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Rational points of bounded height on toric varieties

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Let Σ be a complete d-dimensional regular fan in $N_{\mathbf{R}}$ defining a smooth compact d-dimensional toric variety over a number field F, $\Sigma^{(i)}$ the set of all i-dimensional cones in Σ. Let the elements of $\Sigma^{(1)}$ have integral generators $e_1, \ldots, e_n$. We define some rational function on $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$ associated with the combinatorial structure of the fan Σ

$$f_\Sigma(s) = \sum_{\sigma \in \Sigma^{(d)}} f_\sigma(s),$$

where $f_\sigma(s) = (s_{j_1} \cdots s_{j_d})^{-1}$, if $e_{j_1}, \ldots, e_{j_d}$ are generators of the cone σ.

For arhimedian completions of F, we put

$$f_{\Sigma,\mathbf{R}}(s) = 2^d f_\Sigma(s),\ f_{\Sigma,\mathbf{C}}(s) = (2\pi)^d f_\Sigma(s).$$

Denote by $P_\Sigma(t_1, \ldots, t_n)$ the rational function defined as the the Gilbert-Poincare serie of the $\mathbf{Z}^n_{\geq 0}$-graded Stenley-Reisner ring $R(\Sigma)$ corresponding to Σ.

For any prime $\mathcal{P}$ ideal of F, we denote by $\|\mathcal{P}\|$ the cardinality of the residue field of $\mathcal{P}$, by $\delta_{\mathcal{P}}$ absolute different of the nonarhimedian local field $F_{\mathcal{P}}$, and put

$$f_{\Sigma,\mathcal{P}}(s) = (\frac{1}{\sqrt{\delta_{\mathcal{P}}})^d} P_\Sigma(\|\mathcal{P}\|^{-s_1}, \ldots, \|\mathcal{P}\|^{-s_n}).$$

Denote by $K_\Sigma(s)$ the following product

$$f^{r_1}_{\Sigma,\mathbf{R}}(s) f^{r_2}_{\Sigma,\mathbf{C}}(s) \prod_{\mathcal{P}} f_{\Sigma,\mathcal{P}}(s),$$

where r_1 is the number of real embeddings of F, r_2 is the number of complex embeddings of F.

Let r_F the residue of the Dedekind zeta function $\zeta_F(z)$ at $z = 1$;

$$r_F = \frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{| D_F |} w}.$$

Theorem. *Let* $D(s) = s_1 D_1 + \cdots + s_n D_n$ $(s_i > 0)$ *be an effective divisor on toric variety* V_Σ, $H_\Sigma(s,x)$ *corresponding height function on* F*-rational points* $x \in T(F) \cong F^*$. *Let* $T^1(A_F) = (I^1(F))^d$ *where* $I^1(F)$ *is the group of idele with norm* 1 *of the field* F, $d\mu$ *the standard Haar measure on* $T^1(A_F)$. *Then*

$$\int_{T^1(A_F)} H_\Sigma(s,x)^{-1} d\mu = (2\pi r_F)^{-d} \int_{M_{\mathbf{R}}} K(s+im)dm.$$

This theorem can be applied to the problem of the asymptotic distribution of rational points of bounded height on toric varieties (cf. [1]).

Example. Let Σ defines $\mathbf{P}^d$. Then

$$f_\Sigma(s) = \frac{s_1 + \cdots + s_{d+1}}{s_1 \cdots s_{d+1}}, \; P_\Sigma(t_1, \ldots, t_{d+1}) = \frac{1 - t_1 \cdots t_{d+1}}{(1-t_1)\cdots(1-t_{n+1})},$$

$$K_\Sigma(s) = \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|\, D_F \,|}}\right)^d \left(\frac{s_1 + \cdots + s_{d+1}}{s_1 \cdots s_{d+1}}\right)^{r_1+r_2} \frac{\zeta_F(s_1)\cdots\zeta_F(s_{d+1})}{\zeta_F(s_1 + \cdots + s_{d+1})}.$$

Applying the residue formula to the d-dimensional integral, we get

$$\int_{T^1(A_F)} H_\Sigma(s,x)^{-1} = \left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|\, D_F \,|}}\right)^d \left(\frac{s_1 + \cdots + s_{d+1}}{s_1 + \cdots + s_{d+1} - d}\right)^{r_1+r_2} \frac{\zeta_F(s_1 + \cdots s_{d+1} - d)}{\zeta_F(s_1 + \cdots + s_{d+1})}.$$

The residue of $\int_{T^1(A_F)} H_\Sigma(s,x)^{-1} d\mu$ at $s = (1, \ldots, 1)$ is

$$\left(\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|\, D_F \,|}}\right)^d (d+1)^{r_1+r_2-1} \left(\frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|\, D_F \,|} w}\right) \zeta_F^{-1}(d+1).$$

This number gives the coefficient in the asymptotic formula of Schanuel for the number of rational points in projective spaces [2] .

References

[1] V.V. Batyrev, and Yu. I. Manin, *Sur le nomber des points rationnels de hauter borné des variétés algébriques*, Math. Ann. **286**, 1990, 27-43.

[2] S. Schanuel, *Heights in number fields*, Bull. Soc. Math. France, **107**, 1979, 433-449.