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Correction

Rationality of Moduli Spaces of Stable Bundles

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I am grateful to S. Ramanan for pointing out to me an error in my paper [1]. The error occurs on lines 17, 18 of p. 257, where it is assumed that $ng-d$ is coprime to d; this is of course false if $(g,d)\neq 1$. Since the only restrictions on n and d at this point are

$$(n,d)=1,\ n(g-1)<d<ng,$$

this invalidates the proof of the proposition of [1] and hence also that of the main theorem.

There seems to be no simple way of avoiding this problem, but the methods of [1] do give some positive results, as we shall now show. We adopt the notations and conventions of [1] throughout; in particular L is a line bundle of degree d over the complete non-singular algebraic curve X of genus $g\geqq 2$, and $S_{n,L}(X)$ is the moduli space of stable bundles of rank n and determinant L over X. Note that, up to isomorphism, $S_{n,L}(X)$ depends only on the residue class of d modulo n; moreover, if L^* denotes the dual of L, then $S_{n,L^*}(X)$ is isomorphic to $S_{n,L}(X)$. For any bundle E over $S\times X$ and any $s\in S$, we write E_s for the bundle over X obtained by restricting E to $\{s\}\times X$.

Definition. A *good* (n,L)*-family* parametrised by a variety S is a bundle E of rank n over $S\times X$ such that

(i) $\dim S=(n^2-1)(g-1)$;
(ii) for all $s,t\in S$, $E_s\cong E_t$ if and only if $s=t$;
(iii) for all $s\in S$, E_s is stable and $\det E_s\cong L$;
(iv) for all $s\in S$, $H^1(X;E_s)=0$;
(v) for all $s\in S$, the infinitesimal deformation map of E at s is injective.

Remark 1. If $(n,d)=1$, there exists a bundle U over $S_{n,L}(X)\times X$ with the obvious universal property. We shall say that the pair (n,L) is *good* if $S_{n,L}(X)$ is a rational variety and there exists a Zariski-open subset T of $S_{n,L}(X)$ such that $U|T\times X$ is a good (n,L)-family. Note that $S_{1,L}(X)$ consists of a single point; it therefore follows trivially from Riemann-Roch that $(1,L)$ is good if $d\geqq 2g-1$. [If $n\geqq 2$ and $d>n(g-1)$, one can show that there exists a Zariski-open subset T of $S_{n,L}(X)$ such that $U|T\times X$ is a good (n,L)-family, but we shall not need to use this fact.]

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Remark 2. The argument in the first paragraph of [1, Sect. 4] shows that, if there exists a good (n, L)-family parametrised by S, then S is birationally equivalent to $S_{n,L}(X)$; if S is a rational variety, it follows that (n, L) is good. Note in particular that, by [2, Theorem 2], a good (n, L)-family can exist only if $(n, d) = 1$.

The arguments of [1, Sects. 3, 4] now give

Proposition 1. *Suppose that $n(g-1) < d < ng$ and that there exists a good $(ng-d, L)$-family E' parametrised by T. Then there exists a good (n, L)-family E parametrised by a Zariski-open subset S of $T \times k^p$ for a suitable integer p; moreover one can choose S so that, for all $s \in S$, the sections of E_s generate a trivial subbundle of E_s.*

Corollary. *Suppose that $n(g-1) < d < ng$, let K denote the canonical line bundle over X, and let M be a line bundle of degree 1 over X such that $H^0(X; M) \neq 0$. Suppose further that either $(ng-d, L)$ or $(d-n(g-1),\ L^* \otimes K^n \otimes M^n)$ is good. Then (n, L) is good.*

Proof. In the first case, we apply the proposition with T as in Remark 1 and $E' = U | T \times X$; the result then follows from Remark 2. In the second case, the proposition gives us a good $(n, L^* \otimes K^n \otimes M^n)$-family E parametrised by a rational variety S; moreover, from the last part of the proposition,

$$H^0(X; E_s \otimes M^*) = 0 \quad \text{for all} \quad s \in S.$$

It follows easily that $E^* \otimes p_X^*(K \otimes M)$ is a good (n, L)-family; hence (n, L) is good by Remark 2.

This corollary allows us to prove that $S_{n,L}(X)$ is rational for certain values of g, n, d. In particular we have

Proposition 2. *$S_{n,L}(X)$ is rational in the following cases:*

(a) $d \equiv \pm 1 \bmod n$;

(b) $(n, d) = 1$ *and g is a prime power;*

(c) $(n, d) = 1$ *and the sum of the two smallest distinct prime factors of g is greater than n.*

Proof. The proposition is trivial when $n = 1$; it is therefore sufficient to prove it for

$$n \geqq 2, \quad n(g-1) < d < ng. \tag{$*$}$$

In this case (a) follows at once from the corollary and the fact that $(1, L')$ is good if $\deg L' \geqq 2g-1$; indeed we have the stronger result that (n, L) is good.

For (b) and (c) we again assume $(*)$ and prove that (n, L) is good by induction on n. The essential point is that the hypotheses imply that either $(ng-d, d) = 1$ or $(d - n(g-1), n(2g-1) - d) = 1$; the inductive step now follows easily from the corollary.

Remark. The Corollary to Proposition 1 applies in many other cases (for example $d \equiv \pm 2$, g odd); the simplest case to which our results do not apply is whan $n = 5$, $d \equiv 2$ or 3 and g is divisible by 6.

References

1. Newstead, P.E.: Rationality of moduli spaces of stable bundles. Math. Ann. **215**, 251–268 (1975)
2. Ramanan, S.: The moduli spaces of vector bundles over an algebraic curve. Math. Ann. **200**, 69–84 (1973)

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