{\substack{j=1 \\ j\neq k}}^{m} \Lambda^{kj}(t,x)\frac{1}{\alpha}\left(e^{\alpha(v^\pi(t,x,j)-v^\pi(t,x,k))} - 1\right) \tag{4.17}$$

with $k = 1, \dots, m$ and terminal condition $v^\pi(T, x, k) = H(x)$.

Obviously, the terminal conditions and the left-hand sides of both PDE-systems coincide. Only the interaction-terms on the right-hand sides are different. We recall that the functions Λ^{kj}, $k \neq j$, describe the $(Q^0, \mathbb{F})$-intensities of η to jump from one state to another and the (constant) functions λ^{kj}, $k \neq j$, describe the corresponding intensities with respect to the objective probability measure P (see Remark 3.5.1 for details).

Note that in both PDEs, the interaction can be interpreted as an attraction between the m individual functions in the respective PDE-system. From this point of view, the attraction in

[1] We note that our framework requires H to be a bounded continuous function while Di Masi et. al. just impose a polynomial growth condition on H.

eq. (4.16) is linear in v and given by the distances between the individual functions multiplied with the P-intensities for the corresponding jumps of η. In comparison, the attraction in eq. (4.17) is non-linear in v and given by the exponentially weighted distances between the individual functions multiplied with the Q^0-intensities for the corresponding jumps of η.
We conclude with an observation that is purely formal: When the risk-aversion α in our model tends to zero, the interaction term on the right-hand side of eq. (4.17) tends to the interaction term of eq. (4.16) – except that λ^{kj} apparently is replaced by $\Lambda^{kj}(t,x)$. However, eq. (4.9) indicates that also $\Lambda^{kj}(t,x)$ tends to $\lambda^{kj}(t,x)$ for $\alpha \downarrow 0$. These formal considerations suggest that in the model from [DiMKR94] the risk-minimization approach constitutes a limiting case of the utility indifference approach for vanishing risk aversion. It seems rewarding to explore the relation between the two approaches in more detail.

4.6 Weather derivatives

Weather derivatives constitute a rather new class of derivatives. The payoff of these contingent claims is linked to the evolution of some weather variable like temperature, rainfall, snowfall or humidity. Companies which face weather related risks in their businesses can hedge some of those risks by using such derivatives.
Recently, these contracts gained much recent attention in the financial industry since a large part of the economy is weather sensitive (cf. [CaW00]). Examples include energy companies, agricultural production, retail business or ski resorts. Overall, the emergence of these new derivatives is caused by the overall trend towards a securitization of risk which results in a convergence of capital markets with classical insurance markets. However, a market in these new products has just started to form.
Although weather derivatives have also gained academic attention recently, the theory is at a very premature stage and there is nothing like a reference model yet. According to [CaW00, D01], in practice these contracts are commonly valued by the expected payoff under the objective measure, discounted at the riskless rate. From a theoretical point of view, this is astounding since such practice seems to ignore the fundamental insights of Black-Scholes and Merton [BlS73, Mer90]) which are at the core of the modern theory of finance.
In principle, incomplete financial market models seem to be appropriate for the valuation and hedging of weather derivatives – basically for two reasons. On the one hand, there is typically little or no liquidity in these contracts (cf. [D01]) which are usually traded over the counter[2]. On the other hand, the underlying of these contingent claims is some weather variable like temperature, rainfall, snowfall or humidity, which is itself typically not tradable

[2] I.e. two counterparts directly negotiate and agree to terms of a contract outside of any exchange or other trade matching system. At present (June 2000), only a very limited market is structured around the exchange-traded futures contracts on the temperature of ten cities in the U.S. at the Chicago Mercantile Exchange (CME).

so far. Therefore, valuation and hedging issues for weather derivatives cannot solely rely on replication arguments, and complete market models – like the Black-Scholes model – are not appropriate.

However, the evolution of the weather clearly affects the prices of some liquid assets in the financial market, e.g. stock prices of weather sensitive companies or prices of agricultural commodities. So, there is already a liquid financial market which is stochastically related to the weather derivative – even if the underlying weather variable itself is not tradable. This should enable an issuer of weather derivatives to hedge his risk from these contracts at least partially. Moreover, the issuer's valuation should be consistent in two aspects. On the one hand, there is some part of the derivative contract which can be hedged and has to be priced in accordance with the financial market prices. But there is also some inevitable risk in the derivative contract which remains with the issuer and should be valued according to his own preferences and attitudes towards risk. However, it is typically not clear how a derivative decomposes in two such parts[3]. Even if this problem were solved, a separate valuation of these qualitatively very different parts by different valuation methods might lead to inconsistencies for the combined valuation. It is therefore desirable to have one approach to the issuer's valuation and hedging problem which does consistently deal with tradable and untradable risks, which is in accordance with well-founded economic valuation principles for both types of risks, and – moreover – still permits for rather explicit and interpretable solutions.

After this motivation, let us consider the incomplete financial market model of Section 4.1 in the context of weather derivatives. The Itô process S models the prices of the tradable financial assets, e.g. the stock prices of some weather sensitive firms. In the present context, the finite-state process η represents some weather variable which is considered as an untradable factor of risk.

The model allows for various stochastic dependencies between S and η. By the nature of the problem, it is clear that η should have an influence on S. At first sight, there seems to be no reason to model also an impact of S on η – provided the weather is not affected by the capital markets. However, there is a subtle distinction between "cause" and "effect" and it is plausible that – for instance – high fuel prices may predict a cold winter. In this sense, the possible mutual dependence between S and η is a useful feature of our modeling framework also in the present context.

Based on the assumption that the issuer has constant absolute risk aversion, Section 4.2 provides an explicit solution in terms of a PDE to both the pricing (4.11) and the hedging problem (4.12) for contingent claims B of the general type (4.5).

Let us consider some examples of contingent claims B which fit into our framework. For instance, let η evolve in the state space $\eta_t \in \{\text{rain, no rain}\}$ and consider a contingent claim

[3] A sceptical reader may try to find and justify such a split in the subsequent examples.

which pays one Euro for every hour of rain at a specific place, i.e.

$$B = \int_0^T 1_{\{\text{rain}\}}(\eta_t)dt$$

with T measured in hours. This claim can be embedded into the framework of Section 4.1 by taking $h = 0$, $f(t, x, k) = 1_{\{\text{rain}\}}(k)$ and $f^{kj} = 0$ for all k, j in (4.5).

At present, the most commonly traded weather variable in the financial industry is temperature (cf. [CaW00, D01]), and the physical temperature is of course not a finite-state variable. However, we can deal with this by taking a discretization of the temperature scale. In this sense, let η model the evolution of the current discrete temperature at a certain place or its deviation from the long-term average, e.g. in the state space

$$\{\ldots, (-5^oC, -2^oC], (-2^oC, +2^oC], (+2^oC, +5^oC], (+5^oC, \infty^oC)\}.$$

In our model, the payoff of the claim B could be related solely to the evolution of the variable η in, say, a season $[0, T]$ by taking the form

$$B = \int_0^T F(\eta_t)\, dt\,; \tag{4.18}$$

or B could also incorporate a combination of weather and market price risk, e.g. by taking the form

$$B = \int_0^T F(t, S_t, \eta_t)\, dt\,. \tag{4.19}$$

Note that both cases can be embedded into the framework of the present section by taking $h = 0$, $f = F$ and $f^{kj} = 0$ for all k, j in (4.5).

For concreteness, consider an energy company which supplies gas to some retail distributor. If the winter is unusually mild, the company will sell less gas. To protect itself against this *volume risk*, the company can buy temperature weather derivatives of the type (4.18) with F monotone increasing. However, the company's earnings are also affected by the commodity *price risk* and if it wants to protect its earnings against the combined volume and price risk a contract of the type (4.19) might serve its needs better.

Again, our main contribution is that our stochastic model permits for various dependencies but nevertheless for fairly explicit solutions.

In comparison, Cao and Wei [CaW00] and Davis [D01] restrict their analysis to a certain class of temperature weather derivatives, which are based on the so-called heating degree days. Overall, their work is more concerned with statistical, economic and econometric questions. These are not the subject matter of the present work.

Moreover, these authors apply a different valuation approach and solely address pricing issues. The valuation approach in [CaW00, D01] follows the idea of "zero marginal rate of substitution" to determine a price $\widehat{p}$ of B (see [D97] and Remark 1.3.6 for details). Basically, the price

$\widehat{p}$ for B is defined such that an agent cannot increase his expected utility by buying or selling an infinitesimal amount of the claim. Intuitively, $\widehat{p}$ is a subjective fair price in the sense that the agent will take a net position of zero in the claim. If the agent is representative for the whole market, $\widehat{p}$ can then be interpreted as an equilibrium price for the claim (cf. [CaW00]). In contrast, our model considers a risk-averse individual investor who acts as a price-taker in the financial market S and wants to issue a contingent claim B. From the perspective of this agent, issuing B for the premium $\widehat{p}$ in general is a bad business since it decreases his expected maximal utility! Actually, our valuation $\pi(B)$ is subjectively fair from the perspective of the issuer in the sense that the issuer feels equally well before and after writing the contract if he receives the premium $\pi(B)$.

Part II

INITIAL ENLARGEMENT OF FILTRATIONS AND THE VALUE OF INVESTMENT INFORMATION

Chapter 5

Quantifying the value of initial investment information

(This chapter is a revised and adapted version of Amendinger, Becherer and Schweizer [ABS00])

In this chapter we consider an investor who trades in a financial market so as to maximize his expected utility of wealth at a prespecified time. The investor faces the opportunity to acquire, in addition to the common information flow $\mathbb{F}$, some extra information $\mathcal{G}$ at a certain cost, e.g. by hiring a good analyst or by doing more research about companies that he can invest in. Acquiring the information $\mathcal{G}$ reduces the initial capital but at the same time enlarges the information flow to $\mathbb{G} = \mathbb{F} \vee \mathcal{G}$ on which the investor can then base his portfolio decisions. Our basic question is then: At what cost is the reduction of the investor's initial wealth offset by the increase in the set of available portfolio strategies? To be more precise, let $u^{\mathbb{F}}(y)$ and $u^{\mathbb{G}}(y)$ be the maximal expected utilities of terminal wealth that can be obtained with initial capital y and portfolio decisions based on the information flow $\mathbb{F}$ and $\mathbb{G}$, respectively. Suppose our investor has initial capital x. In this chapter we concentrate on the *utility indifference value* π of the additional information $\mathcal{G}$, defined as the solution $\pi = \pi(x)$ of the equation

$$u^{\mathbb{F}}(x) = u^{\mathbb{G}}(x - \pi) \ .$$

π can be interpreted as the investor's subjective fair (purchase) value of the additional information $\mathcal{G}$. Our aim is to calculate π for common utility functions in the situation of a complete market and to study the dependence of π on $\mathcal{G}$, on x and on the utility function.

The first rigorous mathematical study of the utility maximization problem under additional initial information is the article of Pikovsky and Karatzas [PiK96]. Subsequent works include Elliott, Geman and Korkie [EGK97], Amendinger, Imkeller and Schweizer [AIS98], Grorud and Pontier [GroP98, GroP01] and Amendinger [A00]. They examined the maximal expected utility under additional information $u^{\mathbb{G}}(x)$ or the expected utility gain from additional information $u^{\mathbb{G}}(x) - u^{\mathbb{F}}(x)$. In comparison to this, the present indifference approach quantifies the informational advantage in terms of money, not utility. Similar methods have been previously applied by various authors to the valuation of options instead of information. For instance, Hodges and Neuberger [HodN89] and Davis, Panas and Zariphopoulou [DPZ93] used utility indifference arguments for the pricing of options in the presence of transaction costs.

The outline of the chapter is as follows. In Section 5.1 we provide the basic notation and discuss the central assumptions underlying this chapter. Basically, we need the existence of the so-called martingale preserving probability measure (MPPM) $\widetilde{P}$ corresponding to the original probability P. This measure was introduced by Föllmer and Imkeller [FI93] and extensively used in [AIS98, GroP98, GroP01, A00]. The main property of the MPPM is that it decouples $\mathbb{F}$ and $\mathcal{G}$ in such a way that $\mathbb{F}$-martingales under P remain $\mathbb{G}$-martingales under $\widetilde{P}$. By means of the MPPM, we then transfer in Section 5.2 the strong predictable representation property for local martingales from $\mathbb{F}$ to the initially enlarged filtration $\mathbb{G}$. This extends prior work of Pikovsky [Pi97], Grorud and Pontier [GroP98] and Amendinger [A99, A00] to the general unbounded semimartingale case and, moreover, contributes a conceptually new proof.

In Section 5.3, standard duality arguments are applied to solve the utility maximization problem in a general complete model when the initial information is non-trivial. We then combine this in Section 5.4 with our martingale representation results to derive the utility indifference value for common utility functions. Finally, we compute closed form solutions for this value in an Itô process model if the additional information basically consists of a noisy signal about the terminal stock price.

5.1 General framework and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and let $T > 0$ be a finite time horizon. For simplicity we assume $\mathcal{F}_0$ to be trivial. The expectation of a random variable Y with respect to a measure Q on $\mathcal{F}$ is denoted by $E_Q[Y]$. If $\mathcal{H}$ is a sub-σ-field of $\mathcal{F}$, the usual conditional expectation $E_Q[Y|\mathcal{H}]$ is well-defined whenever Y^+ or Y^- are Q-integrable. Like Jacod and Shiryaev [JaS87] we use a generalized notion of conditional expectation which is defined for all real-valued variables Y by

$$E_Q[Y|\mathcal{H}] = \begin{cases} E_Q[Y^+|\mathcal{H}] - E_Q[Y^-|\mathcal{H}] & \text{on the set where } E_Q[|Y|\,|\mathcal{H}] < \infty, \\ +\infty & \text{elsewhere.} \end{cases} \tag{5.1}$$

All semimartingales adapted to a complete and right-continuous filtration are taken to have right-continuous paths with left limits. For unexplained terminology from stochastic calculus we refer to Dellacherie and Meyer [DelM80] or He, Wang and Yan [HWY92].

The framework

Let the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be an initial enlargement of $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ by some σ-field $\mathcal{G} \subseteq \mathcal{F}$, i.e.

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{G}\,, \quad t \in [0,T]\,. \tag{5.2}$$

We assume that $\mathcal{G}$ is generated by some random variable G taking values in a general measurable space $(X, \mathcal{X})$, i.e. $\mathcal{G} = \sigma(G)$. This causes no loss of generality since we could always choose $(X, \mathcal{X}) := (\Omega, \mathcal{G})$ and take $G : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{G})$, $\omega \mapsto \omega$. In comparison to related work by Amendinger et al. [AIS98, A99, A00] and Grorud et al. [GroP98, DenGP99, GroP01], we do not assume that the random variable G takes values in a Polish space.
In most parts of this chapter we shall assume that $\mathcal{G}$ satisfies the following decoupling condition.

Assumption 5.1.1 (D):
There exists a probability measure $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G} = \sigma(G)$ are R-independent.

The implications and the significance of this decoupling assumption for our problem will be discussed in detail in Remark 5.1.9. For the moment, let us just emphasize that it is a pure existence condition and should be considered as a condition on $\mathcal{G}$. For our analysis on the effects of additional information in financial markets, the construction of a specific decoupling measure will be a crucial step in the mathematical treatment of the problem.
As the following lemma shows, Assumption 5.1.1 implies that $\mathbb{G}$ satisfies the usual conditions.

Lemma 5.1.2 *Suppose Assumption 5.1.1 (D) is satisfied. Then $\mathbb{G}$ satisfies under P the usual conditions of completeness and right-continuity.*

Proof: Theorem 1 in He and Wang [HW82] shows that if $(\mathcal{F}^1_t)_{t\in[0,T]}$ and $(\mathcal{F}^2_t)_{t\in[0,T]}$ are mutually independent filtrations satisfying the usual conditions then also $(\mathcal{F}^1_t \vee \mathcal{F}^2_t)_{t\in[0,T]}$ satisfies these conditions. Assumption 5.1.1 (D) implies that $\mathbb{F}$ and the constant filtration given by the P-completion of $\mathcal{G}$ are independent under some $R \sim P$. Hence the claim follows. □

We shall see that Assumption 5.1.1 (D) implies the existence of a regular conditional distribution of G given $\mathcal{F}_T$. Moreover, Corollary 5.1.8 will even show that Assumption 5.1.1 (D) is equivalent to

Assumption 5.1.3 (E):
The regular conditional distribution of G given $\mathcal{F}_T$ exists and is P-a.s. equivalent to the law of G, i.e.

$$P[G \in \cdot \,|\, \mathcal{F}_T](\omega) \sim P[G \in \cdot\,] \quad \textit{for } P\textit{-a.a. } \omega \in \Omega\,.$$

Assumption 5.1.3 (E) implies by Théorème V.58 in Dellacherie and Meyer (1980) the existence of a strictly positive $\mathcal{X} \otimes \mathcal{F}_T$-measurable function $p : X \times \Omega \to (0, \infty)$ such that for P-a.a. ω and for all $B \in \mathcal{X}$

$$P[G \in B \mid \mathcal{F}_T](\omega) = \int_B p(x, \omega) P[G \in dx] \,. \tag{5.3}$$

We define $p^G(\omega) := p(G(\omega), \omega)$ and $p^x(\omega) := p(x, \omega)$ for $\omega \in \Omega$, $x \in X$. Note that each p^x is $\mathcal{F}_T$-measurable and p^G is $\mathcal{G}_T$-measurable.

The martingale preserving measure

This section contains results on initially enlarged filtrations that satisfy Assumption 5.1.1 (D) or Assumption 5.1.3 (E). We show that these assumptions are equivalent and ensure the existence of the martingale preserving probability measure. This has several useful consequences. The next result basically shows that we are free to choose the marginals of the decoupling measure R from Assumption 5.1.1 (D).

Lemma 5.1.4 *Let P_1, P_2 be probability measures which are equivalent to P. If Assumption 5.1.1 (D) is satisfied then there is a unique probability measure $P_{\text{dec}}(P_1, P_2) \sim P$ on $\mathcal{F}_T \vee \mathcal{G}$ with the following properties: $P_{\text{dec}}(P_1, P_2) = P_1$ on $\mathcal{F}_T$, $P_{\text{dec}}(P_1, P_2) = P_2$ on $\mathcal{G}$, and $\mathcal{F}_T$ and $\mathcal{G}$ are $P_{\text{dec}}(P_1, P_2)$-independent.*

Proof: By Assumption 5.1.1 (D) there exists $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G}$ are R-independent. Let $Z_T^1 := dP_1/dR|_{\mathcal{F}_T}$, $Z_T^2 := dP_2/dR|_{\mathcal{G}}$ and define $dP_{\text{dec}}(P_1, P_2) := Z_T^1 Z_T^2 \, dR$. For all $A \in \mathcal{F}_T$ and $B \in \mathcal{G}$ we obtain

$$E_{P_{\text{dec}}(P_1,P_2)}[1_A 1_B] = E_R[Z_T^1 1_A] E_R[Z_T^2 1_B] = P_1[A] P_2[B]$$

using the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$. This yields the three required properties of $P_{\text{dec}}(P_1, P_2)$ and it is clear that these properties uniquely determine a probability measure on $\mathcal{F}_T \vee \mathcal{G}$. □

As our subsequent analysis will show, the specific decoupling measure that coincides with P on $\mathcal{G}$ plays a key role in our problem.

Definition 5.1.5 *Let $Q \sim P$. The measure $\widetilde{Q} := P_{\text{dec}}(Q, P)$ is called* martingale preserving probability measure *(corresponding to Q).*

By Lemma 5.1.4, $\widetilde{Q}$ is the unique measure on $\mathcal{F}_T \vee \mathcal{G}$ which has the following three properties:

1. $\widetilde{Q} = Q$ on $\mathcal{F}_T$,
2. $\widetilde{Q} = P$ on $\mathcal{G}$, and

3. $\mathcal{F}_T$ and $\mathcal{G}$ are $\widetilde{Q}$-independent.

An immediate consequence of the equality $Q = \widetilde{Q}$ on $\mathcal{F}_T$ is that any integrability property of an $\mathcal{F}_T$-measurable random variable with respect to Q is inherited by $\widetilde{Q}$. In particular, $(Q, \mathbb{F})$-martingales remain $(\widetilde{Q}, \mathbb{F})$-martingales. But more importantly, the martingale property is even preserved under $\widetilde{Q}$ in $\mathbb{G}$, i.e., under an initial enlargement of the filtration and a simultaneous change to $\widetilde{Q}$. This result was extensively exploited in Amendinger, Imkeller and Schweizer [AIS98] and motivated the terminology *martingale preserving probability measure*. We now summarize some useful properties of $\widetilde{Q}$ for further reference.

Corollary 5.1.6 *Suppose Assumption 5.1.1 (D) is satisfied. Let Q be any probability measure equivalent to P and denote by $\widetilde{Q}$ the corresponding martingale preserving measure. Then*

1. *We have*

$$\mathcal{M}_{(\mathrm{loc})}(Q, \mathbb{F}) = \mathcal{M}_{(\mathrm{loc})}(\widetilde{Q}, \mathbb{F}) \subseteq \mathcal{M}_{(\mathrm{loc})}(\widetilde{Q}, \mathbb{G}) . \tag{5.4}$$

2. *Any $\mathbb{F}$-adapted process L has the same distribution under $\widetilde{Q}$ and Q. If L has in addition $(Q, \mathbb{F})$-independent increments, i.e. $L_t - L_s$ is Q-independent of $\mathcal{F}_s$ for $0 \leq s \leq t \leq T$, then L has also $(\widetilde{Q}, \mathbb{G})$-independent increments and the (non-random) semimartingale characteristics of L are the same for $(Q, \mathbb{F})$ and $(\widetilde{Q}, \mathbb{G})$. In particular a $(Q, \mathbb{F})$-Lévy process (Brownian motion, Poisson process) is also a $(\widetilde{Q}, \mathbb{G})$-Lévy process (Brownian motion, Poisson process).*

3. *Every $(Q, \mathbb{F})$-semimartingale is a $(Q, \mathbb{G})$-semimartingale.*

4. *Let S be a multidimensional $(P, \mathbb{F})$-semimartingale. Then an $\mathbb{F}$-predictable process H is S-integrable with respect to $\mathbb{F}$ if and only if H is S-integrable with respect to $\mathbb{G}$. Moreover, the stochastic integrals of H with respect to S coincide for both filtrations.*

By part 4, we do not have to differentiate between stochastic integrals with respect to $\mathbb{F}$ and $\mathbb{G}$ in our setting – although the definition of the stochastic integral a priori involves the filtration.

Proof: 1.: This was shown in Amendinger et al. [AIS98] (proof of Theorem 2.5).
2.: The first statement is clear since $\widetilde{Q} = Q$ on $\mathcal{F}_T$. If the $\mathcal{F}_T$-measurable random variable $L_t - L_s$ is Q-independent from $\mathcal{F}_s$, it is also $\widetilde{Q}$-independent from $\mathcal{F}_s$ since $Q = \widetilde{Q}$ on $\mathcal{F}_T$; therefore it is also $\widetilde{Q}$-independent from $\mathcal{G}_s = \mathcal{F}_s \vee \mathcal{G}$ since $\mathcal{G}$ is $\widetilde{Q}$-independent from $\mathcal{F}_T$. The distribution of a process with independent increments is determined by its characteristics which are unique and non-random (see [JaS87], Th. II.5.2). Hence, these characteristics remain unaltered if we go from $(Q, \mathbb{F})$ to $(\widetilde{Q}, \mathbb{G})$. This implies in particular that a $(Q, \mathbb{F})$-Lévy process (Brownian motion, Poisson process) remains a $(\widetilde{Q}, \mathbb{G})$-Lévy process (Brownian motion, Poisson process) since these processes are determined by their characteristics.

3.: This follows from (5.4) and $\widetilde{Q} \sim Q$.
4.: As stochastic integrals are unaffected by an equivalent change of measure, we may consider the problem under the measure R (or $\widetilde{Q}$). Under that measure, every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale, i.e. the remark following Proposition 8 in Jacod [Ja80] applies. Hence, the claim follows readily by Theorem 7 and Proposition 8 from Jacod [Ja80]. □

The next result demonstrates how the $\mathcal{G}_T$-density of the martingale preserving measure $\widetilde{Q}$ with respect to the 'true world'-measure P can be constructed from the $\mathcal{F}_T$-density of Q by using p^G.

Proposition 5.1.7 *Let Q be a probability measure equivalent to P and denote by Z_T its $\mathcal{F}_T$-density with respect to P. If Assumption 5.1.3 (E) is satisfied then*

$$\frac{d\widetilde{Q}}{dP} = \frac{Z_T}{p^G} \,. \tag{5.5}$$

The proof of this result uses (5.3) to show that Z_T/p^G is a density with respect to P. Then one can argue almost exactly as in the proof of Proposition 2.3 in Amendinger et al. [AIS98] where a conditional density process appears. But introducing this process and explaining its relation to our p^G would take more time than to give a direct proof in our setting. Hence we opt for the latter because the conditional density process is not needed for our later results.

Proof: Using (5.3) we obtain for $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$

$$\begin{aligned} E_P\left[I_A I_{\{G\in B\}} \frac{Z_T}{p^G} \Big| \mathcal{F}_T\right] &== Z_T I_A E_P\left[I_{\{G\in B\}} \frac{1}{p^G} \Big| \mathcal{F}_T\right] \\ &= Z_T I_A \int_B \frac{1}{p^x} p^x P[G \in dx] = Z_T I_A P[G \in B] \,, \quad A \in \mathcal{F}_T,\, B \in \mathcal{X}. \end{aligned} \tag{5.6}$$

Taking $A = \Omega$ and $B = X$, we obtain that Z_T/p^G is like Z_T a P-density and (5.5) defines a probability measure $\widetilde{Q} \sim P$ because $Z_T/p^G > 0$. From

$$\widetilde{Q}[A \cap \{G \in B\}] = E_P\left[Z_T I_A P[G \in B]\right] = Q[A] P[G \in B] \,, \quad A \in \mathcal{F}_T,\, B \in \mathcal{X}, \tag{5.7}$$

follows that $\widetilde{Q}$ is the martingale preserving probability measure corresponding to Q. □

Proposition 5.1.7 allows us to clarify the relation between Assumption 5.1.1 (D) and Assumption 5.1.3 (E) as follows:

Corollary 5.1.8 *Assumption 5.1.1 (D) and Assumption 5.1.3 (E) are equivalent.*

A version of this result was proven in Grorud and Pontier ([GroP98], Lemmata 3.3 and 3.4). For the reader's convenience, we state the proof in our general framework.

Proof:
By choosing $R = \widetilde{P}$ as the martingale preserving measure corresponding to P, i.e. $dR := 1/p^G\, dP$, Proposition 5.1.7 yields that Assumption 5.1.3 (E) implies Assumption 5.1.1 (D). Conversely, assume that there exists a measure $R \sim P$ such that $\mathcal{F}_T$ and $\mathcal{G} = \sigma(G)$ are R-independent. Since the distributions of (ω, G) on $\mathcal{F}_T \otimes \mathcal{X}$ under P and R are equivalent, there is a strictly positive Radon-Nikodým derivative f of $P \circ (\omega, G)^{-1}|_{\mathcal{F}_T \otimes \mathcal{X}}$ with respect to $R \circ (\omega, G)^{-1}|_{\mathcal{F}_T \otimes \mathcal{X}} = R|_{\mathcal{F}_T} \otimes R[G \in \cdot]$; the last equality uses the decoupling property of R. The function

$$f(x|\omega) := \frac{f(\omega, x)}{\int_X f(\omega, x)\, R[G \in dx]}\,, \quad x \in X,\, \omega \in \Omega,$$

is strictly positive and it is straightforward to verify that for $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$

$$E_P\left[1_A P[G \in B|\mathcal{F}_T]\right] = E_P\left[1_A 1_{\{G \in B\}}\right] = E_P\left[1_A \int_B f(x|\cdot)\, R[G \in dx]\right].$$

Hence a regular conditional distribution of G under P given $\mathcal{F}_T$ exists and is given by

$$P[G \in B|\mathcal{F}_T](\omega) = \int_B f(x|\omega)\, R[G \in dx]\,, \quad B \in \mathcal{X}\,, \omega \in \Omega.$$

This implies that for P-a.a. ω we have $P[G \in \cdot\,|\mathcal{F}_T](\omega) \sim R[G \in \cdot\,] \sim P[G \in \cdot\,]$. □

Remark 5.1.9 1. Let us discuss the relevance of Assumption 5.1.1 for our problem. Technically, it allows us to work on an *implicit* product model by switching from P to the decoupling measure R. This largely facilitates the analysis of our problem. Loosely speaking, we can argue under R "as if" $(\Omega, \mathcal{F}_T \vee \mathcal{G}, R)$ equals $(\Omega \times X, \mathcal{F}_T \otimes \mathcal{X}, R|_{\mathcal{F}_T} \times R[G \in \cdot])$. One might ask if one should then not work right from the start with an *explicit* product model. But for our applications of the present results, the answer is negative because neither the decoupling measure R nor the explicit product structure are given a priori. Instead, a model of a financial market is specified by an adapted asset price process S on a probability space $(\Omega, \mathcal{F}, P)$ where P is the "true world"-probability measure. We are interested in situations where $\mathcal{G}$ provides some relevant additional information about S and, hence, $\mathcal{F}_T$ and $\mathcal{G}$ are not P-independent. The examples in Section 5.5 illustrate that there is a priori no given decoupling measure R – except in the trivial case when $\mathcal{G}$ is P-independent from $\mathcal{F}_T$. In fact, the construction of a specific decoupling measure is a key step to quantify the value that the additional information has for the investor. Furthermore, it is interesting to compare the effects from *different* additional informations $\mathcal{G}$. In Section 5.5 for instance, we quantify the effects of noisy signals about the future stock price for varying levels of noise. Each $\mathcal{G}$ corresponds to a different decoupling and the only reasonable way to deal with that is to exploit the benefits of the *implicit* product structures.

2. In the next section, we extend a main result from Amendinger [A99, A00]. A contribution of the present section is that it provides a simplified approach to the martingale preserving probability measure. To explain this, let us compare our framework with the one in Amendinger [A00]. Assumption 2.1 in [A00] supposes that there are suitable versions of the conditional distributions of G such that

$$P[G \in \cdot|\mathcal{F}_t](\omega) \sim P[G \in \cdot] \quad \text{for } P\text{-a.a. } \omega \text{ and } t \in [0,T]\,. \tag{5.8}$$

   Clearly, this condition implies our Assumption 5.1.3. The opposite implication seems plausible but is not obvious since (5.8) requires a.s. equivalence *simultaneously* for all $t \in [0,T]$.

   The equivalence condition (5.8) was introduced by Föllmer and Imkeller [FI93]. It is a strengthening of a similar absolute-continuity condition by Jacod [Ja85] in the theory of enlargements of filtrations and guarantees the existence of a strictly positive conditional density process $(p_t^G)_{t\in[0,T]}$. In [AIS98, A99, A00] the martingale preserving measure is defined by means of this process, but this approach requires technical measurability considerations. Let us just note here that the relation to our setup is given by $p_T^G = p^G$. The present section shows that the use of the process (p_t^G) can be completely avoided and also presents simpler proofs for some other results from [A00] (compare for instance Lemma 5.1.2 and part 4 of Corollary 5.1.6 with the corresponding results in [A00]).

3. In comparison to related work by Amendinger et al. [AIS98, A99, A00] and Grorud and Pontier [GroP98, DenGP99, GroP01], we do not assume that G takes values in a Polish space. ◇

5.2 Strong predictable representation property

Throughout this section let $S = (S^1, \dots, S^d)^{tr}$ be a d-dimensional $\mathbb{F}$-semimartingale. Our aim is to show that under Assumption 5.1.1 (D) the martingale representation property of S with respect to $\mathbb{F}$ and some measure Q implies the same property with respect to the initially enlarged filtration $\mathbb{G}$ and the corresponding martingale preserving measure $\widetilde{Q}$.

We give a conceptually new proof that yields the result without any further assumptions on the semimartingale S – thereby extending a result of Amendinger [A00] where S is assumed to be locally in $\mathcal{H}^2$ to full generality.

First, we recall a version of a classical martingale representation result. For $d = 1$ this is almost Theorem 13.9 in He, Wang and Yan [HWY92]; the multidimensional case can be proved as in Jacod [Ja79], XI.1.a, with obvious modifications for the situation of a non-trivial initial σ-field.

Proposition 5.2.1 *Suppose the filtration $\mathbb{H}$ satisfies the usual conditions and there is a probability measure $Q^{\mathbb{H}} \sim P$ such that $S \in \mathcal{M}_{\mathrm{loc}}(Q^{\mathbb{H}}, \mathbb{H})$. Denote*

$$\Gamma^{\mathbb{H}} := \left\{ Q \sim Q^{\mathbb{H}} \,\Big|\, \frac{dQ}{dQ^{\mathbb{H}}} \text{ is } \mathcal{H}_T\text{-measurable, } Q = Q^{\mathbb{H}} \text{ on } \mathcal{H}_0,\ S \in \mathcal{M}_{\mathrm{loc}}(Q, \mathbb{H}) \right\}. \tag{5.9}$$

Then the following statements are equivalent:

1. $\Gamma^{\mathbb{H}} = \{Q^{\mathbb{H}}\}$.
2. *The set $\mathcal{M}_{0,\mathrm{loc}}(Q^{\mathbb{H}}, \mathbb{H})$ of local $(Q^{\mathbb{H}}, \mathbb{H})$ martingales null at 0 is equal to the set*

   $$\left\{ \theta \cdot S \,\Big|\, \theta \text{ is } S\text{-integrable w.r.t. } (Q^{\mathbb{H}}, \mathbb{H}) \text{ in the sense of local martingales} \right\}$$

   of stochastic integrals with respect to S.

We say that S has the *strong predictable representation property with respect to* $(Q^{\mathbb{H}}, \mathbb{H})$ *(for short* $(Q^{\mathbb{H}}, \mathbb{H})$*-PRP)* if one of these statements is valid.
The following lemma provides us with a nice measurable version of a conditional expectation which will be crucial for the subsequent martingale representation theorem.

Lemma 5.2.2 *Suppose $\mathcal{F}_T$ and $\mathcal{G}$ are R-independent under some measure $R \sim P$. Let $f : \Omega \times X \to \mathbb{R}$ be $\mathcal{F}_T \otimes \mathcal{X}$-measurable, $f(\cdot, G) \in L^1(R)$ and $t \leq T$. Then there exists an $\mathcal{F}_t \otimes \mathcal{X}$-measurable function $g : \Omega \times X \to \mathbb{R}$ such that*

$$g(\cdot, G) \quad \text{is a version of} \quad E_R\left[f(\cdot, G) | \mathcal{F}_t \vee \mathcal{G}\right] \tag{5.10}$$

and for $R[G \in \cdot]$-a.a. $x \in X$ we have $f(\cdot, x) \in L^1(R)$ and

$$g(\cdot, x) \quad \text{is a version of} \quad E_R\left[f(\cdot, x) | \mathcal{F}_t\right]. \tag{5.11}$$

Under the assumptions of this lemma, there are measurable versions $g(\cdot, G)$ and $g(\cdot, x)$, $x \in X$, of the conditional expectations and choosing these versions yields

$$E_R[f(\cdot, G) | \mathcal{F}_t \vee \mathcal{G}] = E_R[f(\cdot, x) | \mathcal{F}_t] \Big|_{x=G}$$

since $g(\cdot, G) = g(\cdot, x)|_{x=G}$. This will be used in the subsequent theorem.
Proof: (of Lemma 5.2.2)
Let us first show the claim for functions of the form $f(\omega, x) = 1_A(\omega) 1_B(x)$ with $A \in \mathcal{F}_T$ and $B \in \mathcal{X}$. We fix a finite version h of $E_R[1_A | \mathcal{F}_t]$ and define $g(\omega, x) := 1_B(x) h(\omega)$. Then we have

$$E_R[1_A 1_B(x) | \mathcal{F}_t] = 1_B(x) E_R[1_A | \mathcal{F}_t] = g(\cdot, x), \quad x \in X,$$

and the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$ yields

$$E_R[1_A 1_B(G) 1_D] = E_R[E_R[1_A | \mathcal{F}_t] 1_B(G) 1_D] = E_R[g(\cdot, G) 1_D]$$

for any $\mathcal{F}_t \vee \mathcal{G}$-measurable $D \in \mathcal{F}_t \vee \mathcal{G}$. Hence, both (5.10) and (5.11) hold for such functions f. By monotone class arguments, the claim of Lemma 5.2.2 therefore holds for all bounded $\mathcal{F}_T \otimes \mathcal{X}$-measurable functions f, and (5.11) then even holds for all $x \in X$ instead of only $R[G \in \cdot]$-a.a. $x \in X$.

Let us now consider the case when f is unbounded but $f(\cdot, G)$ is R-integrable. Using the R-independence of $\mathcal{F}_T$ and $\mathcal{G}$ and Fubini's theorem, we conclude that $f(\cdot, x) \in L^1(R)$ for $R[G \in \cdot]$-a.a. $x \in \mathcal{X}$ since $f(\cdot, G) \in L^1(R)$ by hypothesis.

Let $f^n(\omega, x) := \min(\max(f(\omega, x), -n), n)$, $n \in \mathbb{N}$. By the foregoing considerations, there are $\mathcal{F}_t \otimes \mathcal{X}$-measurable functions g^n such that

$$E_R\left[f^n(\cdot, G)|\mathcal{F}_t \vee \mathcal{G}\right] = g^n(\cdot, G) \tag{5.12}$$

$$E_R\left[f^n(\cdot, x)|\mathcal{F}_t\right] = g^n(\cdot, x), \quad x \in X. \tag{5.13}$$

Define $g(\omega, x) := \liminf_{n\to\infty} g^n(\omega, x)$ for (ω, x) where this limit is finite and $g(\omega, x) := 0$ elsewhere.

By the dominated convergence theorem for conditional expectations, the left hand sides (LHSs) of (5.12) converge for $n \to \infty$ a.s. to the LHS of (5.10). Analogously, for $R[G \in \cdot]$-a.a. x the random variable $g(\cdot, x)$ is in $L^1(R)$ and the LHSs of (5.13) converge a.s. to the LHS of (5.11). Notifying that these limits are a.s. finite, we obtain that equations (5.10) and (5.11) hold. This yields the claim. □

Now we are ready to prove

Theorem 5.2.3 *Suppose Assumption 5.1.1 (D) is satisfied and S has the strong predictable representation property with respect to $(Q^{\mathbb{F}}, \mathbb{F})$ for some $Q^{\mathbb{F}} \sim P$. Let $Q^{\mathbb{G}} = \widetilde{Q^{\mathbb{F}}}$ denote the martingale preserving probability measure corresponding to $Q^{\mathbb{F}}$. Then S has the strong predictable representation property with respect to $(Q^{\mathbb{G}}, \mathbb{G})$. For short: $\Gamma^{\mathbb{F}} = \{Q^{\mathbb{F}}\}$ implies $\Gamma^{\mathbb{G}} = \{Q^{\mathbb{G}}\}$, or*

$$(Q^{\mathbb{F}}, \mathbb{F})\text{-}PRP \quad \textit{implies} \quad (Q^{\mathbb{G}}, \mathbb{G})\text{-}PRP. \tag{5.14}$$

Proof:

Let $Q' \in \Gamma^{\mathbb{G}}$. Without loss of generality we can assume that $dQ'/dQ^{\mathbb{G}} \in L^\infty(Q^{\mathbb{G}})$ (see He, Wang and Yan (1992), Th. 13.9). We prove that $Q' = Q^{\mathbb{G}}$ and thus show the claim by Proposition 5.2.1. Define $Z'_t := \left(dQ'/dQ^{\mathbb{G}}\right)\big|_{\mathcal{G}_t}$, $t \in [0,T]$. For all $t \in [0,T]$ the density Z'_t is $\mathcal{F}_t \vee \sigma(G)$-measurable and thus of the form $Z'_t(\cdot) = z_t(\cdot, G(\cdot))$ for an $\mathcal{F}_t \otimes \mathcal{X}$-measurable function $z_t(\omega, x)$. If ν denotes the distribution of G under $Q^{\mathbb{G}}$, the process $(z_t(\cdot, x))_{t\in[0,T]}$ is RCLL in t for ν-a.a. x because $(Z'_t(\cdot, G))_{t\in[0,T]}$ is an RCLL-process. Since $S \in \mathcal{M}_{\text{loc}}(Q^{\mathbb{F}}, \mathbb{F})$ there is a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times such that for all n the stopped process S^{τ_n} is a uniformly integrable $(Q^{\mathbb{F}}, \mathbb{F})$-martingale, hence also a uniformly integrable $(Q^{\mathbb{G}}, \mathbb{G})$-martingale by Corollary 5.1.6. Because Z' is bounded we conclude that the local $(Q^{\mathbb{G}}, \mathbb{G})$-martingale $Z'S^{\tau_n}$ is of class (D) under $Q^{\mathbb{G}}$ and therefore a uniformly integrable $(Q^{\mathbb{G}}, \mathbb{G})$-martingale. Expectations with respect to $Q^{\mathbb{G}}$ are denoted by $E^{\mathbb{G}}[\cdot]$. Lemma 5.2.2 then implies for $t \in [0,T]$ and $n \in \mathbb{N}$

$$\begin{aligned} z_t(\cdot, G)S_t^{\tau_n} &= E^{\mathbb{G}}\left[z_T(\cdot, G)S_T^{\tau_n} \mid \mathcal{F}_t \vee \sigma(G)\right] \\ &= E^{\mathbb{G}}\left[z_T(\cdot, x)\, S_T^{\tau_n} \mid \mathcal{F}_t\right]\Big|_{x=G} . \end{aligned}$$

Since $z_t(\cdot,\cdot)$ is $\mathcal{F}_t \otimes \mathcal{X}$-measurable and $\mathcal{F}_T$ and $\sigma(G)$ are $Q^{\mathbb{G}}$-independent this implies that $\left(Q^{\mathbb{G}} \otimes \ \nu\right)$-a.e.

$$z_t(\omega, x)S_t^{\tau_n}(\omega) = E^{\mathbb{G}}\left[z_T(\cdot, x)\, S_T^{\tau_n} \mid \mathcal{F}_t\right](\omega) . \tag{5.15}$$

Analogously we obtain for each $t \in [0,T]$ that $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e.

$$z_t(\omega, x) = E^{\mathbb{G}}[z_T(\cdot, x) \mid \mathcal{F}_t](\omega) . \tag{5.16}$$

Thus (5.15) and (5.16) hold $\left(Q^{\mathbb{G}} \otimes \nu\right)$-a.e. simultaneously for all rational $t \in [0,T]$ and then by right-continuity in t of $z_t(\cdot, x)$ and $z_t(\cdot, x)S_t^{\tau_n}$ even simultaneously for all $t \in [0,T]$. Hence both $(z_t(\cdot, x))_{t\in[0,T]}$ and $(z_t(\cdot, x)S^{\tau_n})_{t\in[0,T]}$ are $(Q^{\mathbb{G}}, \mathbb{F})$-martingales and thus $(Q^{\mathbb{F}}, \mathbb{F})$-martingales by Corollary 5.1.6 for ν-a.a. x and $n \in \mathbb{N}$. But now $(Q^{\mathbb{F}}, \mathbb{F})$-PRP implies by Proposition 5.2.1 for ν-a.a. x that $z_T(\cdot, x) = 1$ $Q^{\mathbb{G}}$-a.s. and the $Q^{\mathbb{G}}$-independence of $\mathcal{F}_T$ and $\sigma(G)$ then yields $Z'_T = z_T(\cdot, G(\cdot)) = 1$. □

The preceding theorem is a generalization of Theorem 4.7 in Amendinger (2000), where S is assumed to be locally in $\mathcal{H}^2(Q^{\mathbb{F}}, \mathbb{F})$. But the proof is different: In Amendinger [A00] an L^2-approximation argument shows that any (local) $(Q^{\mathbb{G}}, \mathbb{G})$-martingale null at 0 can be represented as a stochastic integral of a $\mathbb{G}$-predictable integrand with respect to S if this is possible for (local) $(Q^{\mathbb{F}}, \mathbb{F})$-martingales and $\mathbb{F}$-predictable integrands. Our argument proves that the uniqueness of the equivalent $\mathbb{F}$-martingale measure for S implies the uniqueness of the equivalent $\mathbb{G}$-martingale measure in the sense of Proposition 5.2.1. So the result is obtained by working in Proposition 5.2.1 on level 1 instead of level 2.
We refer to Amendinger [A00] for a detailed discussion of related works [Pi97, GroP98, DenGP99] on martingale representation theorems in more special cases of Brownian and Brownian-Poissonian models. In a model of the latter type, Grorud and Pontier [GroP01] use a weak martingale representation result to describe the sets of equivalent local martingale measures for S with respect to the filtration $\mathbb{F}$ and $\mathbb{G}$ when these sets are not singletons.

5.3 Utility maximization with non-trivial initial information

In this section let $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ be a generic filtration describing the information flow of an investor who maximizes his expected utility by dynamically trading in a complete continuous-time security market with several risky assets. The results in this section are similar but more

general than those by Amendinger [A99, A00] who inspired much of our work: For one thing, our setting covers utility functions that are defined on all of $\mathbb{R}$ and not just those on $\mathbb{R}^+$. By our more general martingale representation theorem, we also can omit a square integrability condition on the asset prices that appears in Amendinger ([A00], end of Section 2).
We assume that $\mathbb{H}$ satisfies the usual conditions and emphasize that $\mathcal{H}_0$ need not be trivial. The results are applied in Section 5.4 to the ordinary information flow $\mathbb{F}$ and the initially enlarged information flow $\mathbb{G}$ in order to quantify the informational advantage that the investor receives from the additional information $\mathcal{G}$.
Let the discounted price process of the risky assets be given by a d-dimensional $\mathbb{H}$-semimartingale $S = (S^1, \dots, S^d)^{tr}$. We assume throughout this section that the financial market given by S is $\mathbb{H}$-complete and free of arbitrage in the following sense:

Assumption 5.3.1 ($\mathbb{H}$-C): *There is a* unique *probability measure $Q^{\mathbb{H}} \sim P$ with $dQ^{\mathbb{H}}/dP \in \mathcal{H}_T$ and $Q^{\mathbb{H}} = P$ on $\mathcal{H}_0$ such that $S \in \mathcal{M}_{\text{loc}}(Q^{\mathbb{H}}, \mathbb{H})$. In other words: $\Gamma^{\mathbb{H}} = \{Q^{\mathbb{H}}\}$.*

We denote by $Z^{\mathbb{H}} = (Z_t^{\mathbb{H}})_{t\in[0,T]}$ the $\mathbb{H}$-density process of $Q^{\mathbb{H}}$ with respect to P and by $E^{\mathbb{H}}[\cdot]$ the expectation with respect to $Q^{\mathbb{H}}$.

A measure Q is called *equivalent local martingale measure for S* if $S \in \mathcal{M}_{\text{loc}}(Q, \mathbb{H})$ and $Q \sim P$. The existence of an equivalent local martingale measure for the discounted asset price process is related to the absence of arbitrage (see Delbaen and Schachermayer [DeS98]). More precisely, it is slightly stronger than the condition of *No Free Lunch with Vanishing Risk*. Proposition 5.2.1 shows that under Assumption 5.3.1 S has the predictable representation property with respect to $\left(Q^{\mathbb{H}}, \mathbb{H}\right)$. Such a financial market is called $\mathbb{H}$-*complete* because every contingent future payoff can be replicated in the following sense: For every $H \in L^1\left(Q^{\mathbb{H}}, \mathcal{H}_T\right)$ there is a process ϑ in the space $L(S, \mathbb{H})$ of $\mathbb{R}^d$-valued $\mathbb{H}$-predictable processes integrable with respect to S, such that $\vartheta \cdot S := \int \vartheta\, dS$ is a $\left(Q^{\mathbb{H}}, \mathbb{H}\right)$-martingale and

$$H = E^{\mathbb{H}}[H \mid \mathcal{H}_0] + (\vartheta \cdot S)_T = E^{\mathbb{H}}[H \mid \mathcal{H}_0] + \int_0^T \vartheta\, dS\,. \tag{5.17}$$

Remark 5.3.2 For the informed reader, we mention here that our results could also be proven under the more general assumption that S satisfies NFLVR and that there is an equivalent σ-martingale measure for S that is unique in the sense of Assumption 5.3.1. Since this only involves rewriting some arguments without providing additional insight, we stick with Assumption 5.3.1 here. In fact, NFLVR is satisfied if and only if $S = \varphi \cdot M$ for a strictly positive one-dimensional predictable process φ and an $\mathbb{R}^d$-valued process M which is a local $(Q, \mathbb{H})$-martingale for some $Q \sim P$; see Delbaen and Schachermayer [DeS98]. We can weaken Assumption 5.3.1 by assuming that S satisfies NFLVR and that there is a unique $Q^{\mathbb{H}} \sim P$ on $\mathcal{H}_T$ with $Q^{\mathbb{H}} = P$ on $\mathcal{H}_0$ such that $M \in \mathcal{M}_{\text{loc}}(Q^{\mathbb{H}}, \mathbb{H})$. Since $\vartheta \in L(S, \mathbb{H})$ is equivalent to

$\vartheta\varphi \in L(M, \mathbb{H})$ and $\vartheta \cdot S = (\vartheta\varphi) \cdot M$, we can rewrite each stochastic integral with respect to S as a stochastic integral with respect to M and vice versa. A careful reading shows that the proofs in the sequel are based on martingale arguments applied to stochastic integrals with respect to S, and that we could apply the same arguments to the rewritten stochastic integrals with respect to M. ◇

To define the investor's optimization problem we first introduce admissible trading strategies.

Definition 5.3.3 *$\vartheta \in L(S, \mathbb{H})$ is called an* admissible strategy *if*

$$(\vartheta \cdot S)_t = E^{\mathbb{H}}\left[(\vartheta \cdot S)_T \middle| \mathcal{H}_t\right] \quad \textit{for all } t \in [0, T]. \tag{5.18}$$

The set of admissible strategies is denoted by $\Theta^{\mathbb{H}}_{Adm}$.

Note that (5.18) requires that the right-hand side is well-defined and finite; this is satisfied for any $t \in [0, T]$ if and only if $E^{\mathbb{H}}[|(\vartheta \cdot S)_T|\,|\mathcal{H}_0] < \infty$. This property is part of the definition. However, our use of generalized conditional expectations as in (5.1) means that $(\vartheta \cdot S)_T$ need not be $Q^{\mathbb{H}}$-integrable. We will discuss Definition 5.3.3 in more detail after Lemma 5.3.5.
Intuitively, ϑ^i_t represents the number of shares of risky asset i held by an investor at time t. The *wealth process* of a strategy $\vartheta \in \Theta^{\mathbb{H}}_{Adm}$ with initial capital x is then given by

$$V_t = x + (\vartheta \cdot S)_t = x + \int_0^t \vartheta \, dS\,, \quad 0 \le t \le T\,.$$

In particular, strategies are self-financing. A *utility function* is a strictly increasing, strictly concave and continuously differentiable function $U : (\ell, \infty) \to \mathbb{R}$, $\ell \in [-\infty, 0]$, which satisfies $\lim_{x\uparrow\infty} U'(x) = 0$ and $\lim_{x\downarrow\ell} U'(x) = +\infty$. We use the convention that $U(x) := -\infty$ for $x \le \ell$. Observe that this setting covers utility functions both on all of $\mathbb{R}$ and on $(0, \infty)$.
An investor with information flow $\mathbb{H}$ and initial capital $x > \ell$ who wants to maximize his expected utility from terminal wealth has to solve the optimization problem

$$u^{\mathbb{H}}(x) := \sup_{V \in \mathcal{V}^{\mathbb{H}}(x)} E_P[U(V_T)] \tag{5.19}$$

on the set $\mathcal{V}^{\mathbb{H}}(x) := \left\{ V \,\middle|\, V = x + \vartheta \cdot S,\ \vartheta \in \Theta^{\mathbb{H}}_{Adm},\ U^-(V_T) \in L^1(P) \right\}$. $u^{\mathbb{H}}$ is called the *indirect utility function.* We also consider the related optimization problem

$$\operatorname*{ess\,sup}_{V \in \mathcal{V}^{\mathbb{H}}(x)} E_P[U(V_T)|\mathcal{H}_0]. \tag{5.20}$$

If this supremum is attained by some element of $\mathcal{V}^{\mathbb{H}}(x)$, then this ω-wise optimum is also a solution to the optimization problem (5.19).

Remark 5.3.4 Note that there is a slight error in Amendinger [A99] (ch. 4.1, p. 64, Remark 1) at this point. The supremum there is taken over a set of strategies for which the expectation of utility from consumption and terminal wealth might be undefined. This error comes from the fact that in Amendinger [A99] the suprema for the unconditional optimization and the corresponding conditional optimization problem are taken over different sets of strategies. Therefore the solution for the conditional optimization problem need not be a solution of the unconditional optimization problem. In order to avoid this problem here, we take the suprema in both optimization problems (5.19) and (5.20) over the same set $\mathcal{V}^{\mathbb{H}}(x)$. $\diamond$

Our goal in the next result is to show that our definition of admissibility is quite natural in the present general context.

Lemma 5.3.5 *1. A measure $Q \sim P$ is a local martingale measure for S if and only if its density with respect to P on $\mathcal{H}_T$ is of the form $\left(\frac{dQ}{dP}\Big|_{\mathcal{H}_0}\right) Z^{\mathbb{H}}_T$.*

2. A process V satisfies $V_t = E^{\mathbb{H}}[V_T|\mathcal{H}_t]$ for all $t \in [0,T]$ if and only if V is a martingale with respect to some equivalent local martingale measure Q for S. In particular, $\vartheta \in L(S,\mathbb{H})$ is an admissible strategy if and only if there exists an equivalent local martingale measure Q for S such that $\vartheta \cdot S$ is a $(Q,\mathbb{H})$-martingale.

3. If the lower bound ℓ for the domain of the utility function U is finite, we could replace $\Theta^{\mathbb{H}}_{Adm}$ by $\{\vartheta \in L(S,\mathbb{H})\,|\,\vartheta \cdot S \geq c \text{ for some } c \in \mathbb{R}\}$ without changing either the supremum in (5.19) (or (5.20)) or the optimal solution, if this exists.

Proof: 1: The product YM of a (local) $(P,\mathbb{H})$-martingale M and a finite $\mathcal{H}_0$-measurable random variable Y is a local $(P,\mathbb{H})$-martingale. To see this, note that with $M \in \mathcal{M}(P,\mathbb{H})$ $\tau^n := T1_{\{Y\leq n\}}$ is a sequence of localizing stopping times such that $1_{\{\tau^n>0\}}YM^{\tau^n}$ is a $(P,\mathbb{H})$-martingale. This will be used in the sequel. The "if"-part then follows from the observation that $dQ/dP|_{\mathcal{H}_0}\,Z^{\mathbb{H}}_T$ is a P-density and $dQ/dP|_{\mathcal{H}_0}\,Z^{\mathbb{H}}S$ is a local $(P,\mathbb{H})$-martingale. For the "only if"-part let $Q \sim P$ with $S \in \mathcal{M}_{\text{loc}}(Q,\mathbb{H})$. Let Z denote the $\mathbb{H}$-density process of Q with respect to P. From $Z_0\frac{Z}{Z_0}S \in \mathcal{M}_{\text{loc}}(P,\mathbb{H})$ follows $\frac{Z}{Z_0}S \in \mathcal{M}_{\text{loc}}(P,\mathbb{H})$. By Proposition 5.2.1, Assumption 5.3.1 ($\mathbb{H}$-C) then implies $\frac{Z}{Z_0} = Z^{\mathbb{H}}$, i.e. $\frac{dQ}{dP}\Big|_{\mathcal{H}_T} = Z_0 Z^{\mathbb{H}}_T$.
2: For the "only if"-part assume $V_t = E^{\mathbb{H}}[V_T|\mathcal{H}_t]$, $t \in [0,T]$, and recall (5.1). It follows that $E^{\mathbb{H}}[|V_T|\,|\mathcal{H}_0] < \infty$, $Y := 1 \wedge \left(1/E^{\mathbb{H}}[|V_T|\,|\mathcal{H}_0]\right)$ is $Q^{\mathbb{H}}$-integrable and $dQ := \left(Y/E^{\mathbb{H}}[Y]\right) dQ^{\mathbb{H}}$ defines a measure Q equivalent to P. By part 1 we obtain $S \in \mathcal{M}_{\text{loc}}(Q,\mathbb{H})$. For the converse let $V \in \mathcal{M}(Q,\mathbb{H})$ for some equivalent local martingale measure Q for S. Let Z denote the $\mathbb{H}$-density process of Q with respect to P. Then part 1 implies $Z/Z_0 = Z^{\mathbb{H}}$ and Bayes' formula leads to $E^{\mathbb{H}}[V_T|\mathcal{H}_t] = E_Q[V_T|\mathcal{H}_t] = V_t$, $t \in [0,T]$. By applying this result to $V := \vartheta \cdot S$ with $\vartheta \in L(S,\mathbb{H})$ we obtain the second part of the claim.

3: We first show that for any $\psi \in L(S,\mathbb{H})$ with $\psi \cdot S$ uniformly bounded from below, there exists $\vartheta \in \Theta^{\mathbb{H}}_{Adm}$ such that $(\vartheta \cdot S)_T \geq (\psi \cdot S)_T$. By Corollaire 3 in Ansel and Stricker [AnS94] (which is also valid for non-trivial initial σ-fields) $\psi \cdot S$ is a local $\big(Q^{\mathbb{H}},\mathbb{H}\big)$-martingale. Hence $\psi \cdot S$ is a $\big(Q^{\mathbb{H}},\mathbb{H}\big)$-supermartingale and $E^{\mathbb{H}}[(\psi \cdot S)_T \mid \mathcal{H}_0] \leq 0$. Assumption 5.3.1 implies the existence of some $\vartheta \in \Theta^{\mathbb{H}}_{Adm}$ such that

$$(\vartheta \cdot S)_t = E^{\mathbb{H}}[(\psi \cdot S)_T \mid \mathcal{H}_t] - E^{\mathbb{H}}[(\psi \cdot S)_T \mid \mathcal{H}_0]\,, \quad t \in [0,T]. \tag{5.21}$$

It follows that $\vartheta \cdot S$ is uniformly bounded from below and satisfies $(\vartheta \cdot S)_T \geq (\psi \cdot S)_T$. In the case $(\vartheta \cdot S)_T = (\psi \cdot S)_T$ we obtain that $\psi \cdot S = \vartheta \cdot S$ and so $\psi \in \Theta^{\mathbb{H}}_{Adm}$.
The claim now easily follows from the observation that by our conventions on U, every wealth process in $\mathcal{V}^{\mathbb{H}}(x)$ is uniformly bounded from below by ℓ. □

Remark 5.3.6 We can now discuss Definition 5.3.3 in more detail.

1. If $\mathcal{H}_0$ is not trivial, there is no unique equivalent local martingale measure for S on $\mathcal{H}_T$ since there is complete freedom in the choice of such a measure on the initial σ-field $\mathcal{H}_0$. At first sight, our definition of admissibility seems to involve the particular measure $Q^{\mathbb{H}}$ via (5.18) in a crucial way. But part 2 of Lemma 5.3.5 shows that that we could equally well require (5.18) with any equivalent local martingale measure Q for S.

2. In the usual setting of utility maximization (see e.g. Cox and Huang [CoH89], Karatzas, Lehoczky and Shreve [KaLS87] or Kramkov and Schachermayer [KrS99]) the domain of the utility function is bounded from below and $\vartheta \in L(S,\mathbb{H})$ is defined as admissible if $\vartheta \cdot S$ is uniformly bounded from below. Although our definition differs from this approach, part 3 of Lemma 5.3.5 shows that it is consistent with the usual setting.

In view of these facts, our concept of admissibility is quite natural in the context of a general complete market and a utility function whose domain might be unbounded from below. We note that Nielsen adopts a definition of admissibility (see [Ni99], Sect. 4.6) that is equivalent to ours if the initial σ-field is trivial. For an alternative definition of admissibility and of the set of random variables over which the optimization problem is formulated we refer the interested reader to Schachermayer [Scha01]. ◇

To solve the optimization problem (5.20) we introduce the (continuous, strictly decreasing) inverse I of the derivative U'. I maps $(0,\infty)$ onto (ℓ,∞) and satisfies

$$\lim_{y\downarrow 0} I(y) = +\infty \quad \text{and} \quad \lim_{y\uparrow\infty} I(y) = \ell\,. \tag{5.22}$$

Proposition 5.3.7 *Suppose Assumption 5.3.1 ($\mathbb{H}$-C) is satisfied and there exists an $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbb{H}}(x) : \Omega \to (0, \infty)$ with*

$$E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right] = x \tag{5.23}$$

and such that

$$V_t^{\mathbb{H}} := E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\right)\Big|\mathcal{H}_t\right], \quad t \in [0, T],$$

satisfies $U^-(V_T^{\mathbb{H}}) \in L^1(P)$. Then $V^{\mathbb{H}}$ is the solution to the optimization problem (5.20), i.e. $V^{\mathbb{H}} \in \mathcal{V}^{\mathbb{H}}(x)$ and $E_P\left[U\left(V_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right] = \operatorname*{ess\,sup}_{V \in \mathcal{V}^{\mathbb{H}}(x)} E_P[U(V_T)|\mathcal{H}_0]$.

Proof: By part 2 of Lemma 5.3.5 we obtain that $V^{\mathbb{H}}$ is an $\mathbb{H}$-martingale with respect to some equivalent local martingale measure Q for S. Assumption 5.3.1 ($\mathbb{H}$-C) implies that there is no other equivalent local martingale measure for S that coincides with Q on $\mathcal{H}_0$. Hence S has the strong predictable representation property with respect to $(Q, \mathbb{H})$ and this yields $V^{\mathbb{H}} \in \mathcal{V}^{\mathbb{H}}(x)$. Since U is concave we have

$$U(y) \geq U(x) + U'(y)(y - x), \quad x, y \in (\ell, \infty).$$

Since $V_T^{\mathbb{H}} = I(\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}})$, the above inequality implies

$$U(V_T^{\mathbb{H}}) \geq U(V_T) + \Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}(V_T^{\mathbb{H}} - V_T), \quad V \in \mathcal{V}^{\mathbb{H}}(x). \tag{5.24}$$

Even if $\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}\left(V_T^{\mathbb{H}} - V_T\right)$ is not integrable, we can take generalized conditional expectations to obtain

$$E_P\left[\Lambda^{\mathbb{H}}(x)Z_T^{\mathbb{H}}(V_T^{\mathbb{H}} - V_T)\Big|\mathcal{H}_0\right] = \Lambda^{\mathbb{H}}(x)E^{\mathbb{H}}\left[V_T^{\mathbb{H}} - V_T\Big|\mathcal{H}_0\right] = 0,$$

since $\Lambda^{\mathbb{H}}(x)$ is $\mathcal{H}_0$-measurable by assumption. In combination with (5.24) this yields

$$E_P\left[U\left(V_T^{\mathbb{H}}\right)\Big|\mathcal{H}_0\right] \geq E_P[U(V_T)\,|\,\mathcal{H}_0], \quad V \in \mathcal{V}^{\mathbb{H}}(x). \tag{5.25}$$

Note that both conditional expectations in (5.25) are well-defined in the usual sense due to the definition of $\mathcal{V}^{\mathbb{H}}(x)$ and the assumption that $U^-(V_T^{\mathbb{H}}) \in L^1(P)$. □

Remark 5.3.8

1. If $\sup_{V \in \mathcal{V}^{\mathbb{H}}(x)} E_P[U(V_T)]$ is finite then the strict concavity of U implies that the solution $V^{\mathbb{H}}$ is unique if it exists.

2. For a trivial initial σ-field $\mathcal{H}_0$ we recover the classical problem of a small investor maximizing his expected utility from terminal wealth in a complete and arbitrage free market; see Karatzas, Lehoczky and Shreve [KaLS87] or Cox and Huang [CoH89] and the remarks following Lemma 5.3.5.

For specific utility functions, $\Lambda^{\mathbb{H}}$ and $V^{\mathbb{H}}$ can often be calculated explicitly in terms of $Z^{\mathbb{H}}$ and x. The following result provides such formulae for common utility functions. The relative entropy of P with respect to $Q^{\mathbb{H}}$ is denoted by $H\left(P|Q^{\mathbb{H}}\right)$.

Corollary 5.3.9 *Suppose Assumption 5.3.1 ($\mathbb{H}$-C) is satisfied. Then the optimal wealth process $V^{\mathbb{H}}(x)$ and the indirect utility function $u^{\mathbb{H}}$ for the utility functions U below are given as follows:*

1. *Logarithmic utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto\log x$: We have*

$$\begin{aligned} V_t^{\mathbb{H}}(x) &= \frac{x}{Z_t^{\mathbb{H}}}, \quad t\in[0,T], \\ u^{\mathbb{H}}(x) &= \log x + E_P\left[\log\frac{1}{Z_T^{\mathbb{H}}}\right] = \log x + H\left(P|Q^{\mathbb{H}}\right). \end{aligned}$$

   $u^{\mathbb{H}}$ is finite if and only if $H\left(P|Q^{\mathbb{H}}\right)$ is finite.

2. *Power utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto x^\gamma/\gamma$, $\gamma\in(0,1)$: If $E_P\left[(Z_T^{\mathbb{H}})^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]<\infty$ then*

$$\begin{aligned} V_t^{\mathbb{H}}(x) &= \frac{x}{E^{\mathbb{H}}\left[(Z_T^{\mathbb{H}})^{\frac{1}{\gamma-1}}\Big|\mathcal{H}_0\right]} E^{\mathbb{H}}\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{1}{\gamma-1}}\Big|\mathcal{H}_t\right], \quad t\in[0,T], \\ u^{\mathbb{H}}(x) &= \frac{x^\gamma}{\gamma} E_P\left[E_P\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]^{1-\gamma}\right] \end{aligned}$$

   and $u^{\mathbb{H}}$ is finite if $E_P\left[(Z^{\mathbb{H}})^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]^{1-\gamma}$ is P-integrable.

3. *Exponential utility $U:\mathbb{R}\to\mathbb{R}$, $x\mapsto -e^{-\alpha x}$, $\alpha>0$: If $H\left(Q^{\mathbb{H}}|P\right)$ is finite then $u^{\mathbb{H}}$ is finite and we have*

$$\begin{aligned} V_t^{\mathbb{H}}(x) &= x + \frac{1}{\alpha}E_P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}\Big|\mathcal{H}_0\right] - \frac{1}{\alpha}E^{\mathbb{H}}\left[\log Z_T^{\mathbb{H}}\Big|\mathcal{H}_t\right], \quad t\in[0,T], \\ u^{\mathbb{H}}(x) &= -\frac{1}{e^{\alpha x}}E_P\left[Z_T^{\mathbb{H}}\exp\left(-E_P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}\Big|\mathcal{H}_0\right]\right)\right]. \end{aligned}$$

Proof: In order to apply Proposition 5.3.7 we first calculate $\Lambda^{\mathbb{H}}$ and verify the required assumptions. Note that a solution $\Lambda^{\mathbb{H}}$ to (5.23) is unique if it exists. Proposition 5.3.7 yields the formulae for the optimal wealth process $V^{\mathbb{H}}$ and the formulae for $u^{\mathbb{H}}$ then follow by straightforward calculations of $E_P[U(V_T^{\mathbb{H}})]$ which are left to the reader.
1: $I(y)=1/y$ and $\Lambda^{\mathbb{H}}(x)=1/x$ is the solution to (5.23). Obviously $x/Z^{\mathbb{H}}$ is a $(Q^{\mathbb{H}},\mathbb{H})$-martingale. Moreover, $E_P\left[\log\left(1/Z_T^{\mathbb{H}}\right)\right]=E^{\mathbb{H}}\left[\left(1/Z_T^{\mathbb{H}}\right)\log\left(1/Z_T^{\mathbb{H}}\right)\right]$ is the relative entropy of P with respect to $Q^{\mathbb{H}}$ on $\mathcal{H}_T$ and thus $E_P\left[\log\left(1/Z_T^{\mathbb{H}}\right)\right]$ is well-defined with values in $[0,\infty]$. By Proposition 5.3.7 we then obtain the formula for $V^{\mathbb{H}}$ and thus $u^{\mathbb{H}}$.

2: $I(y) = y^{\frac{1}{\gamma-1}}$, $E^{\mathbb{H}}\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{1}{\gamma-1}}\Big|\mathcal{H}_0\right] = E_P\left[\left(Z_T^{\mathbb{H}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{H}_0\right]$ is finite by assumption and the solution to (5.23) is given by

$$\Lambda^{\mathbb{H}}(x) = \frac{x^{\gamma-1}}{E^{\mathbb{H}}\left[Z_T^{\mathbb{H}\frac{1}{\gamma-1}}\Big|\mathcal{H}_0\right]^{\gamma-1}}.$$

By Proposition 5.3.7 we obtain the formula for $V^{\mathbb{H}}$ and thus $u^{\mathbb{H}}$.
3: $I(y) = -\frac{1}{\alpha}\log\frac{y}{\alpha}$, $E_P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}\right] < \infty$ by assumption and the solution to (5.23) is given by

$$\Lambda^{\mathbb{H}}(x) = \alpha\exp\left(-\alpha x - E_P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}\Big|\mathcal{H}_0\right]\right).$$

By Proposition 5.3.7 we obtain the formula for $V^{\mathbb{H}}$ and thus $u^{\mathbb{H}}$. Since $E_P\left[Z_T^{\mathbb{H}}\log Z_T^{\mathbb{H}}|\mathcal{H}_0\right]$ is bounded from below, $u^{\mathbb{H}}$ is finite. □

The following lemma provides sufficient conditions for the existence of an $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbb{H}}(x)$ satisfying (5.23). For this we fix one version of the regular conditional distribution $Q^{\mathbb{H}}[Z_T^{\mathbb{H}} \in dz\,|\,\mathcal{H}_0]$ and define for each $\omega \in \Omega$ the function $\Psi_\omega : (0,\infty) \to (\ell,\infty]$ by

$$\Psi_\omega(\lambda) \quad := \quad \int I(\lambda z)\, Q^{\mathbb{H}}\left[Z_T^{\mathbb{H}} \in dz\,|\,\mathcal{H}_0\right](\omega)$$

whenever this integral is finite and $\Psi_\omega(\lambda) := +\infty$ otherwise. Note that the integral is always well-defined in $(-\infty,\infty]$ if $\ell > -\infty$ since $I(\cdot) \geq \ell$ and that $\omega \mapsto \Psi_\omega(\lambda)$ is a version of $E^{\mathbb{H}}[I(\lambda Z_T^{\mathbb{H}})\,|\,\mathcal{H}_0]$ for all λ by (5.1).

Lemma 5.3.10 *There exists an $\mathcal{H}_0$-measurable $\Lambda^{\mathbb{H}}(x)$ taking values in $(0,\infty)$ that satisfies (5.23) if one of the following conditions is valid:*

1. *For P-a.a. ω there are $\lambda_1, \lambda_2 \in (0,\infty)$ such that $\Psi_\omega(\lambda_1) \leq x \leq \Psi_\omega(\lambda_2) < \infty$.*
2. *For P-a.a. ω the functions Ψ_ω are finite on $(0,\infty)$.*

Note that the second condition is in particular satisfied if $E^{\mathbb{H}}\left[\left|I(\lambda Z^{\mathbb{H}})\right|\right] < \infty$ for all $\lambda \in \mathbb{R}$. The latter is a classical condition in the theory of utility maximization, see (5.6) in Karatzas, Lehoczky and Shreve [KaLS87].

Proof: 1: Let domΨ_ω denote the subset of $(0,\infty)$ where Ψ_ω is finite. As I is decreasing on $(0,\infty)$ it follows that Ψ_ω is decreasing on domΨ_ω and domΨ_ω is an interval. Furthermore, dominated convergence implies that Ψ_ω is continuous on domΨ_ω.
By assumption there is $N \subset \Omega$ with $P[N] = 0$ such that for each $\omega \in \Omega \setminus N$ there are $\lambda_1, \lambda_2 \in$ domΨ_ω such that $\Psi_\omega(\lambda_1) \leq x \leq \Psi_\omega(\lambda_2)$; hence continuity of Ψ_ω implies that there is $\lambda \in (0,\infty)$ such that $\Psi_\omega(\lambda) = x$. The set

$$B := \left\{(\omega,\lambda) \in \Omega \times (0,\infty)\,\Big|\, \Psi_\omega(\lambda) = x\right\}$$

is $\mathcal{H}_0 \otimes \mathcal{B}((0,\infty))$-measurable and if we denote by Γ the projection of B onto Ω, we have $\Omega \setminus N \subseteq \Gamma$. By the measurable selection theorem (see e.g. Dellacherie and Meyer (1980), III.44-45), B admits a measurable selector, i.e. there exists a $\mathcal{H}_0$-measurable mapping $\Lambda^{\mathbb{H}}(x) : \Omega \to (0,\infty)$ such that $\left(\omega, \Lambda^{\mathbb{H}}(x)(\omega)\right) \in B$ for all $\omega \in \Gamma$. Hence $E^{\mathbb{H}}\left[I\left(\Lambda^{\mathbb{H}}(x) Z_T^{\mathbb{H}}\right) \mid \mathcal{H}_0\right](\omega) = \Psi_\omega\left(\Lambda^{\mathbb{H}}(x)(\omega)\right) = x$ for P-a.a. ω.
2: We can choose a version of $Q^{\mathbb{H}}\left[Z_T^{\mathbb{H}} \in \cdot \,|\mathcal{H}_0\right]$ such that Ψ_ω is finite for all ω. Then $I(\lambda z)$ is decreasing in λ and $Q^{\mathbb{H}}\left[Z_T^{\mathbb{H}} \in dz \mid \mathcal{H}_0\right](\omega)$-integrable for all λ and ω. In combination with (5.22), monotone convergence and dominated convergence imply for all ω

$$\lim_{\lambda \downarrow 0} \Psi_\omega(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \uparrow \infty} \Psi_\omega(\lambda) = \ell\,. \tag{5.26}$$

Since $\ell < x < \infty$, part 1 implies the claim. □

5.4 Utility indifference value of initial information in a complete market

We now consider an investor with information flow $\mathbb{F}$ trading in a complete financial market where the discounted prices of the risky assets are given by an $\mathbb{F}$-semimartingale $S = (S^1, \ldots, S^d)^{tr}$. The aim of the present section is to introduce and study the subjective monetary value of the additional initial information $\mathcal{G}$ for the investor. Section 5.5 provides explicit calculations of this value in an Itô process model where the additional information basically consists of a noisy signal about the terminal stock price.
We impose the following assumption throughout this section.

Assumption 5.4.1 *Suppose Assumption 5.1.1 (D) is satisfied and Assumption 5.3.1($\mathbb{F}$-C) is satisfied with respect to $\mathbb{F}$.*

It follows by Theorem 5.2.3 that Assumption 5.3.1($\mathbb{G}$-C) is also satisfied with respect to $\mathbb{G}$ and the martingale preserving measure $Q^{\mathbb{G}}$ corresponding to $Q^{\mathbb{F}}$. We denote by $Z_T^{\mathbb{F}}$ and $Z_T^{\mathbb{G}}$ the densities of $Q^{\mathbb{F}}$ and $Q^{\mathbb{G}}$ with respect to P. Recall from Theorem 5.1.7 that we have the relation $Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}} = p^G$ since Assumption 5.1.3 (E) holds by Corollary 5.1.8. For both filtrations we are therefore within the complete market framework of the previous section with $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ and can use the corresponding results and notations.

Definition 5.4.2 *The* utility indifference value *of the additional initial information $\mathcal{G}$ is defined as a solution $\pi = \pi(x)$ of the equation*

$$u^{\mathbb{F}}(x) = u^{\mathbb{G}}(x - \pi)\,. \tag{5.27}$$

Equation (5.27) means that the investor with the goal to maximize his expected utility from terminal wealth is indifferent between the following alternatives:

1. Invest the initial capital x optimally by using the information flow $\mathbb{F}$.

2. Acquire the additional information $\mathcal{G}$ by paying π and then invest the remaining capital $x-\pi$ optimally with the help of the enlarged information flow $\mathbb{G}$.

If the indirect utility functions $u^{\mathbb{F}}$ and $u^{\mathbb{G}}$ are finite, continuous, strictly increasing and satisfy $\lim_{y\downarrow\ell} u^{\mathbb{G}}(y) < u^{\mathbb{F}}(x)$, then the utility indifference value exists and is unique. In fact, part 4 of Corollary 5.1.6 and $Q^{\mathbb{G}} = Q^{\mathbb{F}}$ on $\mathcal{F}_T$ imply that $\Theta^{\mathbb{F}} \subseteq \Theta^{\mathbb{G}}$. Thus, $u^{\mathbb{F}}$ is dominated by $u^{\mathbb{G}}$. From $\lim_{y\downarrow\ell} u^{\mathbb{G}}(y) < u^{\mathbb{F}}(x) \le u^{\mathbb{G}}(x)$ we then obtain the existence of a nonnegative solution π for (5.27) by the continuity of $u^{\mathbb{G}}$. The strict monotonicity of $u^{\mathbb{G}}$ implies the uniqueness of this solution. We note that these conditions are satisfied in all subsequent examples.
The following theorem provides explicit expressions for π for common utility functions.

Theorem 5.4.3 *Suppose Assumption 5.4.1 is satisfied. Then the utility indifference value π for the respective utility functions below is given as follows.*

1. *Logarithmic utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto \log x$: If $H\left(P|Q^{\mathbb{G}}\right) = E_P\left[\log \frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$ then*

$$\pi = x\left(1-\exp\left(-E_P\left[\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right]\right)\right). \tag{5.28}$$

2. *Power utility $U:(0,\infty)\to\mathbb{R}$, $x\mapsto x^\gamma/\gamma$, $\gamma\in(0,1)$: If $E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty$ then*

$$\pi = x\left(1-\frac{E_P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right]^{\frac{1-\gamma}{\gamma}}}{E_P\left[E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}_0\right]^{1-\gamma}\right]^{\frac{1}{\gamma}}}\right). \tag{5.29}$$

3. *Exponential utility $U:\mathbb{R}\to\mathbb{R}$, $x\mapsto -e^{-\alpha x}$, $\alpha>0$: If $H(Q^{\mathbb{G}}|P) = E_P\left[Z_T^{\mathbb{G}}\log Z_T^{\mathbb{G}}\right] < \infty$ then*

$$\pi = -\frac{1}{\alpha}\log E_P\left[Z_T^{\mathbb{G}}\exp\left(E_P\left[Z_T^{\mathbb{G}}\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right]\right)\right]. \tag{5.30}$$

Note that we can replace $Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}}$ with p^G.

The formulae in (5.28) and (5.30) can be rewritten in terms of relative entropy: By Proposition 5.1.7 $d\widetilde{P} := 1/p^G\,dP$ defines the martingale preserving measure corresponding to P. Now observe that the expression $E_P\left[\log\left(Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}}\right)\right]$ in (5.28) is equal to $E_P\left[\log p^G\right] = H(P|\widetilde{P})$. Similarly, the expression $E_P\left[Z_T^{\mathbb{G}}\log\left(Z_T^{\mathbb{F}}/Z_T^{\mathbb{G}}\right)|\mathcal{G}_0\right]$ in (5.30) is equal to

$$\begin{aligned} -E^{\mathbb{G}}\left[\log\frac{Z_T^{\mathbb{G}}}{Z_T^{\mathbb{F}}}\Big|\mathcal{G}_0\right] &= -E^{\mathbb{F}}\left[\frac{Z_T^{\mathbb{G}}}{Z_T^{\mathbb{F}}}\log\frac{Z_T^{\mathbb{G}}}{Z_T^{\mathbb{F}}}\Big|\mathcal{G}_0\right] \\ &=: -H_{|\mathcal{G}_0}\left(Q^{\mathbb{G}}|Q^{\mathbb{F}}\right), \end{aligned}$$

with $H_{|\mathcal{G}_0}\left(Q^{\mathbb{G}}|Q^{\mathbb{F}}\right)$ denoting the relative entropy of $Q^{\mathbb{G}}$ with respect to $Q^{\mathbb{F}}$ conditional on $\mathcal{G}_0$.

Proof: By Theorem 5.2.3, Assumption 5.4.1 implies Assumption 5.3.1 ($\mathbb{G}$-C). For each part we show also that the integrability assumptions from Corollary 5.3.9 are satisfied in both $\mathbb{G}$ and $\mathbb{F}$. Then $u^{\mathbb{F}}$ and $u^{\mathbb{G}}$ are given explicitly by Corollary 5.3.9 and we just have to verify (5.27). Since $u^{\mathbb{G}}$ is strictly increasing, the solution π is unique.

1.: Jensen's inequality yields $E_P\left[\log\frac{1}{Z_T^{\mathbb{F}}}\right] = E_P\left[\log\frac{1}{E_P\left[Z_T^{\mathbb{G}}|\mathcal{F}_T\right]}\right] \le E_P\left[\log\frac{1}{Z_T^{\mathbb{G}}}\right] < \infty$. Part 1 of Corollary 5.3.9 implies

$$u^{\mathbb{G}}(x-\pi) - u^{\mathbb{F}}(x) = \log(x-\pi) + E_P\left[\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\right] - \log(x)\,, \tag{5.31}$$

and inserting π given by (5.28) gives (5.27).

2.: From $E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty$ follows $E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}_0\right]^{1-\gamma} \in L^1(P)$ since $\gamma \in (0,1)$. Moreover, by Jensen's inequality we obtain

$$E_P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right] = E_P\left[E_P\left[Z_T^{\mathbb{G}}|\mathcal{F}_T\right]^{\frac{\gamma}{\gamma-1}}\right] \le E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\right] < \infty\,.$$

Thus part 2 of Corollary 5.3.9 shows that (5.27) is equivalent to

$$\frac{x^\gamma}{\gamma}E_P\left[E_P\left[\left(Z_T^{\mathbb{F}}\right)^{\frac{\gamma}{\gamma-1}}\right]^{1-\gamma}\right] = \frac{(x-\pi)^\gamma}{\gamma}E_P\left[E_P\left[\left(Z_T^{\mathbb{G}}\right)^{\frac{\gamma}{\gamma-1}}\Big|\mathcal{G}_0\right]^{1-\gamma}\right]$$

and solving for π leads to (5.29).

3.: Again Jensen's inequality shows $E_P\left[Z_T^{\mathbb{F}}\log Z_T^{\mathbb{F}}\right] \le E_P\left[Z_T^{\mathbb{G}}\log Z_T^{\mathbb{G}}\right] < \infty$. Using the properties that define the martingale preserving measure $Q^{\mathbb{G}}$, we calculate

$$\begin{aligned}
-E_P\left[Z_T^{\mathbb{G}}\log Z_T^{\mathbb{G}}\Big|\mathcal{G}_0\right] &= E^{\mathbb{G}}\left[\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}} - \log Z_T^{\mathbb{F}}\Big|\mathcal{G}_0\right] \\
&= E^{\mathbb{G}}\left[\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right] - E^{\mathbb{G}}\left[\log Z_T^{\mathbb{F}}\Big|\mathcal{G}_0\right] \\
&= E_P\left[Z_T^{\mathbb{G}}\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right] - E_P\left[Z_T^{\mathbb{F}}\log Z_T^{\mathbb{F}}\right].
\end{aligned}$$

In combination with part 3 of Corollary 5.3.9 we get

$$\begin{aligned}
u^{\mathbb{G}}(y) &= -e^{-\alpha y}E^{\mathbb{G}}\left[\exp\left(-E_P\left[Z_T^{\mathbb{G}}\log Z_T^{\mathbb{G}}\Big|\mathcal{G}_0\right]\right)\right] \\
&= -\exp\Big(-\alpha y - E_P\left[Z_T^{\mathbb{F}}\log Z_T^{\mathbb{F}}\right]\Big)E^{\mathbb{G}}\left[\exp\left(E_P\left[Z_T^{\mathbb{G}}\log\frac{Z_T^{\mathbb{F}}}{Z_T^{\mathbb{G}}}\Big|\mathcal{G}_0\right]\right)\right]
\end{aligned} \tag{5.32}$$

for $y \in \mathbb{R}$. For $y = x - \pi$ with π given by (5.30) this leads to

$$u^{\mathbb{G}}(x-\pi) = -\exp\left(-\alpha x - E_P\left[Z_T^{\mathbb{F}}\log Z_T^{\mathbb{F}}\right]\right) = u^{\mathbb{F}}(x)$$

by part 3 of Corollary 5.3.9. □

Remark 5.4.4 1. By definition the utility indifference value π depends on both the initial capital x and the utility function U which incorporates the investor's preferences and attitude towards risk. However, it is not affected by a change to an equivalent utility function $\widetilde{U}$ which is a positive linear transformation of U, i.e., $\widetilde{U} = bU + c$ with $b > 0$, $c \in \mathbb{R}$. In fact the corresponding indirect utility functions u and $\tilde{u}$ are then linked by the same positive linear transformation and so the utility indifference values with respect to U and $\widetilde{U}$ coincide. Furthermore, we note that the utility indifference value π does not depend on the initial capital when the utility function is exponential.

2. We emphasize that our results are derived under the assumption of a small investor since asset prices are exogenously given and not affected by the investor's trading strategy. This assumption is quite usual in the literature on utility maximization but becomes questionable if the optimal strategy takes very large positions in the available securities. In our situation, this might happen in particular for a $\mathbb{G}$-investor who receives some highly relevant additional information. A prime example discussed in more detail in Section 5.5 is a signal about the terminal stock price distorted by only very small noise. However, a systematic analysis of large investor effects and equilibrium questions is beyond the scope of this work and left to future research.

$\diamond$

5.5 Examples: Terminal information distorted by noise

We now calculate closed form expressions for the utility indifference value for logarithmic and exponential utility functions when the financial market is given by the standard complete Itô process model and the additional initial information consists as in Amendinger, Imkeller and Schweizer [AIS98], Section 4.3, of a noisy signal about the terminal value of the Brownian motion driving the asset prices. We have not yet found a closed form solution in the power utility case where an explicit computation of (5.29) appears more cumbersome.

Let the (discounted) prices of the risky assets be given by the stochastic differential equations

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij}\, dW_t^j, \quad S_0^i > 0 \quad \text{for } i = 1, \dots, d, \tag{5.33}$$

where $W = (W^j)_{j=1,\dots,d}$ is a d-dimensional Brownian motion and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the P-augmentation of the filtration generated by W. The excess return vector $\mu = (\mu^i)_{i=1,\dots,d}$ and the volatility matrix $\sigma = (\sigma^{ij})_{i,j=1,\dots,d}$ are assumed predictable with $\int_0^T \left(|\mu_t| + |\sigma_t|^2\right) dt < \infty$ P-a.s. and σ_t has full rank P-a.s. for all $t \in [0, T]$. The *relative risk process* is given by $\lambda_t := \sigma_t^{-1}\mu_t$ and we suppose that $\int_0^T |\lambda_t|^2\, dt$ is finite and the stochastic exponential $Z^{\mathbb{F}} := \mathcal{E}(-\int \lambda\, dW)$ is a $(P, \mathbb{F})$-martingale. Then $dQ^{\mathbb{F}} := Z_T^{\mathbb{F}}\, dP$ is the unique equivalent local martingale measure for S on $\mathcal{F}_T$ so that Assumption 5.3.1($\mathbb{F}$-C) is satisfied.

Suppose the additional information is a noisy signal about the outcome of W_T, i.e., $\mathcal{G} = \sigma(G)$ with

$$G := (\delta_i W_T^i + (1-\delta_i)\varepsilon_i)_{i=1,\dots,d}\,,$$

where the ε_i are i.i.d. $\mathcal{N}(0,1)$-distributed and independent of $\mathcal{F}_T$ and the δ_i are constant numbers in $[0,1)$. If all δ_i are strictly positive, the additional information is also generated by

$$\widetilde{G} := \left(W_T^i + \frac{1-\delta_i}{\delta_i}\varepsilon_i \right)_{i=1,\dots,d}$$

which is an unbiased signal for W_T. The regular conditional distribution of G given $\mathcal{F}_t$ exists for all $t \in [0,T]$ and is multivariate normal with mean vector $\left(\delta_i W_t^i\right)_{i=1,\dots,d}$ and covariance matrix $\operatorname{diag}\Big(\delta_i^2(T-t) + (1-\delta_i)^2\Big)_{i=1,\dots,d}$. Hence the conditional distribution of G given $\mathcal{F}_T$ is a.s. equivalent to the distribution of G which is also normal. Assumption 5.1.3 (E) is therefore satisfied and a straightforward computation gives

$$p^G = \prod_{i=1}^d \sqrt{\frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2}} \exp\left(\frac{1}{2}\left(\frac{G_i^2}{\delta_i^2 T + (1-\delta_i)^2} - \frac{(G_i - \delta_i W_T^i)^2}{(1-\delta_i)^2} \right) \right). \tag{5.34}$$

As in previous sections we denote by $Q^{\mathbb{G}}$ the martingale preserving measure corresponding to $Q^{\mathbb{F}}$ and by $Z_T^{\mathbb{G}}$ its density with respect to P. We recall the relation $Z_T^{\mathbb{G}} = Z_T^{\mathbb{F}}/p^G$.

Logarithmic utility indifference value

Let the utility function of the investor be given by $U(x) = \log x$ and assume $E_P\left[\int_0^T |\lambda_t|^2\,dt\right]$ is finite so that $H\left(P|Q^{\mathbb{F}}\right) = E_P\left[\log(1/Z_T^{\mathbb{F}})\right] = \frac{1}{2}E_P\left[\int_0^T |\lambda_t|^2\,dt\right] < \infty$. In combination with

$$E_P\left[\log p^G\right] = \frac{1}{2}\sum_{i=1}^d \log\frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2} < \infty$$

this leads to $E_P\left[\log\frac{1}{Z_T^{\mathbb{G}}}\right] = E_P\left[\log\frac{1}{Z_T^{\mathbb{F}}}\right] + E_P\left[\log p^G\right] < \infty$. By (5.28) we obtain that the logarithmic utility indifference value is given by

$$\pi = x\left(1 - \prod_{i=1}^d \sqrt{\frac{(1-\delta_i)^2}{\delta_i^2 T + (1-\delta_i)^2}}\right)$$

and we can analyze the behavior of this quantity as the parameters vary. If all δ_i converge to zero, then π tends to zero and in particular $\pi = 0$ if $\delta_i = 0$ for all i. Intuitively this shows that the information delivered by G becomes useless when increasing noise hides all information about W_T. Furthermore π is increasing in T, increasing in each δ_i and converges to x if $\delta_i \uparrow 1$ for one i with all other parameters fixed. For very small noise, the insider information intuitively almost offers an arbitrage opportunity; this is best seen in the case of constant coefficients μ and σ where S_T is a function of W_T. In fact, the value of the information for the ordinary investor then comes close to his total initial capital x and the investor can not pay more than x since the logarithmic utility function enforces an strictly positive remaining initial capital $x-\pi$ by requiring an a.s. strictly positive final wealth. Note that the limiting case $\delta_i = 1$ is not included in our framework since it would violate Assumption 5.1.3 (E).

Exponential utility indifference value

Now consider the case where the investor's utility function is given by $U(x) = -\exp(-\alpha x)$ with $\alpha > 0$ and assume $H(Q^{\mathbb{F}}|P) < \infty$. By Girsanov's theorem, $\widetilde{W} := W + \int \lambda_t \, dt$ is a $(Q^{\mathbb{F}}, \mathbb{F})$-Brownian motion. The relative entropy $H(Q^{\mathbb{F}}|P)$ is given by

$$E_P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right] = E^{\mathbb{F}}\left[-\int_0^T \lambda_t \, d\widetilde{W}_t + \frac{1}{2}\int_0^T |\lambda_t|^2 \, dt\right] = \frac{1}{2} E^{\mathbb{F}}\left[\int_0^T |\lambda_t|^2 \, dt\right]$$

and finite by assumption. From $E^{\mathbb{F}}\left[\left(\int_0^T \lambda_s^i \, ds\right)^2\right] \leq T E^{\mathbb{F}}\left[\int_0^T |\lambda_s^i|^2 \, ds\right] < \infty$ we conclude that W_T^i is in $L^2(Q^{\mathbb{F}})$ and hence in $L^2(Q^{\mathbb{G}})$ for all $i = 1, \ldots, d$. Since $Q^{\mathbb{G}} = P$ on $\sigma(G)$, the random variables G_i, $i = 1, \ldots, d$ are independent and normally distributed under $Q^{\mathbb{G}}$. By (5.34) we obtain $\log p^G \in L^1(Q^{\mathbb{G}})$ and

$$E_P\left[\frac{Z_T^{\mathbb{F}}}{p^G} \log \frac{Z_T^{\mathbb{F}}}{p^G}\right] = E_P\left[Z_T^{\mathbb{F}} \log Z_T^{\mathbb{F}}\right] + E^{\mathbb{G}}\left[\log p^G\right] < \infty .$$

Straightforward calculation yields

$$\begin{aligned}
(5.35) \qquad & \exp\left(E^{\mathbb{G}}\left[\log p^G \middle| \mathcal{G}_0\right]\right) = \exp\left(E^{\mathbb{G}}\left[\log p^g\right]\Big|_{g=G}\right) \\
&= \prod_{i=1}^d \exp\left(\frac{1}{2}\left(\log \frac{\delta_i^2 T + (1-\delta_i)^2}{(1-\delta_i)^2} + \frac{G_i^2}{\delta_i^2 T + (1-\delta_i)^2} - \frac{E^{\mathbb{G}}\left[(g_i - \delta_i W_T^i)^2\right]\big|_{g=G}}{(1-\delta_i)^2}\right)\right) .
\end{aligned}$$

As $Q^{\mathbb{G}} = Q^{\mathbb{F}}$ on $\mathcal{F}_T$, we obtain

$$(5.36) \qquad E^{\mathbb{G}}\left[(g_i - \delta_i W_T^i)^2\right]\Big|_{g=G} = G_i^2 - 2\delta_i E^{\mathbb{F}}\left[W_T^i\right] G_i + \delta_i^2 E^{\mathbb{F}}\left[\left(W_T^i\right)^2\right]$$

Further calculation yields

$$\begin{aligned}
& E^{\mathbb{G}}\left[\exp\left(\frac{-\delta_i^2 T}{2\left(\delta_i^2 T + (1-\delta_i)^2\right)(1-\delta_i)^2} G_i^2 + \frac{\delta_i E^{\mathbb{F}}[W_T^i]}{(1-\delta_i)^2} G_i - \frac{\delta_i^2 E^{\mathbb{F}}\left[(W_T^i)^2\right]}{2(1-\delta_i)^2}\right)\right] \\
(5.37) \qquad &= \sqrt{\frac{(1-\delta_i)^2}{\delta_i^2 T + (1-\delta_i)^2}} \exp\left(\frac{-\delta_i^2}{2(1-\delta_i)^2} \mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\right) .
\end{aligned}$$

The d factors in the product from (5.35) are $Q^{\mathbb{G}}$-independent since $Q^{\mathbb{G}} = P$ on $\sigma(G)$. Hence $E^{\mathbb{G}}\left[\exp\left(E^{\mathbb{G}}[\log p^G|\mathcal{G}_0]\right)\right]$ is equal to the product of the $Q^{\mathbb{G}}$-expectations of the d single factors;

computing $E^{\mathbb{G}}\left[(g_i-\delta_i W_T^i)^2\right]\big|_{g=G}$ by (5.36) and using (5.37) then leads to

$$E^{\mathbb{G}}\left[\exp\left(E^{\mathbb{G}}\left[\log p^G|\mathcal{G}_0\right]\right)\right]=\prod_{i=1}^d \exp\left(\frac{-\delta_i^2}{2(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\right).$$

By (5.30) we obtain that the exponential utility indifference value is given by

$$\begin{aligned}\pi &= \frac{1}{2\alpha}\sum_{i=1}^d \frac{\delta_i^2}{(1-\delta_i)^2}\mathrm{Var}_{Q^{\mathbb{F}}}[W_T^i]\\ &= \frac{1}{2\alpha}\sum_{i=1}^d \frac{\delta_i^2}{(1-\delta_i)^2}\left(T-2\mathrm{Cov}_{Q^{\mathbb{F}}}\left[\widetilde{W}_T^i,\int_0^T \lambda_s^i\,ds\right]+\mathrm{Var}_{Q^{\mathbb{F}}}\left[\int_0^T \lambda_s^i\,ds\right]\right).\end{aligned}\tag{5.38}$$

π is decreasing in the risk-aversion coefficient α, increasing in δ_i and tends to zero if all δ_i converge to zero. π tends to infinity if $\delta_i\uparrow 1$ for one i with all other parameters being fixed. Again this is precisely what intuition suggests should happen.

If the relative risk process λ is deterministic, (5.38) yields the closed form solution

$$\pi=\frac{T}{2\alpha}\sum_{i=1}^d\frac{\delta_i^2}{(1-\delta_i)^2}.$$

Part III

ON THE NUMERAIRE PORTFOLIO

Chapter 6

The numeraire portfolio for unbounded semimartingales

(This chapter is an extended and adapted version of Becherer [Be01])

We consider a financial market with one riskless asset (savings account) and d risky assets (stocks). For simplicity assume the value of the savings account to be identically one. The (discounted) prices of the risky assets are described by a d-dimensional semimartingale $S = (S^i)_{i=1,\dots,d}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ where T is a finite time horizon. For a predictable S-integrable process θ the value process of the (self-financing) trading strategy θ with initial capital 1 is given by $1 + \int \theta\, dS$. If this value process is strictly positive, it is called a tradable numeraire. $\mathcal{N}$ denotes the set of all tradable numeraires.
To motivate the idea of the numeraire portfolio, let us consider the pricing of a European option modelled by an $\mathcal{F}_T$-measurable random variable $H \geq 0$. It is well known that any arbitrage-free valuation is essentially of the form $H \mapsto E^Q[H/N_T]$ where N is an appropriate tradable numeraire and Q a probability measure equivalent to P such that all asset prices discounted by N are (local) Q-martingales. If the state price density $\frac{dQ}{dP}\frac{1}{N_T}$ is not unique, arbitrage arguments alone typically only give an interval of arbitrage-free prices. So one has to choose Q and N in order to decide on one of the possible arbitrage-free valuations of H. There are two dual approaches for making this choice:

(6.1) Fix $N \in \mathcal{N}$ and choose $Q \sim P$ such that
$1/N$ and S/N become (local) Q-martingales.

(6.2) Fix $Q \sim P$ and choose $N \in \mathcal{N}$ such that
$1/N$ and S/N become (local) Q-martingales.

In the first approach one fixes a priori some appropriate tradable numeraire N, e.g., the riskless numeraire $S^0 \equiv 1$. As the set of equivalent martingale measures for the asset prices discounted by N is not a singleton, a choice of Q has to be made and this is typically done according to

some optimality criterion involving the original measure P. Examples are the minimal entropy martingale measure [Ft00], the minimal martingale measure [FSchw91, Schw95] or the variance optimal martingale measure [Schw96]. Several pricing problems, e.g. for the exchange option studied by Margrabe [Ma78], become more convenient under another numeraire than $N = S^0$ (say). Given a martingale measure Q corresponding to N, the methodology of Geman, El Karoui and Rochet [GER95] provides an efficient way for a change to another numeraire by a simultaneous change of the martingale measure.
In the second approach a certain probability Q is fixed a priori - a natural choice is the original probability P. Here the situation is somewhat simpler because one can show that there is at most one tradable numeraire satisfying the requirement (6.2). So this numeraire is the unique 'right' tradable numeraire with respect to a given probability measure Q. In the situation of $Q := P$, Long [Lo90] called this numeraire the *(P-)numeraire portfolio*. The original probability measure P becomes an equivalent martingale measure with respect to the P-numeraire portfolio if this satisfies (6.2). The first appearance of such an idea seems to be in Vasiček [V77] (on p. 184). Suggested applications of this concept use the following two aspects. From a technical point of view, well-known results can be proved from another perspective without the usual change to some risk-neutral measure. On the empirical side the original probability measure P models the 'true world' probability which can be investigated by statistical methods. Long [Lo90], for example, studied the application of measuring abnormal stock returns by discounting NYSE stock returns by empirical proxies of the numeraire portfolio.
Long [Lo90] proved existence of a tradable numeraire N satisfying (6.2) in the case of finite Ω and discrete time and in the case where the asset prices S evolve according to a sufficiently regular multidimensional diffusion model. In these cases the numeraire portfolio turns out to be growth-optimal More properties and applications in the diffusion case, where the numeraire portfolio is locally mean-variance efficient and therefore related to the CAPM-theory, are shown in Bajeux-Besnainou and Portait [BP97] and Johnson [Jo96]. A close connection to the concept of value preservation was very recently pointed out by Korn and Schäl [Ko00, KoS00]. For a survey discussion and further references see Artzner [Ar97].
The basic goal of the current chapter is to study the idea of the numeraire portfolio under minimal assumptions on the model. Section 6.1 contains the general setting. Section 6.2 states essential results on the *growth-optimal* numeraire that is known to be closely related to the P-numeraire portfolio. As examples in Section 6.4 show, the (local) martingale condition (6.2) is too stringent to obtain a general existence result. For this reason we define the (generalized) P-numeraire portfolio in Section 6.3 by a weaker requirement such that our definition is a consistent extension of the original numeraire portfolio concept introduced by Long [Lo90]. By using general duality results from Kramkov and Schachermayer [KrS99] on utility maximization, the existence of the P-numeraire portfolio N^P can be shown under very mild conditions which are of 'no-arbitrage' type. N^P is characterized as the solution to

several optimization problems. In the case of continuous asset prices $1/N^P$ coincides with the density process of the minimal martingale measure. It turns out that pricing by the numeraire portfolio corresponds to the idea of pricing by "zero marginal rate of substitution" (cf. [D01]) with respect to the logarithmic utility function. If this method in general leads to a valuation which can be considered as arbitrage-free, is a delicate question which has to be discussed.

6.1 General framework and preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a filtered probability space with a filtration satisfying the usual conditions, where the time horizon $T > 0$ is finite. For simplicity, we assume $\mathcal{F}_T = \mathcal{F}$. All stochastic processes will be indexed by $t \in [0, T]$. All semimartingales considered in the sequel are assumed to have right-continuous paths with left limits. For an $\mathbb{R}^d$-valued semimartingale S and an S-integrable predictable $\mathbb{R}^d$-valued process θ we denote by $\theta \cdot S$ the semimartingale defined by $(\theta\cdot S)_t = \int_0^t \theta_u \, dS_u$. Let $L(S)$ denote the space of all $\mathbb{R}^d$-valued predictable processes that are integrable with respect to S. For unexplained terminology from stochastic analysis we refer to Dellacherie/Meyer [DelM80] and Jacod [Ja79].
We consider a model of a financial market consisting of $d + 1$ assets: one savings account and d stocks with prices $S = (S^i)_{i=1,\dots,d}$. S is assumed to be a semimartingale. For simplicity we assume the value of the savings account to be constant at one so that the asset prices are expressed in units of the savings account.
A (self-financing) portfolio strategy is defined by a pair $(V_0, \theta) \in \mathbb{R} \times L(S)$, where V_0 specifies the initial capital and θ^i the number of shares of each stock i held in the dynamically varying portfolio. The corresponding value process is given by $V = V_0 + \theta \cdot S$. We denote by

$$\mathcal{N} := \{N \mid N = 1 + \theta \cdot S\,,\ \theta \in L(S),\ N > 0\} \tag{6.3}$$

the family of all strictly positive value processes starting at one. These processes are called *tradable numeraires.*

Definition 6.1.1 *A probability measure $Q \sim P$ is called an* equivalent local martingale measure for $\mathcal{N}$ *if all $N \in \mathcal{N}$ are local Q-martingales. The set of all such measures is denoted by $\mathcal{M}^e(\mathcal{N})$.*

Remark 6.1.2 By Corollaire 3.5 from [AnS94] every equivalent local martingale measure for S (in the usual sense) is an element of $\mathcal{M}^e(\mathcal{N})$, i.e. $\mathcal{M}^e(S) \subseteq \mathcal{M}^e(\mathcal{N})$. Moreover, a simple argument shows that $Q \in \mathcal{M}^e(\mathcal{N})$ if and only if every stochastic integral $\theta \cdot S$ that is locally bounded from below is a local Q-martingale. This implies by taking $\theta := 1$ that $\mathcal{M}^e(\mathcal{N})$ coincides with the set $\mathcal{M}^e(S)$ of equivalent local martingale measures for S if S is locally bounded from below. If S fails to be locally bounded from below the situation may be more complicated: If S is a local Q-martingale, then Q is an equivalent local martingale measure for $\mathcal{N}$ (see [AnS94]), but the converse is not true in general (see [DeS98]). ◇

The next proposition shows that $\mathcal{M}^e(\mathcal{N})$ is not empty if and only if the financial market S satisfies the condition of *No Free Lunch with Vanishing Risk* (NFLVR), i.e.,

$$\bar{C} \cap L^\infty_+(\Omega, \mathcal{F}, P) = \{0\},$$

where $L^\infty_+(\Omega, \mathcal{F}, P)$ is the set of nonnegative P-a.s. bounded random variables and $\bar{C}$ denotes the L^∞-closure of

$$C := \Big\{ f \in L^\infty(\Omega, \mathcal{F}, P) \;\Big|\; \exists\, \theta \in L(S) \text{ s.t. } \theta \cdot S \text{ is } P\text{-a.s. uniformly} \tag{6.4}$$

$$\text{bounded from below and } f \le (\theta \cdot S)_T \Big\}. \tag{6.5}$$

This 'no-arbitrage' type concept was introduced by Delbaen and Schachermayer; see [DeS94, DeS98] for details and an economic interpretation.

Proposition 6.1.3 *S satisfies (NFLVR) if and only if $\mathcal{M}^e(\mathcal{N}) \neq \emptyset$.*

Proof: $\Longleftarrow$: Let $Q \in \mathcal{M}^e(\mathcal{N})$. Let $f^n \in C$ such that $f^n \stackrel{L^\infty}{\longrightarrow} f$ where $f \in L^\infty_+$. By the definition of C there exist $\theta^n \in L(S)$ such that $f^n \le (\theta^n \cdot S)_T$ and $\theta^n \cdot S$ is uniformly bounded from below. Let $Q \in \mathcal{M}^e(\mathcal{N})$. Then $\theta^n \cdot S$ is a local Q-martingale by Remark 6.1.2 and bounded from below, hence a Q-supermartingale. So we obtain $E^Q[f^n] \le E^Q[(\theta^n \cdot S)_T] \le 0$. Hence $E^Q[f] \le 0$ and it follows that $f = 0$.
$\Longrightarrow$: By Theorem 1.1 in [DeS98], S satisfies (NFLVR) if and only if there is a probability measure $Q \sim P$ such that S is a *Q-sigma-martingale*. The latter means that there exist an $\mathbb{R}^d$-valued (local) Q-martingale $\widetilde{M}$ and a strictly positive $\widetilde{M}$-integrable predictable process φ such that $S = S_0 + \varphi \cdot \widetilde{M}$. For any $N = 1 + \theta \cdot S$ in $\mathcal{N}$ we have $N = 1 + \theta \cdot (\varphi \cdot \widetilde{M}) = 1 + (\theta\varphi) \cdot \widetilde{M}$. Applying Corollaire 3.5 from [AnS94] we obtain $Q \in \mathcal{M}^e(\mathcal{N})$. $\square$

Motivated by Proposition 6.1.3 we impose throughout the remainder of this chapter the

$$\textbf{STANDING ASSUMPTION:} \quad \mathcal{M}^e(\mathcal{N}) \neq \emptyset. \tag{6.6}$$

If $\mathcal{M}^e(\mathcal{N}) = \{Q\}$ is a singleton, we have a martingale representation result even though S need not be a local Q-martingale:

Proposition 6.1.4 *Assume $\mathcal{M}^e(\mathcal{N}) = \{Q\}$. Then for every local Q-martingale M there is $\theta \in L(S)$ such that $M = M_0 + \theta \cdot S$.*

Proof: By Proposition 6.1.3 S satisfies the condition (NFLVR) and this implies ([DeS98], Theorem 1.1) $S = S_0 + \varphi \cdot \widetilde{M}$, where $\widetilde{M}$ is an $\mathbb{R}^d$-valued (local) martingale with respect to some probability measure equivalent to P and φ is a strictly positive $\widetilde{M}$-integrable predictable process. Without loss of generality we can assume $\widetilde{M}_0 = 0$. Corollaire 3.5 from

[AnS94] implies that every equivalent local martingale measure for $\widetilde{M}$ is an element of $\mathcal{M}^e(\mathcal{N})$. Hence Q is the unique equivalent local martingale measure for $\widetilde{M}$. Therefore $\widetilde{M}$ has the strong predictable representation property ([Ja79] Corollaire 11.4). By an additional localization argument ([Ja79] Proposition 2.38) this implies that every local Q-martingale M has a representation $M = M_0 + H \cdot \widetilde{M}$, where H is integrable with respect to $\widetilde{M}$ in the sense of local martingales. But $1/\varphi$ is S-integrable since φ is $\widetilde{M}$-integrable and $\varphi > 0$. Hence $H \cdot \widetilde{M} = H \cdot (1/\varphi \cdot S) = (H/\varphi) \cdot S$ and $\theta := H/\varphi$ is an element of $L(S)$. □

The following lemma summarizes results about the structure of $\mathcal{N}$ that are needed in the sequel.

Lemma 6.1.5 *1. $\mathcal{N}$ is convex.*

2. For $N^1, N^2 \in \mathcal{N}$ and for stopping times σ, τ with $0 \le \sigma \le \tau \le T, A \in \mathcal{F}_\sigma$, the process

$$N := I_{[\![0,\sigma]\!]} N^1 + I_{]\!]\sigma,\tau]\!]} \left(I_A \frac{N^1_\sigma}{N^2_\sigma} N^2 + I_{A^c} N^1 \right) + I_{]\!]\tau,T]\!]} \left(I_A \frac{N^1_\sigma}{N^2_\sigma} \frac{N^2_\tau}{N^1_\tau} + I_{A^c} \right) N^1$$

is an element of $\mathcal{N}$.

3. Every $N \in \mathcal{N}$ can be written as a stochastic exponential, and the set

$$\mathcal{L} := \left\{ L \,\middle|\, L \text{ semimartingale}, \; L_0 = 0, \mathcal{E}(L) \in \mathcal{N} \right\}$$

contains 0 and is predictably convex, i.e., for any predictable $[0,1]$-valued process H and $L^1, L^2 \in \mathcal{L}$, the process $L := H \cdot L^1 + (1-H) \cdot L^2$ is an element of $\mathcal{L}$.

Proof: 1. This is obvious.

3.: Every $N \in \mathcal{N}$ is a local Q-martingale for $Q \in \mathcal{M}^e(\mathcal{N})$ and strictly positive, hence $N_- > 0$ by the minimum principle for supermartingales. Taking $L := (1/N_-) \cdot N$ we get $N = \mathcal{E}(L) \equiv 1 + N_- \cdot L$. $0 \in \mathcal{L}$ follows from $1 \in \mathcal{N}$. In order to show predictable convexity of $\mathcal{L}$ let $N^1, N^2 \in \mathcal{N}$ and $N^i = 1 + \theta^i \cdot S = \mathcal{E}(L^i)$ for $i = 1, 2$. Define $L := H \cdot L^1 + (1-H) \cdot L^2$ and $N := \mathcal{E}(L)$. From the equality $\theta^i dS = dN^i = N^i_- dL^i$ for $i = 1, 2$ it follows

$$dN = N_- \left(H dL^1 + (1-H) dL^2 \right) = \left(\frac{N_- H}{N^1_-} \theta^1 + \frac{N_-(1-H)}{N^2_-} \theta^2 \right) dS \,. \tag{6.7}$$

From $N^i > 0$ follows $\Delta L^i > -1$ for $i = 1, 2$, hence $N > 0$ and by (6.7) we obtain $N \in \mathcal{N}$. This implies $L \in \mathcal{L}$.

2.: Let $N^i = \mathcal{E}(L^i)$ for $i = 1, 2$, $H := I_{[\![0,\sigma]\!]} + I_{]\!]\sigma,\tau]\!]} I_{A^c} + I_{]\!]\tau,T]\!]}$ and $L := H \cdot L^1 + (1-H) \cdot L^2$. We show that $N = \mathcal{E}(L)$. Since $dL = I_{[\![0,\sigma]\!]} dL^1 + I_{]\!]\sigma,\tau]\!]} \left(I_{A^c} dL^1 + I_A dL^2 \right) + I_{]\!]\tau,T]\!]} dL^1$, we obtain $dN_t = dN^1_t = N^1_{t-} \, dL^1_t = N_{t-} \, dL_t$ for $(\omega, t) \in [\![0,\sigma]\!]$. For $(\omega, t) \in]\!]\sigma,\tau]\!]$ we have

$$\begin{aligned} dN_t &= I_A \frac{N^1_\sigma}{N^2_\sigma} N^2_{t-} \, dL^2_t + I_{A^c} N^1_{t-} \, dL^1_t \\ &= N_{t-} d(I_A \cdot L^2)_t + N_{t-} d(I_{A^c} \cdot L^1)_t \; = \; N_{t-} dL_t \,, \end{aligned}$$

and finally for $(\omega, t) \in]\!]\tau, T]\!]$ we observe

$$dN_t = \left(I_A \frac{N^1_\sigma}{N^2_\sigma} \frac{N^2_\tau}{N^1_\tau} + I_{A^c} \right) N^1_{t-} dL^1_t = N_{t-} dL_t\,.$$

So N solves $N = 1 + N_- \cdot L$. Hence $N = \mathcal{E}(L)$ and by part 3 this proves the claim. □

6.2 The growth-optimal numeraire

It is well known that there is a close relationship between the numeraire portfolio and the growth-optimal numeraire. In fact, the two coincide in the cases studied in [Lo90]. We shall see in the next section that this remains true in general.

Definition 6.2.1 *We call a tradable numeraire* growth-optimal *and denote it by $N^{\log}$ if it solves the maximization problem*

$$u := \sup_{N \in \mathcal{N}} E[\log N_T]\,. \tag{6.8}$$

Justification for the name is given by Proposition 6.2.3 which summarizes the essential properties of $N^{\log}$.

Remark 6.2.2 The supremum in (6.8) is a priori only taken over all $N \in \mathcal{N}$ for which $E[\log N_T]$ is defined. If $u < \infty$ then $\log^+ N_T \in L^1(P)$ for every $N \in \mathcal{N}$. To see this, observe that for $\varepsilon \in (0,1)$ $N^\varepsilon := \varepsilon + (1-\varepsilon)N \geq \varepsilon$ is bounded away from zero, $N^\varepsilon \in \mathcal{N}$ by Lemma 6.1.5, and the concavity of log implies $E[\log^+ N^\varepsilon_T] \geq (1-\varepsilon)E[\log^+ N_T]$. Hence $E[\log N_T]$ is well-defined with values in $\mathbb{R} \cup \{-\infty\}$ if u is finite. If $0 \leq \sigma \leq \tau \leq T$ then part 2 of Lemma 6.1.5 (with $N^1 = 1$, $N^2 = N$, $A = \Omega$) gives an element $N' \in \mathcal{N}$ such that $N'_T = N_\tau / N_\sigma$. The same argument as above shows that $E[\log(N_\tau/N_\sigma)]$ is also well-defined with values in $\mathbb{R} \cup \{-\infty\}$ if u is finite. This will be used later. ◇

Proposition 6.2.3 *Assume $u < \infty$.*

1. *The growth-optimal numeraire $N^{\log}$ is unique if it exists.*
2. *If it exists, $N^{\log}$ maximizes simultaneously all intertemporal conditional expected growth rates: For any stopping times σ, τ with $0 \leq \sigma \leq \tau \leq T$ we have*

$$E\left[\log \frac{N^{\log}_\tau}{N^{\log}_\sigma} \middle| \mathcal{F}_\sigma \right] \geq E\left[\log \frac{N_\tau}{N_\sigma} \middle| \mathcal{F}_\sigma \right] \quad \forall N \in \mathcal{N}\,. \tag{6.9}$$

Proof: 2.: Using Remark 6.2.2 we have for $0 \leq \sigma \leq \tau \leq T$

$$E\left[\log N^{\log}_T\right] = E\left[\log N^{\log}_\sigma\right] + E\left[\log \frac{N^{\log}_\tau}{N^{\log}_\sigma}\right] + E\left[\log \frac{N^{\log}_T}{N^{\log}_\tau}\right].$$

The finiteness of the left-hand side now implies $\log\left(N_\tau^{\log}/N_\sigma^{\log}\right) \in L^1(P)$. Define $N \in \mathcal{N}$ according to part 2 of Lemma 6.1.5 with $N^1 = N^{\log}$ and arbitrary $N^2 \in \mathcal{N}, A \in \mathcal{F}_\sigma$. Since $E[\log N_T]$ is well-defined by Remark 6.2.2, the log-optimality of $N^{\log}$ implies

$$\begin{aligned} 0 \le E\left[\log N_T^{\log} - \log N_T\right] &= E\left[-\log\left(I_A \frac{N_\sigma^{\log}}{N_\sigma^2}\frac{N_\tau^2}{N_\tau^{\log}} + I_{A^c}\right)\right] \\ &= E\left[I_A\left(\log\frac{N_\tau^{\log}}{N_\sigma^{\log}} - \log\frac{N_\tau^2}{N_\sigma^2}\right)\right]. \end{aligned}$$

Using the P-integrability of $\log\left(N_\tau^{\log}/N_\sigma^{\log}\right)$ we obtain part 2.

1.: The second part implies in particular the equality $E\left[\log N_t^{\log}\right] = \sup_{N\in\mathcal{N}} E[\log N_t]$ for all $t \in [0,T]$. Hence the uniqueness of $N_t^{\log}$ for every $t \in [0,T]$ is a consequence of the convexity of $\mathcal{N}$ and the strict concavity of the logarithm. □

6.3 The numeraire portfolio

Definition 6.3.1 *Let $Q \sim P$. A tradable numeraire in $\mathcal{N}$ is called the Q-*numeraire portfolio[1] *and denoted by N^Q if N/N^Q is a Q-supermartingale for all $N \in \mathcal{N}$.*

There can be at most one Q-numeraire portfolio. To see this, assume N^1 and N^2 are both Q-numeraire portfolios. Then N^1/N^2 and also the reciprocal N^2/N^1 are Q-supermartingales. Using Jensen's inequality we conclude that N^1 and N^2 are equal.

Remark 6.3.2 The definition is a consistent extension of the numeraire portfolio concept (6.2) introduced by Long [Lo90]. In fact, a tradable numeraire N satisfying condition (6.2) (i.e. S/N and $1/N$ are (local) Q-martingales for some $Q \sim P$) defines (locally) a martingale measure $d\widetilde{Q} := 1/N_{T(n)}\, dQ$ for S. By Corollaire 3.5 in [AnS94] this implies that any tradable numeraire N' is a local $\widetilde{Q}$-martingale. Thus N'/N is a local Q-martingale for all $N' \in \mathcal{N}$ and hence N is the unique Q-numeraire portfolio. ◇

We shall confine ourselves in the sequel to the P-numeraire portfolio N^P. This is without loss of generality since we could choose Q instead of P as the original measure of our probability space.

Existence and characterization

The following proposition shows that the condition defining the P-numeraire portfolio is equivalent to growth-optimality. Hence as in the cases studied in the literature the numeraire

[1] This is a slight abuse of terminology: we do not distinguish here between a portfolio and the corresponding value process in order to keep the usual term 'numeraire portfolio'.

portfolio is equal to the growth optimal portfolio.

Proposition 6.3.3 *Assume $u < \infty$. A numeraire is growth-optimal if and only if it is equal to the P-numeraire portfolio, i.e., $N^P = N^{\log}$.*

Note that this result shows in particular that the existence of the P-numeraire portfolio implies the existence of the growth-optimal portfolio and vice versa.

Proof: Assume N^P exists. From $\log a \leq a - 1$ for $a > 0$ we get that $\log y \geq \log x + 1 - x/y$ for all $x, y > 0$. Hence $\log N_T^P \geq \log N_T + 1 - N_T/N_T^P$ for all $N \in \mathcal{N}$. Taking $N = 1$ and using the property defining N^P shows $(\log N_T^P)^- \in L^1(P)$. Hence $E[\log N_T^P]$ is well-defined with values in $\mathbb{R} \cup \{\infty\}$. So we get $E\left[\log N_T^P\right] \geq E\left[\log N_T\right]$ for any $N \in \mathcal{N}$. This proves the growth-optimality of the numeraire portfolio. To prove the converse assume now that $N^{\log}$ exists. We proceed as Kramkov and Schachermayer [KrS99] (proof of (3.17) in Lemma 3.9) in order to obtain (6.10). Let $N \in \mathcal{N}, \varepsilon \in [0,1]$ and define $N^\varepsilon := (1-\varepsilon)N^{\log} + \varepsilon N \in \mathcal{N}$. Since $u < \infty$ we get

$$\begin{aligned} 0 \ &\leq \ E\left[\log N_T^{\log} - \log N_T^\varepsilon\right] = E\left[\int_{N_T^\varepsilon}^{N_T^{\log}} \frac{1}{z}\, dz\right] \\ &\leq \ E\left[\frac{N_T^{\log} - N_T^\varepsilon}{N_T^\varepsilon}\right] = E\left[\frac{\varepsilon(N_T^{\log} - N_T)}{(1-\varepsilon)N_T^{\log} + \varepsilon N_T}\right]. \end{aligned}$$

Thus $E\left[N_T/N_T^\varepsilon\right] \leq E\left[N_T^{\log}/N_T^\varepsilon\right]$, and for $\varepsilon \downarrow 0$ we get by Fatou's lemma

$$E\left[\frac{N_T}{N_T^{\log}}\right] \leq 1 \quad \text{for all } N \in \mathcal{N}. \tag{6.10}$$

For $N' \in \mathcal{N}$ and $0 \leq s \leq t \leq T$ define $A := \left\{\left(N_s^{\log}/N_s'\right) E\left[N_t'/N_t^{\log}\middle|\mathcal{F}_s\right] > 1\right\}$. By part 2 of Lemma 6.1.5 (take $N^2 = N'$, $N^1 = N^{\log}$), (6.10) implies

$$1 \ \geq \ E\left[I_A \frac{N_s^{\log} N_t'}{N_s' N_t^{\log}} + I_{A^c}\right] = E\left[I_A \frac{N_s^{\log}}{N_s'} E\left[\frac{N_t'}{N_t^{\log}}\middle|\mathcal{F}_s\right] + I_{A^c}\right].$$

By the definition of A we obtain $P[A] = 0$ and conclude that $N^{\log}$ is the P-numeraire portfolio. □

Definition 6.3.4 *We denote by*

$$\mathcal{SM} := \left\{Z \,\middle|\, Z \geq 0,\ Z_0 = 1,\ ZN \text{ is a } P\text{-supermartingale for all } \ N \in \mathcal{N}\right\} \tag{6.11}$$

*the family of all P-*supermartingale densities *for $\mathcal{N}$.*

$\mathcal{SM}$ is not empty because for any $Q \in \mathcal{M}^e(\mathcal{N})$ the density process of Q with respect to P is in $\mathcal{SM}$. Note that each $Z \in \mathcal{SM}$ is actually a P-supermartingale because $1 \in \mathcal{N}$. This implies $\log^+ Z_T \in L^1(P)$ and hence $E[\log 1/Z_T]$ is always well-defined and takes values in $\mathbb{R} \cup \{+\infty\}$. In the spirit of Kramkov and Schachermayer [KrS99] we can now set up the dual problem to (6.8):

$$h := \inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_T}\right] \tag{6.12}$$

If $Z_T = dQ/dP$ is the density of a probability measure $Q \sim P$, the right-hand side is the relative entropy $H_{\mathcal{F}_T}(P|Q) := E^Q[(1/Z_T)\log(1/Z_T)]$ of P with respect to Q on $(\Omega, \mathcal{F}_T)$. The full duality picture is described by the following theorem, which is an application of a general duality theorem on utility maximization due to Kramkov and Schachermayer [KrS99]; see also Karatzas et al. [KaLSX91] for previous convex duality results for such problems.

Theorem 6.3.5 *Assume (6.6) (NFLVR) and $u < \infty$ (or equivalently $h < \infty$). Then*

1. *There exists a unique solution $N^{log} \in \mathcal{N}$ to the log-optimization problem (6.8) which is also optimal with respect to any previous time horizon, i.e.*

$$\exists!\, N^{\log} \in \mathcal{N} \quad \textit{such that} \quad E\left[\log N_\tau^{\log}\right] = \sup_{N \in \mathcal{N}} E[\log N_\tau]$$

for all stopping times $\tau \le T$.

2. *There exists a unique solution to the 'reverse entropy'-minimization problem (6.12) which also minimizes the 'reverse entropy' with respect to any previous time horizon, i.e.*

$$\exists!\, \widehat{Z} \in \mathcal{SM} \quad \textit{such that} \quad E\left[\log \frac{1}{\widehat{Z}_\tau}\right] = \inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_\tau}\right]$$

for all stopping times $\tau \le T$.

3. *The solutions to the problems (6.8) and (6.12) are related by $1/N^{\log} = \widehat{Z}$.*

4. *For any stopping time $\tau \le T$ have*

$$\inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_\tau}\right] = \inf_{Q \in \mathcal{M}^e(\mathcal{N})} E\left[\log \frac{1}{\left(\frac{dQ}{dP}\,|_{\mathcal{F}_\tau}\right)}\right] = \inf_{Q \in \mathcal{M}^e(\mathcal{N})} H_{\mathcal{F}_\tau}(P|Q).$$

Proof: Applying Theorem 2.2 of [KrS99] for the case of a logarithmic utility function, we obtain part 4 and the existence of the unique solutions $N^{\log}$ and $\widehat{Z}$ to the optimization problems (6.8) and (6.12). Using Proposition 6.2.3 we obtain part 1. Note that in the log-utility case, we have the duality in part 3 not just for the terminal values $N_T^{\log}$ and $\widehat{Z}_T$ as in [KrS99] but for the processes on the entire time interval. To see this, observe that Proposition 6.3.3 implies $1/N^{\log} \in \mathcal{SM}$. Since we know from [KrS99] that $1/N_T^{\log} = \widehat{Z}_T$, the uniqueness of $\widehat{Z}$ implies the assertion of part 3. Using part 1 this implies part 2. □

Remark 6.3.6 Note that the two assumptions of Theorem 6.3.5 are of no-arbitrage character: a) there exists an equivalent local martingale measure for $\mathcal{N}$ such that b) the relative entropy of P with respect to this martingale measure is finite. The first assumption is equivalent to (NFLVR) by Proposition 6.1.3 and the second assumption is equivalent to the assumption that the optimal expected growth up to the finite time horizon T is not infinite. $\diamond$

As immediate consequences we obtain the following results about the numeraire portfolio:

Corollary 6.3.7 *(existence) Under the assumptions of Theorem 6.3.5, the P-numeraire portfolio N^P exists.*

Corollary 6.3.8 *(equivalent characterizations) Suppose the assumptions of Theorem 6.3.5 are satisfied. Then $\widehat{N} \in \mathcal{N}$ is the P-numeraire portfolio if and only if it solves one (hence every) of the optimization problems*

$$E[\log \widehat{N}_T] = \sup_{N \in \mathcal{N}} E[\log N_T] \tag{6.13}$$

$$E[\log \widehat{N}_T] = \inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_T}\right] \tag{6.14}$$

$$E[\log \widehat{N}_T] = \inf_{Q \in \mathcal{M}^e(\mathcal{N})} H(P|Q) \tag{6.15}$$

This allows us to state a necessary and sufficient condition for the reciprocal of the numeraire portfolio to define an equivalent local martingale measure:

Corollary 6.3.9 *(equivalent local martingale measure condition) Suppose the assumptions of Theorem 6.3.5 are satisfied. Then $1/N^P_T$ is the density of an equivalent local martingale measure $\widehat{Q} \in \mathcal{M}^e(\mathcal{N})$ with respect to P if and only if $\inf_{Q \in \mathcal{M}^e(\mathcal{N})} H(P|Q)$ is attained by $\widehat{Q}$. In this case $1/N^P$ is the density process of $\widehat{Q}$ with respect to P.*

For continuous price processes S, a result of Schweizer [Schw99] shows that the minimal martingale measure (in the sense of Föllmer and Schweizer, see e.g. [FSchw91, Schw95]), if it exists, solves the above reverse entropy minimization problem. Hence we can identify in this case the discount factor given by the P-numeraire portfolio as the minimal martingale density:

Corollary 6.3.10 *(relation to the minimal martingale measure) Assume S is continuous, the minimal martingale measure $\widehat{P}$ exists and $h < \infty$. Then $1/N^P_t = \frac{d\widehat{P}}{dP}\Big|_{\mathcal{F}_t}$ for all $t \in [0, T]$.*

Finally we provide a sufficient condition for the numeraire portfolio to match the requirement in approach (6.2), i.e., for $1/N^P$ to be a 'strict martingale density' (see Schweizer [Schw95]). The proof of this result consists of a simple stopping argument that uses Remark 6.1.2.

Corollary 6.3.11 *(strict martingale density condition) Assume S is locally bounded, N^P exists and $1/N^P$ is a local P-martingale. Then S/N^P is a local P-martingale.*

Note that the last condition is necessary since $1/N^P$ actually can fail to be a local P-martingale (see Example 6 and the subsequent remark in Section 6.4).

Possible generalizations:

1. Since growth-optimality and the property that defines the numeraire portfolio are local properties, the theory could be generalized to the infinite horizon setting $[0, \infty)$ by localization; the assumptions and results then have to be formulated with respect to every T^n from an increasing sequence $(T^n)_{n \in \mathbb{N}}$ of bounded stopping times.

2. For simplicity the present chapter assumes the value of the savings account to be constant. This assumption can be considerably relaxed. The basic idea is the following: If the savings account is modelled by $S^0 > 0$, one can show (in analogy to Proposition 1 in Geman et al. [GER95], see the proof of Theorem 11 in Delbaen and Schachermayer [DeS95], or Proposition 2.1 in Goll and Kallsen [GoK00]) that every (suitably defined) tradable numeraire discounted by the numeraire S^0 is also a tradable numeraire (with the same trading strategy in terms of numbers of assets) with respect to the discounted market. This brings us back to the present setting. Since the discount factor $1/S^0$ cancels out in the condition that defines the numeraire portfolio, we obtain that the numeraire portfolio in the original market and the numeraire portfolio in the discounted market are equal up to the discount factor and use the same strategy. Hence the numeraire portfolio property is invariant under a change of numeraire. If $E[\log S^0_T]$ is finite, this implies that the growth-optimal strategy with respect to the original prices and the one with respect to discounted prices also coincide.

3. Let us note that the duality relation in Proposition 6.3.3 only relies on the properties of $\mathcal{N}$ which are given by Lemma 6.1.5. Thus, the result remains true in a more general market model if the set $\mathcal{N}$ of tradable numeraires in this market satisfies those properties. For example, one could put convex constraints on $\mathcal{L}$.

Pricing property

We conclude this section with some comments on a question that motivated the idea of the numeraire portfolio in the introduction. To what extent does the quantity $\hat{p} := E[H/N^P_T]$ provide a reasonable valuation of an European option with maturity T, modelled by an $\mathcal{F}_T$-measurable random variable $H \geq 0$?

Examples in Section 6.4 will show that in several cases

$$\widehat{p} := E_P \left[\frac{H}{N_T^P} \right] \tag{6.16}$$

is equal to the expectation of H under an equivalent local martingale measure Q. But other examples show that in the general semimartingale case $1/N_T^P$ can fail to be the density of a local martingale measure.

Therefore the question arises what meaning can be given to the quantity $\widehat{p}$ in general. Suppose the option that pays H at time T has price $p > 0$ at time zero. Then an investor who has invested a fraction $\delta \leq 1$ of his initial capital $x > 0$ in the option and wants to maximize expected log-utility faces the optimization problem

$$w(\delta, x, p) := \sup_{N \in \mathcal{N}} E\left[\log\left((1-\delta) x N_T + \frac{\delta x}{p} H \right) \right] . \tag{6.17}$$

The next result shows that the investor is unable to increase his achievable expected utility by diverting some of his initial capital into a buy (respectively sell) and hold strategy with respect to the option when the option price at time zero is $\widehat{p}$.

Lemma 6.3.12 *Assume (6.6) (NFLVR), $u < \infty$ and $0 < \widehat{p} < \infty$. Then the function $\delta \mapsto w(\delta, x, \widehat{p})$ attains its maximum on $(-\infty, 1]$ at $\delta = 0$.*

Proof: The P-numeraire portfolio N^P exists by Corollary 6.3.7. From $\log a \geq \log b + 1 - b/a$ for $a, b > 0$ it follows

$$\log(x N_T^P) \geq \log\left((1-\delta) x N_T + \frac{\delta x}{\widehat{p}} H \right) + 1 - \frac{1}{N_T^P} \left((1-\delta) N_T + \frac{\delta}{\widehat{p}} H \right) .$$

Taking expectations on both sides and using the property that defines N^P then proves the claim. □

Assuming differentiability it follows

$$\frac{\partial w}{\partial \delta}(0, x, \widehat{p}) = 0 . \tag{6.18}$$

This leads to the concept of Davis [D97] who defined a 'fair option price' as the solution $\widehat{p}(x)$ to equation (6.18). This means that transferring a little of his funds into the option contract has no effect on the investor's achievable expected utility. In the case of a logarithmic utility function it follows that the 'fair option price' is given by (6.16).

Clearly, $\hat{p}$ provides an arbitrage-free valuation if $1/N_T^P$ is the density of an equivalent local martingale measure. But the examples in Section 6.4 show that the latter is not the case in general. In particular, Example 6 illustrates that $\hat{p}$ can fail to be within the interval

$\left[\inf_{Q\in\mathcal{M}^e} E^Q[H]\,,\, \sup_{Q\in\mathcal{M}^e} E^Q[H]\right]$ even for $H = 1$. Kramkov and Schachermayer suggested an arbitrage-free interpretation of $\hat{p}$ with respect to a new numeraire, namely N^P; see p. 911-912 in [KrS99]. Note, however, that their set of admissible strategies is defined by those portfolio strategies which lead to a wealth process that is uniformly bounded from below. Since this set is not invariant under a change of numeraire, changing the numeraire implies changing the set of admissible strategies for the investor.

6.4 Examples

To illustrate the theory, we now provide some examples. Example 1 shows the situation of a complete market. Examples 2 and 3 are those cases for which Long [Lo90] introduced and studied the idea of the numeraire portfolio. In these three cases the P-numeraire portfolio is the reciprocal of the density of an equivalent (local) martingale measure for S. So the asset prices discounted by the P-numeraire portfolio are (local) martingales with respect to the original measure P. But other examples show that the general semimartingale case can lead to less well behaved situations. In particular, the numeraire portfolio may give only a local martingale or supermartingale discount factor even with respect to the constant savings account.

Example 1 ('complete' market):
Assume there is a unique equivalent local martingale measure Q for $\mathcal{N}$ (respectively S). Then the P-numeraire portfolio N^P exists and $1/N^P$ is the density process of Q with respect to P.

Proof: Let Q denote the unique equivalent martingale measure for $\mathcal{N}$ (or S, respectively) and Z the density process of P with respect to Q. Note that the uniqueness of Q implies that $\mathcal{F}_0$ is trivial. By Proposition 6.1.4 (or the strong predictable representation property of S, respectively) there exists $\theta \in L(S)$ such that $Z = 1 + \theta \cdot S$. Hence $Z \in \mathcal{N}$ and $1/Z$ is the density process of an equivalent local martingale measure for $\mathcal{N}$ (respectively S). This implies $Z = N^P$. □

Example 2 (finite discrete case):
Assume (6.6) (no arbitrage) and card $(\Omega) < \infty$. Then the P-numeraire portfolio N^P exists and is log-optimal, and $1/N^P, S/N^P$ are both P-martingales.

Proof: We give a direct proof to illustrate how the martingale property follows from the first order condition of the log-optimization problem. For $Q \in \mathcal{M}^e(\mathcal{N})$ and $\theta \in L(S)$, the process $V := 1+\theta\cdot S$ is a Q-martingale, hence $E^Q[V_T] = 1$. Denote by $\mathcal{V}$ the set of all such processes V which are in addition nonnegative. $\mathcal{V}$ is bounded (a.s.) and closed (identifying $L^0(\Omega, \mathcal{F}, P)$ with some $\mathbb{R}^m$, we can view $\mathcal{V}$ as the intersection of an affine subspace with the positive quadrant). Hence $\sup_{V\in\mathcal{V}} E[\log V_T]$ is attained by a growth-optimal $N^{\log} \in \mathcal{V}$. The optimality implies that $N^{\log}$ is strictly positive and hence an element of $\mathcal{N}$. Since $N^{\log} =: 1 + \theta^{\log} \cdot S$ is bounded away from zero, the first order condition yields for $\theta \in L(S)$, $A \in \mathcal{F}_s$ and $0 \leq s < t \leq T$

$$0 = \frac{\partial}{\partial \varepsilon} E\left[\log\left(1 + \int_0^T \left(\theta_u^{\log} + \varepsilon I_A I_{]s,t]}(u)\theta_u\right) dS_u\right)\right]\Big|_{\varepsilon=0} = E\left[I_A \frac{1}{N_T^{\log}} \int_s^t \theta_u \, dS_u\right].$$

Hence $E\left[(1/N_T^{\log}) \int_s^t \theta_u \, dS_u \mid \mathcal{F}_s\right] = 0$ for all $\theta \in L(S)$. By taking $\theta = \theta^{\log}$ and $t = T$ this implies in particular that $1/N^{\log}$ and hence $S/N^{\log} = (S_0 + (S - S_0))/(N^{\log})$ are P-martingales. □

Example 3 (multidimensional Itô-process model):
Assume that asset prices evolve according to the standard multidimensional Itô-process model. Then under suitable regularity conditions the P-numeraire portfolio N^P exists and is growth-optimal, and $1/N^P, S/N^P$ are (local) P-martingales.
To be more precise, let the (discounted) prices of the risky assets be given by the stochastic differential equations

$$\text{(6.19)} \qquad \frac{dS_t^i}{S_t^i} = (\mu_t^i - r_t)dt + \sum_{j=1}^n \sigma_t^{ij} \, dW_t^j, \quad S_0^i > 0 \quad \text{for } i = 1, \ldots, d,$$

where $W = (W^j)_{j=1,\ldots,n}$ is an n-dimensional Brownian motion with respect to P and some filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions (e.g. the P-augmentation of the filtration generated by W). Let the coefficients $r, \mu = (\mu^j)_{j=1,\ldots,n}$ and $\sigma = (\sigma^{ij})_{i=1,\ldots,d; j=1,\ldots,n}$ be predictable. Assume S is well-defined, $n \geq d$ and σ_t has full rank a.s. for every t.
In order to eliminate the drift terms by a Girsanov transformation, define (with tr denoting transposition and $\mathbb{1} = (1, \ldots, 1)^{\text{tr}} \in \mathbb{R}^d$) $\widehat{\lambda}_t := \sigma_t^{\text{tr}}(\sigma_t \sigma_t^{\text{tr}})^{-1}(\mu_t - r_t \mathbb{1})$ and (assuming $\int_0^T |\widehat{\lambda}_t|^2 dt < \infty$ a.s.) $\widehat{Z} := \mathcal{E}\left(-\widehat{\lambda} \cdot W\right)$. If the local P-martingale $\widehat{Z}$ is a true P-martingale, it defines an equivalent local martingale measure $d\widehat{P} = \widehat{Z}_T dP$ which is the minimal martingale measure. By Corollary 6.3.10 we have $1/N^{\log} = \widehat{Z}$ so that $S/N^{\log}$ is a local P-martingale. Under additional conditions (e.g. boundedness of the coefficients in (6.19)), we can obtain that $S/N^{\log}$ is even a true P-martingale.
On the other hand, $\widehat{Z}$ might be just a local martingale but not a martingale. In that case, it is straitforward to show that $1/\widehat{Z}$ is in $\mathcal{N}$ and satisfies the property which defines the numeraire portfolio N^P. Hence, $1/N^P$ is just a local martingale but not a martingale then.

Example 4 ($1/N^P$ is a local martingale but fails to be a martingale):
We recall just the results of a well-known counterexample of mathematical finance (see Delbaen and Schachermayer [DeS98a], Theorem 2.1, take $S := 1/X$). The incomplete market contains just one continuous stock price process S, modelled by the reciprocal of the stochastic exponential of a suitably stopped Brownian motion. Then $1/N^P = 1/N^{\log} = \widehat{Z}$ is a local

P-martingale but not a martingale. So it is not the density process of any martingale measure, although an equivalent local martingale measure for S (and hence for $\mathcal{N}$) exists.

Example 5 (discrete time model with multiplicative i.i.d. returns):
Let $S^i_k = s^i_0 \prod_{j=1}^k Y^i_j$ for $k = 0, \dots, T$, $i = 1, \dots, d$ where $s^i_0 > 0$ and $Y_1, \dots, Y_T$ are i.i.d. $\mathbb{R}^d_+$-valued random variables. Let $\mathbb{F}$ be the filtration generated by S. Assume (6.6) (no arbitrage) and

$$\sup_{\lambda \in \mathcal{B}} E[\log(1 + \lambda^{\mathrm{tr}}(Y_1 - \mathbb{1}))] < \infty$$

where $\mathcal{B} := \left\{ \lambda \in \mathbb{R}^d \middle| \, E\left[\log^-\left(1 + \lambda^{\mathrm{tr}}(Y_1 - \mathbb{1})\right)\right] < \infty \right\}$. Then

1. The P-numeraire portfolio N^P exists, N^P is growth-optimal and a corresponding optimal strategy always invests a constant fraction $\hat{\lambda}^i$ of current wealth in asset S^i for $i = 1, \dots, d$, i.e., there is $\hat{\lambda} \in \mathcal{B}$ such that for $k = 0, \dots, T$

$$N^P_k = \prod_{j=1}^k (1 + \hat{\lambda}^{\mathrm{tr}}(Y_j - \mathbb{1})). \tag{6.20}$$

2. If the growth-optimal strategy invests a strictly positive fraction of wealth in each asset, i.e., $1 - \sum_{i=1}^d \hat{\lambda}^i > 0$ and $\hat{\lambda}^i > 0$ for $i = 1, \dots, d$, then $1/N^P$ and S/N^P are P-martingales.

3. Let $d = 1$. If the risky asset has higher expected returns than the riskless asset but also a positive probability for default, i.e., $E[Y_1] > 1$ and $P[Y_1 = 0] > 0$, then $1/N^P$ and S/N^P are P-martingales.

We note that $\sup_{\lambda \in \mathcal{B}} E[\log(1 + \lambda^{\mathrm{tr}}(Y_1 - \mathbb{1}))]$ is always finite if Ω is finite and if there is no arbitrage (see Example 2). To give also an non-finite example, we could take $X \sim \mathcal{N}(\mu, \Sigma)$ be multidimensional normally distributed with $\det \Sigma \neq 0$ and define $Y^i := \exp(X^i)$, $i = 1, \dots, d$. The proof is fairly standard.

Proof: 1.: Let $N_\cdot := 1 + \sum_{k=1}^{\cdot} \theta^{\mathrm{tr}}_k (S_k - S_{k-1})$ be a tradable numeraire. We could parametrize the corresponding strategy in terms of portfolio proportions π instead of numbers of assets θ by taking $\pi^i_k := \theta^i_k S^i_{k-1}/N_{k-1}$ and get

$$N_\cdot = \prod_{k=1}^{\cdot} \left(1 + \pi^{\mathrm{tr}}_k (Y_k - \mathbb{1})\right). \tag{6.21}$$

On the other hand, any predictable $\mathbb{R}^d$-valued process $\pi = (\pi^i)_{i=1,\dots,d}$ for which the right-hand side in (6.21) is strictly positive defines a tradable numeraire $N \in \mathcal{N}$. Using this we get

$$\begin{aligned} E\left[\log \frac{N_k}{N_{k-1}} \middle| \mathcal{F}_{k-1}\right] &= E\left[\log(1 + \lambda^{\mathrm{tr}}(Y_1 - \mathbb{1}))\right] \Big|_{\lambda = \pi_k(\omega)} \\ &\leq \sup_{\lambda \in \mathcal{B}} E\left[\log(1 + \lambda^{\mathrm{tr}}(Y_1 - \mathbb{1}))\right] < \infty . \end{aligned}$$

This implies $\sup_{N\in\mathcal{N}} E[\log N_T] < \infty$ and so by Theorem 6.3.5 the P-numeraire portfolio N^P exists and is growth-optimal. As in (6.21) we have $N^P_\cdot = \prod_{k=1}^{\cdot}\left(1+\hat{\pi}_k^{\mathrm{tr}}(Y_k - 1\!\!1)\right)$ for some predictable process $\hat{\pi}$. Now we obtain

$$(6.22)\qquad E\left[\log(1+\lambda^{\mathrm{tr}}(Y_1 - 1\!\!1))\right]|_{\lambda=\hat{\pi}_k} = \sup_{\lambda\in\mathcal{B}} E\left[\log(1+\lambda^{\mathrm{tr}}(Y_1 - 1\!\!1))\right] \quad P\text{-a.s.}$$

because otherwise there would be some $\tilde{\lambda}\in\mathcal{B}$ such that the event

$$A := \left\{E\left[\log(1+\lambda^{\mathrm{tr}}(Y_1 - 1\!\!1))\right]|_{\lambda=\hat{\pi}_k} < E\left[\log(1+\tilde{\lambda}^{\mathrm{tr}}(Y_1 - 1\!\!1))\right]\right\} \in \mathcal{F}_{k-1}$$

satisfies $P[A] > 0$ and for $\widetilde{N}\in\mathcal{N}$ corresponding to $\tilde{\pi}_j := I_{\{j=k\}}(\tilde{\lambda}I_A + \hat{\pi}_k I_{A^c}) + I_{\{j\neq k\}}\hat{\pi}_j$ via (6.21) we would get the contradiction $E[\log \widetilde{N}_k] > E[\log N^P_k]$. From equation (6.22) it follows that $\sup_{\lambda\in\mathcal{B}} E\left[\log(1+\lambda^{\mathrm{tr}}(Y_1 - 1\!\!1))\right]$ is attained for some $\hat{\lambda}\in\mathcal{B}$. (6.22) and (6.21) then yield formula (6.20).

2.: Since $S^i > 0$ we have $\prod_{k=1}^{\cdot} Y^i_k = S^i/S^i_0 \in \mathcal{N}$ for all $i = 1,\dots,d$. By part 1 the P-numeraire portfolio exists and so we have for $k = 1,\dots,T$

$$(6.23)\qquad 1 \;\geq\; E\left[\frac{N^P_{k-1}}{N^P_k}\middle|\mathcal{F}_{k-1}\right] = E\left[\frac{1}{1+\hat{\lambda}^{\mathrm{tr}}(Y_1 - 1\!\!1)}\right],$$

$$(6.24)\qquad 1 \;\geq\; E\left[\frac{N^P_{k-1}S^i_k}{S^i_{k-1}N^P_k}\middle|\mathcal{F}_{k-1}\right] = E\left[\frac{Y^i_1}{1+\hat{\lambda}^{\mathrm{tr}}(Y_1 - 1\!\!1)}\right] \text{ for all } i = 1,\dots,d.$$

Since all coefficients in

$$1 = (1-\hat{\lambda}^{\mathrm{tr}}1\!\!1)E\left[\frac{1}{1+\hat{\lambda}^{\mathrm{tr}}(Y_1 - 1\!\!1)}\right] + \sum_{i=1}^{d}\hat{\lambda}^i E\left[\frac{Y^i_k}{1+\hat{\lambda}^{\mathrm{tr}}(Y_1 - 1\!\!1)}\right]$$

are strictly positive we must have equality in (6.23) and (6.24). This proves the claim.

3.: From $P[Y_1 = 0] > 0$ follows $\hat{\lambda} < 1$ and $E[Y_1] > 1$ implies $\hat{\lambda}\neq 0$. Assuming $\hat{\lambda} < 0$ we would get $E\left[\log\left(1-\hat{\lambda}+\hat{\lambda}Y_1\right)\right] \leq \log E\left[1-\hat{\lambda}+\hat{\lambda}Y_1\right] < 0$, which is a contradiction to the growth-optimality of the P-numeraire portfolio since $1\in\mathcal{N}$ implies $u \geq 0$. Hence $\hat{\lambda}\in(0,1)$ and the result from part 3 follows from part 2. □

Example 6 ($1/N^P$ (or S/N^P) is a supermartingale but not a local martingale):
We take the setting of Example 5 with one stock ($d = 1$) and one period ($T = 1$). For notational simplicity, we write Y instead of Y^1_1. Let $S^1_0 = 1$ and Y be lognormally distributed, i.e., $\log Y \sim \mathcal{N}(\mu,\sigma^2)$ with $\sigma > 0$. By the first result of Example 5 we have $N^P_1 = 1+\hat{\lambda}(Y-1)$ for some $\hat{\lambda}\in\mathcal{B} = [0,1]$. Let $N, N'\in\mathcal{N}$. Since $N = 1+\theta\cdot S$ is a tradable numeraire if and only if $\theta\in[0,1]$, we obtain that all coefficients in

$$E\left[\frac{N_1}{N'_1}\right] = (1-\theta)E\left[\frac{1}{N'_1}\right] + \theta E\left[\frac{Y}{N'_1}\right]$$

are nonnegative. It follows that N' is the P-numeraire portfolio if and only if $E[1/N'_1] \leq 1$ and $E[Y/N'_1] \leq 1$. Simple calculation yields

$$E[Y] = \exp\left(\mu + \frac{\sigma^2}{2}\right) \quad \text{and} \quad E\left[\frac{1}{Y}\right] = \exp\left(-\mu + \frac{\sigma^2}{2}\right) . \tag{6.25}$$

We conclude that

$$\begin{array}{rclcrcl} \hat{\lambda} & = & 1 & \text{if} & \mu & \geq & \sigma^2/2\,, \\ \hat{\lambda} & = & 0 & \text{if} & \mu & \leq & -\sigma^2/2\,, \\ \hat{\lambda} & \in &]0,1[& \text{if} & |\mu| & < & \sigma^2/2\,. \end{array}$$

Hence $1/N^P$ (respectively S/N^P) is a P-supermartingale but not a martingale if $\mu > \sigma^2/2$ (respectively $\mu < -\sigma^2/2$). If $|\mu| \leq \sigma^2/2$ then $1/N^P$ and S/N^P are both P-martingales. This follows from (6.25) if $|\mu| = \sigma^2/2$ and from the results of Example 5 otherwise.

With somewhat more effort, Example 5.1' in Kramkov and Schachermayer [KrS99] shows that the same effect as in Example 6 can occur even on a countable probability space with bounded asset prices.

Part IV

REACTION DIFFUSION EQUATIONS

Chapter 7

On interacting systems of semi-linear PDEs

In this chapter, we use stochastic methods to derive existence and uniqueness results for classical solutions of interacting systems of semi-linear parabolic partial differential equations. These systems – also known as reaction-diffusion equations – play a key role in our solution to a valuation and hedging problem from mathematical finance in Chapter 3. There, an Itô process S models the prices of the tradable assets, and further untraded factors of risk are represented by a finite-state process η. Similarly as in the Black-Scholes model, the solution to the valuation and hedging problem can be described via a partial differential equation (PDE). However, the untradable factors of risk lead to an interacting system of PDEs; each single PDE corresponds to a possible state of η and the interaction between the PDEs reflects the impact of the evolution of η on the valuation.

We first derive results for systems where the interaction term satisfies a Lipschitz condition and extend these to some type of exponential interaction. The latter is relevant for our applications to indifference valuation and hedging problems with respect to exponential utility. We strive for rather general assumptions on the coefficients of the PDEs which are satisfied in applications for typical financial models.

7.1 General framework

Let $m \in \mathbb{N}$, $T \in (0, \infty)$ be a fixed time horizon, and D a domain in $\mathbb{R}^d$, i.e. an open connected subset. For $(t, x, k) \in [0, T] \times D \times \{1, \dots, m\}$, consider the following stochastic differential equation (SDE) in $\mathbb{R}^d$

$$(7.1) \qquad X_t^{t,x,k} = x \in D\,, \qquad dX_s^{t,x,k} = b_k(s, X_s^{t,x,k})\, ds + \sum_{j=1}^{r} \Sigma_{k,j}(s, X_s^{t,x,k})\, dW_s^j \quad , s \in [t, T]\,,$$

for continuous functions $b_k : [0,T] \times D \to \mathbb{R}^d$ and $\Sigma_{k,j} : [0,T] \times D \to \mathbb{R}^d$, $j = 1, \dots, r$, with an r-dimensional Brownian motion $W = (W^j)_{j=1,\dots,r}$. We write b_k and each $\Sigma_{k,j}$ as a $d \times 1$ column vector and define the matrix-valued function $\Sigma_k : [0,T] \times D \to \mathbb{R}^{d\times r}$ by $\Sigma_k^{ij} := (\Sigma_{k,j})^i$.

For any k, we assume that the coefficients b_k and $\Sigma_{k,j}$, $j = 1, \dots, r$, of the SDE (7.1) are locally Lipschitz continuous in x, uniformly in t, i.e.,

(7.2) for each compact subset K of D, there is a constant $c_K < \infty$ such that

$$|G(t,x) - G(t,y)| \le c_K |x-y| \text{ for all } t \in [0,T],\ x,y \in K \text{ and } G \in \{b_k, \Sigma_{k,1}, \dots, \Sigma_{k,r}\}.$$

By Theorem II.5.2 of Kunita [Ku84] or Theorem V.38 in Protter[Pr90] the Lipschitz condition (7.2) implies that (7.1) has a unique (strong) solution on any given setting $(\Omega, \mathcal{F}, \mathbb{F}, P, W)$ up to a possibly finite random explosion time. We impose the additional assumption that for all k, t and x the solution $X^{t,x,k}$ neither explodes to infinity nor leaves D before T, i.e.

$$(7.3) \qquad P\left[\sup_{s\in[t,T]} \left|X_s^{t,x,k}\right| < \infty\right] = 1 \quad \text{and} \quad P\left[X_s^{t,x,k} \in D \ \text{ for all } s \in [t,T]\right] = 1\,.$$

By Theorem II.5.2 of Kunita [Ku84], (7.2) and (7.3) imply that $X^{t,x,k}$ has a version such that

$$(7.4) \qquad (t,x,s) \mapsto X_s^{t,x,k} \quad \text{is } P\text{-a.s. continuous.}$$

7.2 Fixed points of the Feynman-Kac representation

For uniqueness and existence problems of non-linear partial differential equations, it is a common approach to consider generalized solutions that are solutions of a corresponding integral equation. Typically, the solution to this integral problem requires less regularity, and various additional assumptions are needed to ensure that the solution to the integral equation is also a classical solution to the PDE. We refer to Pazy [Pa83], 6.1, for an analytic version of this approach and to Freidlin [Fl85, Fl90] for a stochastic version.
For the PDE-problem (7.8) that we are going to consider next, the integral form of the stochastic approach to the problem is the well-known Feynman-Kac representation. Since our PDE is non-linear, the solution function itself appears in the expectation of the Feynman Kac representation so that we have to look for a fixed point. We now show that a unique fixed point exists.

For given continuous bounded functions $h : D \to \mathbb{R}^m$, $g : [0,T] \times D \times \mathbb{R}^m \to \mathbb{R}^m$ and a continuous non-positive function $c : [0,T] \times D \to (-\infty, 0]^m$ we define the operator F by

$$(7.5) \qquad (Fv)^k(t,x) := E\left[h^k(X_T^{t,x,k}) e^{\int_t^T c^k(s,X_s^{t,x,k})\,ds} + \int_t^T g^k(s, X_s^{t,x,k}, v(s, X_s^{t,x,k}))\, e^{\int_t^s c^k(u,X_u^{t,x,k})du}\, ds\right]$$

with $k = 1, \dots, m$, $(t,x) \in [0,T] \times D$. Then $(t,x) \mapsto (Fv)(t,x)$ is well-defined, bounded and continuous due to (7.3), (7.4) and the assumptions imposed on h, g and c.
We want to show that F has a unique fixed point in the space $C_b([0,T] \times D, \mathbb{R}^m)$ of continuous bounded functions. To this end, we prove that F is a contraction.

Proposition 7.2.1 *Assume (7.2) and (7.3) hold. Suppose in addition that* $(t,x,y) \mapsto g(t,x,y)$ *is Lipschitz-continuous in* y*, uniformly in* t *and* x*, i.e. there exists* $L < \infty$ *such that*

$$|g(t,x,y_1) - g(t,x,y_2)| \le L|y_1 - y_2| \quad \text{for all } t \in [0,T],\ x \in D \text{ and } y_1, y_2 \in \mathbb{R}^m \,. \tag{7.6}$$

Then F *is a contraction on the space* $C_b([0,T] \times D, \mathbb{R}^m)$ *with respect to the norm*

$$||v||_\beta := \sup_{(t,x)\in[0,T]\times D} e^{-\beta(T-t)}|v(t,x)| \tag{7.7}$$

for $\beta < \infty$ *large enough. In particular,* F *has a unique fixed point* $\widehat{v} \in C_b([0,T] \times D, \mathbb{R}^m)$.

Proof: The norm (7.7) is equivalent to the common supremum-norm $||v||_\infty$ on the space $C_b([0,T] \times D, \mathbb{R}^m)$. By (7.6), we obtain for $v, w \in C_b([0,T] \times D, \mathbb{R}^m)$ and $\beta > 0$

$$\begin{aligned}
&e^{-\beta(T-t)} \left|(Fv)^k(t,x) - (Fw)^k(t,x)\right| \\
&= \frac{1}{e^{\beta(T-t)}} \left| E\left[\int_t^T \Big(g^k(s, X_s^{t,x,k}, v(s, X_s^{t,x,k})) - g^k(s, X_s^{t,x,k}, w(s, X_s^{t,x,k}))\Big)\, e^{\int_t^s c^k(u, X_u^{t,x,k})du}\, ds\right] \right| \\
&\le \frac{1}{e^{\beta(T-t)}} E\left[\int_t^T \left|g(s, X_s^{t,x,k}, v(s, X_s^{t,x,k})) - g(s, X_s^{t,x,k}, w(s, X_s^{t,x,k}))\right| e^{-\beta(T-s)} e^{\beta(T-s)}\, ds\right] \\
&\le \frac{1}{e^{\beta(T-t)}} L||v - w||_\beta \int_t^T e^{\beta(T-s)}\, ds \\
&\le \frac{L}{\beta} ||v - w||_\beta
\end{aligned}$$

for all $(t,x) \in [0,T] \times D$ and $k = 1, \dots, m$. Thus, F is a contraction with respect to the norm $||\cdot||_\beta$ for $\beta > L$. □

7.3 Classical solutions to interacting systems of semi-linear PDEs

We define the operators $\mathcal{L}^k$, $k = 1, \dots, m$, on sufficiently smooth functions $f : [0,T] \times D \to \mathbb{R}$ by

$$(\mathcal{L}^k f)(t,x) = \sum_{i=1}^d b_k^i(t,x) \frac{\partial f}{\partial x^i}(t,x) + \frac{1}{2} \sum_{i,j=1}^d a_k^{ij}(t,x) \frac{\partial^2 f}{\partial x^i \partial x^j}(t,x)$$

with

$$a_k(t,x) = (a_k^{ij}(t,x))_{i,j\in\{1,\dots,d\}} := \Sigma_k(t,x)\Sigma_k^{\mathrm{tr}}(t,x)\,.$$

Our goal is to show that the fixed point $\widehat{v}$ from Proposition 7.2.1 is the unique classical solution to the following system of semi-linear partial differential equations (PDEs) with boundary conditions at terminal time T and $k = 1, \dots, m$:

$$(7.8)\quad \frac{\partial}{\partial t} v^k(t,x) + \mathcal{L}^k v^k(t,x) + c^k(t,x)v^k(t,x) + g^k(t,x,v(t,x)) = 0 \quad \text{for } (t,x) \in [0,T) \times D$$
$$\text{and} \quad v^k(T,x) = h^k(x) \quad \text{for } x \in D\,.$$

These m partial differential equations are interacting via the g-term which may depend on all components of $v(t,x) = (v^k(t,x))_{k=1,\dots,m}$.
The next proposition asserts the existence of a unique classical solution to the PDE-problem (7.8). The proof applies a general Feynman-Kac type theorem from Heath and Schweizer [HeaS00] to verify that the fixed point from Proposition 7.2.1 solves the PDE-system under consideration. This result requires only weak assumptions on the SDE-coefficients which are satisfied in typical financial models. We impose the following conditions on the coefficients of the SDE (7.1) and the PDE (7.8):

(7.9) There exists a sequence $(D_n)_{n\in\mathbb{N}}$ of bounded domains with closure $\bar{D_n} \subseteq D$ such that $\cup_{n=1}^{\infty} D_n = D$, each D_n has a C^2-boundary, and for each n and $k = 1, \dots, m$

(7.10) the functions b_k and $a_k = \Sigma_k \Sigma_k^{\text{tr}}$ are uniformly Lipschitz-continuous on $[0,T] \times \bar{D_n}$,

(7.11) $\det a_k(t,x) \neq 0$ for all $(t,x) \in [0,T] \times D$,

(7.12) $(t,x) \mapsto c(t,x)$ is uniformly Hölder-continuous on $[0,T] \times \bar{D_n}$,

(7.13) $(t,x,v) \mapsto g(t,x,v)$ is uniformly Hölder-continuous on $[0,T] \times \bar{D_n} \times \mathbb{R}^m$.

Remark 7.3.1 Let us outline the crucial points of our contribution. We do *not* assume that the coefficients of the SDE are bounded or satisfy a linear growth condition. For our purposes, we have to work under more general assumptions on the SDE-coefficients since boundedness assumptions on a_k and b_k would exclude important applications in common financial models. Even in the standard Black-Scholes model the coefficients are unbounded. Other financial models would even violate linear growth conditions on the coefficients. For examples, we refer to Heath and Schweizer [HeaS00]. Another part of our contribution is that we aim for a classical solution of the PDE and not only for a generalized solution – which is for our problem already provided by the foregoing Proposition 7.2.1. To the best of our knowledge, the result of Proposition 7.3.2 has not been available so far under these general assumptions on the coefficients of the PDE. ◇

To formulate our result, we introduce some notation:

Notation: $C^{1,2}_{(b)}([0,T) \times D, \mathbb{R}^m)$ denotes the space of continuous (bounded) functions v :

$[0,T]\times D\to\mathbb{R}^m$ which are of class $C^{1,2}$ with respect to $(t,x)\in[0,T)\times D$. Note that the $C^{1,2}$-condition is imposed only on $[0,T)\times D$ while continuity is required on all of $[0,T]\times D$.

Proposition 7.3.2 *Assume that (7.9)–(7.13) hold in addition to the assumptions of Proposition 7.2.1. Then there is a unique classical solution in $C_b^{1,2}([0,T)\times D,\mathbb{R}^m)$ for the system (7.8) of interacting semi-linear partial differential equations. It is given by the fixed point $\hat v$ from Proposition 7.2.1.*

Proof: Let $w\in C_b^{1,2}([0,T)\times D,\mathbb{R}^m)$ and $v:=Fw$. By the definition of F, we already have $v\in C_b([0,T]\times D,\mathbb{R}^m)$. The claim of Proposition 7.3.2 readily follows from Proposition 7.2.1 if we can show that $v=(v^k)_{k\in\{1,\dots,m\}}$ is in $C_b^{1,2}([0,T)\times D,\mathbb{R}^m)$ and satisfies the following system of m partial differential equations with terminal conditions:

$$\text{(7.14)}\quad \frac{\partial}{\partial t}v^k(t,x)+\mathcal{L}^k v^k(t,x)+c^k(t,x)v^k(t,x)+g^k(t,x,w(t,x))=0\,,\ (t,x)\in[0,T)\times D\,,$$
$$\text{and}\quad v^k(T,x)=h^k(x)\,,\quad x\in D\,.$$

It is evident from the definitions of v and F that v satisfies the terminal condition. To prove the claim, it suffices to show for any $\varepsilon>0$ that v is of class $C^{1,2}([0,T-\varepsilon)\times D,\mathbb{R}^m)$ and satisfies (7.14) on $[0,T-\varepsilon)\times D$ instead of $[0,T)\times D$. To this end, we fix an arbitrary $\varepsilon\in(0,T)$ and $k\in\{1,\dots,m\}$. Let $T':=T-\varepsilon$. For any $(t,x)\in[0,T']\times D$ we have

$$\begin{aligned}
&v^k(t,x)=(Fw)^k(t,x)\\
&=E\left[h^k(X_T^{t,x,k})e^{\int_t^T c^k(s,X_s^{t,x,k})\,ds}+\int_t^T g^k(s,X_s^{t,x,k},w(s,X_s^{t,x,k}))\,e^{\int_t^s c^k(u,X_u^{t,x,k})du}\,ds\right]\\
&=E\left[E\left[h^k(X_T^{t,x,k})e^{\int_t^T c^k(s,X_s^{t,x,k})\,ds}+\int_t^T g^k(s,X_s^{t,x,k},w(s,X_s^{t,x,k}))\,e^{\int_t^s c^k(u,X_u^{t,x,k})du}\,ds\Big|\mathcal{F}_{T'}\right]\right]\\
&=E\left[v^k(T',X_{T'}^{t,x,k})e^{\int_t^{T'} c^k(s,X_s^{t,x,k})\,ds}+\int_t^{T'} g^k(s,X_s^{t,x,k},w(s,X_s^{t,x,k}))\,e^{\int_t^s c^k(u,X_u^{t,x,k})du}\,ds\right],
\end{aligned}$$

using the strong Markov property of $X^{t,x,k}$ for the last equality. Detailed references for this property are given in the proof of Theorem 1 in Heath and Schweizer [HeaS00]. Using the above representation of $v^k(t,x)$ for $(t,x)\in[0,T']\times D$, we obtain via Theorem 1 in Heath and Schweizer [HeaS00] that v^k is in $C^{1,2}([0,T')\times D,\mathbb{R}^m)$ and satisfies the PDE

$$\text{(7.15)}\quad \frac{\partial}{\partial t}v^k(t,x)+\mathcal{L}^k v^k(t,x)+c^k(t,x)v^k(t,x)+g^k(t,x,w(t,x))=0\,,\quad (t,x)\in[0,T')\times D\,,$$

provided we can verify the assumptions [A1], [A2] and [A3] for their general Feynman-Kac type result. [A1] and [A2] are exactly (7.2) and (7.3). We check the list [A3'] of conditions in [HeaS00] whose combination implies [A3]. Conditions [A3'], [A3a'] and [A3c'] in [HeaS00] are exactly (7.9), (7.10), (7.12). By Lemma 3 in [HeaS00], the continuity of Σ_k in combination with (7.11) implies their condition [A3b'], and $v\in C_b([0,T]\times D,\mathbb{R}^m)$ implies [A3e']. In order to verify [A3d'], first note that w is continuously differentiable on $[0,T) \times D$ and therefore Lipschitz-continuous on bounded closed subsets $[0,T'] \times \bar{D}_n$. Thus, $(t,x)^{\mathrm{tr}} \mapsto (t,x,w(t,x))^{\mathrm{tr}}$ is Lipschitz-continuous on $[0,T'] \times \bar{D}_n$ and via (7.13) this implies that the composition

$$\widetilde{g} : (t,x) \mapsto g(t,x,w(t,x)) \quad \text{is uniformly Hölder-continuous on } [0,T'] \times \bar{D}_n. \tag{7.16}$$

So, [A3d'] in [HeaS00] holds as well and we can apply their Theorem 1 and obtain that v^k is in $C^{1,2}([0,T') \times D, \mathbb{R}^m)$ and satisfies the PDE (7.15) on $[0,T') \times D$. Since $\varepsilon > 0$ was arbitrary this implies that $v = (v^k)_{k\in\{1,\dots,m\}} = Fw$ is in $C^{1,2}([0,T) \times D, \mathbb{R}^m)$ and satisfied the PDE (7.14) on $[0,T) \times D$. In particular, we have shown that

$$F \quad \text{maps} \quad C_b^{1,2}([0,T) \times D, \mathbb{R}^m) \quad \text{into} \quad C_b^{1,2}([0,T) \times D, \mathbb{R}^m) \tag{7.17}$$

and conclude that the fixed point $\widehat{v}$ from Proposition 7.2.1 is in $C_b^{1,2}([0,T) \times D, \mathbb{R}^m)$ and satisfies the PDE (7.8).
The uniqueness of this solution follows by the usual Feynman-Kac argument. Basically by applying Itô's formula to the process $v^k(s, X_s^{t,x,k}) \exp(\int_t^s c(u, X_u^{t,x,k})du)$, $s \in [t,T)$, and using the PDE one can show that any solution $v \in C_b^{1,2}([0,T) \times D, \mathbb{R}^m)$ to the PDE (7.8) satisfies the Feynman-Kac representation, i.e., v is a fixed point of F. Since the fixed point is unique, this yields $v = \widehat{v}$. □

For our application in the context of utility indifference pricing we are interested in a solution to the PDE-system (7.8) with a function $g : [0,T] \times D \times \mathbb{R}^m \to \mathbb{R}^m$ of the form

$$g^k(t,x,v) := f^k(t,x) + \sum_{\substack{j=1 \\ j\neq k}}^{m} \lambda^{kj}(t,x) \frac{1}{\alpha} \left(e^{\alpha(v^j - v^k + f^{kj}(t,x))} - 1 \right), \quad k = 1,\dots,m, \tag{7.18}$$

with $\alpha > 0$ and functions f^k, $f^{kj} \in C_b^1([0,T] \times D, \mathbb{R})$ and $\lambda^{kj} \in C_b^1([0,T] \times D, [0,\infty))$, $k,j \in \{1,\dots,m\}$ which are of class C^1 with respect to both arguments on $[0,T] \times D$. We cannot apply Proposition 7.3.2 directly since $g = (g^k)_{k\in\{1,\dots,m\}}$ is not bounded and does not satisfy condition (7.13) in general. So we need some additional arguments. Let us recall that h is in $C_b(D, \mathbb{R}^m)$.

Theorem 7.3.3 *Assume that (7.2), (7.3) and (7.9)–(7.12) hold. Suppose g is of the form (7.18). Then there is a unique classical solution $\widehat{v} \in C_b^{1,2}([0,T) \times D, \mathbb{R}^m)$ to the system (7.8) of interacting partial differential equations. $\widehat{v}$ satisfies the Feynman-Kac representation*

$$\widehat{v}^k(t,x) = E\left[h^k(X_T^{t,x,k}) e^{\int_t^T c^k(s, X_s^{t,x,k})\, ds} + \int_t^T g^k(s, X_s^{t,x,k}, \widehat{v}(s, X_s^{t,x,k}))\, e^{\int_t^s c^k(u, X_u^{t,x,k}) du}\, ds \right] \tag{7.19}$$

for $k = 1,\dots,m$ and $(t,x) \in [0,T] \times D$.

Proof: Let $K, L \in \mathbb{R}_+$ such that $\sup_{x\in D, k\in\{1,\dots,m\}} |h^k(x)| \le K$ and

$$\sup_{(t,x)\in[0,T]\times D, k\in\{1,\dots,m\}} \left| f^k(t,x) + \sum_{\substack{j=1\\ j\neq k}}^m \lambda^{kj}(t,x)\frac{1}{\alpha}\left(e^{\alpha f^{kj}(t,x)} - 1\right)\right| \le L\,.$$

We define a truncation function

$$\kappa : [0,T]\times\mathbb{R}\to\mathbb{R}\,,\quad (t,x)\mapsto \min\Big(\max\Big(x\,,\,-K-(T-t)L\Big)\,,\,+K+(T-t)L\Big)\Big)\,, \tag{7.20}$$

and then $\widetilde{g} : [0,T]\times D\times\mathbb{R}^m\to\mathbb{R}^m$ by means of g with a truncated third argument:

$$\widetilde{g}(t,x,v) = \widetilde{g}\left(t,x,(v^k)_{k=1,\dots,m}\right) := g\left(t,x,\left(\kappa(t,v^k)\right)_{k=1,\dots,m}\right)\,. \tag{7.21}$$

$\widetilde{g}$ is bounded and Lipschitz-continuous on $[0,T]\times\bar{D}_n\times\mathbb{R}^m$ for any bounded closed set $\bar{D}_n\subset D$ since g is of class C^1. So Proposition 7.3.2 can be applied and there is a unique bounded solution $\widehat{v}$ for the PDE (7.8) with $\widetilde{g}$ instead of g. Moreover, $\widehat{v}$ is the fixed point of F defined with $\widetilde{g}$ instead of g in (7.5). We will show below that

$$|\widehat{v}^k(t,x)| \le K+(T-t)L \quad \text{for } (t,x)\in[0,T]\times D \text{ and } k=1,\dots,m. \tag{7.22}$$

Admitting this result for a moment, we get $\kappa(t,\widehat{v}^k(t,x)) = \widehat{v}(t,x)$ and therefore $g(t,x,\widehat{v}(t,x)) = \widetilde{g}(t,x,\widehat{v}(t,x))$ for all $(t,x)\in[0,T]\times D$. Hence, $\widehat{v}$ also solves the PDE (7.8) with g and satisfies (7.19). To see that $\widehat{v}$ is the unique bounded solution to (7.8), let w denote some other bounded solution. By taking K larger if necessary, we can assume $|w^k(t,x)|\le K$ for all k,t,x. Then, both $\widehat{v}$ and w solve (7.8) not just with g but also with $\widetilde{g}$ and this implies $\widehat{v}=w$ by Proposition 7.3.2.
To finish the proof, it remains to establish (7.22). To this end, we fix arbitrary $(t,x)\in[0,T]\times D$ and $k\in\{1,\dots,m\}$ and define the stopping time

$$\tau := \inf\left\{s\in[t,T]\,\middle|\,\widehat{v}^k(s,X_s^{t,x,k}) < K+(T-s)L\right\}\wedge T\,.$$

By the definition of τ, we have

$$\widehat{v}^k(s,X_s^{t,x,k}(\omega)) \ge K+(T-s)L \quad \text{for all} \quad (\omega,s)\in[\![t,\tau[\![$$

and $\widehat{v}^k(\tau,X_\tau^{t,x,k})\le K+(T-\tau)L$. The latter assertion uses the terminal condition at time T in the PDE (7.8). The form of g and the definition of $\widetilde{g}$ then imply

$$\widetilde{g}^k(t,X_s^{t,x,k}(\omega),\widehat{v}(s,X_s^{t,x,k}(\omega))) \le L \quad \text{for} \quad (\omega,s)\in[\![t,\tau[\![\,. \tag{7.23}$$

This yields

$$
\begin{aligned}
&\widehat{v}^k(t,x)\\
&= E\left[h^k(X_T^{t,x,k})e^{\int_t^T c^k(s,X_s^{t,x,k})\,ds} + \int_t^T \widetilde{g}^k(s,X_s^{t,x,k},\widehat{v}(s,X_s^{t,x,k}))\, e^{\int_t^s c^k(u,X_u^{t,x,k})du}\, ds\right]\\
&= E\left[E\left[h^k(X_T^{t,x,k})e^{\int_t^T c^k(s,X_s^{t,x,k})\,ds} + \int_t^T \widetilde{g}^k(s,X_s^{t,x,k},\widehat{v}(s,X_s^{t,x,k}))\, e^{\int_t^s c^k(u,X_u^{t,x,k})du}\, ds \middle| \mathcal{F}_\tau\right]\right]\\
&= E\left[\widehat{v}^k(\tau,X_\tau^{t,x,k})e^{\int_t^\tau c^k(s,X_s^{t,x,k})\,ds} + \int_t^\tau \widetilde{g}^k(s,X_s^{t,x,k},v(s,X_s^{t,x,k}))\, e^{\int_t^s c^k(u,X_u^{t,x,k})du}\, ds\right]\\
&\le E\,[\, K + (T-\tau)L + (\tau-t)L\,]\\
&\le K + (T-t)L
\end{aligned}
$$

where the third equality holds due to the strong Markov property of $X^{t,x,k}$. Detailed references on this property have been given in the proof of Theorem 1 in Heath and Schweizer [HeaS00]. This proves the upper bound in (7.22). Analogously, we can prove the lower bound. In fact, for

$$\tau := \inf\left\{ s\in[t,T] \,\middle|\, v^k(s,X_s^{t,x,k}) > -K-(T-s)L \right\} \wedge T$$

we have

$$\widehat{v}^k(s,X_s^{t,x,k}) \le -K-(T-s)L \quad \text{ for } (\omega,s)\in [\![t,\tau[\![$$

and $\widehat{v}^k(\tau,X_\tau^{t,x,k}) \ge -K-(T-\tau)L$. This gives

$$\widetilde{g}^k(t,X_s^{t,x,k},\widetilde{v}(s,X_s^{t,x,k})) \ge -L \quad \text{for} \quad (\omega,s)\in [\![t,\tau[\![$$

and we obtain $\widehat{v}^k(t,x) \ge -K-(T-t)L$ almost by the same lines as above; we just have to turn around the inequalities and change the signs in the last two steps of calculation. This ends the proof. □

Index of Notation

General	
$:=$	definition
$\inf \emptyset := +\infty$	
$\sup \emptyset := -\infty$	
$a \wedge b := \min\{a, b\}$	minimum ($a, b \in \mathbb{R}$)
$a \vee b := \max\{a, b\}$	maximum ($a, b \in \mathbb{R}$)
$\mathcal{F}^1 \vee \mathcal{F}^2 := \sigma(\mathcal{F}^1, \mathcal{F}^2)$	spanned σ-field
1_A	indicator function on a set A
$1_k := 1_{\{k\}}$	
$(\Omega, \mathcal{F}, P)$	underlying probability space
P	objective probability measure
E	expectation with respect to P
Q	(martingale) probability measure
E_Q	expectation with respect to Q
$H(Q\|P)$	relative entropy of Q w.r.t. P
T	finite time horizon
$\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$	filtration
$S = (S_t)_{t\in[0,T]}$	$\mathbb{R}^d$-valued process: (discounted) asset prices
$L(S)$	set of S-integrable predictable processes
Θ	subset of $L(S)$: set of portfolio strategies
ϑ	portfolio strategy
$\int \vartheta\, dS \equiv \vartheta \cdot S$	stochastic integral: gains from trade
$\int_0^t \vartheta\, dS \equiv \int_{(0,t]} \vartheta\, dS \equiv (\vartheta \cdot S)_t$	gains from trade until time t
τ, τ^n, T^n	stopping times
U	utility function
u	maximal expected utility function
PDE	partial differential equation
SDE	stochastic differential equation
PRP	strong predictable representation property
LHS (RHS)	left(right)-hand side

Part I

Chapters 1-4	
$\mathbb{P}_a, \mathbb{P}_e, \mathbb{P}_f$	sets of martingale measures
$\Theta_{\mathcal{M}}, \Theta_1, \dots, \Theta_4$	certain sets of strategies
B	random variable: contingent claim
$\alpha \in (0, \infty)$	risk aversion parameter
$u(x - B; \alpha)$	maximal expected exponential utility from initial capital x with liability B
Q^B, Q^0	elements of $\mathbb{P}_e$
Q^B	solution to the dual problem
Q^0	entropy minimizing martingale measure
$\vartheta^B \in L(S),\ c^B \in \mathbb{R}$	appear in a representation of Q^B
ϑ^B	optimal investment strategy under liability B
$\pi(B)$	utility indifference (sell) price
(π_t)	utility indifference price process
(c_t^B)	certainty equivalence process
$\psi(B)$	utility indifference hedging strategy
Chapter 2	
$\mathbb{F}^0, \mathbb{I}, \mathbb{I}^k$	sub-filtrations of $\mathbb{F}$
$\mathbb{F}^0$	information flow in a (complete) sub-market
$\mathbb{I}, \mathbb{I}^k$	additional information flows
Q^*	(unique) element of $\mathbb{P}_e$
E^*	expectation with respect to Q^*
Chapters 3-4	
D	domain in $\mathbb{R}^d$
$\Gamma, \Sigma\ (\gamma, \sigma)$	coefficient functions in the SDE for S
$a := \Sigma\Sigma^{\text{tr}}$	
$\tilde{a} := \sigma\sigma^{\text{tr}}$	
$N = (N_t^{kj})$	point process
λ^{kj}	P-intensity function of N^{kj}
η	$\{1, \dots, m\}$-valued finite-state process
m	number of states for η
h, f, f^{kj}	payoff functions of the contingent claim B
$v = v(t, x, k)$	solution to a PDE
v_x	gradient of v with respect to x
v^B, v^0, v^π	solutions to specific PDEs
$C^{1,2}([0,T) \times D \times \{1, \dots, m\}, \mathbb{R})$, $C^{1,2}([0,T) \times D, \mathbb{R}^m)$	spaces of functions

Part II

Chapter 5

G	random variable
$(X, \mathcal{X})$	range of values for G
$\mathcal{G} := \sigma(G)$	additional information
$\mathbb{G}$	initially enlarged filtration
$p^x(\omega)$	conditional density function
$p^G(\omega) := p^{G(\omega)}(\omega)$	random variable
MPPM	martingale preserving probability measure
$\widetilde{Q}$	MPPM corresponding to Q
$\mathbb{H}$	either $\mathbb{F}$ or $\mathbb{G}$
$Q^{\mathbb{H}}$	P-equivalent martingale measure w.r.t $\mathbb{H}$
$Z^{\mathbb{H}}$	P-density process of $Q^{\mathbb{H}}$ with respect to $\mathbb{H}$.
$E^{\mathbb{H}}$	expectation with respect to $Q^{\mathbb{H}}$.
$Q^{\mathbb{G}} := \widetilde{Q^{\mathbb{F}}}$	
U	utility function
I	inverse of U'
$\mathrm{dom}(U)$	domain of U
ℓ	lower bound of $\mathrm{dom}(U) = (\ell, \infty)$
$u^{\mathbb{H}}$	maximal expected utility with respect to $\mathbb{H}$
$V^{\mathbb{H}}$	optimal wealth process
$\mathcal{V}^{\mathbb{H}}$	set of attainable wealth processes
π	utility indifference (buy) value of the additional information

Part III

Chapter 6

$\mathcal{N}$	set of tradable numeraires
$N \in \mathcal{N}$	tradable numeraire
$N^{\log}$	growth optimal numeraire
N^P	wealth process of the P-numeraire portfolio
$\mathcal{M}(\mathcal{N})$	P-equivalent martingale measures for $\mathcal{N}$
$\mathcal{SM}$	supermartingale densities

Part IV

Chapter 7

m, d, r	elements of $\mathbb{N}$
D	domain in $\mathbb{R}^d$
$X^{t,x,k}$	solution to a diffusion SDE
b_k, Σ_k	coefficient functions in the SDE for $X^{t,x,k}$
$a_k := \Sigma_k \Sigma_k^{\mathrm{tr}}$	
$v = (v^k(t,x))_{k=1,\dots,m}$	function with values in $\mathbb{R}^m$
$v_t^k, v_{x^i}^k, v_{x^i x^j}^k$	partial derivatives of v^k
F	operator on a space of functions
$C_{(b)}([0,T) \times D, \mathbb{R}^m)$, $C_{(b)}^{1,2}([0,T) \times D, \mathbb{R}^m)$	spaces of functions

Bibliography

[AP94] Aase, K. K., and S. A. Persson (1994): "Pricing of Unit-Linked Life Insurance Policies", *Scandinavian Actuarial Journal* 1, 26-52.

[A99] Amendinger, J. (1999): "Initial Enlargement of Filtrations and Additional Information in Financial Markets", Doctoral Dissertation, Technical University of Berlin.

[A00] Amendinger, J. (2000): "Martingale Representation Theorems for Initially Enlarged Filtrations", *Stochastic Processes and their Applications* 1, 101-116.

[ABS00] Amendinger, J., D. Becherer and M. Schweizer (2000): "Quantifying the Value of Initial Investment Information", Discussion Paper 41/2000, Sonderforschungsbereich 373, Humboldt University, Berlin.

[AIS98] Amendinger, J., P. Imkeller and M. Schweizer (1998): "Additional Logarithmic Utility of an Insider", *Stochastic Processes and their Applications* 75, 263-286.

[AnS94] Ansel, J. P., and C. Stricker (1994): "Couverture des Actifs Contingents", *Annales de l'Institut Henri Poincaré*, 30, 303-315.

[Ar97] Artzner, P. (1997): "On the Numeraire Portfolio", in M. A. H. Dempster, S. R. Pliska (eds.), *Mathematics of Derivative Securities.* Cambridge University Press, Cambridge, 53-60.

[BP97] Bajeux-Besnainou, I., and R. Portait (1997): "The Numeraire Portfolio: A New Perspective on Financial Theory", *The European Journal of Finance* 3, 291-309.

[Be01] Becherer, D. (2001): "The Numeraire Portfolio for Unbounded Semimartingales", *Finance and Stochastics* 5, 327-341.

[B1738] Bernoulli, D. (1738): "Specimen theoriae novae de mensura sortis", Commentarii Academiae Sientiarum Imperialis Petropolitanae 5, 175-192. Translated by L. Sommer (1954): *Econometrica* 22, 23-36.

[BlS73] BLACK, F., AND M. SCHOLES (1973): "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy* 81, 637-654.

[Bj97] BJÖRK, T. (1997): "Interest Rate Theory", in W. J. Runggaldier (ed.) *Financial Mathematics*, Lecture Notes in Mathematics 1656, 53-122. Springer, Berlin.

[BoL89] BOULEAU, N., AND D. LAMBERTON (1989): "Residual Risk and Hedging Strategies in Markovian Markets", *Stochastic Processes and their Applications* 33, 131-150.

[Br81] BRÉMAUD, P. (1981): *Point Processes and Queues*. Springer, Berlin.

[BrS79] BRENNAN, M. J., AND E. S. SCHWARZ (1979): "Pricing and Investment Strategies for Guaranteed Equity-linked Life Insurance", Monograph no. 7, Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Philadelphia.

[Bw95] BROWNE, S (1995): "Optimal Investment Policies For a Firm With A Random Risk Process: Exponential Utility and Minimizing The Probability Of Ruin", *Mathematics of Operations Research* 20, 937-958.

[CaW00] CAO, M., J. WEI (2000): "Equilibrium Valuation of Weather Derivatives", Preprint, Queens University, Kingston, Ontario.

[CoH89] COX, J. C., AND C.-F. HUANG (1989): "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process", *Journal of Economic Theory*, 49, 33-83.

[CoR76] COX, J. C., AND S. A ROSS (1976): "The Valuation of Options for alternative Stochastic Processes", *Journal of Financial Economics* 3, 145-166.

[Cs75] CSISZÁR, I. (1975): "*I*-Divergence Geometry of Probability Distributions", *Annals of Probability* 3, 146-158.

[CvK99] CVITANIĆ, J., AND I. KARATZAS (1999): "On Dynamic Measures of Risk", *Finance and Stochastics* 3, 451-482.

[CvSW01] CVITANIĆ, J., W. SCHACHERMAYER AND H. WANG (2001): "Utility Maximization in Incomplete Markets with Random Endowment", *Finance and Stochastics*, 5, 259-272.

[D97] DAVIS, M. H. A. (1997): "Option Pricing in Incomplete Markets", in M. A. H. Dempster, S. R. Pliska (eds.), *Mathematics of Derivative Securities*. Cambridge University Press, Cambridge, 227-254.

[D00] Davis, M. H. A. (2000): "Optimal Hedging with Basis Risk", Preprint, Vienna University of Technology.

[D01] Davis, M. H. A. (2001): "Pricing Weather Derivatives by Marginal Value", *Quantitative Finance* 1, 1-4.

[DPZ93] Davis, M. H. A., V. G. Panas and T. Zariphopoulou (1993): "European Option Pricing with Transaction Costs", *SIAM Journal on Control and Optimization* 31, 470-493.

[DGRS³00] Delbaen, F., P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer and C. Stricker (2000): "Exponential Hedging and Entropic Penalties", to appear in *Mathematical Finance*.

[DeS94] Delbaen, F., and W. Schachermayer (1994): "A General Version of the Fundamental Theorem of Asset Pricing", *Mathematische Annalen* 300, 463-520.

[DeS95] Delbaen, F., and W. Schachermayer (1995): "The No Arbitrage Property under a Change on Numéraire", *Stochastics and Stochastic Reports* 53, 213-226.

[DeS98] Delbaen, F., and W. Schachermayer (1998): "The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes", *Mathematische Annalen* 312, 215-250.

[DeS98a] Delbaen, F., and W. Schachermayer (1998): "A Simple Counter-Example to Several Problems in the Theory of Asset Pricing which Arises in Many Incomplete Markets", *Mathematical Finance* 8, 1-11.

[DelM80] Dellacherie, C., and P.-A. Meyer (1980): *Probabilités et Potentiel*, Chapitres V à VIII, Théorie des Martingales. Hermann, Paris.

[DenGP99] Denis, L., A. Grorud, and M. Pontier (1999): "Formes de Dirichlet sur un Espace de Wiener-Poisson. Application au Grossissement de Filtration" in *Séminaire de Probabilité XXXIV*, Lecture Notes in Mathematics 1729, 198-214. Springer, Berlin.

[DiMKR94] Di Masi, G. B., Yu. M. Kabanov and W. J. Runggaldier (1994): "Mean-Variance Hedging of Options on Stocks with Markov Volatilities", *Theory of Probability and Its Applications* 39, 172-182.

[DuFSZ97] Duffie, D., W. Flemming, M. Soner and T. Zariphopoulou (1997): "Hedging in Incomplete Markets with HARA Utility", *Journal of Economic Dynamics and Control* 21, 753-782.

[DuR91] DUFFIE, D.,AND H. R. RICHARDSON (1991): "Mean-Variance Hedging in Continuous Time", *Annals of Applied Probability* 1, 1-15.

[DuSS96] DUFFIE, D., M. SCHRODER AND C. SKIADAS (1996): "Recursive Valuation of Defaultable Securities and the Timing of Resolution of Uncertainty", *Annals of Applied Probability* 6, 1075-1090.

[EGK97] ELLIOTT, R. J., H. GEMAN AND B. M. KORKIE (1997): "Portfolio Optimization and Contingent Claim Pricing with Differential Information", *Stochastics and Stochastics Reports*, 60, 185-203.

[ElKQ95] EL KAROUI, N., AND M. QUENEZ (1995): "Dynamic Programming and Pricing of Contingent Claims in an Incomplete Market", *SIAM Journal on Control and Optimization* 33, 29-66.

[EtK86] ETHIER, S. N., AND T. G. KURTZ (1986): *Markov Processes. Characterization and Convergence.* Wiley, New York.

[FI93] FÖLLMER, H., AND P. IMKELLER (1993): "Anticipation Cancelled by a Girsanov Transformation: A Paradox on Wiener Space", *Annales de l'Institut Henri Poincaré*, 29, 569-586.

[FL99] FÖLLMER, H., AND P. LEUKERT (1999): "Quantile Hedging", *Finance and Stochastics* 3, 251-273.

[FL00] FÖLLMER, H., AND P. LEUKERT (2000): "Efficient Hedging: Cost Versus Shortfall Risk", *Finance and Stochastics* 4, 117-146.

[FK98] FÖLLMER, H., AND Y. KABANOV (1998): "Optional Decomposition and Lagrange Multipliers", *Finance and Stochastics* 2, 69-81.

[FSchi01] FÖLLMER, H., AND A. SCHIED (2001): "Convex Measures of Risk and Trading Constraints", to appear in *Finance and Stochastics*

[FSchw91] FÖLLMER, H., AND M. SCHWEIZER (1991): "Hedging of Contingent Claims under Incomplete Information", in M. H. A. Davis and R. J. Elliott (eds.), *Applied Stochastic Analysis. Stochastics Monographs*, vol. 5, Gordon and Breach, 389-414.

[FSo86] FÖLLMER, H., AND D. SONDERMANN (1988): "Hedging of Non-Redundant Contingent Claims", in W. Hildenbrand and A. Mas-Colell (eds.), *Contributions to Mathematical Economics*, North-Holland, 205-223.

[Fl85] FREIDLIN, M. (1985): *Functional Integration and Partial Differential Equations*, Annals of Mathematics Studies 109, Princeton University Press, Princeton.

[Fl90] FREIDLIN, M. (1990): "Semi-Linear PDE's and Limit Theorems for Large Deviations", in *École d'Été de Probabilités de Saint-Flour XX-1990*, Lecture Notes in Mathematics 1527, 1-114. Springer, Berlin.

[Fm75] FRIEDMAN, A. (1975): *Stochastic Differential Equations and Applications*, Volume 1, Academic Press, New York.

[Ft00] FRITTELLI, M. (2000): "The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets", *Mathematical Finance* 10, 39-52.

[Ft00a] FRITTELLI, M. (2000): "Introduction to a Theory of Value Coherent with the No-Arbitrage Principle", *Finance and Stochastics* 4, 275-297.

[FtB00] FRITTELLI, M., AND F. BELLINI (2000): "Certainty Equivalent and No Arbitrage - a Reconciliation via Duality Theory", Preprint, Università degli Studi di Milano.

[GER95] GEMAN, H., N. EL KAROUI AND J.-C. ROCHET (1995): "Changes of Numeraire, Changes of Probability Measure and Option Pricing", *Journal of Applied Probability* 32, 443-458.

[Ge79] GERBER, H. U. (1979): *An Introduction to Mathematical Risk Theory.* Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, Phiadelphia.

[GoK00] GOLL, T., AND J. KALLSEN (2000): "Optimal Portfolios for Logarithmic Utility", *Stochastic Processes and their Applications* 89, 31-48.

[GraR99] GRANDITS, P., AND T. RHEINLÄNDER (1999): "On the Minimal Entropy Martingale Measure", to appear in *Annals of Probability.*

[GroP98] GRORUD, A., AND M. PONTIER (1998): "Insider Trading in a Continuous Time Market Model", *International Journal of Theoretical and Applied Finance* 1, 331-347.

[GroP01] GRORUD, A., AND M. PONTIER (2001): "Asymmetrical Information and Incomplete Market", *International Journal of Theoretical and Applied Finance* 4, 1-18.

[HK79] HARRISON, J. M., AND D. M. KREPS (1979): "Martingales and Arbitrage in Multiperiod Securities Markets", *Journal of Economic Theory* 20, 381-408.

[HP81] HARRISON, J. M., AND S. PLISKA (1981): "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and Applications* 11, 215-260.

[HW82] He, S. W., and J. G. Wang (1982): “The Property of Predictable Representation of the Sum of Two Semimartingales”, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 61, 141-152.

[HWY92] He, S. W., J. G. Wang and J. A. Yan (1992): *Semimartingale Theory and Stochastic Calculus*. Science Press, CRC Press, New York.

[HeaS00] Heath, D., and M. Schweizer (2001): “Martingales versus PDEs in Finance: an Equivalence Result with Examples”, *Journal of Applied Probability* 37, 947-957.

[Hen01] Henderson, V. (2001): “Valuation of Claims on Non-Traded Assets using Utility Maximization”, Preprint, University of Warwick.

[HenH00] Henderson, V., and D. Hobson (2000): “Real Options with Constant Relative Risk Aversion”, FORC Preprint 2000/110, University of Warwick. To appear in *Journal of Economic Dynamics and Control*.

[HerM53] Herstein, I., and J. Milnor (1953): “An Axiomatic Approach to Measurable Utility”, *Econometrica* 21 (April), 291-307.

[HodN89] Hodges, S. D., and A. Neuberger (1989): “Optimal Replication of Contingent Claims under Transaction Costs”, *Review of Futures Markets*, 8, 222-239.

[Hoe69] Hoem, J. (1969): “Markov Chain Models in Life Insurance”, *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* 9, 91-107.

[Ja79] Jacod, J. (1979): *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics 714. Springer, Berlin

[Ja80] Jacod, J. (1980): “Intégrales Stochastiques par rapport à une Semimartingale Vectorielle et Changement de Filtration” in *Séminaire de Probabilités XIV*, Lecture Notes in Mathematics 784, 161-172. Springer, Berlin.

[Ja85] Jacod, J. (1985): “Grossissement Initial, Hypothèse (H′) et Théorème de Girsanov”, in *Séminaire de Calcul Stochastique 1982/83*, Lecture Notes in Mathematics 1118, 15-35. Springer, Berlin.

[JaS87] Jacod, J., and A. N. Shiryaev (1987): *Limit Theorems for Stochastic Processes*. Springer, Berlin.

[Jo96] Johnson, B. E. (1996): “The Pricing Property of the Optimal Growth Portfolio: Extensions and Applications”, Working Paper, Department of Engineering-Economic-Systems, Stanford University.

[KS00a] KABANOV, Y., AND C. STRICKER (2000): "On Equivalent Martingale Measures with Bounded Densities", Preprint, Université de Franche-Comté, Besançon.

[KS00b] KABANOV, Y., AND C. STRICKER (2000): "On the Optimal Portfolio for the Exponential Utility Maximization - Remarks to the Six-Authors Paper", Preprint, Université de Franche-Comté, Besançon.

[KaLS87] KARATZAS, I., J. P. LEHOCZKY AND S. E. SHREVE (1987): "Optimal Portfolio and Consumption Decisions for a Small Investor on a Finite Horizon", *SIAM Journal on Control and Optimization*, 27, 1557-1586.

[KaLSX91] KARATZAS, I., J. P. LEHOCZKY, S. E. SHREVE AND G.-L. XU (1991): "Martingale and Duality Methods for Utility Maximization in an Incomplete Market", *SIAM Journal on Control and Optimization* 29, 702-730.

[Ko00] KORN, R. (2000): "Value Preserving Portfolio Strategies and a General Framework for Local Approaches to Optimal Portfolios", *Mathematical Finance* 10, 227-241.

[KoS00] KORN, R., AND M. SCHÄL (2000): "On Growth Optimal and Value Preserving Portfolios", *Mathematical Methods in Operations Research* 50, 189-218.

[KoSO01] KORN, R., M. SCHÄL AND F. OERTEL (2001): "The Numeraire Portfolio in Financial Markets Modeled by a Multi Dimensional Jump Diffusion Process", Preprint, Universität Bonn.

[Kp88] KREPS, D. M. (1988): *Notes on the Theory of Choice*, Underground Classics in Economics, Westview Press, Boulder.

[Kr96] KRAMKOV, D. O. (1996): "Optional Decomposition of Supermartingales and Hedging Contingent Claims in Incomplete Security Markets", *Probability and Related Fields* 105, 459-479.

[KrS99] KRAMKOV, D. O., AND W. SCHACHERMAYER (1999): "The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets", *Annals of Applied Probability*, 9, 904-950.

[Ku84] KUNITA, H. (1984): "Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms", in *École d'Été de Probabilitiés de Saint-Flour XII-1982*, Lecture Notes in Mathematics 1097, 142-303. Springer, Berlin

[La97] LANDO, D. (1997): "Modelling Bonds and Derivatives with Default Risk", in M. A. H. Dempster, S. R. Pliska (eds.), *Mathematics of Derivative Securities*. Cambridge University Press, Cambridge, 369-393.

[La98] LANDO, D. (1998): "On Cox Processes and Credit Risky Securities", *Review of Derivatives Research* 2, 99-120.

[Lo90] LONG, J. B. (1990): "The Numeraire Portfolio", *Journal of Financial Economics* 26, 29-69.

[Ma78] MARGRABE, W. (1978): "The Value of an Option to Exchange one Asset for Another", *Journal of Finance* 33, 177-186.

[Mem80] MÉMIN, J. (1980): "Espaces de Semi-Martingales et Changement de Propabilité", *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 52, 9-39.

[Mer69] MERTON, R. C. (1969): "Lifetime Protfolio Seclection under Uncertainty: The Continuous Case", *Review of Economics and Statistics* 51, 247-257.

[Mer71] MERTON, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous-Time Model", Journal of Economic Theory 3, 373-413.

[Mer73] MERTON, R. C. (1973): "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science* 4, 141-183.

[Mer74] MERTON, R. C. (1974): "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates", Journal of Finance 2, 449-470.

[Mer90] MERTON, R. C. (1990): *Continuous Time Finance*. Blackwell Publishers, Cambridge MA.

[Mi96] MIYAHARA, Y. (1996): "Canonical Martingale Measures of Incomplete Asset Markets", in *Probability Theory and Mathematical Statistics*, Proceedings of the Seventh Japan-Russia Symposium, Tokyo 1995, 343-352.

[Mi00] MIYAHARA, Y. (2000): " Geometric Lévy Processes & MEMM - Pricing Model and Related Estimation Problems", Preprint, Nagoya City University.

[Mø00] MØLLER, T. (2000): "Quadratic Hedging Approaches and Indifference Pricing in Insurance", Ph.D. thesis, University of Copenhagen.

[Mø01] MØLLER, T. (2001): "On Transformations of Actuarial Valuation Principles", *Insurance: Mathematics & Economics* 28, 281-303.

[Ni99] NIELSEN, L. T. (1999): *Pricing and Hedging of Derivative Securities*. Oxford University Press, Oxford.

[No92] NORBERG, R. (1992): "Hattendorff's Theorem and Thiele's Differential Equation Generalized", *Scandinavian Actuarial Journal* 1, 2-14.

[Ow01] OWEN, M. P. (2001): "Utility Based Optimal Hedging in Incomplete Markets", Preprint, University of Technology, Vienna.

[Pa83] PAZY, A. (1983): *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, Berlin

[PiK96] PIKOVSKY, I., AND I. KARATZAS (1996): "Anticipative Portfolio Optimization", *Advances in Applied Probability*, 28, 1095-1122.

[Pi97] PIKOVSKY, I. (1997): "Market Equilibrium with Differential Information", Preprint, Columbia University, New York.

[Pr90] PROTTER, P. (1990): *Stochastic Integration and Differential Equations*. Springer, Berlin

[Rh99] RHEINLÄNDER, T. (1999): "Optimal Martingale Measures and Their Applications in Mathematical Finance", Doctoral Dissertation, Technical University of Berlin.

[RoW87] ROGERS, L. C. G., AND D. WILLIAMS (1987): *Diffusions, Markov Processes and Martingales*, Volume 2. Wiley, New York

[RolSST98] ROLSKI, T., H. SCHMIDLI, V. SCHMIDT AND J. TEUGELS (1998): *Stochastic Processes for Insurance and Finance*. Wiley, New York.

[RouE00] ROUGE, R., AND N. EL KAROUI (2000): "Pricing via Utility Maximization and Entropy", *Mathematical Finance*, 10, 259-276.

[Sa64] SAMUELSON, P. (1964): "Rational Theory of Warrant Pricing", Industrial Management Review 6, 13-31.

[Scha00] SCHACHERMAYER, W. (2000): "How Potential Investments May Change the Optimal Portfolio for Exponential Utility", Preprint, Vienna University of Technology.

[Scha00a] SCHACHERMAYER, W. (2000): "What is the Good Definition of the Optimal Investment Strategy with Respect to Exponential Utility", Preprint, Vienna University of Technology.

[Scha01] SCHACHERMAYER, W. (2001): "Optimal Investments when Wealth May Become Negative", to appear in *Annals of Applied Probability.*

[Schw91] SCHWEIZER, M. (1991): "Option hedging for Semimartingales", *Stochastic Processes and their Applications* 37, 339-363.

[Schw95] SCHWEIZER, M. (1995): "On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition", *Stochastic Analysis and Applications* 13, 573-599.

[Schw96] SCHWEIZER, M. (1996): "Approximation Pricing and the Variance-Optimal Martingale Measure", *Annals of Probability* 24, 206-236.

[Schw99] SCHWEIZER, M. (1999): "A Minimality Property of the Minimal Martingale Measure", *Statistics and Probability Letters* 42, 27-31.

[Schw01] SCHWEIZER, M. (2001): "From Actuarial to Financial Valuation Principles", *Insurance: Mathematics & Economics* 28, 31-47.

[Schw01a] SCHWEIZER, M. (2001): "A Guided Tour through Quadratic Hedging Approaches", in E. Jouini, J. Cvitanić, M. Musiela (eds.), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, Cambridge, 538-574.

[V77] VASIČEK, O. (1977): "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics* 5, 177-188.

[vNM44] VON NEUMANN, J., AND O. MORGENSTERN (1944): *Theory of Games and Economic Behaviour*, Princeton University Press, Princeton.

[Z01] ZARIPHOPOULOU, T. (2001): "A Solution Approach to Valuation with Unhedgeable Risks", *Finance and Stochastics*, 5, 61-82.