Rationality of vertex operator algebra V_L^+ : higher rank

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Abstract

The lattice vertex operator V_L associated to a positive definite even lattice L has an automorphism of order 2 lifted from -1 isometry of L. It is established that the fixed point vertex operator algebra V_L^+ is rational.

1 Introduction

The notion of rational vertex operator algebra is an analogue to that of semisimple Lie algebra or semisimple associative algebra. Rational vertex operator algebras whose admissible module category is semisimple form a fundamental class of vertex operator algebras. Familiar examples of rational vertex operator algebras include the vertex operator algebras V_L associated with even lattices [D1], [DLM1], vertex operator algebras associated to the irreducible vacuum representations for affine Kac-Moody algebras with positive levels [FZ], [DL], [LL], vertex operator algebras associated to the minimal series for the Virasoro algebra [W].

Let V be a vertex operator algebra and G a finite automorphism group of V, then the space of G-invariants V^G is itself a vertex operator algebra. A well known conjecture in the orbifold conformal field theory states that if V is rational then V^G is rational. Solving this conjecture has significant applications in the theory of vertex operator algebras and conformal field theory.

Let V_L be a lattice vertex operator algebra associated with a positive definite even lattice L which plays an important role in shaping the theory of vertex

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operator algebras. The vertex operator algebra V_L has an automorphism θ of order 2 lifted from the -1 isometry of L and we denote the θ-fixed points of V_L by V_L^+. The vertex operator algebras V_L^+ have been studied extensively. The irreducible modules for V_L^+ were classified in [DN2] and [AD]. The fusion rules among irreducible V_L^+-modules were computed in [ADL]. The C_2-cofiniteness of V_L^+ was obtained in [Y] and [ABD]. But the rationality of V_L^+ has only been established if L has rank one [A2], or L is a special lattice [DGH]. In this paper, we extend the rationality result to any lattice. That is, V_L^+ is rational for any positive definite even lattice L.

We prove the rationality of V_L^+ in three steps. First, we show that the Zhu algebra $A(V_L^+)$ is a finite dimensional semisimple associative algebra. Although $A(V_L^+)$ was studied in [DN2] and [AD] in great detail for the purpose of classification of irreducible modules, there is still a distance to claim the semi-simplicity. Second, we use the rationality of V_L^+ in the case that L has rank one and the fusion rules to deal with the rationality of V_L^+ if L has an orthogonal base. Last, we use the rationality of $V_{L_1}^+$ to prove the rationality of V_L^+ for any L where L_1 is a sublattice of L with the same rank and has an orthogonal base. The main idea in the proof of rationality is to prove that if there is a V_L^+-module exact sequence

$$0 \to M^1 \to M \to M^2 \to 0$$

for any irreducible V_L^+-modules M^1 and M^2, then M is a direct sum of M^1 and M^2.

The representation theory of V_L^+ is complete in some sense after this paper. But the structure theory of V_L^+ is far from over. Determining the derivation Lie algebra and the automorphism group of V_L^+ for an arbitrary positive definite even L remains a major problem. This has been achieved when the rank of L is one, two or three [DG1], [DG2], [S1], [S2], or L is a special lattice which is either unimodular or does not have roots [S1], [S2]. Extending these results to general lattice seems a big challenge.

The paper is organized as follows: In Section 2, we recall definitions of vertex operator algebra, module, intertwining operator and fusion rules. We also give some basic facts on vertex operator algebras in this section. In Section 3, we present the results on vertex operator algebras $M(1)^+$ and V_L^+ and the irreducible modules. Section 4 is devoted to the proof of the semi-simplicity of $A(V_L^+)$. We deal with the rationality of V_L^+ when L has an orthogonal base in Section 5. The rationality of V_L^+ for any positive definite even lattice L is given in Section 6.

2 Preliminaries

In this section we briefly review the definitions of twisted modules and rationality from [FLM], [DLM2]. We present the Zhu algebra, the tensor product and fusion rules from [FHL] and [Z]. We also discuss the extensions of modules following

[A2] and give a sufficient condition under which the extensions are trivial. This result will be used extensively in latter sections.

A *vertex operator algebra* V is a $\mathbb{Z}$-graded vector space $V=\bigoplus_{n\in\mathbb{Z}}V_n$ equipped with a linear map $Y:V\to(\mathrm{End}V)[[z,z^{-1}]]$, $a\mapsto Y(a,z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}$ for $a\in V$ such that $\dim V_n$ is finite for all integer n and that $V_n=0$ for sufficiently small n (see [FLM]). There are two distinguished vectors, the *vacuum vector* $\mathbf{1}\in V_0$ and the *Virasoro element* $\omega\in V_2$. By definition $Y(\mathbf{1},z)=\mathrm{id}_V$, and the component operators $\{L(n)|n\in\mathbb{Z}\}$ of $Y(\omega,z)=\sum_{n\in\mathbb{Z}}L(n)z^{-n-2}$ give a representation of the Virasoro algebra on V with central charge c. Each homogeneous subspace $V_n\,(n\geq 0)$ is an eigenspace for $L(0)$ with eigenvalue n.

An automorphism g of a vertex operator algebra V is a linear isomorphism of V satisfying $g(\omega)=\omega$ and $gY(a,z)g^{-1}=Y(g(a),z)$ for any $a\in V$. We denote by $\mathrm{Aut}(V)$ the group of all automorphisms of V. For a subgroup $G<\mathrm{Aut}(V)$, the fixed point set $V^G=\{a\in V\,|\,g(a)=a,\,g\in G\}$ has a canonical vertex operator algebra structure.

Let g be an automorphism of a vertex operator algebra V of order T. Then V is a direct sum of eigenspaces for g:

$$V=\bigoplus_{r=0}^{T-1}V^r,\ V^r=\{\,a\in V\,|\,g(a)=e^{-\frac{2\pi ir}{T}}a\,\}.$$

Definition 2.1. A *weak g-twisted V-module* M is a vector space equipped with a linear map

$$\begin{aligned}&Y_M:V\to(\mathrm{End}M)\{z\},\\ &\qquad a\mapsto Y_M(a,z)=\sum_{n\in\mathbb{Q}}a_nz^{-n-1},\ a_n\in\mathrm{End}M\end{aligned}$$

such that the following conditions hold for $0\leq r\leq T-1$, $a\in V^r$, $b\in V$ and $u\in M$:

(1) $b_mu=0$ if m is sufficiently large,

(2) $Y_M(a,z)=\sum_{n\in\mathbb{Z}+\frac{r}{T}}a_nz^{-n-1}$,

(3) $Y_M(\mathbf{1},z)=\mathrm{id}_M$,

(4) (the twisted Jacobi identity)

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(a,z_1)Y_M(b,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(b,z_2)Y_M(a,z_1)\\ &\quad=z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-\frac{r}{T}}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(a,z_0)b,z_2).\end{aligned}$$

A weak g-twisted V-module is denoted by $(M,\,Y_M)$, or simply by M. In the case g is the identity, any weak g-twisted V-module is called a *weak V-module.* A *g-twisted weak V-submodule* of a g-twisted weak module M is a subspace N

of M such that $a_n N \subset N$ hold for all $a \in V$ and $n \in \mathbb{Q}$. If M has no g-twisted weak V-submodule except 0 and M, M is called *irreducible* or *simple*.

Set $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. Then $\{L(n)\,|\,n \in \mathbb{Z}\}$ give a representation of the Virasoro algebra on M with central charge c and the $L(-1)$-derivative property

$$Y_M(L(-1)a, z) = \frac{d}{dz}Y_M(a, z) \text{ for all } a \in V \tag{2.1}$$

holds for any $a \in V$ (see [DLM1]).

Definition 2.2. An *admissible g-twisted V-module* M is a weak g-twisted V-module which has a $\frac{1}{T}\mathbb{Z}_{\geq 0}$-gradation $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_{\geq 0}} M(n)$ such that

$$a_m M(n) \subset M(\mathrm{wt}a + n - m - 1) \tag{2.2}$$

for any homogeneous $a \in V$ and m, $n \in \mathbb{Q}$.

In the case g is the identity, any admissible g-twisted V-module is called an *admissible V-module*. Any g-twisted weak V-submodule N of a g-twisted admissible V-module is called a *g-twisted admissible V-submodule* if $N = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_{\geq 0}} N \cap M(n)$.

A g-twisted admissible V-module M is said to be *irreducible* if M has no non-trivial admissible weak V-submodule. When a g-twisted admissible V-module M is a direct sum of irreducible admissible submodules, M is called *completely reducible*.

Definition 2.3. A vertex operator algebra V is said to be *g-rational* if any g-twisted admissible V-module is completely reducible. If V is id_V-rational, then V is called *rational*.

Definition 2.4. A *g-twisted V-module* $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$ is a $\mathbb{C}$-graded weak g-twisted V-module with $M_\lambda = \{u \in M | L(0)u = \lambda u\}$ such that M_λ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M_{\lambda+n/T} = 0$ for sufficiently small integer n. A vector $w \in M_\lambda$ is called a weight vector of weight λ, and we write $\lambda = \mathrm{wt}w$.

We next define Zhu's algebra $A(V)$ which is an associative algebra following [Z]. For any homogeneous vectors $a, b \in V$, we define

$$a * b = Res_z \frac{(1+z)^{wta}}{z} Y(a, z)b,$$

$$a \circ b = Res_z \frac{(1+z)^{wta}}{z^2} Y(a, z)b$$

and extend the two operations to $V \times V$ bilinearly. Denote by $O(V)$ the subspace of V linearly spanned by $a \circ b$, for all $a, b \in V$ and set $A(V) = V/O(V)$. The following theorem is due to [Z].

Theorem 2.5. *(1) The bilinear operation $*$ induces $A(V)$ an associative algebra structure. The vector $[\mathbf{1}]$ is the identity and $[\omega]$ is in the center of $A(V)$.*

(2) Let $M = \bigoplus_{n=0}^{\infty} M(n)$ be an admissible V-module with $M(0) \neq 0$. Then the linear map

$$o : V \to \mathrm{End}M(0),\ a \mapsto o(a)|_{M(0)}$$

induces an algebra homomorphism from $A(V)$ to $\mathrm{End}M(0)$. Thus $M(0)$ is a left $A(V)$-module.

(3) The map $M \mapsto M(0)$ induces a bijection between the set of equivalence classes of irreducible admissible V-modules and the set of equivalence classes of irreducible $A(V)$-modules.

Now we consider the tensor product vertex algebra and the tensor product modules for tensor product vertex operator algebra. The tensor product of vertex operator algebras $(V^1, Y, \mathbf{1}, \omega^1), \cdots, (V^p, Y, \mathbf{1}, \omega^p)$ is constructed on the tensor product vector space

$$V = V^1 \otimes \cdots \otimes V^p$$

where the vertex operator $Y(\cdot, z)$ is defined by

$$Y(v^1 \otimes \cdots \otimes v^p, z) = Y(v^1, z) \otimes \cdots \otimes Y(v^p, z)$$

for $v^i \in V^i$ $(1 \leq i \leq p)$, the vacuum vector is

$$1 = 1 \otimes \cdots \otimes 1$$

and the Virasoro element is

$$\omega = \omega^1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \omega^p.$$

then $(V, Y, 1, \omega)$ is a vertex operator algebra (see [FHL], [LL]).

Let M^i be an admissible V^i-module for $i = 1, ..., p$. We may construct the tensor product admissible module $M^1 \otimes \cdots \otimes M^p$ for the tensor product vertex operator algebra $V^1 \otimes \cdots \otimes V^p$ by

$$Y(v^1 \otimes \cdots \otimes v^p) = Y(v^1, z) \otimes \cdots \otimes Y(v^p, z).$$

Then $(M^1 \otimes \cdots \otimes M^p, Y)$ is an admissible $V^1 \otimes \cdots \otimes V^p$-module. The following result was given in [FHL] and [DMZ].

Theorem 2.6. *Let $V^1, \cdots, V^p$ be rational vertex operator algebras, then $V^1 \otimes \cdots \otimes V^p$ is rational and any irreducible $V^1 \otimes \cdots \otimes V^p$-module is a tensor product $M^1 \otimes \cdots \otimes M^p$, where $M^1, \cdots, M^p$ are some irreducible modules for the vertex operator algebras $V^1, \cdots, V^p$, respectively.*

Let $M = \bigoplus_{r\in\mathbb{C}} M(r)$ be a V-module. Set $M' = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda^*$, the restricted dual of M. It was proved in [FHL] that M' is naturally a V-module where the vertex operator map denoted by Y' is defined by the property

$$\langle Y'(a,z)u', v\rangle = \langle u', Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v\rangle$$

for $a \in V, u' \in M'$ and $v \in M$. The V-module M' is called the contragredient module of M. It was proved that if M is irreducible , then so is M'. A V-module M is said to be self dual if M and M' are isomorphic V-modules.

We now recall the notion of intertwining operators and fusion rules from [FHL].

Definition 2.7. *Let M^1, M^2, M^3 be weak V-modules. An intertwining operator $\mathcal{Y}(\cdot, z)$ of type $\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right)$ is a linear map*

$$\mathcal{Y}(\cdot, z) : M^1 \to Hom(M^2, M^3)\{z\}$$

$$v^1 \mapsto \mathcal{Y}(v^1, z) = \sum_{n\in\mathbb{C}} v_n^1 z^{-n-1}$$

satisfying the following conditions:

(1) For any $v^1 \in M^1, v^2 \in M^2$ and $\lambda \in \mathbb{C}, v^1_{n+\lambda}v^2 = 0$ for $n \in \mathbb{Z}$ sufficiently large.

(2) For any $a \in V, v^1 \in M^1$,

$$z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y_{M^3}(a, z_1)\mathcal{Y}(v^1, z_2) - z_0^{-1}\delta(\frac{z_1 - z_2}{-z_0})\mathcal{Y}(v^1, z_2)Y_{M^2}(a, z_1)$$

$$= z_2^{-1}\delta(\frac{z_1 - z_0}{z_2})\mathcal{Y}(Y_{M^1}(a, z_0)v^1, z_2).$$

(3) For $v^1 \in M^1$, $\frac{d}{dz}\mathcal{Y}(v^1, z) = \mathcal{Y}(L(-1)v^1, z)$.

All of the intertwining operators of type $\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right)$ form a vector space denoted by $I_V\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right)$. The dimension of $I_V\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right)$ is called the fusion rule of type $\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right)$ for V.

We now have the following result which was essentially proved in [ADL].

Theorem 2.8. *Let V^1, V^2 be rational vertex operator algebras. Let M^1, M^2, M^3 be V^1-modules and N^1, N^2, N^3 be V^2-modules such that*

$$dim I_{V^1}\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right) < \infty,\ dim I_{V^2}\left(\begin{array}{c} N^3 \\ N^1\ N^2 \end{array}\right) < \infty.$$

Then the linear map

$$\sigma : I_{V^1}\left(\begin{array}{c} M^3 \\ M^1\ M^2 \end{array}\right) \otimes I_{V^2}\left(\begin{array}{c} N^3 \\ N^1\ N^2 \end{array}\right) \to I_{V^1\otimes V^2}\left(\begin{array}{c} M^3\otimes N^3 \\ M^1\otimes N^1\ \ M^2\otimes N^2 \end{array}\right)$$

$$\mathcal{Y}_1(\cdot, z)\otimes\mathcal{Y}_2(\cdot, z)\mapsto(\mathcal{Y}_1\otimes\mathcal{Y}_2)(\cdot, z)$$

is an isomorphism, where $(\mathcal{Y}_1\otimes\mathcal{Y}_2)(\cdot, z)$ *is defined by*

$$(\mathcal{Y}_1\otimes\mathcal{Y}_2)(\cdot, z)(u^1\otimes v^1, z)u^2\otimes v^2 = \mathcal{Y}_1(u^1, z)u^2\otimes\mathcal{Y}_2(v^1, z)v^2.$$

Now let M^1, M^2 be weak V-modules, we call a weak V-module M an extension of M^2 by M^1 if there is a short exact sequence

$$0\to M^1\to M\to M^2\to 0.$$

Then we could define the equivalence of two extensions, and then define the extension group $Ext^1_V(M^2, M^1)$. Recall that V is called C_2-cofinite if the subspace $C_2(V)$ of V spanned by $u_{-2}v$ for $u, v\in V$ has finite codimension. We have the following facts [A1].

Theorem 2.9. *Let M and N be irreducible V-modules, then $Ext^1_V(N, M) = 0$ if and only if $Ext^1_V(M', N') = 0$.*

Theorem 2.10. *Let V be a C_2-cofinite vertex operator algebra, then V is rational if and only if $Ext^1_V(N, M) = 0$ for any pair of irreducible V-modules M and N.*

The following result will be extensively used in Sections 5 and 6.

Lemma 2.11. *Let V be a vertex operator algebra and U a rational vertex operator subalgebra of V with the same Virasoro element. Let M^1, M^2 be irreducible V-modules. Assume that*

$$I_U\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0$$

for any irreducible U-submodules N^1, N, N^2 of M^1, V, M^2, respectively. Then $Ext^1_V(M^2, M^1) = 0$.

Proof: Let M be an extension of M^2 by M^1. Then $M = M^1\oplus M^2$ as U-modules as U is rational. Let N^1, N, N^2 be any irreducible U-submodules of M^1, V, M^2, respectively. Then $P_{N^1}Y(u, z)|_{N^2}$ for $u\in N$ is an intertwining operator of type $I_U\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right)$ where P_{N^1} is the projection from M to N_1. From the assumption, $P_{N^1}Y(u, z)|_{N^2} = 0$. Since N^1, N, N^2 are arbitrary, we see that $u_nM^2\subset M^2$ for any $u\in V$ and $n\in\mathbb{Z}$. As a result, M^2 is a V-module and $Ext^1_V(M^2, M^1) = 0$. The proof is complete. $\square$

3 Vertex operator algebras $M(1)^+$ and V_L^+

In this section we recall vertex operator algebras $M(1)^+$ and V_L^+ [FLM] and related results [DN1], [DN2], [DN3], [A1], [A2], [AD], [ADL]. In particular, the irreducible modules, fusions and contragredient modules of irreducible modules for V_L^+ are discussed.

Let L be a positive definite even lattice in the sense that L has a symmetric positive definite $\mathbb{Z}$-valued $\mathbb{Z}$-bilinear form $(\cdot,\cdot)$ such that $(\alpha,\alpha)\in 2\mathbb{Z}$ for any $\alpha\in L$. We set $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}L$ and extend $(\cdot,\cdot)$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}}=\mathbb{C}[t,t^{-1}]\otimes\mathfrak{h}\oplus\mathbb{C}C$ be the affinization of commutative Lie algebra $\mathfrak{h}$ defined by

$$[\beta_1\otimes t^m,\,\beta_2\otimes t^n]=m(\beta_1,\beta_2)\delta_{m,-n}C \text{ and } [C,\hat{\mathfrak{h}}]=0$$

for any $\beta_i\in\mathfrak{h}$, $m,\,n\in\mathbb{Z}$. Then $\hat{\mathfrak{h}}^{\geq 0}=\mathbb{C}[t]\otimes\mathfrak{h}\oplus\mathbb{C}C$ is a commutative subalgebra. For any $\lambda\in\mathfrak{h}$, we can define a one dimensional $\hat{\mathfrak{h}}^{\geq 0}$-module $\mathbb{C}e^\lambda$ by the actions $\rho(h\otimes t^m)e^\lambda=(\lambda,h)\delta_{m,0}e^\lambda$ and $\rho(C)e^\lambda=e^\lambda$ for $h\in\mathfrak{h}$ and $m\geq 0$. Now we denote by

$$M(1,\lambda)=U(\hat{\mathfrak{h}})\otimes_{U(\hat{\mathfrak{h}}^{\geq 0})}\mathbb{C}e^\lambda\cong S(t^{-1}\mathbb{C}[t^{-1}]),$$

the $\hat{\mathfrak{h}}$-module induced from $\hat{\mathfrak{h}}^{\geq 0}$-module $\mathbb{C}e^\lambda$. Set $M(1)=M(1,0)$. Then there exists a linear map $Y: M(1)\to(\mathrm{End}M(1,\lambda)[[z,z^{-1}]]$ such that $(M(1),\,Y,\,\mathbf{1},\,\omega)$ has a simple vertex operator algebra structure and $(M(1,\lambda),Y)$ becomes an irreducible $M(1)$-module for any $\lambda\in\mathfrak{h}$ (see [FLM]). The vacuum vector and the Virasoro element are given by $\mathbf{1}=e^0$ and $\omega=\frac{1}{2}\sum_{a=1}^d h_a(-1)^2\mathbf{1}$ respectively, where $\{h_a\}$ is an orthonormal basis of $\mathfrak{h}$.

Let $\widehat{L}$ be the canonical central extension of L by $\langle\kappa\rangle=\langle\kappa|\kappa^2=1\rangle$:

$$1\to\langle\kappa\rangle\to\widehat{L}\xrightarrow{\bar{}}L\to 1$$

with the commutator map $c(\alpha,\beta)=\kappa^{(\alpha,\beta)}$ for $\alpha,\beta\in L$. Let $e: L\to\widehat{L}$ be a section such that $e_0=1$ and $\epsilon: L\times L\to\langle\kappa\rangle$ the corresponding 2-cocycle. We may assume that ϵ is bimultiplicative. Then $\epsilon(\alpha,\beta)\epsilon(\beta,\alpha)=\kappa^{(\alpha,\beta)}$,

$$\epsilon(\alpha,\beta)\epsilon(\alpha+\beta,\gamma)=\epsilon(\beta,\gamma)(\alpha,\beta+\gamma),$$

and $e_\alpha e_\beta=\epsilon(\alpha,\beta)e_{\alpha+\beta}$ for $\alpha,\beta,\gamma\in L$. Let θ denote the automorphism of $\widehat{L}$ defined by $\theta(e_\alpha)=e_{-\alpha}$ and $\theta(\kappa)=\kappa$. Set $K=\{a^{-1}\theta(a)|a\in\widehat{L}\}$. Note that if $(\alpha,\beta)\in 2\mathbb{Z}$ for all $\alpha,\beta\in L$ then the $\widehat{L}=L\times\langle\kappa\rangle$ is a direct product of abelian groups. In this case we can and do choose $\epsilon(\alpha,\beta)=1$ for all $\alpha,\beta\in L$.

The lattice vertex operator algebra associated to L is given by

$$V_L=M(1)\otimes\mathbb{C}^\epsilon[L],$$

where $\mathbb{C}^\epsilon[L]$ is the twisted group algebra of L with a basis e^α for $\alpha \in L$ and is an $\widehat{L}$-module such that $e_\alpha e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}$. Note that if $(\alpha, \beta) \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$ then $\mathbb{C}^\epsilon[L] = \mathbb{C}[L]$ is the usual group algebra.

Recall that $L^\circ = \{ \lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z} \}$ is the dual lattice of L. There is an $\widehat{L}$-module structure on $\mathbb{C}[L^\circ] = \bigoplus_{\lambda \in L^\circ} \mathbb{C}e^\lambda$ such that κ acts as -1 (see [DL]). Let $L^\circ = \cup_{i \in L^\circ/L}(L+\lambda_i)$ be the coset decomposition such that (λ_i, λ_i) is minimal among all (λ, λ) for $\lambda \in L+\lambda_i$. In particular, $\lambda_0 = 0$. Set $\mathbb{C}[L+\lambda_i] = \bigoplus_{\alpha \in L} \mathbb{C}e^{\alpha+\lambda_i}$. Then $\mathbb{C}[L^\circ] = \bigoplus_{i \in L^\circ/L} \mathbb{C}[L+\lambda_i]$ and each $\mathbb{C}[L+\lambda_i]$ is an $\widehat{L}$-submodule of $\mathbb{C}[L^\circ]$. The action of $\widehat{L}$ on $\mathbb{C}[L+\lambda_i]$ is as follows:

$$e_\alpha e^{\beta+\lambda_i} = \epsilon(\alpha, \beta)e^{\alpha+\beta+\lambda_i}$$

for α, $\beta \in L$. On the surface, the module structure on each $\mathbb{C}[L+\lambda_i]$ depends on the choice of λ_i in $L+\lambda_i$. It is easy to prove that different choices of λ_i give isomorphic $\widehat{L}$-modules.

Set $\mathbb{C}[M] = \bigoplus_{\lambda \in M} \mathbb{C}e^\lambda$ for a subset M of L°, and define $V_M = M(1) \otimes \mathbb{C}[M]$. Then V_L is a rational vertex operator algebra and $V_{L+\lambda_i}$ for $i \in L^\circ/L$ are the irreducible modules for V_L (see [B], [FLM], [D1], [DLM1]).

Define a linear isomorphism $\theta : V_{L+\lambda_i} \to V_{L-\lambda_i}$ for $i \in L^\circ/L$ by

$$\theta(\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)e^{\alpha+\lambda_i}) = (-1)^k\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)e^{-\alpha-\lambda_i}$$

for $\beta_i \in \mathfrak{h}$, $n_i \geq 1$ and $\alpha \in L$ if $2\lambda_i \notin L$, and

$$\begin{aligned}&\theta(\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)e^{\alpha+\lambda_i})\\ &\qquad = (-1)^k c_{2\lambda_i}\epsilon(\alpha, 2\lambda_i)\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)e^{-\alpha-\lambda_i}\end{aligned}$$

if $2\lambda_i \in L$ where $c_{2\lambda_i}$ is a square root of $\epsilon(2\lambda_i, 2\lambda_i)$. Then θ defines a linear isomorphism from V_{L° to itself such that

$$\theta Y(u, z)v = Y(\theta u, z)\theta v$$

for $u \in V_L$ and $v \in V_{L^\circ}$. In particular, θ is an automorphism of V_L which induces an automorphism of $M(1)$.

For any θ-stable subspace U of V_{L°, let $U^\pm$ be the ±1-eigenspace of U for θ. Then V_L^+ is a simple vertex operator algebra.

Also recall the θ-twisted Heisenberg algebra $\mathfrak{h}[-1]$ and its irreducible module $M(1)(\theta)$ from [FLM]. Let χ be a central character of $\widehat{L}/K$ such that $\chi(\kappa) = -1$ and T_χ the irreducible $\widehat{L}/K$-module with central character χ. Note that if $(\alpha, \beta) \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$ then $\widehat{L}/K = L/K \times \langle\kappa\rangle$ and each T_χ is, in fact, an $L/2L$-module. In particular, T_χ is one-dimensional. In this case, let $\beta_1, \cdots, \beta_d$ be a basis of L, then $T_\chi = T_{\chi_1} \otimes \cdots \otimes T_{\chi_d}$ where each T_{χ_i} is an irreducible $\mathbb{Z}\beta_i/2\mathbb{Z}\beta_i$-module such that e_{β_i} acts as $\chi(e_{\beta_i})$. This fact will be used later.

It is well known that $V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi$ is an irreducible θ-twisted V_L-module (see [FLM], [D2]). We define actions of θ on $M(1)(\theta)$ and $V_L^{T_\chi}$ by

$$\theta(\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)) = (-1)^k \beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)$$

$$\theta(\beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)t) = (-1)^k \beta_1(-n_1)\beta_2(-n_2)\cdots\beta_k(-n_k)t$$

for $\beta_i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}_+$ and $t \in T_\chi$. We denote the ± 1-eigenspace of $M(1)(\theta)$ and $V_L^{T_\chi}$ under θ by $M(1)(\theta)^\pm$ and $(V_L^{T_\chi})^\pm$ respectively. We have the following results proved in [DN1], [DN3] and [AD]:

Theorem 3.1. *Any irreducible module for the vertex operator algebra $M(1)^+$ is isomorphic to one of the following modules:*

$$M(1)^+, M(1)^-, M(1,\lambda) \cong M(1,-\lambda) \ (0 \neq \lambda \in \mathfrak{h}), M(1)(\theta)^+, M(1)(\theta)^-.$$

Theorem 3.2. *Let $\{\lambda_j\}$ be the set of representatives of L°/L, then any irreducible V_L^+-module is isomorphic to one of the following modules:*

$$V_L^\pm, V_{\lambda_j+L}(2\lambda_j \notin L), V_{\lambda_j+L}^\pm(2\lambda_j \in L), (V_L^{T_\chi})^\pm.$$

Now we consider some decompositions of the modules for the vertex operator algebras $M(1)^+$ and V_L^+. We denote $M_\mathfrak{h}(1)$ for the vertex operator algebra $M(1)$ associated with $\mathfrak{h}$ and similarly for the modules. It is clear that if $\mathfrak{h}'$ is a subspace of $\mathfrak{h}$ such that the restriction of the bilinear form on $\mathfrak{h}$ to $\mathfrak{h}'$ is non-degenerate, then $M_{\mathfrak{h}'}^+$ is a simple vertex operator subalgebra of $M_\mathfrak{h}^+$. Furthermore, if $\mathfrak{h} = \mathfrak{h}_1 \bigoplus \mathfrak{h}_2$ such that $(\mathfrak{h}_1, \mathfrak{h}_2) = 0$, then the modules in Theorem 3.1 viewed as $M_{\mathfrak{h}_1}^+ \otimes M_{\mathfrak{h}_2}^+$-modules can be decomposed as follows:

$$M_\mathfrak{h}^+ \cong (M_{\mathfrak{h}_1}^+ \otimes M_{\mathfrak{h}_2}^+) \bigoplus (M_{\mathfrak{h}_1}^- \otimes M_{\mathfrak{h}_2}^-),$$

$$M_\mathfrak{h}^- \cong (M_{\mathfrak{h}_1}^+ \otimes M_{\mathfrak{h}_2}^-) \bigoplus (M_{\mathfrak{h}_1}^- \otimes M_{\mathfrak{h}_2}^+),$$

$$M_\mathfrak{h}(1,\lambda) \cong M_{\mathfrak{h}_1}(1,\lambda_1) \otimes M_{\mathfrak{h}_2}(1,\lambda_2),$$

$$M_\mathfrak{h}(1)(\theta)^+ \cong (M_{\mathfrak{h}_1}(1)(\theta)^+ \otimes M_{\mathfrak{h}_2}(1)(\theta)^+) \bigoplus (M_{\mathfrak{h}_1}(1)(\theta)^- \otimes M_{\mathfrak{h}_2}(1)(\theta)^-),$$

$$M_\mathfrak{h}(1)(\theta)^- \cong (M_{\mathfrak{h}_1}(1)(\theta)^+ \otimes M_{\mathfrak{h}_2}(1)(\theta)^-) \bigoplus (M_{\mathfrak{h}_1}(1)(\theta)^- \otimes M_{\mathfrak{h}_2}(1)(\theta)^+).$$

Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice of rank one. Then all irreducible V_L^+-modules are decomposed into direct sums of irreducible $M(1)^+$-modules as follows (cf. [DG1] and [A1])

$$V_L^\pm \cong M(1)^\pm \bigoplus_{m=1}^{\infty} M(1, m\alpha),$$

$$V_{\lambda+L} \cong \bigoplus_{m\in\mathbb{Z}} M(1, \lambda + m\alpha),$$

$$V^{\pm}_{\frac{\alpha}{2}+L} \cong \bigoplus_{m=0}^{\infty} M(1, \frac{\alpha}{2} + m\alpha),$$

$$(V_L^{T_i})^{\pm} \cong M(1)(\theta)^{\pm},\ i = 1, 2$$

where T_i is the one dimensional $\widehat{L}/K$-module T_χ with $\chi(e_\alpha) = \pm 1$ respectively.

We also have the following result given in [Y] and [ABD].

Theorem 3.3. *V_L^+ is C_2-cofinite.*

The following results were obtained in [A2].

Theorem 3.4. *The vertex operator algebra V_L^+ is rational, if L is a positive definite even lattice of rank one.*

Proposition 3.5. *Let L be a positive definite even lattice such that V_L^+ is C_2-cofinite and $A(V_L^+)$ is semisimple. Let M^1, M^2 be irreducible V_L^+-modules. If the difference of the lowest weight of M^1 and M^2 is not a nonzero integer, then*

$$Ext^1_{V_L^+}(M^2, M^1) = 0.$$

Remark 3.6. *In the next section, we will prove that for a positive definite even lattice L, the Zhu's algebra $A(V_L^+)$ is semisimple. Thus Proposition 3.5 is true for any positive definite even lattice.*

We now consider the fusion rules for the vertex operator algebra V_L^+. For any $\lambda \in L^\circ$ and a central character χ of $\widehat{L}/K$, let $\chi^{(\lambda)}$ be the central character defined by $\chi^{(\lambda)}(a) = (-1)^{(\overline{a},\lambda)}\chi(a)$. We set $T_\chi^{(\lambda)} = T_{\chi^{(\lambda)}}$. We call a triple (λ, μ, ν) for $\lambda, \mu, \nu \in L^\circ$ an admissible triple modulo L, if $p\lambda + q\mu + r\nu \in L$ for some $p, q, r \in \{\pm 1\}$.

The following result on part of fusion rules for the vertex operator algebra V_L^+ when L is of rank one comes from [A1]. This result will be used in Section 5 to deal with the rationality of V_L^+ when L has an orthogonal base.

Theorem 3.7. *Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice and $L^\circ/L = \{\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_k\}$ such that $\lambda_0 = 0$ and $\lambda_k = \alpha/2$. Let $W^i, i = 1, 2, 3$ be irreducible V_L^+-modules. Then*

(1) the fusion rule of type $\left(\begin{array}{c} W^3 \\ W^1\ W^2 \end{array}\right)$ is either 0 or 1.

(2) the fusion rule of type $\left(\begin{array}{c} W^3 \\ W^1\ W^2 \end{array}\right)$ is non-zero if and only if $W^i (i = 1, 2, 3)$ satisfy one of the following cases:

(i) $W^1 = V_L^+$ and $W^2 \cong W^3$.

(ii) $W^1 = V_L^-$ *and the pair* (W^2, W^3) *is one of the following pairs*

$$(V_L^{\pm}, V_L^{\mp}),\ (V_{\alpha/2+L}^{\pm}, V_{\alpha/2+L}^{\mp}),\ (V_L^{T_1,\pm}, V_L^{T_1,\mp}),\ (V_L^{T_2,\pm}, V_L^{T_2,\mp}),\ (V_{\lambda_i+L}, V_{\lambda_i+L}),$$

for $i = 1, 2, \cdots, k-1$.

The fusion rules for V_L^+ for any L was obtained in [ADL] and will be exploited in Section 6. Recall the number $\pi_{\lambda,\mu}$ for $\lambda, \mu \in L^\circ$ from [ADL].

Theorem 3.8. *Let L be a positive definite even lattice , for any irreducible V_L^+-modules M^i $(i = 1, 2, 3)$, the fusion rule of type* $\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix}$ *is either 0 or 1. Furthermore, the fusion rule of type* $\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix}$ *with M^1 being one of $V_{\lambda+L}$ for $(2\lambda \notin L)$, $V_{\lambda+L}^+$ for $(2\lambda \in L)$, $V_{\lambda+L}^-$ for $(2\lambda \in L)$ is 1 if and only if M^i $(i = 1, 2, 3)$ satisfy one of the following conditions:*

(1) $M^1 = V_{\lambda+L}$ for $\lambda \in L^\circ$ such that $2\lambda \notin L$ and M^2, M^3 is one of the following pairs:

$(V_{\mu+L}, V_{\nu+L})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \notin L$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_{\mu+L}^{\pm}, V_{\nu+L}), ((V_{\nu+L})', (V_{\mu+L}^{\pm})')$ for $\mu, \nu \in L^\circ$ such that $2\mu \in L, 2\nu \notin L$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_L^{T_\chi,\pm}, V_L^{T_\chi^{(\lambda)},\pm}), (V_L^{T_\chi,\pm}, V_L^{T_\chi^{(\lambda)},\mp})$ for any irreducible $\widehat{L}/K$-module T_χ.

(2) $M^1 = V_{\lambda+L}^+$ for $\lambda \in L^\circ$ such that $2\lambda \in L$ and M^2, M^3 is one of the following pairs:

$(V_{\mu+L}, V_{\nu+L})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \notin L$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_{\mu+L}^{\pm}, V_{\nu+L}^{\pm})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \in L, \pi_{\lambda,2\mu} = 1$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_{\mu+L}^{\pm}, V_{\nu+L}^{\mp})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \in L, \pi_{\lambda,2\mu} = -1$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_L^{T_\chi,\pm}, V_L^{T_\chi^{(\lambda)},\pm}), ((V_L^{T_\chi^{(\lambda)},\pm})', (V_L^{T_\chi,\pm})')$ for any irreducible $\widehat{L}/K$-module T_χ such that $c_\chi(\lambda) = 1$.

$(V_L^{T_\chi,\pm}, V_L^{T_\chi^{(\lambda)},\mp}), ((V_L^{T_\chi^{(\lambda)},\pm})', (V_L^{T_\chi,\mp})')$ for any irreducible $\widehat{L}/K$-module T_χ such that $c_\chi(\lambda) = -1$.

(3) $M^1 = V_{\lambda+L}^-$ for $\lambda \in L^\circ$ such that $2\lambda \in L$ and M^2, M^3 is one of the following pairs:

$(V_{\mu+L}, V_{\nu+L})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \notin L$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_{\mu+L}^{\pm}, V_{\nu+L}^{\pm})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \in L, \pi_{\lambda,2\mu} = -1$ and (λ, μ, ν) is an admissible triple modulo L.

$(V_{\mu+L}^{\pm}, V_{\nu+L}^{\mp})$ for $\mu, \nu \in L^\circ$ such that $2\mu, 2\nu \in L, \pi_{\lambda,2\mu} = 1$ and (λ, μ, ν) is an admissible triple modulo L such that $c_\chi(\lambda) = 1$.

$(V_L^{T_\chi,\pm}, V_L^{T_\chi^{(\lambda)},\pm}), ((V_L^{T_\chi^{(\lambda)},\pm})', (V_L^{T_\chi,\pm})')$ *for any irreducible* $\widehat{L}/K$*-module* T_χ *such that* $c_\chi(\lambda) = -1$.

$(V_L^{T_\chi,\pm}, V_L^{T_\chi^{(\lambda)},\mp}), ((V_L^{T_\chi^{(\lambda)},\pm})', (V_L^{T_\chi,\mp})')$ *for any irreducible* $\widehat{L}/K$*-module* T_χ *such that* $c_\chi(\lambda) = 1$.

Next we identify the contragredient modules of the irreducible V_L^+-modules [ADL]:

Proposition 3.9. *The irreducible* V_L^+*-modules* $V_L^\pm$ *and* $V_{\lambda+L}$ *for* $\lambda \in L^\circ$ *with* $2\lambda \notin L$ *are self dual. For* $\lambda \in L^\circ$ *with* $2\lambda \in L$*,* $V_{\lambda+L}^\pm$ *are self dual if* $2(\lambda,\lambda)$ *is even, and* $(V_{\lambda+L}^\pm)' \cong V_{\lambda+L}^\mp$ *if* $2(\lambda,\lambda)$ *is odd. Let* χ *be a central character of* $\widehat{L}/K$ *such that* $\chi(\kappa) = -1$*, then the irreducible modules* $(V_L^{T_\chi,\pm})'$ *are isomorphic to* $(V_L^{T'_\chi,\pm})'$ *respectively, where* χ' *is a central character of* $\widehat{L}/K$ *defined by* $\chi'(a) = (-1)^{\frac{(\overline{a},\overline{a})}{2}}\chi(a)$ *for any* $a \in Z(\widehat{L}/K)$.

4 Semisimplicity of $A(V_L^+)$

Motivated by Proposition 3.5, we prove the semisimplicity of $A(V_L^+)$ for any positive definite even lattice L in this section. In the case that the rank of L is 1, this result has previously been obtained in [DN2]. The semisimplicity of $A(V_L^+)$ enables us to establish that if the two V_L-modules have the same lowest weight then the extension of one module by the other is always trivial.

First recall that the irreducible $A(V_L^+)$-modules are the top levels $W(0)$ of irreducible admissible V_L^+-modules W. So by Theorem 3.2 (also see [DN2]), we have

Lemma 4.1. *The irreducible* $A(V_L^+)$*-modules are given as follows:*

$$V_L^+(0) = \mathbb{C}\mathbf{1}, \quad V_L^-(0) = \mathfrak{h}(-1)\bigoplus(\textstyle\bigoplus_{\alpha\in L_2}\mathbb{C}(e^\alpha - e^{-\alpha})),$$
$$V_{\lambda_i+L}(0) = \textstyle\bigoplus_{\alpha\in\Delta(\lambda_i)}\mathbb{C}e^{\lambda_i+\alpha} \quad (2\lambda_i \notin L),$$
$$V_{\lambda_i+L}^\pm(0) = \textstyle\sum_{\alpha\in\Delta(\lambda_i)}\mathbb{C}(e^{\lambda_i+\alpha} \pm \theta e^{\lambda_i+\alpha}) \quad (2\lambda_i \in L),$$
$$V_L^{T_\chi,+}(0) = T_\chi, \quad V_L^{T_\chi,-}(0) = \mathfrak{h}(-1/2)\otimes T_\chi,$$

where $L_2 = \{\alpha \in L | (\alpha,\alpha) = 2\}$*,* $\mathfrak{h}(-1) = \{h(-1)\mathbf{1} | h \in \mathfrak{h}\} \subset M(1)$ *and* $\mathfrak{h}(-1/2) = \{h(-1/2)\mathbf{1} | h \in \mathfrak{h}\} \subset M(1)(\theta)$.

Let $\{h_1, \cdots, h_d\}$ be an orthonormal basis of $\mathfrak{h}$. Recall from [DN2] and [AD] the following vectors in V_L^+ for $a, b = 1, \cdots, d$ and $\alpha \in L$

$$
\begin{aligned}
S_{ab}(m,n) &= h_a(-m)h_b(-n), \\
E^u_{ab} &= 5S_{ab}(1,2) + 25S_{ab}(1,3) + 36S_{ab}(1,4) + 16S_{ab}(1,5) \; (a \neq b), \\
\bar{E}^u_{ba} &= S_{ab}(1,1) + 14S_{ab}(1,2) + 41S_{ab}(1,3) + 44S_{ab}(1,4) + 16S_{ab}(1,5) \; (a \neq b), \\
E^u_{aa} &= E^u_{ab}E^u_{ba}, \\
E^t_{ab} &= -16(3S_{ab}(1,2) + 14S_{ab}(1,3) + 19S_{ab}(1,4) + 8S_{ab}(1,5)) \; (a \neq b), \\
\bar{E}^t_{ba} &= -16(5S_{ab}(1,2) + 18S_{ab}(1,3) + 21S_{ab}(1,4) + 8S_{ab}(1,5)) \; (a \neq b), \\
E^t_{aa} &= E^t_{ab}E^t_{ba}, \\
\Lambda_{ab} &= 45S_{ab}(1,2) + 190S_{ab}(1,3) + 240S_{ab}(1,4) + 96S_{ab}(1,5), \\
E^\alpha &= e^\alpha + e^{-\alpha}.
\end{aligned}
$$

For $v \in V_L^+$, we denote $v + O(V_L^+)$ by $[v]$. Let A^u and A^t be the linear subspace of $A(M(1)^+)$ spanned by $E^u_{ab} + O(M(1)^+)$ and $E^t_{ab} + O(M(1)^+)$ respectively for $1 \leq a,\, b \leq d$. Then A^t and A^u are two sided ideals of $A(M(1)^+)$. Note that the natural algebra homomorphism from $A(M(1)^+)$ to $A(V_L^+)$ gives embedding of A^u and A^t into $A(V_L^+)$. We should remark that the A^u and A^t are independent of the choice of the orthonormal basis $\{h_1, \cdots, h_d\}$.

By Lemma 7.3 of [AD] we know that

$$V_L^-(0) = \mathfrak{h}(-1) \bigoplus (\sum_{\alpha \in L_2} \mathbb{C}[E^\alpha]\alpha(-1)),$$

where $L_2 = \{\alpha \in L | (\alpha, \alpha) = 2\}$. Let $L_2 = \{\pm\alpha_1, \cdots, \pm\alpha_r, \pm\alpha_{r+1}, \cdots, \pm\alpha_{r+l}\}$ be such that $\{\alpha_1, \cdots, \alpha_r\}$ are linearly independent and $\{\alpha_{r+1}, \cdots, \alpha_{r+l}\} \subseteq \bigoplus_{i=1}^r \mathbb{Z}_+\alpha_i$. We can choose the orthonormal basis $\{h_i |\ i = 1, \cdots, d\}$ so that $h_i \in \mathbb{C}\alpha_1 + \cdots + \mathbb{C}\alpha_i$, for $i = 1, \cdots, r$. Then we have

$$\alpha_i(-1) = a_{i1}h_1(-1) + \cdots + a_{ii}h_i(-1),\ i = 1, \cdots, r,$$

$$\alpha_j(-1) = a_{j1}h_1(-1) + \cdots + a_{jr}h_r(-1),\ j = r+1, \cdots, r+l,$$

where $a_{ii} \neq 0, i = 1, \cdots, r$. For $i \in \{1, 2, \cdots, l\}$, let k_i be such that

$$a_{r+i,k_i} \neq 0,\ a_{r+i,k_i+1} = \cdots = a_{r+i,r} = 0.$$

We know from [AD] that $e^i = h_i(-1)$ for $i = 1, \cdots, d$ and $e^{d+j} = [E^{\alpha_j}]\alpha_j(-1)$ for $j = 1, \cdots, r+l$ form a basis of $V_L^-(0)$. We first construct a two-sided ideal of $A(V_L^+)$ isomorphic to $\mathrm{End}(V_L^-(0))$. Recall E^u_{ij} for $i, j = 1, \cdots, d$. We now extend the definition of E^u_{ij} to all $i, j = 1, \cdots, d+r+l$ and the linear span of E^u_{ij} will be the ideal of $A(V_L^+)$ isomorphic to $\mathrm{End}V_L^-(0)$ (with respect to the basis $\{e^1, \cdots, e^{d+r+l}\}$).

For the notational convenience, we also write $E^u_{i,j}$ for E^u_{ij} from now on. Define

$$[E^u_{j,d+i}] = \frac{1}{4\epsilon(\alpha_i, \alpha_i)a_{ii}}[E^u_{ji} * E^{\alpha_i}],\ i = 1, \cdots, r, j = 1, \cdots, d,$$

$$[E^u_{j,d+r+i}] = \frac{1}{4\epsilon(\alpha_{r+i},\alpha_{r+i})a_{r+i,k_i}}[E^u_{j,k_i} * E^{\alpha_{r+i}}],\ i=1,\cdots,l, j=1,\cdots,d.$$

Define

$$[E^u_{d+i,j}] = \sum_{k=1}^{r} a_{ik}[E^{\alpha_i}] * [E^u_{kj}],\ i=1,\cdots,r+l, j=1,\cdots,d,$$

where $a_{ij}=0$, for $1\le i<j\le r$. Recall from [DN2] and [AD] that $[E^u_{ab}]h_c(-1) = \delta_{c,b}h_a(-1)$ for $a,b,c=1,\cdots,d$.

Lemma 4.2. *The following holds:*

$$[E^u_{ij}]e^k = \delta_{k,j}e^i,\ [E^u_{st}]e^k = \delta_{t,k}e^s$$

for $i,t=1,\cdots,d,\ j,s=d+1,\cdots,d+r+l$ *and* $k=1,\cdots,d+r+l$.

Proof: Let $h\in\mathfrak{h}$ such that $(h,h)\neq 0$. Then $\omega_h = \frac{1}{2(h,h)}h(-1)^2$ is a Virasoro element with central charge 1. Note that $\omega_h\beta(-1) = \frac{(\beta,h)^4}{2(h,h)}h(-1)$ for any $\beta\in\mathfrak{h}$. For $\alpha\in L_2$ then $[E^\alpha]*[E^\alpha] = 4\epsilon(\alpha,\alpha)[\omega_\alpha]$ in $A(V_L^+)$ by Proposition 4.9 of [AD]. Then for $i=1,\cdots,d, j=1,\cdots,r$, we have

$$\begin{aligned}[E^u_{i,d+j}]e^{d+j} &= [E^u_{i,d+j}]([E^{\alpha_j}]\alpha_j(-1))\\ &= \frac{1}{4\epsilon(\alpha_j,\alpha_j)a_{jj}}([E^u_{ij}]*[E^{\alpha_j}]*[E^{\alpha_j}])\alpha_j(-1)\\ &= \frac{1}{a_{jj}}[E^u_{ij}]\alpha_j(-1) = h_i(-1).\end{aligned}$$

Let $k\in\{1,\cdots,r+l\}$ such that $k\neq j$. Then

$$\begin{aligned}[E^u_{i,d+j}]e^{d+k} &= [E^u_{i,d+j}]([E^{\alpha_k}]\alpha_k(-1))\\ &= \frac{1}{4\epsilon(\alpha_j,\alpha_j)a_{jj}}([E^u_{ij}]*[E^{\alpha_j}]*[E^{\alpha_k}])\alpha_k(-1).\end{aligned}$$

By Proposition 5.4 of [AD], we have

$$[E^{\alpha_j}]*[E^{\alpha_k}] = \sum_p [v^p]*[E^{\alpha_j+\alpha_k}]*[w^p] + \sum_q [x^q]*[E^{\alpha_j-\alpha_k}]*[y^q],$$

where $v^p,w^p,x^q,y^q\in M(1)^+$. Since A^u is an ideal of $A(M(1)^+)$, we have $[E^u_{ji}]*[v^p], [E^u_{ji}]*[x^q]\in A^u$. By the proof of Proposition 7.2 of [AD], we know that $A^u[E^\alpha]\alpha(-1)=0$, for any $\alpha\in L_2$. So by Lemma 7.1 and Proposition 7.2 of [AD], we have

$$[E^u_{i,d+j}]e^{d+k}=0,\ i=1,\cdots,d, j=1,\cdots,r, k=1,\cdots,r+l, j\neq k.$$

It follows from the proof of Proposition 7.2 of [AD] that

$$E^u_{i,j+d}e^s = [E^u_{ij}] * [E^{\alpha_j}]h_s(-1) = 0$$

for $s = 1, \cdots, d$ as $[E^{\alpha_j}]h_s(-1) \in \sum_{p=1}^{r+l} \mathbb{C}[e^{\alpha_p} - e^{-\alpha_p}]$. This completes the proof for $E^u_{i,j+d}$ for $i = 1, \cdots, d$, $j = 1, \cdots, r$. The other cases can be done similarly. □

Recall H_a and $\omega_a = \omega_{h_a}$ for $a = 1, \cdots, d$ from [DN2] and [AD]. The following lemma collects some formulas from Propositions 4.5, 4.6, 4.8 and 4.9 of [AD].

Lemma 4.3. *For any indices a, b, c, d,*

$$[\omega_a] * [E^u_{bc}] = \delta_{ab}[E^u_{bc}], \tag{4.1}$$

$$[E^u_{bc}] * [\omega_a] = \delta_{ac}[E^u_{bc}]. \tag{4.2}$$

$$[E^u_{ab}] * [E^t_{cd}] = [E^t_{cd}] * [E^u_{ab}] = 0, \tag{4.3}$$

$$[\Lambda_{ab}] * [E^u_{cd}] = [\Lambda_{ab}] * [E^t_{cd}] = [E^u_{cd}] * [\Lambda_{ab}] = [E^t_{cd}] * [\Lambda_{ab}] = 0 \ (a \neq b). \tag{4.4}$$

For distinct a, b, c,

$$\big(70[H_a] + 1188[\omega_a]^2 - 585[\omega_a] + 27\big) * [H_a] = 0, \tag{4.5}$$

$$([\omega_a] - 1) * \left([\omega_a] - \frac{1}{16}\right) * \left([\omega_a] - \frac{9}{16}\right) * [H_a] = 0, \tag{4.6}$$

$$-\frac{2}{9}[H_a] + \frac{2}{9}[H_b] = 2[E^u_{aa}] - 2[E^u_{bb}] + \frac{1}{4}[E^t_{aa}] - \frac{1}{4}[E^t_{bb}], \tag{4.7}$$

$$\begin{aligned} &-\frac{4}{135}(2[\omega_a] + 13) * [H_a] + \frac{4}{135}(2[\omega_b] + 13) * [H_b] \\ &= 4([E^u_{aa}] - [E^u_{bb}]) + \frac{15}{32}([E^t_{aa}] - [E^t_{bb}]), \end{aligned} \tag{4.8}$$

$$[\omega_b] * [H_a] = -\frac{2}{15}([\omega_a] - 1) * [H_a] + \frac{1}{15}([\omega_b] - 1) * [H_b], \tag{4.9}$$

$$[\Lambda_{ab}]^2 = 4[\omega_a] * [\omega_b] - \frac{1}{9}([H_a] + [H_b]) - ([E^u_{aa}] + [E^u_{bb}]) - \frac{1}{4}([E^t_{aa}] + [E^t_{bb}]), \tag{4.10}$$

$$[\Lambda_{ab}] * [\Lambda_{bc}] = 2[\omega_b] * [\Lambda_{ac}]. \tag{4.11}$$

For $\alpha \in L$ such that $(\alpha, \alpha) = 2k \neq 2$,

$$[H_\alpha] * [E^\alpha] = \frac{18(8k-3)}{(4k-1)(4k-9)} \left([\omega_\alpha] - \frac{k}{4}\right) \left([\omega_\alpha] - \frac{3(k-1)}{4(8k-3)}\right) [E^\alpha], \tag{4.12}$$

$$\left([\omega_\alpha] - \frac{k}{4}\right) \left([\omega_\alpha] - \frac{1}{16}\right) \left([\omega_\alpha] - \frac{9}{16}\right) [E^\alpha] = 0. \tag{4.13}$$

If $\alpha \in L_2$,

$$[E^\alpha] * [E^\alpha] = 4\epsilon(\alpha, \alpha)[\omega_\alpha], \tag{4.14}$$

$$[H_\alpha] * [E^\alpha] + [E^\alpha] * [H_\alpha] = -12[\omega_\alpha] * \left([\omega_\alpha] - \frac{1}{4}\right) * [E^\alpha], \tag{4.15}$$

$$([\omega_\alpha] - 1) * \left([\omega_\alpha] - \frac{1}{4}\right) * \left([\omega_\alpha] - \frac{1}{16}\right) * \left([\omega_\alpha] - \frac{9}{16}\right) * [E^\alpha] = 0. \tag{4.16}$$

For any $\alpha \in L$,

$$I^t * [E^\alpha] = [E^\alpha] * I^t, \tag{4.17}$$

where I^t is the identity of the simple algebra A^t.

Lemma 4.4. *For any $\alpha \in L_2$, we have*

$$A^u * [E^\alpha] * A^u = 0.$$

Proof: Let $\alpha \in L_2$ and $\{h_1, \cdots, h_d\}$ be an orthonormal basis of $\mathfrak{h}$ such that $h_1 \in \mathbb{C}\alpha$. (A^u is independent of the choice of orthonormal basis.) By (4.1)-(4.2) and (4.16), we have $[E^\alpha] = f([\omega_\alpha]) * [E^\alpha] = [E^\alpha] * f([\omega_\alpha])$ for some polynomial $f(x)$ with $f(0) = 0$. Note that $\omega_\alpha = \omega_1$. By (4.1)-(4.2), we only need to prove that

$$[E^u_{i1}] * [E^\alpha] * [E^u_{1s}] = 0, \;\; i, s = 1, 2, \cdots, d.$$

Let $a = 1$, $b \neq 1$ in (4.9). Multiplying (4.9) by $[E^u_{i1}]$ on left and using (4.2) and (4.3), we have

$$[E^u_{i1}] * [H_b] = 0, \;\; b \neq 1.$$

Then setting $a = 1$, $b \neq 1$ in (4.7) and multiplying (4.7) by $[E^u_{i1}]$ on left yields

$$[E^u_{i1}] * [H_1] = -9[E^u_{i1}].$$

Let $a = 1$, $b \neq 1$ in (4.10). Multiplying (4.10) by $[E^u_{1s}]$ on right and using (4.1) and (4.4), we have

$$-\frac{1}{9}[H_1] * [E^u_{1s}] - \frac{1}{9}[H_b] * [E^u_{1s}] = [E^u_{1s}].$$

On the other hand, multiplying (4.7) by $[E^u_{1s}]$ on right yields

$$-\frac{1}{9}[H_1] * [E^u_{1s}] + \frac{1}{9}[H_b] * [E^u_{1s}] = [E^u_{1s}].$$

Comparing the above two formulas, we have

$$[H_1] * [E^u_{1s}] = -9[E^u_{1s}], \;\; [H_a] * [E^u_{1s}] = 0, \;\; a \neq 1.$$

So

$$[E^u_{i1}] * [H_1] * [E^\alpha] * [E^u_{1s}] + [E^u_{i1}] * [E^\alpha] * [H_1] * [E^u_{1s}] = -18[E^u_{i1}] * [E^\alpha] * [E^u_{1s}].$$

But by (4.1)-(4.2) and (4.15) we have

$$[E^u_{i1}] * [H_1] * [E^\alpha] * [E^u_{1s}] + [E^u_{i1}] * [E^\alpha] * [H_1] * [E^u_{1s}] = -9[E^u_{i1}] * [E^\alpha] * [E^u_{1s}].$$

This implies that $[E^u_{i1}] * [E^\alpha] * [E^u_{1s}] = 0$, as required. $\square$

We now define $E^u_{i,j}$ for all $i, j = 1, \cdots, d+r+l$. Set

$$[E^u_{d+i,d+j}] = [E^u_{d+i,1}] * [E^u_{1,d+j}],\ \ i, j = 1, \cdots, r+l.$$

It is easy to see that $[E^u_{d+i,1}] * [E^u_{1,d+j}] = [E^u_{d+i,k}] * [E^u_{k,d+j}]$, $k = 2, \cdots, d$.

Denote by A^u_L the subalgebra of $A(V^+_L)$ generated by

$$\{[E^u_{ij}], [E^u_{d+p,j}], [E^u_{i,d+p}] | i, j = 1, \cdots, d,\ \ p = 1, \cdots, r+l\}.$$

From Lemma 4.4, (4.14) and the definition of $[E^u_{ij}]$, $i, j = 1, \cdots, d+r+l$, we can easily deduce the following result.

Lemma 4.5. *A^u_L is a matrix algebra over $\mathbb{C}$ with basis $\{[E^u_{ij}] | i, j = 1, \cdots, d+r+l\}$ such that*

$$[E^u_{ij}] * [E^u_{ks}] = \delta_{j,k}[E^u_{is}],\ \ [E^u_{ij}]e^k = \delta_{j,k}e^i,\ \ i, j, k, s = 1, 2, \cdots, d+r+l.$$

Lemma 4.6. *Let $\alpha \in L$ be such that $\alpha \notin L_2$, then*

$$[E^\alpha] * A^u = 0.$$

Proof: Let $\{h_1, \cdots, h_d\}$ be an orthonormal basis of $\mathfrak{h}$ such that $h_1 \in \mathbb{C}\alpha$. If $|\alpha|^2 = 2k$ and $k \neq 4$, the lemma follows from (4.1)-(4.2) and (4.13).

If $|\alpha|^2 = 8$, by (4.1)-(4.2) and (4.13) we have

$$[E^u_{ab}] * [E^\alpha] = [E^\alpha] * [E^u_{ba}] = 0,$$

for all $1 \leq a, b \leq d$ and $b \neq 1$. By (4.1) and (4.12), we have

$$[E^u_{a1}] * [H_1] * [E^\alpha] = 0. \tag{4.18}$$

On the other hand, for $a \neq 1$, by (4.7)-(4.8) and (4.3), we have

$$-\frac{2}{9}[E^u_{a1}] * ([H_a] - [H_1]) * [E^\alpha] = -2[E^u_{a1}] * [E^\alpha],$$

$$\frac{4}{135}[E^u_{a1}] * (-13[H_a] + 15[H_1]) * [E^\alpha] = -4[E^u_{a1}] * [E^\alpha].$$

Therefore by (4.18), we have

$$\frac{1}{9}[E^u_{a1}] * [H_a] * [E^\alpha] = [E^u_{a1}] * [E^\alpha],\ \ a \neq 1,$$

$$\frac{13}{135}[E^u_{a1}] * [H_a] * [E^\alpha] = [E^u_{a1}] * [E^\alpha],\ \ a \neq 1.$$

This means that

$$[E^u_{a1}] * [E^\alpha] = 0.$$

Since $[H_\alpha] * [E^\alpha] = [E^\alpha] * [H_\alpha]$, we similarly have

$$[E^\alpha] * [E^u_{1a}] = 0.$$

This completes the proof. □

Lemma 4.7. *A_L^u is an ideal of $A(V_L^+)$.*

Proof: By Proposition 5.4 of [AD], (4.3), (4.17) and Lemmas 4.4-4.6, it is enough to prove that $[E^{\alpha_i}]*[E^u_{jk}],[E^u_{jk}]*[E^{\alpha_i}]\in A_L^u,\ j,k=1,\cdots,d,\ i=1,\cdots,r+l$.

Let $\alpha\in L_2$. For convenience, let $a_{ij}=0$, for $1\le i<j\le r$ and $k_i=i$ for $1\le i\le r$. Since $\alpha_i(-1)=\sum_{k=1}^{d}a_{ik}h_k(-1)$, we have

$$\omega_{\alpha_i}=\frac{1}{4}\sum_{k=1}^{d}a_{ik}^2h_k(-1)^2+\frac{1}{4}\sum_{p\neq q}a_{ip}a_{iq}h_p(-1)h_q(-1).$$

Recall from [AD] that

$$[S_{ab}(1,1)]=[E^u_{ab}]+[E^u_{ba}]+[\Lambda_{ab}]+\frac{1}{2}[E^t_{ab}]+\frac{1}{2}[E^t_{ba}],\ a\neq b.$$

So from (4.3) and (4.4) we have

$$[E^u_{jk}]*[\omega_{\alpha_i}]=\frac{1}{2}a_{ik}^2[E^u_{jk}]+\frac{1}{2}\sum_{p\neq k}a_{ik}a_{ip}[E^u_{jp}],\ j,k=1,\cdots,d,\ i=1,\cdots,r+l.$$

Then it can easily be deduced that

$$(a_{ik}[E^u_{jk_i}]-a_{ik_i}[E^u_{jk}])*[\omega_{\alpha_i}]=0,\ j,k=1,\cdots,d,\ i=1,\cdots,r+l.$$

Then by (4.16), we have

$$(a_{ik}[E^u_{jk_i}]-a_{ik_i}[E^u_{jk}])*[E^{\alpha_i}]=0,\ j,k=1,\cdots,d,\ i=1,\cdots,r+l.$$

So $[E^u_{jk}]*[E^{\alpha_i}]\in A_L^u$, for all $j,k=1,\cdots,d,i=1,\cdots,r+l$. Similarly, we have

$$[E^{\alpha_i}]*[E^u_{jk}]=0,\ k=1,\cdots,d,\ i=1,\cdots,r+l,\ j=r+1,\cdots,d,$$

$$[E^{\alpha_i}]*[\sum_{b=1}^{r}a_{kb}E^u_{bj}]=\frac{(\alpha_i,\alpha_k)}{2}[E^{\alpha_i}]*[\sum_{b=1}^{r}a_{ib}E^u_{bj}],$$

for $j=1,\cdots,d,\ k=1,\cdots,r,\ i=1,\cdots,r+l$. Since both $\{\alpha_1,\cdots,\alpha_r\}$ and $\{h_1,\cdots,h_r\}$ are linearly independent, it follows that for each $i=1,\cdots,r,\ j=1,\cdots,d$, $[E^u_{ij}]$ is a linear combination of $a_{11}[E^u_{1j}]$, $[a_{21}E_{1j}+a_{22}E^u_{2j}],\cdots,[a_{r1}E^u_{1j}+\cdots+a_{rr}E^u_{rj}]$. Therefore $[E^{\alpha_i}]*[E^u_{jk}]\in A_L^u,\ j,k=1,\cdots,d,\ i=1,\cdots,r+l$. $\square$

For $0\neq\alpha\in L$, let $\{h_1,\cdots,h_d\}$ be an orthonormal basis of $\mathfrak{h}$ such that $h_1\in\mathbb{C}\alpha$. Define

$$[B_\alpha]=2^{|\alpha|^2-1}([I^t]*[E^\alpha]-\frac{2|\alpha|^2}{2|\alpha|^2-1}[E^t_{11}]*[E^\alpha]),$$

and $[B_0]=[I^t]$ (see formula (6.5) of [AD]).

Lemma 4.8. *For $\alpha \in L$, $[E^t_{ij}] \in A^t$, $[B_\alpha] * [E^t_{ij}] = [E^t_{ij}] * [B_\alpha]$.*

Proof: It is enough to prove that

$$[B_\alpha] * [E^t_{ij}] = [E^t_{ij}] * [B_\alpha],$$

for $i = 1$ or $j = 1$. By the definition of $[E^t_{ab}]$ and the fact that

$$[I^t] * [E^\alpha] = [E^\alpha] * [I^t]$$

and

$$[I^t] * [\Lambda_{ab}] = [I^t] * [E^u_{ab}] = 0, \;\; a \neq b,$$

we have

$$\begin{aligned}
[B_\alpha] * [E^t_{ab}] &= [B_\alpha] * (-[S_{ab}(1,1)] - 2[S_{ab}(1,2)]), \\
[E^t_{ab}] * [B_\alpha] &= (-[S_{ab}(1,1)] - 2[S_{ab}(1,2)]) * [B_\alpha].
\end{aligned}$$

Let $b \neq 1$. Similar to the proof of Lemma 7.5 of [AD], we have

$$\begin{aligned}
&(2|\alpha|^2 - 1)([E^t_{1b}] + 3[E^u_{1b}] + [\Lambda_{1b}]) * [E^\alpha] + [E^\alpha] * ([E^t_{1b}] + 3[E^u_{1b}] + [\Lambda_{1b}]) \\
&= -([E^t_{b1}] - [E^u_{1b}] + [\Lambda_{1b}]) * [E^\alpha] - (2|\alpha|^2 - 1)[E^\alpha] * ([E^t_{b1}] - [E^u_{1b}] + [\Lambda_{1b}])
\end{aligned}$$

$$\begin{aligned}
&(2|\alpha|^2 - 1)(\frac{1}{16}[E^t_{1b}] * [E^\alpha] + [\omega_b] * [\Lambda_{1b}] * [E^\alpha]) \\
&\quad + \frac{1}{16}[E^\alpha] * [E^t_{1b}] + [E^\alpha] * [\omega_b] * [\Lambda_{1b}] \\
&= \frac{9}{16}[E^t_{b1}] * [E^\alpha] + [\omega_b] * [\Lambda_{1b}] * [E^\alpha] \\
&\quad - (2|\alpha|^2 - 1)(\frac{9}{16}[E^\alpha] * [E^t_{b1}] + [E^\alpha] * [\omega_b] * [\Lambda_{1b}]).
\end{aligned}$$

So we have

$$(2|\alpha|^2 - 1)[E^t_{1b}] * [E^\alpha] = -[E^\alpha] * [E^t_{1b}] + x,$$

where $x \in A^u_L + \mathbb{C}[\Lambda_{1b}] * [E^\alpha] + \mathbb{C}[E^\alpha] * [\Lambda_{1b}] + \mathbb{C}[\omega_b] * [\Lambda_{1b}] * [E^\alpha] + \mathbb{C}[E^\alpha] * [\Lambda_{1b}] * [\omega_b]$. Since $y * x = 0$ for any $y \in A^t$, we have

$$\begin{aligned}
&[B_\alpha] * [E^t_{1b}] \\
&= 2^{|\alpha|^2 - 1}\left(-(2|\alpha|^2 - 1)[E^t_{1b}] * [E^\alpha] + 2|\alpha|^2[E^t_{11}] * [E^t_{1b}] * [E^\alpha]\right) \\
&= 2^{|\alpha|^2 - 1}[E^t_{1b}] * [E^\alpha] = [E^t_{1b}] * [B_\alpha].
\end{aligned}$$

Similarly,

$$[B_\alpha] * [E^t_{b1}] = [E^t_{b1}] * [B_\alpha],$$

completing the proof. □

Lemma 4.9. *A_L^t is an ideal of $A(V_L^+)$ and $A_L^t \cong A^t \otimes_{\mathbb{C}} \mathbb{C}[\widehat{L}/K]/J$, where $\mathbb{C}[\widehat{L}/K]$ is the group algebra of $\widehat{L}/K$ and J is the ideal of $\mathbb{C}[\widehat{L}/K]$ generated by $\kappa K + 1$.*

Proof: By Proposition 5.4 of [AD] and Lemmas 4.7-4.8, it is easy to check that A_L^t is an ideal of $A(V_L^+)$. Similar to the proof of Proposition 7.6 of [AD], we have

$$[B_\alpha] * [B_\beta] = \epsilon(\alpha, \beta)[B_{\alpha+\beta}],$$

for $\alpha, \beta \in L$ where $\epsilon(\alpha, \beta)$ is understood to be ± 1 by identifying κ with -1. Then the lemma follows from Proposition 7.6 of [AD] and Lemma 4.8. $\square$

It is clear that $A_L^u \cap A_L^t = 0$. Let

$$\bar{A}(V_L^+) = A(V_L^+)/(A_L^u \oplus A_L^t),$$

and for $x \in A(V_L^+)$, we still denote the image of x in $\bar{A}(V_L^+)$ by x.

Lemma 4.10. *In $\bar{A}(V_L^+)$, we have*

$$[H_a] = [H_b], \ 1 \le a, b \le d, \tag{4.19}$$

$$([\omega_a] - \frac{1}{16}) * [H_a] = 0, \ 1 \le a \le d, \tag{4.20}$$

$$\frac{128}{9}[H_a] * \frac{128}{9}[H_a] = \frac{128}{9}[H_a], \ 1 \le a \le d, \tag{4.21}$$

$$[\Lambda_{ab}] * [H_c] = 0, \ 1 \le a, b, c \le d, \ a \ne b. \tag{4.22}$$

Proof: (4.19) follows from (4.7) and (4.20) follows from (4.8) and (4.9). Then from (4.5) we can get (4.21). By (4.10), we have

$$[\Lambda_{ab}]^2 * [H_c] = 0, \ a \ne b.$$

If $d \ge 3$, then by (4.19) we can let $c \ne a$, $c \ne b$. So by (4.11) and (4.20),

$$\begin{aligned} &[\Lambda_{ab}] * [H_c] = 16[\Lambda_{ab}] * [\omega_c] * [H_c] \\ &= 8[\Lambda_{ac}] * [\Lambda_{cb}] * [H_c] = 128[\Lambda_{ac}] * [\Lambda_{cb}] * [\omega_a] * [H_a] \\ &= 64[\Lambda_{ac}] * [\Lambda_{ca}] * [\Lambda_{ab}] * [H_a] = 0. \end{aligned}$$

If $d = 2$. Notice that $[\Lambda_{ab}] = [S_{ab}(1,1)]$. By Remark 4.1.1 of [DN2] and the fact that $[\omega_a * S_{ab}(m,n)] = [S_{ab}(m,n) * \omega_a]$ in $\bar{A}(V_L^+)$ for $m, n \ge 1$, we have

$$[S_{ab}(m+1, n)] + [S_{ab}(m,n)] = 0. \tag{4.23}$$

By the proof of Lemma 6.1.2 of [DN2], we know that

$$[H_a] = -9[S_{aa}(1,3)] - \frac{17}{2}[S_{aa}(1,2)] + \frac{1}{2}[S_{aa}(1,1)]. \tag{4.24}$$

Direct calculation yields

$$[S_{ab}(1,1)] * [S_{aa}(1,3)] = h_b(-1)h_a(-3)h_a(-1)^2,$$

$$[S_{ab}(1,1)] * [S_{aa}(1,2)] = h_b(-1)h_a(-2)h_a(-1)^2,$$

$$[S_{ab}(1,1)] * [S_{aa}(1,1)] = h_b(-1)h_a(-1)h_a(-1)^2.$$

Here we have used (4.23). Then (4.22) immediately follows from Lemma 4.2.1 of [DN2], (4.23) and (4.24). The proof is complete. $\square$

For $0 \neq \alpha \in L$, let $\{h_1, \cdots, h_d\}$ be an orthonormal basis of $\mathfrak{h}$ such that $h_1 \in \mathbb{C}\alpha$. Define

$$[\bar{B}_\alpha] = 2^{|\alpha|^2-1}\frac{128}{9}[H_1] * [E^\alpha].$$

We also set $[\bar{B}_0] = \frac{128}{9}[H_1]$.

Lemma 4.11. *The subalgebra A_H of $\bar{A}(V_L^+)$ spanned by $[\bar{B}_\alpha]$, $\alpha \in L$ is an ideal of $\bar{A}(V_L^+)$ isomorphic to $\mathbb{C}[\widehat{L}/K]/J$.*

Let

$$\widehat{A}(V_L^+) = \bar{A}(V_L^+)/A_H.$$

Lemma 4.12. *Any $\widehat{A}(V_L^+)$-module is completely reducible. That is, $\widehat{A}(V_L^+)$ is a semisimple associative algebra.*

Proof: Let M be an $\widehat{A}(V_L^+)$-module. For $\alpha \in L$, by [DN2] M is a direct sum of irreducible $A(V_{\mathbb{Z}\alpha}^+)$-modules. Following the proof of Lemma 6.1 of [AD] one can prove that the image of any vector from $M(1)^+$ in $\widehat{A}(V_L^+)$ is semisimple on M. By Table 1 of [AD], we can assume that

$$M = \bigoplus_{\lambda \in \mathfrak{h}/(\pm 1)} M_\lambda,$$

where $M_\lambda = \{w \in M |\ [\frac{1}{2}h(-1)^2 1]w = \frac{1}{2}(\lambda, h)^2 w, h \in \mathfrak{h}\}$. So $\omega_a w = \frac{1}{2}(\lambda, h_a)^2 w$, for $w \in M_\lambda$. By (4.10) and (4.11), we have

$$\Lambda_{ab} w = (\lambda, h_a)(\lambda, h_b)w,$$

for $a \neq b$, $w \in M_\lambda$. For any $u \in M_\lambda$, $\lambda \neq 0$, set $M(u) = \sum_{\alpha \in L} \mathbb{C}[E^\alpha]u$. By (4.12)-(4.13) and (4.15)-(4.17), if $[E^\alpha]u \neq 0$, then $\alpha \in \Delta(\lambda)$ or $-\alpha \in \Delta(\lambda)$, where $\Delta(\lambda) = \{\alpha \in L |\ |\lambda + \alpha|^2 = |\lambda|^2\}$. So

$$M(u) = \bigoplus_{\alpha \in \Delta(\lambda)} \mathbb{C}[E^\alpha]u.$$

Since L is positive-definite, there are finitely many $\alpha \in L$ which belong to $\Delta(\lambda)$. Thus $M(u)$ is finite-dimensional. Similar to the proof of Lemma 6.4 of [AD], we can deduce that $\Lambda_{ab}M(u) \subseteq M(u)$, $\omega_a M(u) \subseteq M(u)$. By Proposition 5.4 of [AD], $[E^\alpha] * [E^\beta] = [x] * [E^{\alpha+\beta}]$ for some $x \in M(1)^+$. We deduce that $M(u)$ is an $\widehat{A}(V_L^+)$-submodule of M. Suppose $[E^\alpha]u \neq 0$, for some $\alpha \in \Delta(\lambda)$. If $(\alpha, \alpha) = 2$, then by (4.14), we have $0 \neq [E^\alpha][E^\alpha]u \in \mathbb{C}u$. If $(\alpha, \alpha) = 2k \neq 2$. Let $\{h_1, \cdots, h_d\}$ be an orthonormal basis of $\mathfrak{h}$ such that $h_1 \in \mathbb{C}\alpha$. By the fact that $[H_1] = [J_1] + [\omega_1] - 4[\omega_1^2] = 0$ and (4.13) we know that $[\omega_1]u = \frac{k}{4}u$. Then by Lemma 5.5 of [DN2], we have

$$[E^\alpha][E^\alpha]u = \frac{2k^2}{(2k)!}(k^2-1)(k^2-2^2)\cdots(k^2-(k-1)^2)u \neq 0.$$

Therefore $M(u)$ is irreducible. We prove that M is a direct sum of finite-dimensional irreducible $\widehat{A}(V_L^+)$-modules. $\square$

Theorem 4.13. *$A(V_L^+)$ is a finite dimensional semisimple associative algebra.*

Proof: Clearly $A(V_L^+)$ is finite dimensional. By Lemmas 4.5, 4.7, 4.9 we know that $A_L^u \oplus A_L^t$ is a semisimple ideal of $A(V_L^+)$. Thus $A(V_L^+)$ is semisimple if and only if $\bar{A}(V_L^+) = A(V_L^+)/(A_L^u \oplus A_L^t)$ is semisimple. By Lemma 4.11, A_H is a semisimple ideal of $\bar{A}(V_L^+)$. So $\bar{A}(V_L^+)$ is semisimple if and only if $\widehat{A}(V_L^+) = \bar{A}(V_L^+)/A_H$ is semisimple. The result now follows from Lemma 4.12. $\square$

5 Rationality of V_L^+ when L has an orthogonal base

In this section we assume that L has an orthogonal base $\{\beta_i, 1 \leq i \leq d\}$ in the sense that $(\beta_i, \beta_j) = 0$, for $i \neq j$. Then we have $L = \bigoplus_{i=1}^d \mathbb{Z}\beta_i$ and this induces the relations

$$\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+ \subseteq (V_{\bigoplus_{i=1}^d \mathbb{Z}\beta_i})^+ = V_L^+$$

between vertex operator algebras. By Theorems 3.4 and 2.6, $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$ is a rational vertex operator subalgebra of V_L^+. This is a crucial fact in our discussion of rationality of V_L^+ in this section.

The following lemma is trivial:

Lemma 5.1. *The vertex operator algebras $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}$ and $V_{\bigoplus_{i=1}^d \mathbb{Z}\beta_i}$ are isomorphic.*

By Lemma 5.1, a $V_{\bigoplus_{i=1}^d \mathbb{Z}\beta_i}$-module $V_{\bigoplus_{i=1}^d \mathbb{Z}\beta_i + c_i\beta_i}$ can be viewed as a $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}$-module, and is isomorphic to $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i + c_i\beta_i}$. On the other hand, by Theorem 2.6 and Theorem 3.4, $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$ is rational, then any admissible V_L^+-module is completely reducible as an admissible $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$-module. Thus we may decompose all the irreducible V_L^+-modules as a direct sum of irreducible $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$-modules.

Lemma 5.2. *The following are $\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i}^{+}$-module isomorphisms.*

(1) V_L^+ is isomorphic to a direct sum of

$$V_{\mathbb{Z}\beta_1}^{\epsilon_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d}^{\epsilon_d}$$

with sign $\epsilon_i = \{\pm\}, 1 \leq i \leq d$ such that the number of i with $\epsilon_i = -$ is even.

(2) V_L^- is isomorphic to a direct sum of

$$V_{\mathbb{Z}\beta_1}^{\epsilon_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d}^{\epsilon_d}$$

with sign $\epsilon_i = \{\pm\}, 1 \leq i \leq d$ such that the number of i with $\epsilon_i = -$ is odd.

(3) V_{λ_j+L} for $\lambda_j = k_1\beta_1 + \cdots + k_d\beta_d \in L^\circ/L, k_1, \cdots, k_d \in \mathbb{C}$ and $2\lambda_j \notin L$ is isomorphic to

$$V_{\mathbb{Z}\beta_1+k_1\beta_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d+k_d\beta_d},$$

where $V_{\mathbb{Z}\beta_i+k_i\beta_i} = V_{\mathbb{Z}\beta_i+k_i\beta_i}^{+} \bigoplus V_{\mathbb{Z}\beta_i+k_i\beta_i}^{-}$ for i such that $2k_i\beta_i \in \mathbb{Z}\beta_i$.

(4) $V_{\lambda_j+L}^{+}$ for $\lambda_j = k_1\beta_1 + \cdots + k_d\beta_d \in L^\circ/L, k_1, \cdots, k_d \in \mathbb{C}$ and $2\lambda_j \in L$ is isomorphic to a direct sum of

$$\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i+k_i\beta_i}^{\epsilon_i}$$

with sign $\epsilon_i = \{\pm\}, 1 \leq i \leq d$ such that the number of i with $\epsilon_i = -$ is even.

(5) $V_{\lambda_j+L}^{-}$ for $\lambda_j = k_1\beta_1 + \cdots + k_d\beta_d, k_1, \cdots, k_d \in \mathbb{C}$ and $2\lambda_j \in L$ is isomorphic to a direct sum of

$$\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i+k_i\beta_i}^{\epsilon_i}$$

with sign $\epsilon_i = \{\pm\}, 1 \leq i \leq d$ such that the number of i with $\epsilon_i = -$ is odd.

(6) $(V_L^{T_\chi})^+$ is isomorphic to a direct sum of

$$(V_{\mathbb{Z}\beta_1}^{T_{\chi_1}})^{\epsilon_1} \otimes \cdots \otimes (V_{\mathbb{Z}\beta_d}^{T_{\chi_d}})^{\epsilon_d}$$

with signs $\epsilon_i \in \{\pm\}, i = 1, \cdots, d$ such that the number of i with $\epsilon_i = -$ is even.

(7) $(V_L^{T_\chi})^-$ is isomorphic to a direct sum of

$$(V_{\mathbb{Z}\beta_1}^{T_{\chi_1}})^{\epsilon_1} \otimes \cdots \otimes (V_{\mathbb{Z}\beta_d}^{T_{\chi_d}})^{\epsilon_d}$$

with signs $\epsilon_i \in \{\pm\}, i = 1, \cdots, d$ such that the number of i with $\epsilon_i = -$ is odd.

Proof: (1) By Lemma 5.1, we have $V_L \cong \otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i}$ and the corresponding θ is changed to $\theta_1 \otimes \cdots \otimes \theta_d$, where θ_i is the restriction of θ to $V_{\mathbb{Z}\beta_i}$, then V_L^+ is isomorphic to $(\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i})^+$ as $\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i}^{+}$-modules. The decomposition of $(\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i})^+$ into direct sum of irreducible $\otimes_{i=1}^{d} V_{\mathbb{Z}\beta_i}^{+}$-modules is now obvious.

The proof of (2) is similar to that of (1). (3) is obvious. For (4), note that

$$V_{\lambda_j+L} = V_{\mathbb{Z}\beta_1+k_1\beta_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d+k_d\beta_d}.$$

Since $2\lambda_j \in L$ and $\{\beta_1, \cdots, \beta_d\}$ is a basis of L, it follows that $2k_i\beta_i \in \mathbb{Z}\beta_i$ and $V_{\mathbb{Z}\beta_i + k_i\beta_i} = V_{\mathbb{Z}\beta_i - k_i\beta_i}$, for $i = 1, 2, \cdots, d$. The decomposition then follows easily. The proof of (5) is similar.

Now we consider the last two cases. Note that $(\alpha, \beta) \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$. From the discussion given in Section 3 (before Theorem 3.1) we see that

$$V_L^{T_\chi} = V_{\mathbb{Z}\beta_1}^{T_{\chi_1}} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d}^{T_{\chi_d}}.$$

(6) and (7) follows immediately. $\square$

By Lemma 4.1 we have:

Lemma 5.3. *The lowest weights of irreducible V_L^+-modules are given by*

V_L^+	V_L^-	V_{λ_i+L}	$V_{\mu_j+L}^{\pm}$	$(V_L^{T_{\chi_i}})^+$	$(V_L^{T_{\chi_j}})^-$
0	*1*	$\frac{\langle\lambda_i,\lambda_i\rangle}{2}$	$\frac{\langle\mu_j,\mu_j\rangle}{2}$	$\frac{d}{16}$	$\frac{d+8}{16}$

where $2\lambda_i \notin L$, $2\mu_j \in L$ and $\mu_j \neq 0$.

From Lemma 5.3, Theorem 3.3, Proposition 3.5 and Theorem 4.13, we get the following result:

Lemma 5.4. $Ext^1_{V_L^+}(N, M) = 0$ *for the following irreducible V_L^+-module pairs (M, N):*

$$(M, N) = (M, M) \text{ (i.e. } M = N), \ (V_{\lambda_j+L}^{\pm}, \ V_{\lambda_j+L}^{\mp}), \ \lambda_j \neq 0,$$

$$((V_L^{T_{\chi_i}})^{\mp}, (V_L^{T_{\chi_j}})^{\pm}), \ ((V_L^{T_{\chi_i}})^{\pm}, (V_L^{T_{\chi_j}})^{\pm}).$$

Furthermore, we have:

Lemma 5.5. *The extension groups* $Ext^1_{V_L^+}(V_L^{\pm}, V_L^{\mp}) = 0$.

Proof: By Theorem 2.9 and Proposition 3.9, we only need to prove

$$Ext^1_{V_L^+}(V_L^-, V_L^+) = 0.$$

We consider an exact sequence

$$0 \to V_L^- \to M \to V_L^+ \to 0$$

for a weak V_L^+-module M. By the rationality of $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$, there exists a $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$-submodule M^1 of M such that $M^1 \cong V_L^+$ as $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$-modules. Since $\otimes_{i=1}^d V_{\mathbb{Z}\beta_i}^+$ and V_L^+ have the same Virasoro element, then there is a vector u in M^1 such that $L(-1)u = 0$ and $L(0)u = 0$, this implies that u generates a V_L^+-submodule isomorphic to V_L^+, then we have $M \cong V_L^- \bigoplus V_L^+$ and $Ext^1_{V_L^+}(V_L^-, V_L^+) = 0$. $\square$

We next prove $Ext^1_{V_L^+}(M^2, M^1) = 0$ for the remaining pairs (M^1, M^2).

Lemma 5.6. *The extension groups* $Ext^1_{V_L^+}(M^2, M^1) = 0$ *for the following pairs* (M^1, M^2)*:*

$$(M^1, M^2) = (V_{\lambda_i+L}, V_{\lambda_j+L})\ \lambda_i \neq \lambda_j,\ (V_{\lambda_i+L}, V^{\pm}_{\lambda_j+L}),\quad (V^{\pm}_{\lambda_i+L}, V_{\lambda_j+L}),$$

$$(V^{\pm}_{\lambda_i+L}, V^{\pm}_{\lambda_j+L}),\quad (V^{\pm}_{\lambda_i+L}, V^{\mp}_{\lambda_j+L}),\ \lambda_i \neq \lambda_j,$$

$$((V_L^{T_\chi})^{\pm}, V^{\pm}_{\lambda_j+L}),\quad ((V_L^{T_\chi})^{\mp}, V^{\pm}_{\lambda_j+L}),\ (V^{\mp}_{\lambda_j+L}, (V_L^{T_\chi})^{\pm}),\quad (V^{\pm}_{\lambda_j+L}, (V_L^{T_\chi})^{\pm}),$$

$$(V_{\lambda_i+L}, (V_L^{T_\chi})^{\pm}),\quad ((V_L^{T_\chi})^{\pm}, V_{\lambda_i+L}).$$

Proof: Let (M^1, M^2) be one of the following pairs

$$((V_L^{T_\chi})^{\pm}, V^{\pm}_{\lambda_j+L}),\quad ((V_L^{T_\chi})^{\mp}, V^{\pm}_{\lambda_j+L}),\ (V^{\mp}_{\lambda_j+L}, (V_L^{T_\chi})^{\pm}),\quad (V^{\pm}_{\lambda_j+L}, (V_L^{T_\chi})^{\pm}),$$

$$(V_{\lambda_i+L}, (V_L^{T_\chi})^{\pm}),\quad ((V_L^{T_\chi})^{\pm}, V_{\lambda_i+L}).$$

Let $U = \otimes_{i=1}^d V^+_{\mathbb{Z}\beta_i}$. Then by Theorem 2.6 and Theorem 3.4, U is rational. It is obvious that U has the same Virasoro element with V. Then by Lemma 2.11, it is enough to show that

$$I_U\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0$$

for any irreducible U-submodules N^1, N, N^2 of M^1, V_L^+, M^2, respectively. By (1)-(2) and (6)-(7) of Lemma 5.2, we know that there is exactly one N^i that has the form

$$(V_{\mathbb{Z}\beta_1}^{T_{\chi_1}})^{\epsilon_1} \otimes \cdots \otimes (V_{\mathbb{Z}\beta_d}^{T_{\chi_d}})^{\epsilon_d}.$$

Then by Theorem 3.7 (2) and Theorem 2.8, the fusion rule of type

$$\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right)$$

for the vertex operator algebra $U = \otimes_{i=1}^d V^+_{\mathbb{Z}\beta_i}$ is zero. Thus $Ext^1_{V_L^+}(M^2, M^1) = 0$.

Consider the pair $(M^1, M^2) = (V_{\lambda_i+L}, V_{\lambda_j+L})$, where $\lambda_i = k_1\beta_1 + \cdots + k_d\beta_d$, $\lambda_j = l_1\beta_1 + \cdots + l_d\beta_d$ such that $\lambda_i \neq \lambda_j$ and $2\lambda_i, 2\lambda_j \notin L$. Without loss of generality, we may assume that $k_1 \neq l_1$. By (3) of Lemma 5.2, we have

$$V_{\lambda_i+L} = V_{\mathbb{Z}\beta_1+k_1\beta_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d+k_d\beta_d},$$

$$V_{\lambda_j+L} = V_{\mathbb{Z}\beta_1+l_1\beta_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d+l_d\beta_d}.$$

Note that $V_{\mathbb{Z}\beta_r+n_r\beta_r}$ is an irreducible $V_{\mathbb{Z}\beta_r}$-module if $2n_r\beta_r \notin \mathbb{Z}\beta_r$ and $V_{\mathbb{Z}\beta_r+n_r\beta_r} = V^+_{\mathbb{Z}\beta_r+n_r\beta_r} \bigoplus V^-_{\mathbb{Z}\beta_r+n_r\beta_r}$ is a sum of two irreducible $V_{\mathbb{Z}\beta_r}$-modules if $2n_r\beta_r \in \mathbb{Z}\beta_r$. Let N^1, N, N^2 be any irreducible U-submodules of M^1, V_L^+, M^2, respectively. It follows immediately from Theorem 3.7 (2) and Theorem 2.8 that

$$I_U\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0.$$

So by Lemma 2.11, $Ext^1_{V_L^+}(M^2, M^1) = 0$ in this case.

If $(M^1, M^2) = (V_{\lambda_i+L}, V^{\pm}_{\lambda_j+L})$, $(V^{\pm}_{\lambda_i+L}, V_{\lambda_j+L})$, where $2\lambda_i \notin L$, $2\lambda_j \in L$, by Proposition 3.9 and Theorem 2.9, we only need to consider the pairs $(M^1, M^2) = (V^{\pm}_{\lambda_j+L}, V_{\lambda_i+L})$. Let $\lambda_j = k_1\beta_1 + \cdots + k_d\beta_d$ and $\lambda_i = l_1\beta_1 + \cdots + l_d\beta_d$. Then $2k_i\beta_i \in \mathbb{Z}\beta_i$, for $i = 1, 2, \cdots, d$. Since $2\lambda_i \notin L$, it follows that there exists $1 \le s \le d$ such that $2l_s\beta_s \notin \mathbb{Z}\beta_s$. By (3)-(5) of Lemma 5.2,

$$V_{\lambda_i+L} = V_{\mathbb{Z}\beta_1+l_1\beta_1} \otimes \cdots \otimes V_{\mathbb{Z}\beta_d+l_d\beta_d},$$

where $V_{\mathbb{Z}\beta_r+l_r\beta_r} = V^+_{\mathbb{Z}\beta_r+l_r\beta_r} \bigoplus V^-_{\mathbb{Z}\beta_r+l_i\beta_r}$ if $2l_r\beta_r \in \mathbb{Z}\beta_r$. Let N^1, N, N^2 be any irreducible U-submodules of M^1, V_L^+, M^2, respectively. Then N^1 has the form:

$$\otimes_{i=1}^d V^{\epsilon_i}_{\mathbb{Z}\beta_i+k_i\beta_i}.$$

We know from Theorem 3.7 (2) and Theorem 2.8 that

$$I_U\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0.$$

Again by Lemma 2.11, we have $Ext^1_{V_L^+}(M^2, M^1) = 0$.

Finally we deal with the pairs (M^1, M^2):

$$(V^{\pm}_{\lambda_i+L}, V^{\pm}_{\lambda_j+L}),\ (V^{\pm}_{\lambda_i+L}, V^{\mp}_{\lambda_j+L}),\ \lambda_i \neq \lambda_j.$$

Since $\lambda_i \neq \lambda_j$, it follows that there exists $1 \le s \le d$ such that $k_s \neq l_s$. Let $\lambda_i = k_1\beta_1 + \cdots + k_d\beta_d$ and $\lambda_j = l_1\beta_1 + \cdots + l_d\beta_d$. Then $2k_i\beta_i, 2l_i\beta_i \in \mathbb{Z}\beta_i$, for $i = 1, 2, \cdots, d$. Let N^1, N, N^2 be any irreducible U-submodules of M^1, V_L^+, M^2, respectively. By (3)-(4) of Lemma 5.2, N^1 and N^2 have the form:

$$V^{\epsilon_i}_{\mathbb{Z}\beta_1+k_1\beta_1} \otimes \cdots \otimes V^{\epsilon_i}_{\mathbb{Z}\beta_d+k_d\beta_d},$$

and

$$V^{\epsilon_i}_{\mathbb{Z}\beta_1+l_1\beta_1} \otimes \cdots \otimes V^{\epsilon_i}_{\mathbb{Z}\beta_d+l_d\beta_d}$$

respectively. By Theorem 3.7 (2) and Theorem 2.8,

$$I_U\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0.$$

This implies that $Ext^1_{V_L^+}(M^2, M^1) = 0$. The proof is complete. □

We are now in a position to state the main result of this section.

Theorem 5.7. *Let L be a positive definite even lattice with an orthogonal base, then V_L^+ is rational.*

Proof: The theorem follows from Lammas 5.4-5.6 and Theorem 2.10. □

6 Rationality of V_L^+ : general case

We are now in a position to deal with V_L^+ for any positive definite even lattice L by using the rationality result obtained in Section 5.

Note that L has a sublattice $L_1 = \bigoplus_{i=1}^d \mathbb{Z}\beta_i$ of the same rank, where $\{\beta_i, 1 \le i \le d\}$ is an orthogonal subset in L in the sense that $\langle \beta_i, \beta_j \rangle = 0$, $i \ne j$. This induces an embedding

$$V_{L_1}^+ = (V_{\bigoplus_{i=1}^d \mathbb{Z}\beta_i})^+ \subset V_L^+$$

of vertex operator algebras. Let $\{\gamma_s\}$ be a set of representatives of L/L_1. In Section 5, we prove that $V_{L_1}^+$ is rational. Then we can decompose all the irreducible V_L^+-modules as a direct sum of irreducible $V_{L_1}^+$-modules. Specifically, we have

Lemma 6.1. *(1) V_L^+ is isomorphic to a direct sum of*

$$V_{\gamma_s+L_1}^+, (2\gamma_s \in L_1), V_{\gamma_s+L_1}, (2\gamma_s \notin L_1)$$

(one of the two irreducible modules may not exist in the direct sum).

(2) V_L^- is isomorphic to a direct sum of

$$V_{\gamma_s+L_1}^-, (2\gamma_s \in L_1), V_{\gamma_s+L_1}, (2\gamma_s \notin L_1)$$

(one of the two irreducible modules may not exist in the direct sum).

(3) $V_{\lambda_j+L}^+$ for $(2\lambda_j \in L)$ is isomorphic to a direct sum of

$$V_{\lambda_j+\gamma_s+L_1}^+ \ (2(\lambda_j+\gamma_s) \in L_1), \ V_{\lambda_j+\gamma_s+L_1}(2(\lambda_j+\gamma_s) \notin L_1)$$

(one of the two irreducible modules may not exist in the direct sum).

(4) $V_{\lambda_j+L}^-$ for $(2\lambda_j \in L)$ is isomorphic to a direct sum of

$$V_{\lambda_j+\gamma_s+L_1}^- \ (2(\lambda_j+\gamma_s) \in L_1), \ V_{\lambda_j+\gamma_s+L_1} \ (2(\lambda_j+\gamma_s) \notin L_1)$$

(one of the two irreducible modules may not exist in the direct sum).

(5) V_{λ_j+L} for $(2\lambda_j \notin L)$ is isomorphic to a direct sum of

$$V_{\lambda_j+\gamma_s+L_1}.$$

(6) $(V_L^{T_\chi})^+$ is isomorphic to a direct sum of irreducible $V_{L_1}^+$-modules

$$(V_{L_1}^{T_{\chi_i}})^+$$

for some irreducible $\widehat{L_1}/K_1$-module T_{χ_i} with central character χ_i such that $\chi_i(\kappa) = -1$, where $K_1 = \{\theta(a)a^{-1} | a \in \widehat{L_1}\}$ and $\widehat{L_1} = \{a \in \hat{L} | \bar{a} \in L_1\}$.

(7) $(V_L^{T_\chi})^-$ is isomorphic to a direct sum of

$$(V_{L_1}^{T_{\chi_i}})^-$$

for some irreducible $\widehat{L_1}/K_1$-module T_{χ_i} with central character χ_i such that $\chi_i(\kappa) = -1$.

Proof: (1)-(4) are obvious. (5) is immediate by noting that $2(\lambda_j+\gamma_s)\notin L_1$ as L_1 is a sublattice of L. Now we prove (6)-(7). Let $\widehat{L_1}=\{a\in\widehat{L}|\bar{a}\in L_1\}$ and $K_1=\{\theta(a)a^{-1}|a\in\widehat{L_1}\}$. Then $\widehat{L_1}$ is an abelian group isomorphic to $L_1\times\langle\kappa\rangle$ and K_1 is isomorphic to $2L_1$. As a result, $\widehat{L_1}/K_1$ is isomorphic to $L_1/2L_1\times\langle\kappa\rangle$. Since $\widehat{L_1}/K_1$ is a subgroup of $\widehat{L}/K_1$, T_χ is a direct sum of one-dimensional irreducible $\widehat{L}/K_1$-modules T_{χ_i} such that κ acts as -1. Since $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}L_1=\mathbb{C}\otimes_{\mathbb{Z}}L$, we see that

$$V_L^{T_\chi}=M(1)(\theta)\otimes T_\chi=\bigoplus_i M(1)(\theta)\otimes T_{\chi_i}=\bigoplus_i V_{L_1}^{T_{\chi_i}}.$$

That is, $V_L^{T_\chi}$ is a direct sum of irreducible θ-twisted V_{L_1}-modules $V_{L_1}^{T_{\chi_i}}$. (6)-(7) are evident now. $\square$

By Lemma 5.3, Theorem 3.3, Proposition 3.5 and Theorem 4.13, we have the following result similar to Lemma 5.4.

Lemma 6.2. *The extension groups $Ext^1_{V_L^+}(N,M)=0$ for the following irreducible V_L^+-module pairs (M,N),*

$$(M,M)\ (\text{i.e } M=N),\ (V_{\lambda_j+L}^{\pm},V_{\lambda_j+L}^{\mp}),\ \lambda_j\neq 0,$$

$$((V_L^{T_{\chi_i}})^{\mp},(V_L^{T_{\chi_j}})^{\pm}),\ ((V_L^{T_{\chi_i}})^{\pm},(V_L^{T_{\chi_j}})^{\pm}).$$

Note that $V_{L_1}^+$ and V_L^+ have the same Virasoro element. An analogue of Lemma 5.5 with the same proof is the following:

Lemma 6.3. *The extension groups $Ext^1_{V_L^+}(V_L^{\pm},V_L^{\mp})=0$.*

For the remaining pairs (M^1,M^2), we also have the following result:

Lemma 6.4. *$Ext^1_{V_L^+}(M^2,M^1)=0$ for the following irreducible V_L^+-module pairs (M^1,M^2):*

$$(M^1,M^2)=(V_{\lambda_i+L},V_{\lambda_j+L})\ \lambda_i\neq\lambda_j,\ (V_{\lambda_i+L},V_{\lambda_j+L}^{\pm}),\quad (V_{\lambda_i+L}^{\pm},V_{\lambda_j+L}),$$

$$(V_{\lambda_i+L}^{\pm},V_{\lambda_j+L}^{\pm}),\quad (V_{\lambda_i+L}^{\pm},V_{\lambda_j+L}^{\mp}),\ \lambda_i\neq\lambda_j,$$

$$((V_L^{T_\chi})^{\pm},V_{\lambda_j+L}^{\pm}),\quad ((V_L^{T_\chi})^{\mp},V_{\lambda_j+L}^{\pm}),\ (V_{\lambda_j+L}^{\mp},(V_L^{T_\chi})^{\pm}),\quad (V_{\lambda_j+L}^{\pm},(V_L^{T_\chi})^{\pm}),$$

$$(V_{\lambda_i+L},(V_L^{T_\chi})^{\pm}),\quad ((V_L^{T_\chi})^{\pm},V_{\lambda_i+L}).$$

Proof: Let (M^1,M^2) be one of the following pairs:

$$((V_L^{T_\chi})^{\pm},V_{\lambda_j+L}^{\pm}),\quad ((V_L^{T_\chi})^{\mp},V_{\lambda_j+L}^{\pm}),\ (V_{\lambda_j+L}^{\mp},(V_L^{T_\chi})^{\pm}),\quad (V_{\lambda_j+L}^{\pm},(V_L^{T_\chi})^{\pm}),$$

$$(V_{\lambda_i+L},(V_L^{T_\chi})^{\pm}),\quad ((V_L^{T_\chi})^{\pm},V_{\lambda_i+L}).$$

By Theorem 5.7 and Lemma 2.11, it is enough to show that

$$I_{V_{L_1}^+}\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0$$

for any irreducible $V_{L_1}^+$-submodules N^1, N, N^2 of M^1, V_L^+, M^2, respectively. By (6)-(7) of Lemma 6.1, there is exactly one N^i which has the form

$$(V_{L_1}^{T_{\chi_i}})^+ \quad or \quad (V_{L_1}^{T_{\chi_i}})^-.$$

By Theorem 3.8, we have

$$I_{V_{L_1}^+}\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0.$$

Now consider the pair $(M^1, M^2) = (V_{\lambda_i+L}, V_{\lambda_j+L})$ with $2\lambda_i \notin L, 2\lambda_j \notin L, \lambda_i \neq \lambda_j$. By the choice of $\{\lambda_k | k \in L^\circ/L\}$ (see Section 3), we have $L+\lambda_i \neq L+\lambda_j$. If $L+\lambda_i = L-\lambda_j$, then by Lemma 5.3 V_{λ_i+L} and $V_{-\lambda_j+L}$ have the same lowest weight. By Proposition 3.5 and Theorem 4.13, $Ext^1_{V_L^+}(M^2, M^1) = 0$. So we now assume that $L+\lambda_i \neq L \pm \lambda_j$. Let N^1, N, N^2 be any irreducible $V_{L_1}^+$-submodules of M^1, V_L^+, M^2, respectively. Then by (5) of Lemma 6.1,

$$N^1 = V_{\lambda_i+\gamma_s+L_1},\ N^2 = V_{\lambda_j+\gamma_r+L_1},$$

for some γ_r, γ_s, where $2(\lambda_i+\gamma_s) \notin L_1$, $2(\lambda_j+\gamma_r) \notin L_1$. By (1) of Lemma 6.1, N is of the form $V_{\gamma_l+L_1}^+(2\gamma_l \in L_1)$ or $V_{\gamma_l+L_1}(2\gamma_l \notin L_1)$. We claim that $(\gamma_l, \lambda_i+\gamma_s, \lambda_j+\gamma_r)$ is not an admissible triple modulo L_1. Otherwise, since $\gamma_r, \gamma_l, \gamma_s \in L$ and $L_1 \subseteq L$, this forces $\lambda_i+\lambda_j \in L$ or $\lambda_i - \lambda_j \in L$, a contradiction. Since $(\gamma_l, \lambda_i+\gamma_s, \lambda_j+\gamma_r)$ is not an admissible triple modulo L_1, Theorem 3.8 asserts that

$$I_{V_{L_1}^+}\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0.$$

For the pairs $(M^1, M^2) = (V_{\lambda_i+L},\ V_{\lambda_j+L}^{\pm}), (V_{\lambda_j+L}^{\pm}, V_{\lambda_i+L})$, where $2\lambda_i \notin L$, $2\lambda_j \in L$, by Proposition 3.9 and Theorem 2.9, we only need to consider the pairs $(M^1, M^2) = (V_{\lambda_j+L}^{\pm}, V_{\lambda_i+L})$. Let N^1, N, N^2 be any irreducible U-submodules of M^1, V_L^+, M^2, respectively. Then by (3)-(5) of Lemma 6.1, $N^2 = V_{\lambda_i+\gamma_r+L_1}$, for some γ_r and N^1 is one of the following:

$$V_{\lambda_j+\gamma_s+L_1}^{\pm}\ (2(\lambda_j+\gamma_s) \in L_1),\ V_{\lambda_j+\gamma_s+L_1}\ (2(\lambda_j+\gamma_s) \notin L_1).$$

By (1) of Lemma 6.1, N is of the form $V_{\gamma_l+L_1}^+(2\gamma_l \in L_1)$ or $V_{\gamma_l+L_1}(2\gamma_l \notin L_1)$. Note that $(\gamma_l, \lambda_j+\gamma_s, \lambda_i+\gamma_r)$ is not an admissible triple modulo L_1. Otherwise, either $\lambda_i+\lambda_j$ or $\lambda_i-\lambda_j \in L$. In either case we conclude that $2\lambda_i \in L$, a contradiction. Again by Theorem 3.8,

$$I_{V_{L_1}^+}\left(\begin{array}{c} N^1 \\ N\ N^2 \end{array}\right) = 0.$$

The proof for the remaining pairs (M^1, M^2):

$$(V^{\pm}_{\lambda_i+L}, V^{\pm}_{\lambda_j+L}),\ (V^{\pm}_{\lambda_i+L}, V^{\mp}_{\lambda_j+L})$$

is quite similar. We omit it. □

By Lemmas 6.2-6.4 and Theorem 2.10, together with Theorem 5.7, we have the following main result of the paper.

Theorem 6.5. *Let L be a positive definite even lattice. Then V_L^+ is rational.*

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