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by

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ABSTRACT

The minimization model concept is defined, and its applications to nonlinear optimization are described. Necessary conditions and sufficient conditions are established for functions to have minimization models of certain types. These necessary conditions may also be thought of as properties of the optimal value functions of certain optimization problems subject to linear RHS perturbations.

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1. Introduction

The primary purpose of this report is to establish a broad class of optimization problems that may be "re-formulated" in such a way that they may be solved by MIP techniques. (If irrational coefficients are allowed in the MIP formulations, a significantly broader class of optimization problems can be handled (see [9]), but such formulations have little value since irrational coefficients are computationally intractable.) As a specific illustration of the type of problem that can be dealt with, consider a production cost minimization problem of the following form:

$$\min_{x,z} f(x) + \tilde{f}(z)$$

$$\text{(1.1)} \qquad \text{s.t.} \quad B_1x + B_2z = d$$

$$x \geq 0, \; z \geq 0,$$

where x is a real variable representing the quantity of item A purchased, $z = (z_1, \ldots, z_p)^T$ is a vector variable representing the quantities of other items purchased, the functions f and $\tilde{f}$ are cost functions whose sum gives the total cost of purchasing x units of item A and

$z_1, \ldots, z_p$ units of the remaining items, and the constraints represent the production technology and production goals. We further assume for the purposes of this example that f (see figure 1) is given by the relations

$$(1.2) \qquad f(x) \equiv \begin{cases} 2x & \text{if } x \in [0, 1] \\ 2 & \text{if } x \in [1, 2] \\ x & \text{if } x \geq 2 . \end{cases}$$

(From an economic point of view, this could be thought of as an "economy of scale" purchase function in which the initial unit of item A costs \$2, and volume discounts are made such that a "two for the price of one" policy allows the second unit of item A to be obtained for no additional cost, while further units may be purchased at a cost of \$1 per unit after the initial two units have been purchased.) The function $\tilde{f}$ will be assumed to consist of a sum of piecewise-linear cost functions for the other items, but we shall not be concerned with its specific form at this time.

The cost minimization problem as stated in the form (1.1) is not amenable to solution by traditional linear programming techniques since its objective function is piecewise-linear and non-convex. Although problems involving non-convex piecewise-linear functions can often be re-formulated as mixed-integer programs by well-known techniques (see, for example, [4], [5], [6], [7], or [13]), these standard techniques do <u>not</u> apply in this case because f is defined for <u>all non-negative reals</u>. (A re-formulation employing the "special ordered sets" of Beale and Tomlin [2] is possible, however, and the relationship between such sets and MIP's will be discussed below.) However, by using the results to be developed below, the problem (1.1) first is re-formulated as the "equivalent" problem

$$\min_{x,z,y_i,v_i,u} \quad 2y_1 + v_2 + \tilde{f}(z)$$

(1.3)

$$\text{s.t.} \quad B_1x + B_2z = d$$

$$x,z \geq 0$$

$$0 \leq y_2 \leq y_3 \leq y_1 \leq 1$$

$$v_1 = y_1 + y_2$$

$$x = v_1 + v_2$$

$$0 \leq v_2 \leq 2u \leq x$$

$$y_3 \text{ integer}, \quad u \text{ integer}$$

and then, if the function $\tilde{f}$ is also sufficiently well-behaved, similar re-formulation steps can be used to "eliminate" $\tilde{f}$ from the objective function of (1.3), so that the resulting problem will be an MIP, which (storage space and money permitting) may be solved by using a commercial MIP code.

On the other hand, suppose that the function f, instead of being given by (1.2), is defined by (see Figure 2)

$$(1.4) \qquad f(x) \equiv \begin{cases} 2x & \text{if } x \in [0,1] \\ 2 & \text{if } x \in [1,2] \\ 2+.9(x-2) & \text{if } x \geq 2 . \end{cases}$$

The only difference between the functions defined in (1.2) and (1.4) is the difference of .1 in the slopes of their "unbounded" segments. However, this difference is critical, since by again using the results to be developed below, we will show that no re-formulation involving linear relations with integer variables and rational coefficients can be used to "eliminate" f, as given by (1.4), from the problem (1.1). A re-formulation using "special ordered sets", however, is possible, as will be shown in section 4.

In order to make these notions precise, we develop in section 2 a general re-formulation procedure and describe in what sense it yields equivalent problems, and in section 3 we consider re-formulating problems as MIP's with rational coefficients. Section 4 gives necessary and sufficient conditions for re-formulating problems in terms of unions of polyhedral sets, and section 5 summarizes the results.

2. General Minimization Models

Consider the following minimization problem (in which $x \in \mathbb{R}^n$ is considered as a vector of parameters):

$$(\mathrm{MM}(x)) \qquad \begin{array}{ll} \min\limits_{y} & g(y) \\ \text{s.t.} & y \in Y \cap H(x) \end{array}$$

where $y \in \mathbb{R}^m$, $Y \subseteq \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^1$, H is a mapping from $\mathbb{R}^n$ into the subsets of $\mathbb{R}^m$, and it is assumed that MM(x) has an optimal solution if it is not unbounded or infeasible. Given an extended real-valued function f defined on $S \subseteq \mathbb{R}^n$, MM(x) is said to be a minimization model for f on S (which we will write as $\mathrm{MM}(x) \in \mathcal{M}(f, S)$) if $\omega(x)$, the optimal value function of MM(x), has the property:

$$(2.1) \qquad \omega(x) = f(x) \text{ for every } x \in S .$$

(The optimal value function $\omega(x)$ is assumed to be an extended-real-valued function defined on $\mathbb{R}^n$, with the understanding that $\omega(x) = +\infty$ if $Y \cap H(x) = \emptyset$, and $\omega(x) = -\infty$ if MM(x) is an unbounded problem.)

On the one hand, it is trivial to show that given an arbitrary $f : \mathbb{R}^n \to [-\infty, +\infty]$ and an arbitrary $S \subseteq \mathbb{R}^n$, there exists a minimization model for f on S (i.e., $\mathcal{M}(f, S) \neq \emptyset$). For, such a minimization model (or MM) may be obtained by defining

$$S^- \equiv \{x \mid x \in S,\ f(x) = -\infty\}, \quad S^+ \equiv \{x \mid x \in S,\ f(x) = +\infty\},$$

$$H(x) \equiv \begin{cases} \{f(x)\} & \text{if } x \in S/(S^+ \cup S^-) \\ \mathbb{R}^1 & \text{if } x \in S^- \\ \emptyset & \text{if } x \in S^+ \end{cases},$$

$Y \equiv \mathbb{R}^1$ and $g(y) \equiv y$. Although the question of *nonemptiness* of $\mathcal{M}(f, S)$ is thus answered, there remains the more interesting question of characterizing f and S for which $\mathcal{M}(f, S)$ contains minimization problems of certain *special forms.* From the standpoint of applications, the answer to the latter question is significant because of Theorem 2.1 below, which generalizes a result in [9]. This obvious theorem says that when a function appearing as a *term* in the *objective function* of a minimization problem has an MM, then that function may be "replaced" by its MM. Hence, if the MM is given in terms of functions that are *simpler* than f (e.g., f may be nonlinear, whereas MM may have a linear objective function and linear constraints), this reformulation will yield an equivalent problem that is, perhaps, easier to deal with than the original problem.

Theorem 2.1. If $f : S \to \mathbb{R}^1$, $\hat{f} : S \times \mathbb{R}^p \to \mathbb{R}^1$, and $MM(x) \in \mathcal{M}(f, S)$, then the problems

$$\text{(P)} \qquad \min_{(x,z)} \; f(x) + \hat{f}(x, z) \quad \text{s.t.} \quad x \in \tilde{S}, \; (x, z) \in T$$

and

$$\text{(P')} \qquad \min_{(x,y,z)} \; g(y) + \hat{f}(x, z) \quad \text{s.t.} \quad x \in \tilde{S}, \; (x, z) \in T, \; y \in Y \cap H(x)$$

where $\tilde{S} \subseteq S$, are equivalent in the sense that (a) (P) is feasible if and only if (P') is feasible, (b) x^* is an optimal solution of (P) if and only if there exists a y^* such that (x^*, y^*) solves (P'), and (c) (P) is unbounded if and only if (P') is unbounded.

Proof: See [11].

3. MIMM's and Rational MIMM's

In this section we will consider minimization models of two special types, namely, mixed-integer minimization models (MIMM's) and rational MIMM's. The notions of a MIMM and a rational MIMM for a function on $\mathbb{R}^n$ were defined by Meyer [9], and here we will generalize the definitions to allow subsets of $\mathbb{R}^n$ to be considered, and obtain some additional properties of rational MIMM's. Consider the parametrically defined (x is the vector of parameters) mixed-integer program:

$$
\begin{aligned}
&\min_{z,u,v} \quad z \\
(3.1) \qquad &\text{s.t.} \quad A_0 z + A_1 u + A_2 v = b - A_3 x \\
&\qquad\quad u \geq 0 \text{ and } u \text{ integer} \\
&\qquad\quad v \geq 0,
\end{aligned}
$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^1$, $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, b is a given vector in $\mathbb{R}^m$, and A_0, A_1, A_2, and A_3 are given matrices of the appropriate dimensions, and define $m(x)$ to the optimal value function of (3.1). The parametrically defined problem (3.1) is said to be a *standard-form MIMM for the function* f *on* S *if*

$$m(x) = f(x) \quad \text{for every } x \in S. \tag{3.2}$$

(Put another way, the function f on S is representable by (3.1) and we write $f \in \mathcal{M}_M(S)$.) By "standard-form" we mean that the objective function and constraints are of the format of (3.1). *Any* mixed-integer program in which the RHS is a linear affine function of x may be converted into a standard-form MIMM by well-known techniques.) If, in addition, the matrices A_i $(i = 0, \ldots, 3)$ of (3.1) are all comprised of *rational* data, we say that (3.1) is a *rational* MIMM for f on S and we write $f \in \mathcal{M}_R(S)$ to indicate that f is representable on S by a rational MIMM.

Note that every *piecewise-linear convex* function $h(x)$ of the form $h(x) = \max_{1 \le i \le k} \{c^{(i)}x + d^{(i)}\}$ is in $\mathcal{M}_M(\mathbb{R}^n)$, since we have:

$$\begin{aligned} h(x) = \min\ & z \\ \text{s.t.}\ & z \ge c^{(i)}x + d^{(i)} \quad (i = 1, \ldots, k)\ . \end{aligned}$$

(In fact, note that this is a *linear programming* minimization model for h, and it thus may be converted to a standard-form MIMM with A_1 being the 0 matrix). In particular, every *linear* function is in $\mathcal{M}_M(\mathbb{R}^n)$, and every function $h(x)$ of the above form in which all of the $c^{(i)}$ are rational vectors is in $\mathcal{M}_R(\mathbb{R}^n)$. In [9] and [10] it is shown that some rather broad classes of lower semi-continuous piecewise-linear non-convex functions are in the family $\mathcal{M}_R(\mathbb{R}^n)$, and the classes $\mathcal{M}_R(S)$ are also studied for various sets S in the case in which the projection of the feasible set of (3.1) on the space of the integer variables u is contained in some compact set for all $x \in \mathbb{R}^n$. (Such models are termed bounded-integer MIMM's.) Here we shall avoid such compactness hypotheses, and, as a consequence, the results obtained will not be as strong as those for bounded domains.

In [9] it was shown that the uncapacitated fixed-charge function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ +\infty & \text{otherwise} \end{cases}$$

has no rational MIMM on $\mathbb{R}^1$. The following theorem generalizes this result, and in particular, establishes a convexity property that a class of functions with rational MIMM's must have.

Theorem 3.1: Let $\theta : S \to \mathbb{R}^1$ with the property that there exists an $M > 0$ and a rational c such that

$$\theta(x) = cx + d \quad \text{for} \quad x \geq M.$$

If there exists an $\bar{x} \in S$ such that $\theta(\bar{x}) < c\bar{x} + d$ then $\theta \notin \mathcal{M}_R(S)$.

Proof: See [11].

From a geometrical viewpoint, this result says that the line $cx + d$ must be a supporting hyperplane for the epigraph of θ in order for θ to have a rational MIMM on S. This may be thought of as a convexity property of rational MIMM's. (Jeroslow has shown [8] that the convex hull of the epigraph of the optimal value function of a rational MIMM must be closed, and observed that Theorem 3.1 also follows from his result.) The following results show that this property is also sufficient if the function is well-enough behaved on its nonlinear portion.

Theorem 3.2: Let $f_1(x) \equiv \min\{g(y) \mid y \in Y \cap H(x)\}$, and assume that $f_1(x) \geq CX + d$ for $x \in [\alpha,\beta]$, where $\alpha \geq 0$. If $f_1(\beta) = c\beta + d$, then on $[\alpha, +\infty)$, the function f_2 defined by

$$f_2(x) = \begin{cases} f_1(x) & \text{if } x \in [\alpha,\beta] \\ cx + d & \text{if } x \geq \beta \end{cases}$$

has the minimization model:

$$\min_{(y,u,v_1,v_2)} \quad g(y) + cv_2$$

$$\text{(3.3)} \qquad \begin{aligned} \text{s.t.} \quad & y \in Y \cap H(v_1) \\ & x = v_1 + v_2 \\ & \alpha \le v_1 \le \beta \\ & 0 \le v_2 \le \beta u \le x \\ & u \text{ integer} . \end{aligned}$$

Proof: If $x \in [\alpha, \beta)$, then $u = v_2 = 0$, $x = v_1$, and the minimization model relations reduce to $\min \{g(y) \mid y \in Y \cap H(x)\}$, which by definition is $f_1(x)$.

If $x \in [\beta, +\infty)$, then we may obtain a feasible point for (3.3) by letting $v_1 = \beta$, $v_2 = x-\beta$, $u = [x/\beta]$, and y be the element y^* of $Y \cap H(\beta)$ such that $g(y^*) = c\beta + d$. Note that all the constraints of (3.3) other than $v_2 \le \beta u$ are obviously satisfied, and that $v_2 \le \beta u$ is satisfied since $v_2 = \beta(x-\beta)/\beta = \beta((x/\beta)-1) \le \beta[x/\beta]$. Moreover, at this feasible point, the objective function value is $c\beta + d + c(x - \beta) = cx + d$. Thus, the theorem will be proved if it is shown that the objective function value of any other feasible point is at least $cx + d$ in this case. Since $x = v_1 + v_2$ and $v_1 \le \beta$, $\bar{y} \in Y \cap H(v_1)$ implies $g(\bar{y}) \ge f_1(v_1) \ge cv_1 + d$. But the objective function value is thus at least $cv_1 + d + cv_2 = cx + d$. ∎

Corollary 3.3. Let

$$f_2(x) \equiv \begin{cases} f_1(x) & \text{if } x \in [\alpha, \beta] \\ cx+d & \text{if } x \geq \beta \end{cases},$$

where $\alpha \geq 0$, β is a positive rational, c is rational, and $f_1 \in \mathcal{M}_R[\alpha,\beta]$. Then $f_2 \in \mathcal{M}_R([\alpha, +\infty))$ if and only if $f_1(x) \geq cx + d$ for $x \in [\alpha, \beta)$.

Proof. Clearly these conditions are sufficient, since the method of ~~construction of the minimization model (3.3) will yield a rational MIMM.~~ Conversely, if there exists an $\bar{x} \in [\alpha, \beta)$ such that $f(\bar{x}) < c\bar{x} + d$, Theorem 3.1 shows that $f_2 \notin \mathcal{M}_R([\alpha, +\infty))$. ∎

As examples of the application of Corollary 3.3, we will re-consider the two examples described in section 1. When $f(x)$ is given by (1.2), we can apply Corollary 3.3 by letting $f_2 \equiv f$,

$$f_1(x) \equiv \begin{cases} 2x & \text{if } x \in [0,1] \\ x & \text{if } x \in [1,2] \end{cases}, \quad \alpha = 0, \quad \beta = 2, \quad c = 1, \quad \text{and } d = 0.$$

It is easily seen that $f_1 \in \mathcal{M}_R[0, 2]$ and that f_1 on $[0, 2]$ has the rational MIMM:

$$f_1(x) = \min_{y_1, x} \; 2y_1$$

$$\text{s.t. } 0 \leq y_2 \leq y_3 \leq y_1 \leq 1$$

$$x = y_1 + y_2$$

$$y_3 \text{ integer .}$$

Thus, since $f_1(x) \geq x$ on $[0, 2)$, by Corollary 3.3, $f_2 \in \mathcal{M}_R[0, \infty)$ and has the rational MIMM:

$$
\begin{aligned}
f_2(x) = \min_{x, y_i, v_i, u} \quad & 2y_1 + v_2 \\
\text{s.t.} \quad & 0 \leq y_2 \leq y_3 \leq y_1 \leq 1 \\
& v_1 = y_1 + y_2 \\
& x = v_1 + v_2 \\
& 0 \leq v_2 \leq 2u \leq x \\
& y_3 \text{ integer}, \ u \text{ integer}.
\end{aligned}
$$

By using the substitution procedure of section 2, then, the equivalent problem (1.3) is obtained from (1.1).

On the other hand, if f is given by (1.4), then the condition $f_1(x) \geq 2 + .9(x-2)$ is not satisfied for $x \in [0, 2)$, (in particular, it is violated at $x = 0$) so that no rational MIMM for f exists in this case. Note, however, that by using the general construction procedure of [9], a MIMM with irrational coefficients may be generated for the function defined by (1.4):

$$f(x) = \min_{x, v_i, y_i, u_i, t} \quad 2y_1 + t$$

$$\text{s.t.} \quad 0 \leq y_2 \leq y_3 \leq y_1 \leq 1$$

$$v_1 = y_1 + y_2$$

$$x = v_1 + v_2$$

$$t \geq 0, \quad t \geq .2 + .9v_2 - .2u_1$$

$$0 \leq v_2 \leq u_2, \quad x + 2u_1 \geq 2$$

$$0 \leq \sqrt{2}\,u_2 - u_3 \leq (1-u_1)$$

$$v_1 \leq 2u_1$$

$$y_3 \text{ integer}, \quad u_i \text{ integer } (i = 1, 2, 3).$$

Unfortunately, because of its irrational coefficients, this MIMM cannot be handled by commercial MIP codes.

This leads us to the consideration of another class of minimization models, polyhedral-union minimization models, which do allow a rational representation of the above function.

4. Polyhedral Union Minimization Models

In this section we will consider minimization models in which some of the variables are constrained to be in a finite union of polyhedral sets. Such models will be called polyhedral union minimization models (PUMM's). Optimization problems with logical constraints involving linear inequalities have been studied elsewhere, namely in Balas [1] where cutting planes are developed for these problems. Some examples of functions that have PUMM's but no rational MIMM's (and vice versa) will be given along with a theorem characterizing functions representable as PUMM's. We will also show that any PUMM is equivalent to a minimization model with "special ordered sets". (see Beale and Tomlin [2].)

Formally, consider the program:

$$\min_{z,u,v} \quad z$$

$$\text{(4.1)} \qquad \text{s.t.} \quad A_0z + A_1u + A_2v = b - A_3x$$

$$u \in \bigcup_{j=1}^{N} T_j, \quad v \geq 0,$$

where all data have the same dimension as in (3.1), each T_j is a polyhedral set, and x is a vector of parameters. Let $m(x)$ be the optimal value of (4.1) for a given x. The program

(4.1) is said to be a PUMM for the function f on a subset S of $\mathbb{R}^n$ if:

$$m(x) = f(x) \quad \text{for every } x \in S.$$

To see that special ordered sets can be modelled in the above form, recall that a set $S_1(K) \subseteq \mathbb{R}^K$ is said to be a special ordered set of type 1 if $S_1(K)$ consists of all points in $\mathbb{R}^K$ with at most one nonzero component.

Then

$$S_1(K) = \bigcup_{i=1}^{K} T_i, \quad T_i = \{(u_1,\ldots,u_K) | u_j=0, \ j \neq i\}$$

is a representation of a type 1 special ordered set as a finite union of polyhedral sets. Furthermore, if $u \in S_1(K)$ and $\sum_{i=1}^{K} u_i = 1$, then the u_i can only take the values 0 or 1. Thus bounded integer MIMM's are contained in the class of PUMM's.

As noted in section 3, the uncapacitated fixed charge function

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \\ +\infty & \text{otherwise} \end{cases}$$

has no rational MIMM. However, using PUMM's it follows that

$$
\begin{aligned}
f(x) = \min_{u_1,u_2} \quad & (1-u_1) \\
\text{s.t.} \quad & u_2 = x \\
& u_2 \geq 0 \\
& 0 \leq u_1 \leq 1 \\
& (u_1,u_2) \in S_1(2).
\end{aligned}
$$

Also in section 3, it was shown that example (1.4) has no rational MIMM, but it does have the following PUMM

$$
\begin{aligned}
f(x) = \min_{\lambda_i} \quad & 2\lambda_1 + 2\lambda_2 + .9\lambda_3 \\
\text{s.t.} \quad & x = \lambda_1 + 2\lambda_2 + \lambda_3 \\
& \quad \lambda_0 + \lambda_1 + \lambda_2 = 1 \\
& \quad \lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0 \\
& \quad (\lambda_0,\lambda_1,\lambda_2,\lambda_3) \in T_1 \cup T_2 \cup T_3,
\end{aligned}
$$

$$
\left.
\begin{aligned}
\text{where } T_1 &= \{(\lambda_0,\lambda_1,\lambda_2,\lambda_3) | \lambda_0 \geq 0,\ \lambda_1 \geq 0,\ \lambda_2 = \lambda_3 = 0\} \\
T_2 &= \{(\lambda_0,\lambda_1,\lambda_2,\lambda_3) | \lambda_0 = 0,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 = 0\} \\
T_3 &= \{(\lambda_0,\lambda_1,\lambda_2,\lambda_3) | \lambda_0 = \lambda_1 = 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0\}.
\end{aligned}
\right\}
$$

Finally, we note that the rational MIMM

$$
\begin{aligned}
f(x) = \min_{y_1,y_2} \quad & y_1 + 10y_2 \\
\text{s.t.} \quad & y_1 + 12y_2 \geq x \\
& y_1,\ y_2 \geq 0 \\
& y_2 \ \text{integer}
\end{aligned}
$$

has no PUMM since there are infinitely many breakpoints in the graph of f. That such functions have no PUMM's follows from the next theorem.

<u>Theorem 4.1</u>: Let $f: \mathbb{R}^n \to (-\infty, +\infty]$. Then f has a PUMM if and only if there exist extended affine* functions $g_j(x)$ with polyhedral effective domains D_j, $j = 1,\ldots,N$, such that

$$f(x) = \min_{j=1,\ldots,N} g_j(x)$$

<u>Proof</u>: Assume that f has the PUMM

$$f(x) = \min_{z,u,v} \quad z$$

$$\text{s.t.} \quad (z,u,v) \in \Omega(x)$$

$$\text{where} \quad \Omega(x) = \left\{ (z,u,v) \left| \begin{array}{l} A_0 z + A_1 u + A_2 v = b - A_3 x \\ \\ v \geq 0, \; u \in \bigcup_{j=1}^{N} T_j \end{array} \right. \right\}$$

and each T_j is polyhedral.

$$\text{Let} \quad \Omega_j(x) = \left\{ (z,u,v) \left| \begin{array}{l} A_0 z + A_1 u + A_2 v = b - A_3 x \\ \\ v \geq 0, \; u \in T_j \end{array} \right. \right\}$$

---

* $g: \mathbb{R}^n \to (-\infty,+\infty]$ is said to be an extended affine function if there exists a and b and a polyhedral set $D \subseteq \mathbb{R}^n$ such that $g(x) = ax + b$ if $x \in D$ and $g(x) = +\infty$ if $x \notin D$.

then each $\Omega_j(x)$ is polyhedral and

$$\Omega(x) = \bigcup_{j=1}^{N} \Omega_j(x).$$

Thus $$f(x) = \min_{j=1,\ldots,N} g_j(x)$$

where $$g_j(x) = \min_{z,u,v} \{z \mid (z,u,v) \in \Omega_j(x)\}.$$

Now each $g_j(x)$ is the optimal value function of a parametric linear program, so eff dom g_j which we denote by D_j is polyhedral. By a standard result in linear programming, for each j there exist polyhedral sets $D_j^1,\ldots,D_j^{M_j}$, for some M_j, and affine functions $g_j^i(x)$ $i = 1,\ldots,M_j$ such that $D_j = \bigcup_{i=1}^{M_j} D_j^i$ and

$$g_j(x) = \begin{cases} g_j^i(x) & \text{if } x \in D_j^i \\ +\infty & \text{if } x \notin D_j. \end{cases}$$

Let $$\tilde{g}_j^i(x) = \begin{cases} g_j^i(x) & \text{if } x \in D_j^i \\ +\infty & \text{if } x \notin D_j^i \end{cases}$$

then

$$f(x) = \min \{\tilde{g}_j^i(x) \mid j=1,\ldots,N;\ i=1,\ldots,M_j\}$$

which proves half of the theorem.

Conversely, suppose

$$g_i(x) = \begin{cases} c^i x + d^i & \text{if } x \in D_i \ (i=1,\ldots,N) \\ +\infty & \text{otherwise} \end{cases}$$

where each D_i is polyhedral and

$$f(x) = \min_{i=1,\ldots,N} g_i(x).$$

Then since D_i is polyhedral, there exist integers n_i, m_i and vectors $\eta_{ij} \in \mathbb{R}^n$ $(j=1,\ldots,n_i)$ and $\zeta_{ij} \in \mathbb{R}^n$ $(j=1,\ldots,m_i)$ such that

$$D_i = \{\sum_{j=1}^{n_i} \lambda_{ij}\eta_{ij} + \sum_{j=1}^{m_i} t_{ij}\zeta_{ij} \mid \sum_{j=1}^{n_i} \lambda_{ij}=1,\ \lambda_{ij}\geq 0,\ t_{ij}\geq 0\}$$

(see Rockafellar [12]).

Then $$f(x) = \min_{z,u,v,w,\lambda,t} z$$

s.t. $$z = \sum_{i=1}^{N} (c^i[\sum_{j=1}^{n_i} \lambda_{ij}\eta_{ij} + \sum_{j=1}^{m_i} t_{ij}\zeta_{ij}]+d^i u_i)$$

$$x = \sum_{i=1}^{N} (\sum_{j=1}^{n_i} \lambda_{ij}\eta_{ij} + \sum_{j=1}^{m_i} t_{ij}\zeta_{ij})$$

$$\sum_{j=1}^{n_i} \lambda_{ij} = u_i, \; u_i \in \{0,1\} \quad (i=1,\ldots,N)$$

$$\sum_{i=1}^{N} u_i = 1$$

$$\sum_{j=1}^{m_i} t_{ij} = v_i \quad (i = 1,\ldots,N)$$

$$w_i = 1 - u_i \quad (i = 1,\ldots,N)$$

$$(v_i, w_i) \in S_1(2) \quad (i = 1,\ldots,N)$$

$$\lambda_{ij} \geq 0, \; t_{ij} \geq 0 \quad \text{for all } i, j.$$

which is a PUMM for f. ∎

The representation for the function f used in the last half of the above theorem needed only type 1 special ordered sets, not general polyhedral sets. Thus any function that can be modelled with a PUMM can be represented as a minimization model with type 1 special ordered sets. However, such a model would require knowledge of all extreme points and rays for each T_j. In the absence of this knowledge or of any special structure for the polyhedral sets T_j, it is not clear how to branch effectively and calculate bounds if a branch and bound algorithm is to be used to solve the problem. A cutting plane procedure such as that proposed by Balas [1] could be employed directly on (4.1) though.

Note also that a type 2 special ordered set as described by Beale may be represented as a union of polyhedral sets in which only two "adjacent" variables are allowed to be non-zero, and thus models involving only linear constraints and type 2 special ordered sets are also PUMM's. (It is easy to see that the converse also holds, namely, that any PUMM may be reduced to a model involving only linear constraints and type 2 special ordered sets.) Finally, because of the observations following Theorem 5.6 of [9], every function having a PUMM also has a MIMM, but not necessarily one with rational coefficients.

We close with the observation that Theorem 4.1 may be extended to the case $f\colon \mathbb{R}^n \to [-\infty,+\infty]$ by appropriately generalizing the notion of an extended affine function (to allow $ax + b$ to be replaced by $-\infty$) and by carrying out some minor modifications in the proof.

5. Summary

The results in sections 3 and 4 give necessary conditions and sufficient conditions for the existence of various representations of piecewise-linear functions. Similar conditions have been given in earlier reports ([9], [10]) for other types of MIP representations, and, in Table 1 we summarize results to date specialized to the case of continuous piecewise-linear (with a finite number of segments) functions of a non-negative real variable. For each function of this class there exist a set of breakpoints $0 = a_1 < a_2 < \ldots < a_n$ and a set of slopes $c_1, \ldots, c_n$ such that on the interval $[a_i, a_{i+1}]$ $(i=1,\ldots,n-1)$, the function is a linear function with slope c_i, and on $[a_n, +\infty)$, the function is a linear function with slope c_n.

It should be noted, however, that continuity of f on $[0, +\infty]$, is not a necessary condition for the existence of a MIMM for f. Some of the continuity properties established in this and earlier papers are summarized in Table 2.

These continuity properties again demonstrate the generality of the unrestricted MIMM as well as the ability to handle most cases of interest via special ordered sets (because of the equivalence with PUMM's). Figure 3 shows the relationships between various model classes, e.g., the class of optimal value functions for linear programming minimization models (LPMM's) is contained in the corresponding class for bounded-integer minimization models (BIMM's). (RMIMM (MIMM) denotes the class of optimal value functions for rational (unrestricted) mixed-integer minimization models.)

Model Class	Corresponding Slope Conditions	Remarks
No integer variables	$c_1 \le c_2 \le \cdots \le c_n$	For a linear programming model, the function must be convex.
Integer variables bounded	$c_i \le c_n$, $i = 1, \ldots, n-1$	This condition was established in [10]. Note that it implies the condition $f_1(x) \ge c_n x + d$ of Corollary 3.3.
Rational MIMM	$c_{n-1} \le c_n$	The other slopes are constrained indirectly by the condition $f_1(x) \ge c_n x + d$ from Corollary 3.3.
PUMM or unrestriced MIMM	No restrictions	The construction of the MIMM in this case is given in [9].

Table 1: Summary of Results Specialized to Continuous Functions on $[0, +\infty)$

Model Class	Corresponding Continuity Property	Remarks
No integer variables	Continuous on $[0, +\infty)$	This is a well-known property of linear programs; see Dantzig [4].
Integer variables bounded	Lower semi-continuous on $[0, +\infty)$, and continuous from the right on $[0, +\infty)$	This condition was established in [10].
Rational MIMM	Lower semi-continuous on $[0, +\infty)$, and continuous from the right at a_n.*	This follows from Corollary 3. (lower semi-continuity was established in [9]).
PUMM	Lower semi-continuous	Follows from Theorem 4.1
Unrestricted MIMM	Not established	An example in which lower semi-continuity does <u>not</u> hold is given in [9].

*Here we are assuming that f is linear for $x \geq a_n$.

Table 2: Summary of Continuity Properties

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