RATIOS INVOLVING EXTREME VALUES[1]

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1. Summary. Ratios of the form $(x_n - x_{n-j})/(x_n - x_i)$ for small values of i and j and $n = 3, \cdots, 30$ are discussed. The variables concerned are order statistics, i.e., sample values such that $x_1 < x_2 < \cdots < x_n$. Analytic results are obtained for the distributions of these ratios for several small values of n and percentage values are tabled for these distributions for samples of size $n \leqq 30$.

2. Introduction. There has been interest in the problem of gross errors in data since Chauvenet presented his solution for the problem about 1850. His hypothesis was essentially that in some samples a small portion of the observations were from a population with a different mean value. There has been research from that time up to the present on procedures suitable for treating such data.

If it is assumed that a certain percentage of "gross errors" may occur, then there are two general procedures for treating such data:

(1) A statistical treatment may be given to the data which gives very little weight to such aberrant values as may occur.

(2) A statistical test may be constructed which will indicate such values so that they may be rejected.

The functions to be discussed here were designed for testing the consistency of suspected values with the sample as a whole. Investigation of the performance of these criteria is given in another paper.

3. Critical values for r_{10}. The first statistic to be considered is

$$r_{10} = (x_n - x_{n-1})/(x_n - x_1),$$

where the subscripts on the x's indicate ordered values such that $x_1 < x_2 < \cdots < x_n$. The density function for x_1, x_{n-1}, x_n is

$$(1) \qquad \frac{n!}{(n-3)!} f(x_1)\, dx_1 \left(\int_{x_1}^{x_{n-1}} f(t)\, dt \right)^{n-3} f(x_{n-1})\, dx_{n-1} f(x_n)\, dx_n .$$

Setting $v = x_n - x_1$, $rv = x_n - x_{n-1}$, $x = x_n$, and integrating x and v over their range of definition we have the density function of r_{10} for a sample of size n. (The subscripts on the r's will be dropped when there is no ambiguity.) This function appears as

$$(2) \qquad \frac{n!}{(n-3)!} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(\int_{x-v}^{x-r_{10}v} f(t)\, dt \right)^{n-3} f(x-v) f(x-r_{10}v) f(x) v\, dv\, dx .$$

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There will be no loss in generality by considering the values x_i to have been drawn from a distribution with zero mean and unit variance, since the statistic is the ratio of two differences. It should also be noted that for symmetric populations, the distribution of $(x_n - x_{n-1})/(x_n - x_1)$ will be the same as that of $(x_2 - x_1)/(x_n - x_1)$. For the rectangular distribution the density function is

$$(n-2)(1-r_{10})^{n-3} \qquad (0 < r_{10} < 1), \tag{3}$$

and the cdf is

$$1-(1-R_{10})^{n-2}. \tag{4}$$

If we set this expression equal to $1 - \alpha$ we obtain critical values of R_{10}

$$R_{10\alpha} = 1 - \alpha^{1/n-2}. \tag{5}$$

For the more interesting case of the normal distribution, the operations indicated above are much more arduous.

$n = 3$, *Normal population.* The integral in (2) above can be evaluated to obtain the density function of r_{10} for the assumption of normality

$$g_3(r_{10}) = \frac{3\sqrt{3}}{2\pi}\,\frac{1}{r^2 - r + 1}\,. \tag{6}$$

The integration of this density results in the cdf

$$\frac{3}{\pi}\arc\tan\frac{2}{\sqrt{3}}\left(R_{10} - \tfrac{1}{2}\right) + \tfrac{1}{2}. \tag{7}$$

Upon setting this last expression equal to $1 - \alpha$, we obtain

$$R_{10\alpha} = \tfrac{1}{2} + \frac{\sqrt{3}}{2}\tan\frac{\pi}{3}\left(\tfrac{1}{2} - \alpha\right). \tag{8}$$

$n = 4$, *Normal population.* The density function in this case becomes

$$g_4(r_{10}) = \frac{3}{\pi}\,\frac{1}{r^2 - r + 1}\left[\frac{1-2r}{\sqrt{4r^2 - 4r + 3}} - \frac{r-2}{\sqrt{3r^2 - 4r + 4}}\right]. \tag{9}$$

If we now set the cdf equal to $1 - \alpha$ we obtain

$$5 - \frac{6}{\pi}\left[\arc\tan\sqrt{4R^2 - 4R + 3} + \arc\tan\frac{1}{R}\sqrt{3R^2 - 4R + 4}\right] = 1 - \alpha, \tag{10}$$

which may be written as follows by taking the tangent of both sides of this equation:

$$\frac{\sqrt{4R^2 - 4R + 3} + \frac{1}{R}\sqrt{3R^2 - 4R + 4}}{1 - \frac{1}{R}\sqrt{(4R^2 - 4R + 3)(3R^2 - 4R + 4)}} = \tan\frac{\pi}{6}\,(\alpha + 4). \tag{11}$$

The integration of $g_4(r_{10})$ was performed for the first term by substituting $r = \frac{1}{2} + (1/\sqrt{2})\sqrt{x^2 - 1}$. The second term of $g_4(r_{10})$ is identical with the first if one substitutes $s = 1/r$.

$n = 5$, *Normal population.* For this case it can be shown that the density function has the following form

$$g_5(r_{10}) = \frac{15\left[h(r) + h\left(\frac{1}{r}\right)\right]}{\pi^2(r^2 - r + 1)}, \tag{12}$$

where

$$h(r) = \frac{2 - r}{\sqrt{3r^2 - 4r + 4}} \tan^{-1} \frac{(1 - r)\sqrt{5(3r^2 - 4r + 4)}}{3r^2 - 3r + 4}.$$

The cdf for $n = 5$ has not been obtained in a comparable form to those obtained for $n = 3, 4$. No such expressions were obtained for larger values of n. Various percentage values were computed from the above distributions and are presented in Table I. The percentage values were also obtained by numerical integrations for $n = 5, 7, 10, 15, 20, 25, 30$. Values for other values of n were obtained by interpolation. These percentage values can be obtained by a double quadrature since

$$\begin{aligned} G(R_{10}) &= \int_0^R \int_{-\infty}^{\infty} \int_0^{\infty} g(r, x, v)\, dv\, dx\, dr_{10} = \\ &1 - n(n-1) \int_0^R \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_{x-v}^{x - r_{10}v} f(t)\, dt\right)^{n-2} f(x)f(x - v)\, dv\, dx\, dr_{10}. \end{aligned} \tag{13}$$

This integral was evaluated for all combinations of the values of n indicated above and for $R_{10} = 0, .06, .10, .16, .21, .26, .30, .34, .40, .44, .48, .53, .56, .60, .80, .90$. These values are not regularly spaced since several computations were made before it was possible to select the particular values of R which would be most useful for evaluating $G(R_{10})$. The values of the integral in (13) were used as the base for computations for all the tables included in this paper.

4. Distribution of other ratios. It can be suggested that a ratio to test whether x_n is significantly far from x_{n-1} should avoid x_1. Let us consider $r_{11} = (x_n - x_{n-1})/(x_n - x_2)$. Its cdf is

$$\int_{-\infty}^{\infty} \int_0^{\infty} \frac{n!}{(n-2)!} \int_{-\infty}^{x} f(t)\, dt \left(\int_{x-v}^{x - r_{11}v} f(s)\, ds\right)^{n-3} f(x - v)f(x)\, dv\, dx. \tag{14}$$

For the rectangular distribution we obtain the density function

$$(n - 3)(1 - r_{11})^{n-4}. \tag{15}$$

For the rectangular distribution we can write down the density function of $r_{1,k-1} = (x_n - x_{n-1})/(x_n - x_k)$ as

$$(n - k - 1)(1 - r_{1,k-1})^{n-k-2}, \tag{16}$$

where $k = 0, 1, \cdots, n - 2$.

$n = 4$, *Normal population.* When we assume the normal distribution for our $f(x)$ and consider $k = 2$, the first sample size of interest is $n = 4$, here $r_{11} = (x_4 - x_3)/(x_4 - x_2)$. The density function may be obtained for this ratio by the procedures used above for r_{10}. The helpful substitution here is $r_{11} = (\sqrt{2}/2 + \sqrt{w^2 - 1})^{-1/2}$. The resulting expression is

$$g(r_{11}) = \frac{3\sqrt{3}}{\pi(r^2 - r + 1)}\left[1 + \frac{r - 2}{\sqrt{3}(4 - 4r + 3r^2)^{1/2}}\right] \tag{17}$$

and the cdf is

$$\frac{6}{\pi}\left[\arc\tan\frac{1}{\sqrt{3}}(2R - 1) + \arc\tan\frac{1}{R}(4 - 4R + 3R^2)^{1/2}\right] - 2. \tag{18}$$

If we now set this function equal to $1 - \alpha$, we may solve for the various percentage values for this distribution.

$n = 5$, *Normal population.* The distribution of the similar ratio for samples of size five, $r_{11} = (x_5 - x_4)/(x_5 - x_2)$ is integrable into an expression similar to the distribution of r_{10} for $n = 5$. The percentage values for the distribution of r_{11} for $n = 4, \cdots, 30$ are in Table II. The distribution of r_{11} for samples of size 5 is

$$\alpha\left[\frac{\beta}{\sqrt{3}}\left(\tan^{-1}\frac{\delta}{\sqrt{5}} - 2\tan^{-1}\frac{\beta}{\sqrt{5}}\right) - \frac{\pi\gamma}{6}(\beta + \gamma)\tan^{-1}\frac{\delta'}{\sqrt{5}}\right],$$

where the symbols in this expression and those to follow are

$$\alpha = \frac{15\sqrt{3}}{\pi^2(1 - r + r^2)}, \qquad \beta = (2 - r)/q_1, \qquad \delta = (3r - 2)/q_1$$

$$q_1 = \sqrt{4 - 4r + 3r^2}, \qquad \beta' = (2 + r)/q_1, \qquad \delta' = (3 - 2r)/q_2,$$

$$q_2 = \sqrt{3 - 4r + 4r^2}, \qquad \gamma = (1 - 2r)/q_2, \qquad \eta = (1 + r)/q_3,$$

$$q_3 = \sqrt{3 - 2r + 3r^2}, \qquad \gamma' = (1 + 2r)/q_2, \qquad \eta' = (3 - r)/q_3,$$

$$\eta'' = (3r - 1)/q_3.$$

The percentage values of the distribution of the ratio $r_{12} = (x_n - x_1)/(x_n - x_3)$ are in Table III. The general expression for the cdf is

$$\int_{-\infty}^{\infty}\int_0^{\infty}\frac{n!}{2(n - 4)!}\left(\int_{-\infty}^{x-v} f(t)\,dt\right)^2\left(\int_{x-v}^{x-R_{12}v} f(s)\,ds\right)^{n-4} f(x - v)f(x)\,dv\,dx.$$

The smallest sample size for which this ratio will have meaning is $n = 5$. The density function for $n = 5$ is

$$\frac{\alpha}{2}\left[\frac{\pi}{2} + \tan^{-1}\frac{1}{\sqrt{15}} + \frac{2\beta}{\sqrt{3}}\tan^{-1}\frac{\beta}{\sqrt{5}} - \frac{\pi\beta}{\sqrt{3}}\right].$$

Percentage values have been computed in a similar manner for $r_{20} = (x_n - x_{n-2})/(x_n - x_1)$, $r_{21} = (x_n - x_{n-2})/(x_n - x_2)$, $r_{22} = (x_n - x_{n-2})/(x_n - x_3)$

and are presented in Tables IV, V, and VI. Here again analytic expressions can be obtained for the distribution of a particular ratio for small values of n.

We have the distribution of r_{20} for $n = 4$ since for this sample size $r_{20} + r_{10} = 1$ if we consider $r_{10} = \dfrac{x_2 - x_1}{x_n - x_1}$.

For $n = 5$ the density function of r_{20} is

$$\alpha\left[\frac{\beta}{\sqrt{3}}\left(\tan^{-1}\frac{\delta}{\sqrt{5}} + \tan^{-1}\frac{\beta'}{\sqrt{5}}\right) + \frac{\gamma}{\sqrt{3}}\left(\tan^{-1}\frac{\gamma'}{\sqrt{5}} - 2\tan^{-1}\frac{\gamma}{\sqrt{5}} - \tan^{-1}\frac{\delta'}{\sqrt{5}}\right) + \frac{\eta}{\sqrt{3}}\left(\tan^{-1}\frac{\eta'}{\sqrt{5}} - \tan^{-1}\frac{\eta''}{\sqrt{5}}\right)\right].$$

For $n = 5$ the density function of r_{21} is

$$\alpha\left[\frac{-\beta}{\sqrt{3}}\left(\tan^{-1}\frac{\delta}{\sqrt{5}} + \tan^{-1}\frac{\beta'}{\sqrt{5}}\right) - \frac{\gamma}{\sqrt{3}}\left(\frac{\pi}{2} - \tan^{-1}\frac{\delta'}{\sqrt{5}}\right) + \frac{\eta}{\sqrt{3}}\left(\frac{\pi}{2} - \tan^{-1}\frac{\eta'}{\sqrt{5}}\right)\right].$$

The distribution for the ratio $r_{j,i-1} = (x_n - x_j)/(x_n - x_i)$ is

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{n!}{(i-1)!(n-j-i-1)!(j-1)!}\left(\int_{-\infty}^{x-v} f(t)\,dt\right)^{i-1} f(x-v) \cdot \left(\int_{x-v}^{x-rv} f(t)\,dt\right)^{n-j-i-1} f(x-rv)f(x)\left(\int_{x-rv}^{x} f(t)\,dt\right)^{j-1} dv\,dx.$$

5. Final remarks.

5.1. Accuracy of tables. The goal with respect to accuracy was to obtain three places of accuracy in the percentage values. It is believed that the values in Tables I, II, III are in error by not more than one or two in the third place, while the values in Tables IV, V, and VI are believed to be accurate to within three or four units in the third place.

5.2. Investigation of the performance of the ratios. It is important to know something about the performance of these ratios for various purposes. Reference is made to another paper [1] evaluating the performance of these criteria as well as a number of others.

REFERENCE

[1] W. J. Dixon, "Analysis of extreme values," *Annals of Math. Stat.*, Vol. 21 (1950), pp. 488–506.

TABLE I

$Pr(r_{10} > R) = \alpha$

n \ α	.005	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	α / n
3	.994	.988	.976	.941	.886	.781	.684	.591	.500	.409	.316	.219	.114	.059	3
4	.926	.889	.846	.765	.679	.560	.471	.394	.324	.257	.193	.130	.065	.033	4
5	.821	.780	.729	.642	.557	.451	.373	.308	.250	.196	.146	.097	.048	.023	5
6	.740	.698	.644	.560	.482	.386	.318	.261	.210	.164	.121	.079	.038	.018	6
7	.680	.637	.586	.507	.434	.344	.281	.230	.184	.143	.105	.068	.032	.016	7
8	.634	.590	.543	.468	.399	.314	.255	.208	.166	.128	.094	.060	.029	.014	8
9	.598	.555	.510	.437	.370	.290	.234	.191	.152	.118	.086	.055	.026	.013	9
10	.568	.527	.483	.412	.349	.273	.219	.178	.142	.110	.080	.051	.025	.012	10
11	.542	.502	.460	.392	.332	.259	.208	.168	.133	.103	.074	.048	.023	.011	11
12	.522	.482	.441	.376	.318	.247	.197	.160	.126	.097	.070	.045	.022	.011	12
13	.503	.465	.425	.361	.305	.237	.188	.153	.120	.092	.067	.043	.021	.010	13
14	.488	.450	.411	.349	.294	.228	.181	.147	.115	.088	.064	.041	.020	.010	14
15	.475	.438	.399	.338	.285	.220	.175	.141	.111	.085	.062	.040	.019	.010	15
16	.463	.426	.388	.329	.277	.213	.169	.136	.107	.082	.060	.039	.019	.009	16
17	.452	.416	.379	.320	.269	.207	.165	.132	.104	.080	.058	.038	.018	.009	17
18	.442	.407	.370	.313	.263	.202	.160	.128	.101	.078	.056	.036	.018	.009	18
19	.433	.398	.363	.306	.258	.197	.157	.125	.098	.076	.055	.036	.017	.008	19
20	.425	.391	.356	.300	.252	.193	.153	.122	.096	.074	.053	.035	.017	.008	20
21	.418	.384	.350	.295	.247	.189	.150	.119	.094	.072	.052	.034	.016	.008	21
22	.411	.378	.344	.290	.242	.185	.147	.117	.092	.071	.051	.033	.016	.008	22
23	.404	.372	.338	.285	.238	.182	.144	.115	.090	.069	.050	.033	.016	.008	23
24	.399	.367	.333	.281	.234	.179	.142	.113	.089	.068	.049	.032	.016	.008	24
25	.393	.362	.329	.277	.230	.176	.139	.111	.088	.067	.048	.032	.015	.008	25
26	.388	.357	.324	.273	.227	.173	.137	.109	.086	.066	.047	.031	.015	.007	26
27	.384	.353	.320	.269	.224	.171	.135	.108	.085	.065	.047	.031	.015	.007	27
28	.380	.349	.316	.266	.220	.168	.133	.106	.084	.064	.046	.030	.015	.007	28
29	.376	.345	.312	.263	.218	.166	.131	.105	.083	.063	.046	.030	.014	.007	29
30	.372	.341	.309	.260	.215	.164	.130	.103	.082	.062	.045	.029	.014	.007	30

TABLE II

$Pr(r_{11} > R) = \alpha$

n \ α	.005	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	α / n
4	.995	.991	.981	.955	.910	.822	.737	.648	.554	.459	.362	.250	.131	.069	4
5	.937	.916	.876	.807	.728	.615	.524	.444	.369	.296	.224	.151	.078	.039	5
6	.839	.805	.763	.689	.609	.502	.420	.350	.288	.227	.169	.113	.056	.028	6
7	.782	.740	.689	.610	.530	.432	.359	.298	.241	.189	.140	.093	.045	.022	7
8	.725	.683	.631	.554	.479	.385	.318	.260	.210	.164	.121	.079	.037	.019	8
9	.677	.635	.587	.512	.441	.352	.288	.236	.189	.148	.107	.070	.033	.016	9
10	.639	.597	.551	.477	.409	.325	.265	.216	.173	.134	.098	.063	.030	.014	10
11	.606	.566	.521	.450	.385	.305	.248	.202	.161	.124	.090	.058	.028	.013	11
12	.580	.541	.498	.428	.367	.289	.234	.190	.150	.116	.084	.055	.026	.012	12
13	.558	.520	.477	.410	.350	.275	.222	.180	.142	.109	.079	.052	.025	.012	13
14	.539	.502	.460	.395	.336	.264	.212	.171	.135	.104	.075	.049	.024	.011	14
15	.522	.486	.445	.381	.323	.253	.203	.164	.129	.099	.072	.047	.023	.011	15
16	.508	.472	.432	.369	.313	.244	.196	.158	.124	.095	.069	.045	.022	.011	16
17	.495	.460	.420	.359	.303	.236	.190	.152	.119	.092	.067	.044	.021	.010	17
18	.484	.449	.410	.349	.295	.229	.184	.148	.116	.089	.065	.042	.020	.010	18
19	.473	.439	.400	.341	.288	.223	.179	.143	.112	.087	.063	.041	.020	.010	19
20	.464	.430	.392	.334	.282	.218	.174	.139	.110	.084	.061	.040	.019	.010	20
21	.455	.421	.384	.327	.276	.213	.170	.136	.107	.082	.059	.039	.019	.009	21
22	.446	.414	.377	.320	.270	.208	.166	.132	.104	.081	.058	.038	.018	.009	22
23	.439	.407	.371	.314	.265	.204	.163	.130	.102	.079	.056	.037	.018	.009	23
24	.432	.400	.365	.309	.260	.200	.160	.127	.100	.077	.055	.036	.018	.009	24
25	.426	.394	.359	.304	.255	.197	.156	.124	.098	.076	.054	.036	.017	.009	25
26	.420	.389	.354	.299	.250	.193	.154	.122	.096	.074	.053	.035	.017	.008	26
27	.414	.383	.349	.295	.246	.190	.151	.120	.095	.073	.052	.034	.017	.008	27
28	.409	.378	.344	.291	.243	.188	.149	.118	.093	.072	.051	.034	.016	.008	28
29	.404	.374	.340	.287	.239	.185	.146	.116	.092	.070	.051	.033	.016	.008	29
30	.399	.369	.336	.283	.236	.182	.144	.115	.090	.069	.050	.032	.016	.008	30

TABLE III

$Pr(r_{12} > R) = \alpha$

n \ α	.005	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	α / n
5	.996	.992	.984	.960	.919	.838	.755	.669	.579	.483	.381	.268	.143	.074	5
6	.951	.925	.891	.824	.745	.635	.545	.465	.390	.316	.240	.165	.088	.049	6
7	.875	.836	.791	.712	.636	.528	.445	.374	.307	.245	.183	.123	.064	.031	7
8	.797	.760	.708	.632	.557	.456	.382	.317	.258	.203	.152	.101	.056	.025	8
9	.739	.701	.656	.580	.504	.409	.339	.270	.227	.177	.130	.086	.044	.021	9
10	.694	.655	.610	.537	.454	.373	.308	.258	.204	.158	.116	.075	.038	.019	10
11	.658	.619	.575	.502	.431	.345	.283	.232	.187	.145	.106	.069	.035	.017	11
12	.629	.590	.546	.473	.406	.324	.265	.217	.174	.135	.098	.063	.032	.016	12
13	.612	.554	.521	.451	.387	.307	.250	.204	.163	.126	.092	.059	.030	.015	13
14	.580	.542	.501	.432	.369	.292	.237	.193	.153	.118	.086	.055	.028	.014	14
15	.560	.523	.482	.416	.354	.280	.226	.184	.146	.112	.082	.053	.026	.013	15
16	.544	.508	.467	.401	.341	.269	.217	.177	.139	.107	.078	.050	.025	.013	16
17	.529	.493	.453	.388	.330	.259	.209	.170	.134	.103	.075	.048	.024	.012	17
18	.516	.480	.440	.377	.320	.251	.202	.163	.129	.099	.072	.047	.023	.012	18
19	.504	.469	.429	.367	.311	.243	.196	.157	.125	.096	.069	.045	.022	.011	19
20	.493	.458	.419	.358	.303	.237	.191	.153	.121	.093	.067	.044	.022	.011	20
21	.483	.449	.410	.349	.296	.231	.186	.148	.118	.090	.065	.042	.021	.010	21
22	.474	.440	.402	.342	.290	.225	.181	.145	.114	.088	.063	.041	.020	.010	22
23	.465	.432	.394	.336	.284	.220	.176	.141	.112	.086	.062	.040	.020	.010	23
24	.457	.423	.387	.330	.278	.216	.173	.138	.109	.084	.060	.039	.019	.010	24
25	.450	.417	.381	.324	.273	.212	.169	.135	.107	.082	.059	.038	.019	.009	25
26	.443	.411	.375	.319	.268	.208	.166	.132	.105	.080	.058	.037	.019	.009	26
27	.437	.405	.370	.314	.263	.204	.163	.130	.103	.079	.057	.037	.018	.009	27
28	.431	.399	.365	.309	.259	.201	.160	.128	.101	.077	.056	.036	.018	.009	28
29	.426	.394	.360	.305	.255	.197	.157	.126	.099	.076	.055	.035	.017	.009	29
30	.420	.389	.355	.301	.251	.194	.154	.124	.098	.075	.054	.035	.017	.009	30

TABLE IV

$Pr(r_{20} > R) = \alpha$

n \ α	.005	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	α / n
4	.996	.992	.987	.967	.935	.871	.807	.743	.676	.606	.529	.440	.321	.235	4
5	.950	.929	.901	.845	.782	.694	.623	.560	.500	.440	.377	.306	.218	.155	5
6	.865	.836	.800	.736	.670	.585	.520	.463	.411	.358	.305	.245	.172	.126	6
7	.814	.778	.732	.661	.596	.516	.454	.402	.355	.306	.261	.208	.144	.099	7
8	.746	.710	.670	.607	.545	.468	.410	.361	.317	.274	.230	.184	.125	.085	8
9	.700	.667	.627	.565	.505	.432	.378	.331	.288	.250	.208	.166	.114	.077	9
10	.664	.632	.592	.531	.474	.404	.354	.307	.268	.231	.192	.153	.104	.070	10
11	.627	.603	.564	.504	.449	.381	.334	.290	.253	.217	.181	.143	.097	.065	11
12	.612	.579	.540	.481	.429	.362	.316	.274	.239	.205	.172	.136	.091	.060	12
13	.590	.557	.520	.461	.411	.345	.301	.261	.227	.195	.164	.129	.086	.057	13
14	.571	.538	.502	.445	.395	.332	.288	.250	.217	.187	.157	.123	.082	.054	14
15	.554	.522	.486	.430	.382	.320	.277	.241	.209	.179	.150	.118	.079	.052	15
16	.539	.508	.472	.418	.370	.310	.268	.233	.202	.173	.144	.113	.076	.050	16
17	.526	.495	.460	.406	.359	.301	.260	.226	.195	.167	.139	.109	.074	.049	17
18	.514	.484	.449	.397	.350	.293	.252	.219	.189	.162	.134	.105	.071	.048	18
19	.503	.473	.439	.379	.341	.286	.246	.213	.184	.157	.130	.101	.069	.047	19
20	.494	.464	.430	.372	.333	.279	.240	.208	.179	.152	.126	.098	.067	.046	20
21	.485	.455	.422	.365	.326	.273	.235	.203	.175	.148	.123	.096	.065	.045	21
22	.477	.447	.414	.358	.320	.267	.230	.199	.171	.145	.120	.094	.064	.044	22
23	.469	.440	.407	.352	.314	.262	.225	.195	.167	.142	.117	.092	.062	.043	23
24	.462	.434	.401	.347	.309	.258	.221	.192	.164	.139	.114	.090	.061	.042	24
25	.456	.428	.395	.343	.304	.254	.217	.189	.161	.136	.112	.089	.060	.041	25
26	.450	.422	.390	.338	.300	.250	.214	.186	.158	.134	.110	.087	.059	.041	26
27	.444	.417	.385	.334	.296	.246	.211	.183	.156	.132	.109	.086	.058	.040	27
28	.439	.412	.381	.330	.292	.243	.208	.180	.154	.130	.107	.085	.058	.040	28
29	.434	.407	.376	.326	.288	.239	.205	.177	.151	.128	.106	.083	.057	.039	29
30	.428	.402	.372	.322	.285	.236	.202	.175	.149	.126	.104	.082	.056	.039	30

TABLE V

$Pr(r_{21} > R) = \alpha$

n \ α	.005	.01	.02	.05	.10	.20	.30	.40	.50	60	.70	.80	.90	.95	α / n
5	.998	.995	.990	.976	.952	.902	.850	.795	.735	.669	.594	.501	.374	.273	5
6	.970	.951	.924	.872	.821	.745	.680	.621	.563	.504	.439	.364	.268	.195	6
7	.919	.885	.842	.780	.725	.637	.575	.517	.462	.408	.350	.285	.198	.138	7
8	.868	.829	.780	.710	.650	.570	.509	.454	.402	.352	.298	.240	.166	.117	8
9	.816	.776	.725	.657	.594	.516	.458	.407	.360	.313	.265	.212	.146	.103	9
10	.760	.726	.678	.612	.551	.474	.420	.374	.329	.286	.240	.189	.130	.089	10
11	.713	.679	.638	.576	.517	.442	.391	.348	.305	.265	.221	.173	.118	.080	11
12	.675	.642	.605	.546	.490	.419	.370	.326	.285	.247	.206	.161	.110	.074	12
13	.649	.615	.578	.521	.467	.399	.351	.308	.269	.232	.194	.152	.104	.070	13
14	.627	.593	.556	.501	.448	.381	.334	.293	.256	.219	.184	.144	.099	.066	14
15	.607	.574	.537	.483	.431	.366	.319	.280	.245	.208	.175	.138	.094	.062	15
16	.589	.557	.521	.467	.416	.353	.307	.269	.235	.199	.167	.132	.090	.059	16
17	.573	.542	.507	.453	.403	.341	.296	.259	.225	.192	.161	.127	.086	.057	17
18	.559	.529	.494	.440	.391	.331	.287	.250	.218	.186	.155	.122	.082	.054	18
19	.547	.517	.482	.428	.380	.322	.279	.243	.211	.180	.150	.117	.078	.052	19
20	.536	.506	.472	.419	.371	.314	.271	.236	.205	.174	.145	.113	.075	.050	20
21	.526	.496	.462	.410	.363	.306	.264	.229	.199	.170	.141	.110	.073	.049	21
22	.517	.487	.453	.402	.356	.299	.258	.223	.194	.165	.137	.107	.071	.048	22
23	.509	.479	.445	.395	.349	.293	.252	.218	.189	.161	.133	.105	.069	.046	23
24	.501	.471	.438	.388	.343	.287	.247	.214	.185	.158	.130	.103	.068	.045	24
25	.493	.464	.431	.382	.337	.282	.242	.210	.181	.154	.127	.100	.067	.043	25
26	.486	.457	.424	.376	.331	.277	.238	.206	.178	.151	.125	.098	.066	.042	26
27	.479	.450	.418	.370	.325	.273	.234	.203	.175	.149	.123	.096	.064	.041	27
28	.472	.444	.412	.365	.320	.269	.230	.200	.172	.146	.121	.094	.063	.041	28
29	.466	.438	.406	.360	.316	.265	.227	.197	.170	.144	.119	.092	.062	.040	29
30	.460	.433	.401	.355	.312	.261	.224	.194	.167	.142	.117	.091	.061	.040	30

TABLE VI

$Pr(r_{22} > R) = \alpha$

n \ α	.005	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	.95	α / n
6	.998	.995	.992	.983	.965	.930	.880	.830	.780	.720	.640	.540	.410	.300	6
7	.970	.945	.919	.881	.850	.780	.730	.670	.610	.540	.470	.390	.270	.200	7
8	.922	.890	.857	.803	.745	.664	.602	.546	.490	.434	.375	.309	.218	.156	8
9	.873	.840	.800	.737	.676	.592	.530	.478	.425	.373	.320	.261	.186	.128	9
10	.826	.791	.749	.682	.620	.543	.483	.433	.384	.335	.285	.231	.150	.111	10
11	.781	.745	.703	.637	.578	.503	.446	.397	.351	.305	.258	.208	.142	.099	11
12	.740	.704	.661	.600	.543	.470	.416	.370	.325	.282	.238	.190	.130	.090	12
13	.705	.670	.628	.570	.515	.443	.391	.347	.304	.263	.222	.177	.122	.084	13
14	.674	.641	.602	.546	.492	.421	.370	.328	.287	.247	.208	.166	.115	.079	14
15	.647	.616	.579	.525	.472	.402	.353	.312	.273	.234	.196	.156	.109	.075	15
16	.624	.595	.559	.507	.454	.386	.338	.298	.261	.223	.186	.148	.104	.071	16
17	.605	.577	.542	.490	.438	.373	.325	.286	.250	.214	.178	.142	.099	.067	17
18	.589	.561	.527	.475	.424	.361	.314	.276	.241	.206	.171	.135	.094	.063	18
19	.575	.547	.514	.462	.412	.350	.304	.268	.233	.199	.165	.130	.090	.060	19
20	.562	.535	.502	.450	.401	.340	.295	.260	.226	.193	.160	.125	.086	.057	20
21	.551	.524	.491	.440	.391	.331	.287	.252	.220	.187	.155	.120	.082	.054	21
22	.541	.514	.481	.430	.382	.323	.280	.245	.213	.182	.150	.116	.078	.051	22
23	.532	.505	.472	.421	.374	.316	.274	.239	.207	.177	.146	.113	.075	.049	23
24	.524	.497	.484	.413	.367	.310	.268	.232	.201	.172	.142	.111	.074	.047	24
25	.516	.489	.457	.406	.360	.304	.262	.227	.196	.168	.138	.108	.073	.045	25
26	.508	.486	.450	.399	.354	.298	.257	.222	.192	.164	.135	.106	.072	.044	26
27	.501	.475	.443	.393	.348	.292	.252	.218	.189	.161	.132	.104	.071	.043	27
28	.495	.469	.437	.387	.342	.287	.247	.215	.186	.158	.130	.102	.069	.042	28
29	.489	.463	.431	.381	.337	.282	.243	.211	.183	.155	.128	.100	.068	.041	29
30	.483	.457	.425	.376	.332	.278	.239	.208	.180	.153	.126	.098	.067	.041	30