Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Rayleigh–Bénard convection: Improved bounds on the Nusselt number

by

Felix Otto, and Christian Seis

Preprint no.: 6 2011

Rayleigh–Bénard convection: Improved bounds on the Nusselt number

Felix Otto and Christian Seis*

March 1, 2011

Abstract

We consider Rayleigh–Bénard convection as modelled by the Boussinesq equations in the infinite-Prandtl-number limit. We are interested in the scaling of the average upward heat transport, the Nusselt number Nu, in terms of the non-dimensionalized temperature forcing, the Rayleigh number Ra. Experiments, asymptotics and heuristics suggest that $Nu \sim Ra^{1/3}$.

This work is mostly inspired by two earlier rigorous work on upper bounds of Nu in terms of Ra: 1.) The work of Constantin and Doering establishing $Nu \lesssim Ra^{1/3} \ln^{2/3} Ra$ with help of a (logarithmically failing) maximal regularity estimate in L^∞ on the level of the Stokes equation. 2.) The work of Doering, Reznikoff and the first author establishing $Nu \lesssim Ra^{1/3} \ln^{1/3} Ra$ with help of the background temperature method.

The paper contains two results: 1.) The background temperature method can be slightly modified to yield $Nu \lesssim Ra^{1/3} \ln^{1/15} Ra$. 2.) The estimates behind the temperature background method can be combined with the maximal regularity in L^∞ to yield $Nu \lesssim Ra^{1/3} \ln^{1/3} \ln Ra$ — an estimate that is only a double logarithm away from the supposedly optimal scaling.

1 Introduction

1.1 Model

We consider the infinite-Prandtl-number limit of the Boussinesq equations in a container in $\mathbb{R}^d$, $d \geq 2$, of height H, that is,

$$\partial_t T + u \cdot \nabla T - \Delta T = 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$-\Delta u + \nabla p = Te. \tag{3}$$

*Max-Planck-Institut für Mathematik in den Naturwissenschaften, 04103 Leipzig, Germany

This system is complemented with periodic boundary conditions in the horizontal directions $y \in \mathbb{R}^{d-1}$ for all $d-1$ variables involved, together with Dirichlet boundary conditions for T and u at the bottom ($z = 0$) and top ($z = H$) plates,

$$T = \left\{ \begin{array}{ll} 1 & \text{for } z = 0 \\ 0 & \text{for } z = H \end{array} \right\} \quad \text{and} \quad u = 0 \text{ for } z \in \{0, H\}. \tag{4}$$

Here, T, u, and p denote the temperature, the fluid velocity, and the hydrodynamic pressure, respectively. Moreover, e denotes the upward unit vector. This system is nondimensionalized and owns one dimensionless parameter, the container height H. The size of the horizontal period cell is chosen arbitrary and will not enter in our later results.

The model describes the evolution of an incompressible Newtonian fluid between two horizontal plates with an imposed temperature gradient due to heating of the bottom plate: the Rayleigh–Bénard experiment. Thermal conduction compensates local temperature differences, whereas buoyancy forces drive the formation of dynamic flow pattern over large distances via thermal convection. Convection is limited by inner friction due to viscosity, especially near the plates, where the fluid is at rest. It becomes apparent that the height of the container plays a crucial role for the qualitative behavior of the fluid. Indeed, for small container heights, $H \ll 1$, thermal conduction turns out to be the only relevant transport mechanism, so that the fluid stabilizes with a constant temperature gradient. If the container height exceeds some critical value $H \sim 1$, the purely conducting state becomes unstable and the system undergoes a cascade of bifurcations, during which large convection rolls — so-called Bénard cells — drive the heat transport. Finally, in the regime of large container heights, $H \gg 1$, the flow is less constrained by the container walls and can react more to buoyancy forces so that the flow pattern changes: Along the horizontal plates, thin thermal boundary layers with a high vertical temperature gradient emerge. Due to the rigidity of the container walls, conduction is the dominating heat transport in these layers. In the bulk, the convection rolls break down and the bulk flow becomes turbulent, in the sense that it displays an aperiodic chaotic behavior. In particular, from the laminar thermal boundary layers, small fluid parcels of different temperature than the ambient fluid, so-called plumes, detach and flow rapidly through the bulk to the opposite boundary.

In this paper, we focus on the regime of convective turbulence, and thus suppose that

$$H \gg 1. \tag{5}$$

Convective turbulence becomes important in a variety of problems in applied physics like astrophysics [18], transport in physical oceanography [12] and atmospheric science [15, 6]. The infinite-Prandtl-number limit of the Boussinesq equations is the standard model for earth mantel convection studies in terrestrial geophysics, [13, 20].

Before discussing the main issues of this paper, let us shortly comment on the role of buoyancy in the mathematical setting (1)–(3). Buoyancy forces originate from temperature variations, which are accompanied by density variations and lead to pressure gradients in the presence of gravity. In the Boussinesq equations, buoyancy appears only as a driving force in the equation of motion of the fluid, while the fluid itself is supposed to be incompressible. This approximation is valid when the density variations are sufficiently small. In our setting, we consider the infinite-Prandtl-number limit of the Boussinesq equations, meaning that thermal diffusivity is negligible compared to kinematic viscosity. Mathematically, this is realized in dropping the inertial terms of the incompressible Navier–Stokes equation. This results in the stationary Stokes equation (2), (3). The limit from finite to infinite Prandtl number is discussed in [21]. The evolution equation for the temperature is just a convection-diffusion equation, (1).

1.2 Motivation and results

One of the fundamental quantities of interest in the Rayleigh–Bénard experiment is the Nusselt number Nu, a measure for the average upward heat transport. Over many years, an substantial theoretical and experimental research effort has been devoted to the question how this quantity depends on the system parameter H. Despite the complex details in turbulent flows, it is conjectured that the Nusselt number satisfies a universal scaling law — at least for very high Prandtl numbers in the turbulent regime (5):

$$Nu \sim 1 \quad \text{if } H \gg 1. \tag{6}$$

For further reading, we refer to the recent extensive review article by Ahlers, Großmann, and Lohse [1] and the references therein. To avoid confusion, we have to point out that our scaling differs from the common one in the physical literature. Our dimensionless parameter is the container height H. In the physical literature, the dimensionless parameter mostly appears as a prefactor in front of the buoyancy forcing term in the (Navier–)Stokes-equation, the so-called Rayleigh number Ra. Both quantities are linked through the relation $H = Ra^{1/3}$. Since the Nusselt number describes the *average* upward heat transport, the Nusselt number scaling also differs in the factor H. Therefore, the scaling $Nu \sim 1$ corresponds in the physical language use to $Nu_{phys} \sim Ra^{1/3}$.

Now, we give the mathematical definition of the Nusselt number, being the average upward heat transport (cf. (1)),

$$Nu := \frac{1}{H} \int_0^H \langle (uT - \nabla T) \cdot e \rangle \, dz. \tag{7}$$

Here, the brackets denote the (horizontal) space and time average,

$$\langle f \rangle := \limsup_{t_0 \uparrow \infty} \frac{1}{t_0} \int_0^{t_0} \frac{1}{\Lambda^{d-1}} \int_{[0,\Lambda]^{d-1}} f(t,y) \, dydt,$$

when L is the side length of the horizontal period cell.

In this paper, we prove two new a priori upper bounds on the Nusselt number, which are optimal up to logarithmic factors with respect to the expected scaling (6). For the first bound, Theorem 1, we apply the so-called *background field method*, which will be introduced in Subsection 2.2 below. The analysis follows closely the one of [5]. However, we can slightly improve this result thanks to new a priori bounds on the velocity field.

Theorem 1 (Bound using background field method)**.** *Assume that* $H \gg 1$*. Then*

$$Nu \lesssim \ln^{1/15} H.$$

Remark 1. *In terms of the common physical language use, Theorem 1 states that* $Nu_{phys} \lesssim Ra^{1/3} \ln^{1/15} Ra$.

We use the "$\lesssim$" inequality sign to indicate that the inequality holds in the regime $H \gg 1$ up to a generic constant that may depend on the space dimension only. We conjecture that the bound stated in Theorem 1 above is optimal for the background field method.

The second bound on Nu, Theorem 2, makes in addition extensively use of the *maximum principle* for the temperature field in the sense that $\langle \sup_x T^2 \rangle \leq 1$. This a priori information is converted by an L^∞ maximal regularity estimate (involving logarithms) for the Stokes equation in the spirit of [2]. To our knowledge, this is the best rigorous estimate on the average upward heat transport available for the infinite-Prandtl-number limit.

Theorem 2 (Bound using maximum principle)**.** *Assume that* $H \gg 1$*. Then*

$$Nu \lesssim \ln^{1/3} \ln H.$$

Remark 2. *In terms of the common physical language use, Theorem 2 states that* $Nu_{phys} \lesssim Ra^{1/3} \ln^{1/3} \ln Ra$.

We want to remark that proving upper bounds for the Nusselt number is quite different from proving lower bounds. This is due to the fact that the evolution of the dynamical system depends sensitively on the initial data. Notice that the pure conductive state $T = 1 - z/H$, $u = 0$, $p = z - z^2/(2H)$ is always a trivial solution for the system (1)–(4), for which $Nu = 1/H$ holds. Hence one can only expect to prove a priori *upper bounds* on the Nusselt number. Lower bounds can only be generically true. This is a fact which occurs in several fluid dynamical models and is also similar to coarsening bounds for Cahn–Hilliard-type models, cf. [10].

We like to close this subsection with a short discussion of previous upper bounds on the Nusselt number for infinite-Prandtl-number Rayleigh–Bénard convection in the turbulent regime (5). The Nusselt number scaling $Nu \sim 1$ is predicted by a marginally stable boundary layer argument in [11, 8]. Progress in the rigorous treatment was initiated by Peter Constantin and Charles R. Doering. They establish the first rigorous upper bound [2], $Nu \lesssim \ln^{2/3} H$, which is optimal up to the logarithmic factor. Their proof relies on a maximum principle for the temperature field via an L^∞ maximal regularity estimate for the Stokes equation. In fact, in Theorem 2 we will follow this approach. In [3], the same authors introduce a new approach to the upper bound theory, the background field method (cf. Subsection 2.2), which was introduced by Hopf [7] in the context of the Navier–Stokes equations with inhomogeneous boundary data. However, the first attempts [4] provide only algebraically suboptimal bounds. The use of a monotonic background temperature profile produces the suboptimal bound $Nu \lesssim H^{1/5}$. This result was shown to be sharp for the background field method restricted to monotonic background profiles, [14, 16]. The best rigorous bound on the Nusselt number previously proven, $Nu \lesssim \ln^{1/3} H$ in [5], uses the background field method with a profile that has monotone decreasing boundary layers but a logarithmically increasing bulk. In fact, both in Theorem 1 and Theorem 2 we will appeal to estimates from [5] related to the logarithmic background fields, cf. Lemma 1. Finally, the approach of [17, 9] is a mixture of numerical and analytical methods. The authors conjecture that the background field method with a nonlinear profile is indeed applicable in order to obtain the expected Nusselt number scaling.

The remainder of the paper contains the rigorous analysis. Subsection 2.1 serves as preliminary and reviews previously known estimates. In Subsection 2.2, we introduce the background field method and prove Theorem 1. Finally, Subsection 2.3 provides the proof of Theorem 2.

2 Proofs

2.1 Preliminaries and review

In view of our results stated in Theorem 1 and Theorem 2, it is no restriction to assume that

$$Nu \geq 1.$$

We begin the rigorous part by recalling two representations of the Nusselt number defined in (7). We first notice by applying $\langle\cdot\rangle$ to (1) that the upward heat flux, when averaged in horizontal space and time, is constant in vertical direction, so that,

$$Nu \;=\; \langle Tw\rangle - \partial_z \langle T\rangle \quad \text{for } z \in (0,H). \tag{8}$$

Furthermore, testing (1) with T and using (8) for $z = 0$, we obtain (cf. [4, eq. (2.23)])

$$Nu \;=\; \int_0^H \langle |\nabla T|^2\rangle \, dz. \tag{9}$$

We remark that in the derivation of (8) and (9), we make use of the maximum principle for the temperature. In fact, $T \in [0,1]$ is preserved during the evolution. Even if this condition is not satisfied initially, T attains this temperature range exponentially fast, and, in particular, the condition holds in time average. For this reason, terms which involve time derivatives drop out in the derivation of (8) and (9).

It is convenient to eliminate the pressure term in (3) by the use of the incompressibility condition (2). This leads to a forth-order boundary value problem for the vertical velocity component $w = u \cdot e$:

$$\Delta^2 w = -\Delta_y T \quad \text{and} \quad w = \partial_z w = 0 \text{ for } z \in \{0, H\}. \tag{10}$$

Let us now review the key estimate from [5] (slightly extended by including $|\nabla_y w|^2$):

Lemma 1 ([5])**.** *Let T and w be periodic in y and satisfy* (10)*. Then we have*

$$\int_0^H \frac{\langle w^2 \rangle}{z^3}\,dz \;\leq\; \int_0^H \frac{\langle |\nabla w|^2 \rangle}{z}\,dz \;\lesssim\; \int_0^H \frac{\langle T w \rangle}{z}\,dz. \tag{11}$$

As a byproduct of the bound on the Nusselt number derived in [5] by the background field method, we have the following control of the term on the r.h.s. of (11) in terms of Nu.

Lemma 2 ([5])**.**

$$\int_0^H \frac{\langle T w \rangle}{z}\,dz \;\lesssim\; (\ln H) Nu. \tag{12}$$

For the convenience of the reader, we display a proof of Lemma 2 that is not based on the background field method, but relies on the two representation of Nu stated above.

Proof of Lemma 2. We consider boundary layer and bulk contribution of the term under consideration separately. We set $\theta = T - \langle T \rangle$. Observe that $\langle |\nabla \theta|^2 \rangle \leq \langle |\nabla T|^2 \rangle$ and $\langle T w \rangle = \langle \theta w \rangle$ because of $\langle w \rangle = 0$. Since θ vanishes at $z = 0$, we may estimate the boundary layer term as follows:

$$\begin{aligned}
\int_0^1 \frac{\langle T w \rangle}{z}\,dz \quad &= \quad \int_0^1 \frac{\langle \theta w \rangle}{z}\,dz \\
&\leq \quad \left(\int_0^1 \langle \theta^2 \rangle\,dz \right)^{1/2} \left(\int_0^1 \frac{\langle w^2 \rangle}{z^2}\,dz \right)^{1/2} \\
&\lesssim \quad \left(\int_0^1 \langle (\partial_z \theta)^2 \rangle\,dz \right)^{1/2} \left(\int_0^1 \frac{\langle w^2 \rangle}{z^3}\,dz \right)^{1/2} \\
&\overset{(9)\&(11)}{\lesssim} \quad Nu^{1/2} \left(\int_0^H \frac{\langle T w \rangle}{z}\,dz \right)^{1/2}.
\end{aligned}$$

The estimate of the bulk contribution relies on the Nusselt number representation (8). Dividing (8) by z and integrating over $(1, H)$ yields

$$\begin{aligned}
\int_1^H \frac{\langle T w \rangle}{z}\,dz \;&\leq\; (\ln H) Nu + \left(\int_1^H \frac{1}{z^2}\,dz \right)^{1/2} \left(\int_1^H \langle (\partial_z T)^2 \rangle\,dz \right)^{1/2} \\
&\overset{(9)}{\leq}\; ((\ln H) + 1) Nu + 1.
\end{aligned}$$

It remains to combine both estimates. Using Young's inequality and $H \gg 1$ and $Nu \geq 1$, we find

$$\int_0^H \frac{\langle Tw \rangle}{z}\, dz \;\lesssim\; ((\ln H) + 1)Nu + 1 \;\lesssim\; (\ln H)Nu.$$

□

By maximal regularity in L^2 and (9), the third-order derivatives of the velocity field are controlled by the Nusselt number. In proving Theorem 1 and Theorem 2, we make use of this control in the following sense.

Lemma 3. *Let T and w be periodic in y and satisfy* (10). *Then we have the estimate*

$$\int_0^H \langle |\nabla^3 w|^2 \rangle \, dz \;\lesssim\; \int_0^H \langle |\nabla_y T|^2 \rangle \, dz + \frac{1}{H^4} \int_0^H \langle (\partial_z w)^2 \rangle \, dz. \tag{13}$$

Proof of Lemma 3. We prove the result mode by mode. For this purpose, we introduce the Fourier transform $\mathcal{F}f$ of a function $f = f(y)$ via

$$\mathcal{F}f(k) \;=\; \frac{1}{\Lambda^{d-1}} \int_{[0,\Lambda]^{d-1}} f(y) e^{iy\cdot k}\, dy, \tag{14}$$

for every wave number $k \in \frac{2\pi}{\Lambda}\mathbb{Z}^{d-1}$, assuming a periodic box of side length Λ. Under $\mathcal{F}$, equation (10) transforms to the forth-order ordinary differential equation

$$|k|^4 \mathcal{F}w - 2|k|^2 \frac{d^2}{dz^2}\mathcal{F}w + \frac{d^4}{dz^4}\mathcal{F}w \;=\; |k|^2 \mathcal{F}T, \tag{15}$$

with $\mathcal{F}w = \frac{d}{dz}\mathcal{F}w = 0$ for $z \in \{0, H\}$. For convenience, we neglect time dependence and time averages during this proof. Obviously, the same arguments can be performed when carrying the time average from line to line. We then need to prove:

$$\begin{aligned} &\int_0^H \left(|k|^6 |\mathcal{F}w|^2 + |k|^4 |\frac{d}{dz}\mathcal{F}w|^2 + |k|^2 |\frac{d^2}{dz^2}\mathcal{F}w|^2 + |\frac{d^3}{dz^3}\mathcal{F}w|^2 \right) dz \\ &\quad\lesssim\; \int_0^H |k|^2 |\mathcal{F}T|^2 \, dz + \frac{1}{H^4} \int_0^H |\frac{d}{dz}\mathcal{F}w|^2 \, dz. \end{aligned} \tag{16}$$

The control of the terms up to second order follows immediately from testing equation (15) with $|k|^2 \overline{\mathcal{F}w}$:

$$\int_0^H \left(|k|^6 |\mathcal{F}w|^2 + |k|^4 |\frac{d}{dz}\mathcal{F}w|^2 + |k|^2 |\frac{d^2}{dz^2}\mathcal{F}w|^2 \right) dz \;\lesssim\; \int_0^H |k|^2 |\mathcal{F}T|^2 \, dz. \tag{17}$$

Observe that, as an easy consequence of (17) and the triangle inequality, also the forth-order term $\frac{1}{|k|}\frac{d^4}{dz^4}\mathcal{F}w = |k|\mathcal{F}T - |k|^3 \mathcal{F}w + 2|k|\frac{d^2}{dz^2}\mathcal{F}w$ is bounded:

$$\int_0^H \frac{1}{|k|^2} |\frac{d^4}{dz^4}\mathcal{F}w|^2 \, dz \;\lesssim\; \int_0^H |k|^2 |\mathcal{F}T|^2 \, dz. \tag{18}$$

We now show that we also control the intermediate third-order term

$$\int_0^H |\frac{d^3}{dz^3}\mathcal{F}w|^2\,dz \;\lesssim\; \int_0^H |k|^2|\mathcal{F}T|^2\,dz + \frac{1}{H^4}\int_0^H |\frac{d}{dz}\mathcal{F}w|^2\,dz. \tag{19}$$

In the case of large wave numbers, in the sense of $H|k| \gtrsim 1$, this follows via the interpolation estimate

$$\int_0^H |\frac{d^3}{dz^3}\mathcal{F}w|^2\,dz \;\lesssim\; \int_0^H |k|^2|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz + \int_0^H \frac{1}{|k|^2}|\frac{d^4}{dz^4}\mathcal{F}w|^2\,dz,$$

from of (17) and (18). Rescaling $z = \frac{1}{|k|}\hat{z}$, and accordingly $H = \frac{1}{|k|}\hat{H}$, the above interpolation estimate is equivalent to

$$\int_0^{\hat{H}} |\frac{d^3}{d\hat{z}^3}\mathcal{F}w|^2\,d\hat{z} \;\lesssim\; \int_0^{\hat{H}} |\frac{d^2}{d\hat{z}^2}\mathcal{F}w|^2\,d\hat{z} + \int_0^{\hat{H}} |\frac{d^4}{d\hat{z}^4}\mathcal{F}w|^2\,d\hat{z},$$

which for $\hat{H} \sim 1$ is an elementary estimate for $\frac{d^2}{d\hat{z}^2}\mathcal{F}w$. For $\hat{H} \gg 1$ we just have to decompose the integrals into intervals of the unit length and sum up.

It remains to treat the case of small wave numbers in the sense of $H|k| \lesssim 1$. In this case, we just use (18), which implies

$$H^2\int_0^H |\frac{d^4}{dz^4}\mathcal{F}w|^2\,dz \;\lesssim\; \int_0^H |k|^2|\mathcal{F}T|^2\,dz. \tag{20}$$

We apply successively Poincaré's inequality to infer from (20) the estimates

$$\int_0^H |\frac{d^3}{dz^3}\mathcal{F}w - \frac{d^3}{dz^3}\mathcal{F}w(0)|^2\,dz \;\lesssim\; \int_0^H |k|^2|\mathcal{F}T|^2\,dz \tag{21}$$

and

$$\frac{1}{H^4}\int_0^H |\frac{d}{dz}\mathcal{F}w - (\frac{d^3}{dz^3}\mathcal{F}w(0)\frac{1}{2}z^2 + \frac{d^2}{dz^2}\mathcal{F}w(0)z)|^2\,dz \;\lesssim\; \int_0^H |k|^2|\mathcal{F}T|^2\,dz. \tag{22}$$

We now observe that

$$|\frac{d^3}{dz^3}\mathcal{F}w(0)|^2H^4 + |\frac{d^2}{dz^2}\mathcal{F}w(0)|^2H^2 \;\lesssim\; \frac{1}{H}\int_0^H |\frac{d^3}{dz^3}\mathcal{F}w(0)\frac{1}{2}z^2 + \frac{d^2}{dz^2}\mathcal{F}w(0)z|^2\,dz. \tag{23}$$

This "inverse" estimate is a consequence of the equivalence of norms in finite dimensions, in the sense of

$$\left(a^2+b^2\right)^{1/2} \;\lesssim\; \left(\int_0^1 (a\frac{1}{2}\hat{z}^2 + b\hat{z})^2\,d\hat{z}\right)^{1/2},$$

and of rescaling $z = H\hat{z}$. We use (23) to control the third-order derivative of $\mathcal{F}w$ on the boundary as follows:

$$
\begin{aligned}
&|\frac{d^3}{dz^3}\mathcal{F}w(0)|^2 H \\
&\overset{(23)}{\lesssim} \frac{1}{H^4}\int_0^H |\frac{d}{dz}\mathcal{F}w - (\frac{d^3}{dz^3}\mathcal{F}w(0)\frac{1}{2}z^2 + \frac{d^2}{dz^2}\mathcal{F}w(0)z)|^2\,dz + \frac{1}{H^4}\int_0^H |\frac{d}{dz}\mathcal{F}w|^2\,dz \\
&\overset{(22)}{\lesssim} \int_0^H |k|^2|\mathcal{F}T|^2\,dz + \frac{1}{H^4}\int_0^H |\frac{d}{dz}\mathcal{F}w|^2\,dz.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\int_0^H |\frac{d^3}{dz^3}\mathcal{F}w|^2\,dz &\lesssim \int_0^H |\frac{d^3}{dz^3}\mathcal{F}w - \frac{d^3}{dz^3}\mathcal{F}w(0)|^2\,dz + |\frac{d^3}{dz^3}\mathcal{F}w(0)|^2 H \\
&\overset{(21)}{\lesssim} \int_0^H |k|^2|\mathcal{F}T|^2\,dz + \frac{1}{H^4}\int_0^H |\frac{d}{dz}\mathcal{F}w|^2\,dz.
\end{aligned}
$$

This concludes the proof of (19). □

2.2 Upper bound using the background field method

In this subsection, we derive the upper bound on the Nusselt number, Theorem 1, using the background field method. First of all, let us roughly explain what the background field method is.

We decompose the temperature field T into a steady, horizontally uniform *background temperature field profile* τ, satisfying the imposed temperature boundary conditions, cf. (4),

$$\tau = 1 \text{ at } z = 0, \quad \text{and} \quad \tau = 0 \text{ at } z = H,$$

and into the temperature *fluctuations* θ around this profile, satisfying homogeneous boundary conditions

$$\theta = 0 \text{ at } z = 0, H. \tag{24}$$

Thus

$$T(t,y,z) = \tau(z) + \theta(t,y,z).$$

With this decomposition, the heat equation (1) reads

$$\partial_t\theta + u\cdot\nabla\theta - \Delta\theta = \tau'' - w\tau',$$

where w denotes the vertical velocity $w = v\cdot e$. Recall that w is determined by a forth-order boundary value problem:

$$\Delta^2 w = -\Delta_y\theta \quad \text{and} \quad w = \partial_z w = 0 \quad \text{for } z \in \{0, H\}. \tag{25}$$

It turns out that the Nusselt number (9) can be written as

$$Nu = \int_0^H (\tau')^2 \, dz - \int_0^H \langle |\nabla\theta|^2 + 2\tau'\theta w\rangle \, dz, \tag{26}$$

cf. [4, eq. (2.31)]. The background field method now produces an upper bound on Nu by the following variational principle: Construct a background field τ that satisfies the positivity condition

$$\mathcal{Q}_\tau[\theta] \ := \ \int_0^H \langle |\nabla\theta|^2 + 2\tau'\theta w\rangle \, dz \geq 0 \quad \text{for all } \theta \text{ with (24).} \tag{27}$$

Obviously, in view of (26) and (27), the Dirichlet integral of such a τ is an upper bound on the Nusselt number, i.e.,

$$Nu \leq \int_0^H (\tau')^2 \, dz.$$

Finding the optimal upper bound on Nu in terms of the background field method corresponds therefore to a *saddle point problem* in the variables (τ, θ):

$$Nu \ \leq \ \inf_{\tau \text{ with (27)}} \ \sup_{\theta \text{ with (24)}} \left(\int_0^H (\tau')^2 \, dz - \mathcal{Q}_\tau[\theta] \right).$$

In the literature, the background field method is often interpreted as a *mathematically rigorous version of the marginally stable boundary layer theory*, cf. [5, p. 235-236]. This theory produces heuristically the Nusselt number scaling $Nu \sim 1$ by investigating the stability condition for a purely conducting boundary layer. In consideration of this interpretation, one often refers to the positivity condition (27) as a *stability constraint*: If τ is a steady conduction solution, then (27) is precisely the nonlinear energy stability constraint of the system, which is a sufficient condition for absolute stability, cf. [19]. Combining this condition with the sufficient condition for instabilities in the linear theory, it is possible to determine a unique critical value of H up to which the purely conducting state is stable. Therefore, in the language of stability theory, the Dirichlet integral of an nonlinearly stable steady temperature profile yields an upper bound on the Nusselt number.

We now turn to the proof of Theorem 1, which can be considered as a refinement of the one of [5]. The background temperature field profile introduced in [5] is

$$\tau(z) \ = \ \begin{cases} 1 - \frac{z}{\delta} & \text{for } 0 \leq z \leq \delta, \\ \frac{1}{2} + \lambda(\delta) \ln\left(\frac{z}{H-z}\right) & \text{for } \delta \leq z \leq H - \delta, \\ \frac{1}{\delta}(H - z) & \text{for } H - \delta \leq z \leq H. \end{cases} \tag{28}$$

The value of $\delta \ll H$ has to be chosen such that (27) holds. Furthermore, $\lambda(\delta)$ is the normalizing constant, ensuring continuity of τ,

$$\lambda(\delta) = \frac{1}{2\ln\left(\frac{H-\delta}{\delta}\right)} \overset{\delta \ll H}{\ll} 1. \tag{29}$$

The main step in the proof of Theorem 1 is the following:

Proposition 1. *Let τ be defined by (28) and (29) with*

$$\frac{1}{H} \ll \delta \ll \ln^{-1/15} H. \tag{30}$$

Let θ and w be periodic in y and satisfy (25) with $\theta = 0$ for $z \in \{0, H\}$. Then the positivity condition is satisfied:

$$\int_0^H \langle |\nabla\theta|^2 + 2\tau' w\theta\rangle \, dz \geq 0. \tag{31}$$

The assertion of Theorem 1 follows immediately:

Proof of Theorem 1. We chose the background field τ as defined in (28) such that (30) is satisfied. Now, Proposition 1 yields

$$\mathcal{Q}_\tau[\theta] \overset{(31)}{\geq} 0,$$

and thus

$$Nu \overset{(26)}{\leq} \int_0^H (\tau')^2 \, dz \lesssim \frac{1}{\delta}.$$

Since by (30), δ may be chosen as a small multiple of $\ln^{-1/15} H$, $\frac{1}{\delta}$ is estimated by a large multiple of $\ln^{1/15} H$, which yields Theorem 1. $\square$

Besides the particular choice of τ, the proof of Proposition 1 relies on the above mentioned ingredients Lemma 1 & 3.

Proof of Proposition 1. Following [5] and appealing to Ansatz (28) and definition (27), we write

$$\mathcal{Q}_\tau[\theta] = \mathcal{Q}_{lower}[\theta] + \mathcal{Q}_{upper}[\theta],$$

where

$$\mathcal{Q}_{lower}[\theta] := \frac{1}{2}\int_0^H \langle |\nabla\theta|^2\rangle \, dz + 2\lambda \int_0^H \frac{\langle\theta w\rangle}{z} \, dz - 2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{H-z}\right)\langle\theta w\rangle \, dz,$$

and

$$\mathcal{Q}_{upper}[\theta] := \frac{1}{2}\int_0^H \langle |\nabla\theta|^2\rangle \, dz + 2\lambda \int_0^H \frac{\langle\theta w\rangle}{H-z} \, dz - 2\int_{H-\delta}^H \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{H-z}\right)\langle\theta w\rangle \, dz.$$

For symmetry reasons, it is sufficient to consider $\mathcal{Q}_{lower}[\theta]$ only. Let us first show that

$$\sup_{0\leq z\leq \lambda^{-1/3}} \langle (\partial_z^2 w)^2\rangle \lesssim \lambda^{-1/3}\left(\frac{1}{2}\int_0^H \langle |\nabla\theta|^2\rangle \, dz + 2\lambda\int_0^H \frac{\langle\theta w\rangle}{z} \, dz\right). \tag{32}$$

For this purpose, observe that for $0 \leq L \leq H$, there holds the inequality

$$\sup_{0\leq z\leq L} \langle (\partial_z^2 w)^2\rangle \lesssim L\int_0^L \langle (\partial_z^3 w)^2\rangle \, dz + \frac{1}{L^3}\int_0^L \langle (\partial_z w)^2\rangle \, dz, \tag{33}$$

which in turn is an easy consequence of the elementary inequality

$$\begin{aligned}\sup_{0\le z\le L} f^2 &\lesssim \int_0^L |\partial_z f^2|\,dz + \frac{1}{L}\int_0^L f^2\,dz \\ &\lesssim \left(\int_0^L f^2\,dz \int_0^L (\partial_z f)^2\,dz\right)^{1/2} + \frac{1}{L}\int_0^L f^2\,dz,\end{aligned}$$

when choosing $f = \partial_z^2 w$ and $f = \partial_z w$ successively and then averaging in y and t. From (33) we deduce with the help of Lemma 1 & 3 with T replaced by θ that

$$\begin{aligned}\sup_{0\le z\le L} \langle(\partial_z^2 w)^2\rangle &\overset{(33),(13)}{\lesssim} L\int_0^H \langle|\nabla_y\theta|^2\rangle\,dz + \frac{L}{H^4}\int_0^H \langle(\partial_z w)^2\rangle\,dz + \frac{1}{L^3}\int_0^L \langle(\partial_z w)^2\rangle\,dz \\ &\le L\int_0^H \langle|\nabla_y\theta|^2\rangle\,dz + \left(\frac{L}{H^3}+\frac{1}{L^2}\right)\int_0^H \frac{\langle(\partial_z w)^2\rangle}{z}\,dz \\ &\overset{(11),\,L\le H}{\lesssim} L\int_0^H \langle|\nabla\theta|^2\rangle\,dz + \frac{1}{\lambda L^2}\lambda\int_0^H \frac{\langle\theta w\rangle}{z}\,dz.\end{aligned}$$

We equilibrate both terms by choosing $L \sim \lambda^{-1/3}$. This implies (32).

It remains to estimate the indefinite term of $\mathcal{Q}_{lower}[\theta]$. This can be done quite crudely:

$$\begin{aligned}&\left|2\int_0^\delta \left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{H-z}\right)\langle\theta w\rangle\,dz\right| \\ &\quad\lesssim \sup_{0\le z\le\delta}\frac{\langle\theta^2\rangle^{1/2}}{z^{1/2}} \sup_{0\le z\le\delta}\frac{\langle w^2\rangle^{1/2}}{z^2}\int_0^\delta\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{H-z}\right)z^{5/2}\,dz.\end{aligned}$$

In view of the homogeneous boundary conditions of θ and w, one easily checks the following Poincaré-type estimates. For $0\le z\le\delta$ we have

$$\sup_{0\le z\le\delta}\frac{\langle\theta^2\rangle}{z} \lesssim \int_0^\delta \langle(\partial_z\theta)^2\rangle\,dz,$$

and

$$\sup_{0\le z\le\delta}\frac{\langle w^2\rangle}{z^4} \lesssim \sup_{0\le z\le\delta}\langle(\partial_z^2 w)^2\rangle.$$

Moreover,

$$\int_0^\delta\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{H-z}\right)z^{5/2}\,dz \lesssim \delta^{5/2},$$

provided that $\lambda \ll 1$, so that

$$\left|2\int_0^\delta\left(\frac{1}{\delta}+\frac{\lambda}{z}+\frac{\lambda}{H-z}\right)\langle\theta w\rangle\,dz\right| \lesssim \delta^{5/2}\left(\int_0^\delta\langle|\nabla\theta|^2\rangle\,dz\right)^{1/2}\sup_{0\le z\le\delta}\langle(\partial_z^2 w)^2\rangle^{1/2}. \tag{34}$$

This result, in conjunction with (32) implies a lower bound on $\mathcal{Q}_{lower}[\theta]$:

$$\begin{aligned}\mathcal{Q}_{lower}[\theta] &\overset{(34)}{\geq} \frac{1}{2}\int_0^H \langle|\nabla\theta|^2\rangle\,dz + 2\lambda\int_0^H \frac{\langle\theta w\rangle}{z}\,dz \\ &\quad - C\delta^{5/2}\left(\int_0^H \langle|\nabla\theta|^2\rangle\,dz\right)^{1/2} \sup_{0\leq z\leq\delta}\langle(\partial_z^2 w)^2\rangle^{1/2} \\ &\overset{(32)}{\geq} (1 - C\delta^{5/2}\lambda^{-1/6})\left(\frac{1}{2}\int_0^H \langle|\nabla\theta|^2\rangle\,dz + 2\lambda\int_0^H \frac{\langle\theta w\rangle}{z}\,dz\right)\end{aligned}$$

for some generic constant $C > 0$, provided that $\delta \leq \lambda^{-1/3}$. We deduce that $\mathcal{Q}_{lower}[\theta]$ is nonnegative for

$$\delta \ll \lambda^{1/15} \overset{(29)}{\sim} \ln^{-1/15}(H-\delta) + \ln^{-1/15}\frac{1}{\delta} \overset{\frac{1}{H}\ll\delta\ll H}{\sim} \ln^{-1/15} H.$$

We note that this is consistent with $\delta \leq \lambda^{-1/3}$. Thus (31) holds. □

2.3 Upper Bound using the maximum principle

Throughout this subsection, we denote by θ the temperature fluctuations around the vertical mean temperature, $\theta = T - \langle T\rangle$. As in the previous subsection, the vertical component w of velocity field is determined by a forth-order boundary value problem:

$$\Delta^2 w = -\Delta_y\theta \text{ for } z \in (0,H), \quad w = \partial_z w = 0 \text{ for } z \in \{0,H\}. \tag{35}$$

The first major ingredient in proving Theorem 2 is a maximal regularity estimate for the bi-Laplacian in L^∞ and thus with a logarithmic correction in the spirit of [2].

Lemma 4. *Let w and θ be periodic in y and satisfy (35) and $\theta = 0$ for $z \in \{0,H\}$. Then we have*

$$\sup_{z\in(0,H)}\langle|\nabla^2 w|^2\rangle^{1/2} \lesssim \ln\left(\frac{H\int_0^H\langle|\nabla\theta|^2\rangle\,dz}{\langle\sup_x\theta^2\rangle} + e\right)\langle\sup_x\theta^2\rangle^{1/2}. \tag{36}$$

Due to the maximum principle for the temperature, we may assume $\langle\sup_x\theta^2\rangle \leq 1$ when we apply Lemma 4 to prove Theorem 2.

Note that compared to [2, Theorem 1], we replaced $\sup_x|\nabla^2 w|$ by the weaker $\sup_z\langle|\nabla^2 w|^2\rangle^{1/2}$, since the Dirichlet integral $\int_0^H\langle|\nabla\theta|^2\rangle\,dz$ (which is controlled by the Nusselt number) only dominates the former. Note also that it is the height H, not the possibly larger horizontal period in y, that appears in the logarithm. This will be crucial since Lemma 4 will be applied with H replaced by some $L \ll H$.

The main step in the proof of Theorem 2 is the following estimate on the Hessian $\nabla^2 w$ of w that allows to make best use of the Dirichlet boundary conditions on w, cf. (35);

Proposition 2.

$$\sup_{z\in(0,1)} \langle|\nabla^2 w|^2\rangle \;\lesssim\; Nu + \ln^2(Nu \ln^{1/2} H). \tag{37}$$

Next to the above-mentioned ingredients Lemma 1 & 4, Proposition 2 relies on the following Caccioppoli-type estimate for the bi-Laplacian with Dirichlet boundary condition:

Lemma 5. *Let w be periodic in y and satisfy the homogeneous boundary value problem*

$$\Delta^2 w \;=\; 0 \;\; \textit{for}\; z\in(0,H) \quad \textit{and} \quad w \;=\; \partial_z w \;=\; 0 \;\; \textit{for}\; z=0.$$

Then we have the inverse estimate

$$\sup_{z\in(0,H/8)} \langle|\nabla^2 w|^2\rangle \;\lesssim\; \frac{1}{H^3}\int_0^H \langle|\nabla w|^2\rangle\, dz.$$

Proof of Proposition 2. Fix an $1 \ll L \ll H$ to be chosen at the end of the proof and solve (35) with H replaced by L and with θ replaced by θ_L, the restriction of θ on $\{0 \le z \le L\}$. More precisely, we define $\theta_L = \eta_L\theta$ for some smooth function $\eta_L = \eta_L(z) \in [0,1]$, with the cut-off properties $\eta_L(z) = 1$ for $z \le L/2$, $\eta_L(z) = 0$ for $z \ge L$, and $\sup|\partial_z\eta_L| \lesssim L^{-1}$. Call the solution w_L. We note that

$$\langle\sup_x \theta_L^2\rangle \;\le\; \langle\sup_x \theta^2\rangle \;\le\; 1,$$

and

$$\begin{aligned}
&\left(\int_0^L \langle|\nabla\theta_L|^2\rangle\, dz\right)^{1/2}\\
&\le\; \sup_z \eta \left(\int_0^L \langle|\nabla\theta|^2\rangle\, dz\right)^{1/2} + \langle\sup_x|\theta|^2\rangle^{1/2}\left(\int_0^L (\partial_z\eta_L)^2\, dz\right)^{1/2}\\
&\overset{(9)}{\lesssim}\; Nu^{1/2} + \frac{1}{L^{1/2}}.
\end{aligned}$$

We apply (36) to w_L and with H replaced by L:

$$\sup_{z\in(0,L)} \langle|\nabla^2 w_L|^2\rangle^{1/2} \;\lesssim\; \ln(LNu + e) \;\sim\; \ln(LNu), \tag{38}$$

since $L \gg 1$, and $Nu \ge 1$.

We now consider $\delta w_L := w - w_L$ and note that by definition of w_L it satisfies the homogeneous boundary value problem

$$\Delta^2 \delta w_L \;=\; 0 \;\; \text{for}\; z\in(0,L/2) \quad \text{and} \quad \delta w_L \;=\; \partial_z\delta w_L \;=\; 0 \;\; \text{for}\; z=0.$$

The application of Lemma 5 with H replaced by $L/2$ and w replaced by δw_L thus gives the inverse estimate

$$\sup_{z\in(0,L/16)}\langle|\nabla^2\delta w_L|^2\rangle \;\lesssim\; \frac{1}{L^3}\int_0^{L/2}\langle|\nabla\delta w_L|^2\rangle\,dz. \tag{39}$$

This is helpful since, on the one hand, (11) and (12) imply

$$\frac{1}{L^3}\int_0^L\langle|\nabla w|^2\rangle\,dz \;\lesssim\; \frac{1}{L^2}(\ln H)Nu.$$

On the other hand, because of $w_L=\partial_z w_L=0$ for $z\in\{0,L\}$, we have

$$\frac{1}{L^3}\int_0^L\langle|\nabla w_L|^2\rangle\,dz \;\lesssim\; \sup_{z\in(0,L)}\langle|\nabla^2 w_L|^2\rangle,$$

so that (38) yields an estimate of the same quantity for w_L:

$$\frac{1}{L^3}\int_0^L\langle|\nabla w_L|^2\rangle\,dz \;\lesssim\; \ln^2(LNu).$$

The combination of both estimates yields by the triangle inequality

$$\frac{1}{L^3}\int_0^L\langle|\nabla\delta w_L|^2\rangle\,dz \;\lesssim\; \frac{1}{L^2}(\ln H)Nu+\ln^2(LNu).$$

Because of (39), this upgrades to

$$\sup_{z\in(0,L/16)}\langle|\nabla^2\delta w_L|^2\rangle \;\lesssim\; \frac{1}{L^2}(\ln H)Nu+\ln^2(LNu).$$

The combination with (38) yields by the triangle inequality

$$\sup_{z\in(0,L/16)}\langle|\nabla^2 w|^2\rangle \;\lesssim\; \frac{1}{L^2}(\ln H)Nu+\ln^2(LNu).$$

Choosing $L=\ln^{1/2}H\gg 1$ yields (37). □

Proof of Theorem 2. Let $0<\delta\ll 1$ be arbitrary; to be fixed at the end of the proof. From (8) and the boundary conditions on T and w, as well as the maximum principle on T in the sense of $\langle T\rangle\geq 0$ and $\langle\sup_x\theta^2\rangle\leq 1$, we infer the representation and inequality

$$\begin{aligned}
Nu &= \frac{1}{\delta}\left(\int_0^\delta\langle\theta w\rangle\,dz+1-\langle T_{|z=\delta}\rangle\right)\\
&\leq \sup_{z\in(0,\delta)}\langle w^2\rangle^{1/2}+\frac{1}{\delta}\\
&\leq \delta^2\sup_{z\in(0,1)}\langle(\partial_z^2 w)^2\rangle^{1/2}+\frac{1}{\delta}.
\end{aligned}$$

We now may insert (37) to obtain

$$Nu \;\lesssim\; \delta^2(Nu^{1/2}+\ln(Nu\ln^{1/2}H))+\frac{1}{\delta}.$$

By Young's inequality, this estimate simplifies to

$$Nu \;\lesssim\; \delta^4+\delta^2\ln(Nu\ln^{1/2}H)+\frac{1}{\delta} \;\overset{\delta\ll 1}{\lesssim}\; \delta^2\ln(Nu\ln^{1/2}H)+\frac{1}{\delta}.$$

The choice of $\delta=\ln^{-1/3}(Nu\ln^{1/2}H)$ (which is consistent with $\delta\ll 1$ because of $Nu\geq 1$, $H\gg 1$) yields

$$Nu \;\lesssim\; \ln^{1/3}(Nu\ln^{1/2}H).$$

We rewrite this implicit estimate as

$$(Nu\ln^{1/2}H)\ln^{-1/3}(Nu\ln^{1/2}H) \;\lesssim\; \ln^{1/2}H,$$

to see that it implies as desired the explicit estimate:

$$Nu\ln^{1/2}H \;\lesssim\; (\ln^{1/2}H)(\ln^{1/3}\ln^{1/2}H) \;\sim\; (\ln^{1/2}H)(\ln^{1/3}\ln H).$$

□

Proof of Lemma 5. This is a standard argument that we display for the convenience of the reader. By rescaling we may w. l. o. g. assume that $H=1$. We prove the result on the Fourier level, cf. (14), and neglect the time average as in the proof of Lemma 3. This means, for any $k\in\frac{2\pi}{\Lambda}\mathbb{Z}^{d-1}$, we have to deduce the estimate

$$\sup_{z\in(0,1/8)}\left(|k|^4|\mathcal{F}w|^2+|k|^2|\frac{d}{dz}\mathcal{F}w|^2+|\frac{d^2}{dz^2}\mathcal{F}w|^2\right) \lesssim \int_0^1|k|^2|\mathcal{F}w|^2\,dz+\int_0^1|\frac{d}{dz}\mathcal{F}w|^2\,dz \tag{40}$$

from the ordinary differential equation

$$|k|^4\mathcal{F}w-2|k|^2\frac{d^2}{dz^2}\mathcal{F}w+\frac{d^4}{dz^4}\mathcal{F}w \;=\; 0,\quad \mathcal{F}w=\frac{d}{dz}\mathcal{F}w=0 \text{ for } z=0. \tag{41}$$

The first step consists in establishing the Caccioppoli-type estimates

$$\int_0^{1/2}|k|^4|\mathcal{F}w|^2\,dz+\int_0^{1/2}|k|^2|\frac{d}{dz}\mathcal{F}w|^2\,dz+\int_0^{1/2}|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \lesssim \int_0^1|k|^2|\mathcal{F}w|^2\,dz+\int_0^1|\frac{d}{dz}\mathcal{F}w|^2\,dz. \tag{42}$$

and

$$\int_0^{1/8} |k|^8|\mathcal{F}w|^2\,dz + \int_0^{1/8} |k|^6|\frac{d}{dz}\mathcal{F}w|^2\,dz + \int_0^{1/8} |k|^4|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz$$
$$\lesssim \int_0^1 |k|^2|\mathcal{F}w|^2\,dz + \int_0^1 |\frac{d}{dz}\mathcal{F}w|^2\,dz. \qquad (43)$$

For this purpose, we select a universal smooth function $\eta(z)$ with the following cut-off properties

$$\eta \;=\; 1 \;\text{ for } z \le L \quad \text{and} \quad \eta \;=\; 0 \;\text{ for } z \ge 2L$$

for some $0 < L \ll 1$. Testing (41) with $\eta^2\overline{\mathcal{F}w}$ we obtain by integration by parts since $\eta^2\mathcal{F}w$ vanishes to first order at $z \in \{0,1\}$:

$$\begin{aligned}
&\int_0^1 \eta^2|k|^4|\mathcal{F}w|^2\,dz + 2\int_0^1 \eta^2|k|^2|\frac{d}{dz}\mathcal{F}w|^2\,dz + \int_0^1 \eta^2|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \\
=\;& -4\int_0^1 \eta\eta'|k|^2\overline{\mathcal{F}w}\frac{d}{dz}\mathcal{F}w\,dz - 4\int_0^1 \eta\eta'\overline{\frac{d}{dz}\mathcal{F}w}\frac{d^2}{dz^2}\mathcal{F}w\,dz \\
& - 2\int_0^1 \eta\eta''\overline{\mathcal{F}w}\frac{d^2}{dz^2}\mathcal{F}w\,dz + 4\int_0^1 \eta'\eta''\overline{\mathcal{F}w}\frac{d}{dz}\mathcal{F}w\,dz \\
& + 2\int_0^1 (\eta')^2|\frac{d}{dz}\mathcal{F}w|^2\,dz.
\end{aligned}$$

Invoking successively the Cauchy-Schwarz and Young's inequality, the definition of η, and Poincaré's inequality, this yields

$$\begin{aligned}
&\int_0^L |k|^4|\mathcal{F}w|^2\,dz + \int_0^L |k|^2|\frac{d}{dz}\mathcal{F}w|^2\,dz + \int_0^L |\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \\
\lesssim\;& \frac{1}{L^2}\int_0^{2L} |k|^2|\mathcal{F}w|^2\,dz + \frac{1}{L^2}\int_0^{2L} |\frac{d}{dz}\mathcal{F}w|^2\,dz + \frac{1}{L^4}\int_0^{2L} |\mathcal{F}w|^2\,dz \\
\lesssim\;& \frac{1}{L^2}\int_0^{2L} |k|^2|\mathcal{F}w|^2\,dz + \frac{1}{L^2}\int_0^{2L} |\frac{d}{dz}\mathcal{F}w|^2\,dz.
\end{aligned}$$

The choice $L = 1/2$ yields (42). On the other hand, this estimate can be applied recursively to establish

$$\begin{aligned}
&\int_0^L |k|^8|\mathcal{F}w|^2\,dz + \int_0^L |k|^6|\frac{d}{dz}\mathcal{F}w|^2\,dz + \int_0^L |k|^4|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \\
\lesssim\;& \frac{1}{L^2}\int_0^{2L} |k|^6|\mathcal{F}w|^2\,dz + \frac{1}{L^2}\int_0^{2L} |k|^4|\frac{d}{dz}\mathcal{F}w|^2\,dz \\
\lesssim\;& \frac{1}{L^6}\int_0^{8L} |k|^2|\mathcal{F}w|^2\,dz + \frac{1}{L^6}\int_0^{8L} |\frac{d}{dz}\mathcal{F}w|^2\,dz.
\end{aligned}$$

We obtain (43) when choosing $L = 1/8$. We use equation (41) and the triangle inequality to deduce from (43) that also the forth-order derivatives are bounded:

$$\int_0^{1/8} |\frac{d^4}{dz^4}\mathcal{F}w|^2\,dz \;\lesssim\; \int_0^1 |k|^2|\mathcal{F}w|^2\,dz + \int_0^1 |\frac{d}{dz}\mathcal{F}w|^2\,dz. \qquad (44)$$

We now argue how (42), (43), and (44) imply (40). Thanks to the homogeneous boundary conditions of $\mathcal{F}w$, we have

$$\sup_{z\in(0,1/8)} |k|^4|\mathcal{F}w|^2 \lesssim \int_0^{1/8} |k|^4|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \overset{(43)}{\lesssim} \int_0^1 |k|^2|\mathcal{F}w|^2\,dz + \int_0^1 |\frac{d}{dz}\mathcal{F}w|^2\,dz$$

and

$$\begin{aligned}\sup_{z\in(0,1/8)} |k|^2|\frac{d}{dz}\mathcal{F}w|^2 &\lesssim \int_0^{1/8} |k|^2|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \\ &\lesssim \int_0^{1/8} |k|^4|\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz + \int_0^{1/8} |\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz \\ &\overset{(42)\&(43)}{\lesssim} \int_0^1 |k|^2|\mathcal{F}w|^2\,dz + \int_0^1 |\frac{d}{dz}\mathcal{F}w|^2\,dz.\end{aligned}$$

It thus remains to argue that

$$\sup_{z\in(0,1/8)} |\frac{d^2}{dz^2}\mathcal{F}w|^2 \lesssim \int_0^{1/8} |\frac{d^4}{dz^4}\mathcal{F}w|^2\,dz + \int_0^{1/8} |\frac{d^2}{dz^2}\mathcal{F}w|^2\,dz,$$

from which the desired bound follows via (42) and (44). Introducing $f := \frac{d^2}{dz^2}\mathcal{F}w$ we see that the above estimate reduces to two standard estimates, an interpolation estimate and a trace estimate:

$$\begin{aligned}\int_0^{1/8} |\frac{d}{dz}f|^2\,dz &\lesssim \int_0^{1/8} |\frac{d^2}{dz^2}f|^2 + |f|^2\,dz, \\ \sup_{z\in(0,1/8)} |f|^2 &\lesssim \int_0^{1/8} |\frac{d}{dz}f|^2 + |f|^2\,dz.\end{aligned}$$

These two statements are as well-known as they are elementary to prove. □

Proof of Lemma 4. We start with two reductions. We first argue that it is sufficient to prove the result under the additional assumption that the typical horizontal wave length is smaller than the height H, a condition we express on the level of horizontal Fourier transform, cf. (14):

$$\mathcal{F}w \;=\; 0 \quad \text{for } H|k| \le 1, \tag{45}$$

where k denotes the dual variable of y. To this purpose, we construct a suitable projection: We select a Schwartz function $\phi(\hat{y})$ such that its Fourier transform $\mathcal{F}\phi(\hat{k}) \in [0,1]$ satisfies

$$\mathcal{F}\phi = 1 \;\text{ for } |\hat{k}| \le 1 \quad\text{and}\quad \mathcal{F}\phi = 0 \;\text{ for } |\hat{k}| \ge 2. \tag{46}$$

We rescale according to $\phi_H(y) := H^{-(d-1)}\phi(\frac{y}{H})$, or on the Fourier level $\mathcal{F}\phi_H(k) = \mathcal{F}\phi(Hk)$. We introduce the long wave-length part $(\delta w_H, \delta\theta_H)$ and short wave-length part (w_H, θ_H) of the velocity/temperature pair (w, θ) by using $\mathcal{F}\phi_H$ as a Fourier multiplier:

$$\begin{aligned}(\mathcal{F}\delta w_H, \mathcal{F}\delta\theta_H) &:= (\mathcal{F}w\mathcal{F}\phi_H, \mathcal{F}\theta\mathcal{F}\phi_H),\\ (w_H, \theta_H) &:= (w - \delta w_H, \theta - \delta\theta_H)\\ &= (\mathcal{F}w(1-\mathcal{F}\phi_H), \mathcal{F}\theta(1-\mathcal{F}\phi_H)).\end{aligned}$$

We note that the differential equation and the boundary conditions of (w, θ) are preserved for (w_H, θ_H) and $(\delta w_H, \delta\theta_H)$. Moreover, the L^2-based control of θ is obviously preserved by the bounded Fourier multiplier:

$$\int_0^H \langle|\nabla\theta_H|^2\rangle dz + \int_0^H \langle|\nabla\delta\theta_H|^2\rangle dz \lesssim \int_0^H \langle|\nabla\theta|^2\rangle dz.$$

Because $\delta\theta_H$ is the convolution in y of θ with a function ϕ_H that satisfies $\int|\phi_H|dy = \int|\phi|d\hat{y} \lesssim 1$, also the L^∞ control is preserved for the long wave-length part

$$\langle\sup_x \delta\theta_H^2\rangle \lesssim \langle\sup_x \theta^2\rangle,$$

and thus by the triangle inequality also for the short wave-length part

$$\langle\sup_x \theta_H^2\rangle \lesssim \langle\sup_x \theta^2\rangle.$$

By the property (46) of the Fourier multiplier we gain that

$$\mathcal{F}w_H = 0 \quad \text{for } H|k| \le 1 \quad \text{and} \quad \mathcal{F}\delta w_H = 0 \quad \text{for } H|k| \ge 2. \tag{47}$$

The reduction to (45) thus follows from the triangle inequality provided we show that the long wave-length pair $(\delta w_H, \delta\theta_H)$ satisfies the same estimate as the one we want to prove for the original (w, θ). In fact, we shall show that the pair $(\delta w_H, \delta\theta_H)$ satisfies the stronger estimate

$$\sup_{z\in(0,H)} \langle|\nabla^2\delta w_H|^2\rangle \lesssim \frac{1}{H}\int_0^H \langle\delta\theta_H^2\rangle dz. \tag{48}$$

Since we got rid of the logarithm in (48), we may appeal to a rescaling to assume w. l. o. g. that $H = 1$. We may also disregard the time dependence. Since we got rid of the supremum in x, we may reformulate (48) in terms of the horizontal Fourier transform: We claim that for arbitrary, yet fixed k,

$$|k|^4 \sup_{z\in(0,1)} |\mathcal{F}\delta w|^2 + |k|^2 \sup_{z\in(0,1)} |\frac{d}{dz}\mathcal{F}\delta w|^2 + \sup_{z\in(0,1)} |\frac{d^2}{dz^2}\mathcal{F}\delta w|^2 \lesssim \int_0^1 |\mathcal{F}\delta\theta|^2 dz. \tag{49}$$

We recall that we work under the long wave-length assumption, cf. (47), that after our rescaling takes the form

$$\mathcal{F}\delta w(k) \;=\; 0 \quad \text{for all } |k| \geq 2.$$

Therefore, (49) reduces to

$$\sup_{z\in(0,1)} |\mathcal{F}\delta w|^2 + \sup_{z\in(0,1)} |\frac{d}{dz}\mathcal{F}\delta w|^2 \;\lesssim\; \int_0^1 |\mathcal{F}\delta\theta|^2 dz$$

and

$$\sup_{z\in(0,1)} |\frac{d^2}{dz^2}\mathcal{F}\delta w|^2 \;\lesssim\; |k|^2\int_0^1 |\mathcal{F}\delta\theta|^2 dz + \sup_{z\in(0,1)} |\frac{d}{dz}\mathcal{F}\delta w|^2.$$

In view of the boundary conditions $\mathcal{F}\delta w = \frac{d}{dz}\mathcal{F}\delta w = 0$ for $z \in \{0,1\}$, the first estimate follows from

$$\int_0^1 |\frac{d}{dz}\mathcal{F}\delta w|^2 dz + \int_0^1 |\frac{d^2}{dz^2}\mathcal{F}\delta w|^2 dz \;\lesssim\; \int_0^1 |\mathcal{F}\delta\theta|^2 dz.$$

This last estimate is part of the the energy estimate, i.e., the estimate we obtain testing the equation on the Fourier level, i.e.,

$$|k|^4\mathcal{F}\delta w - 2|k|^2\frac{d^2}{dz^2}\mathcal{F}\delta w + \frac{d^4}{dz^4}\mathcal{F}\delta w \;=\; |k|^2\mathcal{F}\delta\theta,$$

for $|k| \leq 2$ with $\mathcal{F}\delta w$ and using the boundary equations $\mathcal{F}\delta w = \frac{d}{dz}\mathcal{F}\delta w = 0$ for $z \in \{0,1\}$ when performing the integrations by parts in z.

On the other hand, with the help of the boundary condition $\frac{d}{dz}\mathcal{F}\delta w = 0$ for $z \in \{0,1\}$ we obtain the elementary estimate

$$\sup_{z\in(0,1)} |\frac{d^2}{dz^2}\mathcal{F}\delta w|^2 \;\lesssim\; \int_0^1 |\frac{d^3}{dz^3}\mathcal{F}\delta w|^2\, dz.$$

The r.h.s. is controlled thanks to Lemma 3 with w, θ, and H replaced by δw, $\delta\theta$, and 1, respectively, and which we apply on the Fourier level (in the sense of (16)):

$$\begin{aligned}\sup_{z\in(0,1)} |\frac{d^2}{dz^2}\mathcal{F}\delta w|^2 \;&\lesssim\; |k|^2\int_0^1 |\mathcal{F}\delta\theta|^2\, dz + \int_0^1 |\frac{d}{dz}\mathcal{F}\delta w|^2\, dz \\ &\leq\; |k|^2\int_0^1 |\mathcal{F}\delta\theta|^2 dz + \sup_{z\in(0,1)} |\frac{d}{dz}\mathcal{F}\delta w|^2.\end{aligned}$$

This concludes the argument that we may assume (45).

The next reduction consists in passing from the cumbersome no-slip boundary conditions

$$w \;=\; \partial_z w \;=\; 0 \quad \text{for } z \in \{0,H\} \tag{50}$$

to the — as we shall see — more convenient free-slip boundary conditions

$$w \;=\; \partial_z^2 w \;=\; 0 \quad \text{for } z \in \{0, H\}. \tag{51}$$

Indeed, for given θ, let w denote the solution of $\Delta^2 w = -\Delta_y \theta$ with no-slip boundary conditions (50) and $\tilde{w}$ the solution with free-slip boundary-conditions (51). In this step, we will argue that it is sufficient to establish the estimate on $\tilde{w}$:

$$\sup_{z\in(0,H)} \langle|\nabla^2\tilde{w}|^2\rangle^{1/2} \;\lesssim\; \ln\left(\frac{H\int_0^H\langle|\nabla\theta|^2\rangle\,dz}{\langle\sup_x\theta^2\rangle}+e\right)\langle\sup_x\theta^2\rangle^{1/2}. \tag{52}$$

Let us consider $\delta w := w - \tilde{w}$. In order to pass from (52) to the statement of the lemma by the triangle inequality, it is sufficient to show

$$\sup_{z\in(0,H)} \langle|\nabla^2\delta w|^2\rangle^{1/2} \;\lesssim\; \langle|\nabla^2\tilde{w}_{|z=0}|^2\rangle^{1/2} + \langle|\nabla^2\tilde{w}_{|z=H}|^2\rangle^{1/2}, \tag{53}$$

which we will do now.

In establishing (53), we may again neglect the time dependence and pass to the horizontal Fourier transform:

$$\sup_{z\in(0,H)}\left(|k|^4|\mathcal{F}\delta w|^2 + |k|^2|\frac{d}{dz}\mathcal{F}\delta w|^2 + |\frac{d^2}{dz^2}\mathcal{F}\delta w|^2\right) \;\lesssim\; \max_{z=0,H}\left(|k|^2|\frac{d}{dz}\mathcal{F}\tilde{w}|^2\right). \tag{54}$$

Note that the r. h. s. has this simple form due to the free-slip boundary conditions (51) for $\tilde{w}$, i.e., $\mathcal{F}\tilde{w} = \frac{d^2}{dz^2}\mathcal{F}\tilde{w} = 0$ for $z \in \{0, H\}$. We know that δw satisfies the homogeneous equation $\Delta^2\delta w = 0$ with inhomogeneous no-slip boundary conditions. On the Fourier level this reads

$$\begin{aligned} (|k|^2 - \tfrac{d^2}{dz^2})^2\mathcal{F}\delta w \;&=\; 0 && \text{for } z \in (0, H), \\ \mathcal{F}\delta w \;=\; 0,\ \tfrac{d}{dz}\mathcal{F}\delta w \;&=\; -\tfrac{d}{dz}\mathcal{F}\tilde{w} && \text{for } z \in \{0, H\}. \end{aligned} \tag{55}$$

We note that because of (45), $\mathcal{F}w$, and thus also $\mathcal{F}\theta$, $\mathcal{F}\tilde{w}$ and ultimately $\mathcal{F}\delta w$ vanish for $H|k| \le 1$, so that we may assume

$$H|k| \ge 1. \tag{56}$$

By the triangle inequality, we may treat the inhomogeneous boundary condition at the upper boundary $z = H$ and at the lower boundary $z = 0$ separately; by symmetry, it suffices to treat the upper boundary. By a change of variables we may assume $|k| = 1$. Hence the statement that (55) implies (54) reduces to the following statement on a complex-valued function $f(z)$: The equation and boundary condition

$$\begin{aligned} (1 - \tfrac{d^2}{dz^2})^2 f \;&=\; 0 && \text{for } z \in (0, H), \\ f \;=\; \tfrac{df}{dz} \;&=\; 0 && \text{for } z = 0, \\ f \;=\; 0,\ \tfrac{df}{dz} \;&=\; 1 && \text{for } z = H \end{aligned} \tag{57}$$

imply the estimate

$$\sup_{z\in(0,H)}\left(|f|^2+|\frac{df}{dz}|^2+|\frac{d^2f}{dz^2}|^2\right)\ \lesssim\ 1 \tag{58}$$

uniformly in $H\geq 1$. Since this estimate is obvious for $H\sim 1$ (but obviously wrong for $H\ll 1$ — therefore we needed the reduction to (45) that ensured (56)) we shall assume

$$H\ \gg\ 1. \tag{59}$$

We will now argue that (57) implies (58) in the regime (59). We recall that the space of homogeneous solution of (57) is spanned by $\cosh z$, $\sinh z$, $z\cosh z$, and $z\sinh z$. Since it is also translation and reflection invariant, the two expressions $z\sinh(H-z)$ and $(H-z)\sinh z$ are homogeneous solutions that both vanish at $z\in\{0,H\}$. Hence the solution of the boundary value problem (57) is given by

$$f\ =\ -\frac{(\sinh H)(H-z)\ \sinh z-Hz\sinh(H-z)}{(\sinh H)^2-H^2},$$

since

$$\frac{df}{dz}\ =\ -\frac{(\sinh H)((H-z)\ \cosh z-\sinh z)-H(-z\cosh(H-z)+\sinh(H-z))}{(\sinh H)^2-H^2}.$$

has the desired boundary values. We also note that

$$\frac{d^2f}{dz^2}\ =\ -\frac{(\sinh H)((H-z)\ \sinh z-2\cosh z)-H(z\sinh(H-z)-2\cosh(H-z))}{(\sinh H)^2-H^2}.$$

We observe that in the regime (59) we have

$$H\ \ll\ \sinh H\ \approx\ \cosh H,$$

so that for all $z\in[0,H]$:

$$\begin{aligned}
f\ &=\ -\frac{(H-z)\sinh z}{\sinh H}+o(1),\\
\frac{df}{dz}\ &=\ -\frac{(H-z)\cosh z-\sinh z}{\sinh H}+o(1),\\
\frac{d^2f}{dz^2}\ &=\ -\frac{(H-z)\sinh z-2\cosh z}{\sinh H}+o(1).
\end{aligned}$$

These expressions are largest in magnitude for $z\sim H$, so that

$$\begin{aligned}
f\ &=\ -(H-z)\exp(z-H)+O(1),\\
\frac{df}{dz}\ &=\ (1-H+z)\exp(z-H)+O(1),\\
\frac{d^2f}{dz^2}\ &=\ (2-H+z)\exp(z-H)+O(1).
\end{aligned}$$

This yields the desired bounds (58).

We now turn to the proof of (36) under the free-slip boundary conditions (51). By scaling, we may w. l. o. g. assume that $H=\pi$. The boundary condition (51) has the advantage that odd reflection of w and θ w. r. t. $z=0$ and $w=\pi$ preserves the equation $\Delta^2 w = -\Delta_y\theta$. Hence we may think of w and θ as (restrictions of) 2π-periodic functions in z. Because of our assumptions $\theta=0$ for $z\in\{0,H\}$, not only is the L^∞ norm of the extension of θ controlled by the L^∞ norm of the original θ, but the same applies to the (homogeneous) H^1 norm. It thus suffices to establish

$$\sup_z\langle|\nabla^2 w|^2\rangle^{1/2} \;\lesssim\; \ln\left(\frac{\int_0^{2\pi}\langle|\nabla\theta|^2\rangle\,dz}{\langle\sup_x\theta^2\rangle}+e\right)\langle\sup_x\theta^2\rangle^{1/2} \tag{60}$$

for functions w and θ that are 2π-periodic in z and have a common period in y. This allows to use Fourier series $\mathcal{F}$ in *both* z and y; we denote the dual variables by $\omega\in\mathbb{Z}$ and (as before) k, respectively. We note that in dual variables, the relation between θ and $\nabla^2 w$ is given by a 0-homogeneous Fourier multiplier:

$$\mathcal{F}\nabla^2 w \;=\; \begin{pmatrix}k\\ \omega\end{pmatrix}\otimes\begin{pmatrix}k\\ \omega\end{pmatrix}\frac{|k|^2}{(|k|^2+\omega^2)^2}\mathcal{F}\theta. \tag{61}$$

We shall mimic a Littlewood-Paley decomposition. For that purpose, we select a Schwartz function $\phi_0(x)$, $x\in\mathbb{R}^d$, with the property that its Fourier transform satisfies

$$\begin{aligned}\mathcal{F}\phi_0(k,\omega) &= 1 \text{ for } |k|^2+\omega^2\le e^{-2},\\ \mathcal{F}\phi_0(k,\omega) &= 0 \text{ for } |k|^2+\omega^2\ge 1,\\ \mathcal{F}\phi_0(k,\omega) &\in [0,1].\end{aligned} \tag{62}$$

We think of ϕ_0 as the mask of a family $\{\phi_\ell\}_{\ell\in\mathbb{Z}}$ of kernels of length scale $e^{-\ell}$:

$$\phi_\ell(x) \;=\; (e^\ell)^d\phi_0(e^\ell x). \tag{63}$$

We now claim for all $\ell\in\mathbb{N}\cup\{0\}$:

$$\sup_x|\nabla^2(\phi_\ell-\phi_{\ell+1})*w| \;\lesssim\; \sup_x|\theta|, \tag{64}$$

$$\sup_z\langle|\nabla^2(w-\phi_\ell*w)|^2\rangle \;\lesssim\; e^{-\ell}\int_0^{2\pi}\langle|\nabla\theta|^2\rangle\,dz, \tag{65}$$

$$\sup_z\langle|\nabla^2\phi_0*w|^2\rangle \;\lesssim\; \sup_z\langle\theta^2\rangle. \tag{66}$$

We start with the statement on intermediate length scales, i.e., (64). Indeed, it follows from (61) that

$$\mathcal{F}(\nabla^2(\phi_\ell-\phi_{\ell+1})*w) \;=\; \begin{pmatrix}k\\ \omega\end{pmatrix}\otimes\begin{pmatrix}k\\ \omega\end{pmatrix}\frac{|k|^2}{(|k|^2+\omega^2)^2}(\mathcal{F}\phi_\ell-\mathcal{F}\phi_{\ell+1})(\mathcal{F}\theta),$$

which in view of (63) and the 0-homogeneity of the multiplier in (61) turns into

$$\nabla^2(\phi_\ell-\phi_{\ell+1})*w \;=\; \psi_\ell*\theta, \tag{67}$$

where the (tensor-valued) convolution kernel $\psi_\ell(x)$ is given by

$$\psi_\ell(x) \;=\; (e^\ell)^d \psi_0(e^\ell x) \tag{68}$$

and

$$\mathcal{F}\psi_0 \;=\; \begin{pmatrix} k \\ \omega \end{pmatrix} \otimes \begin{pmatrix} k \\ \omega \end{pmatrix} \frac{|k|^2}{(|k|^2+\omega^2)^2} (\mathcal{F}\phi_0(k,\omega) - \mathcal{F}\phi_0(e^{-1}k, e^{-1}\omega)).$$

In view of (62), the factor $(\mathcal{F}\phi_0(k,\omega) - \mathcal{F}\phi_0(e^{-1}k, e^{-1}\omega))$ vanishes for $|k|^2+\omega^2 \;\leq\; e^{-2}$ so that $(\mathcal{F}\psi_0)(k,\omega)$ and thus $\psi_0(x)$ is (componentwise) a Schwartz function. This implies in particular

$$\int |\psi_\ell|\,dx \;\stackrel{(68)}{=}\; \int |\psi_0|\,dx \;<\; \infty,$$

so that (64) follows from (67).

We now turn to the estimate of the small length scales, i.e., (65), which we split into the three statements

$$\int_0^{2\pi} \langle |\nabla^3 w|^2 \rangle\,dz \;\;\leq\;\; \int_0^{2\pi} \langle |\nabla\theta|^2 \rangle\,dz, \tag{69}$$

$$\int_0^{2\pi} \langle |\nabla^3 (w - \phi_\ell * w)|^2 \rangle\,dz \;\;\leq\;\; \int_0^{2\pi} \langle |\nabla^3 w|^2 \rangle\,dz, \tag{70}$$

$$\sup_z \langle |\nabla^2 (w - \phi_\ell * w)|^2 \rangle \;\;\lesssim\;\; e^{-\ell} \int_0^{2\pi} \langle |\nabla^3 (w - \phi_\ell * w)|^2 \rangle\,dz. \tag{71}$$

Inequality (69) follows immediately from the Fourier space representation (61). Inequality (70) also is a straightforward consequence from the Fourier space representation and (62) in form of $\mathcal{F}\phi_\ell \in [0,1]$. We now turn to (71). In view of (62) & (63), we have $\mathcal{F}(w - \phi_\ell * w)(k,\omega) = (1 - \mathcal{F}\phi_0)(e^{-\ell}k, e^{-\ell}\omega)(\mathcal{F}w)(k,\omega) = 0$ for $|k|^2 + \omega^2 \leq e^{2(\ell-1)}$. Treating the dual variable k to y as a parameter and neglecting the time dependence, (71) reduces to the following statement for a 2π-periodic (complex valued) function $f(z)$: It holds

$$\sup_z |f|^2 \;\lesssim\; e^{-\ell} \int_0^{2\pi} (|k|^2 |f|^2 + |\frac{df}{dz}|^2)\,dz \tag{72}$$

provided

$$(\mathcal{F}f)(\omega) \;=\; 0 \quad \text{for } |k|^2 + \omega^2 \leq e^{2(\ell-1)}. \tag{73}$$

In order to show how (73) implies (72), we distinguish the two cases $|k| \geq \frac{1}{\sqrt{2}} e^{\ell-1}$ and $|k| \leq \frac{1}{\sqrt{2}} e^{\ell-1}$.

We first treat the case $|k| \geq \frac{1}{\sqrt{2}} e^{\ell-1}$. In this case, (72) reduces to

$$\sup_z |f|^2 \;\lesssim\; \int_0^{2\pi} (e^\ell |f|^2 + e^{-\ell} |\frac{df}{dz}|^2)\,dz,$$

which because of $e^\ell \ge 1$ ($\ell \ge 0$) follows from the elementary estimate

$$\sup_z |f|^2 \;\lesssim\; \int_0^{2\pi} |f|^2\,dz + \left(\int_0^{2\pi} |f|^2\,dz \int_0^{2\pi} |\frac{df}{dz}|^2\,dz\right)^{1/2}.$$

We now turn to the case of $|k| \le \frac{1}{\sqrt{2}} e^{\ell-1}$ in which case (73) implies

$$(\mathcal{F}f)(\omega) \;=\; 0 \quad \text{for } |\omega| \le \frac{1}{\sqrt{2}} e^{\ell-1}.$$

Therefore we have as desired

$$\begin{aligned}
\sup_z |f|^2 \;&\lesssim\; \left(\sum_{\omega\in\mathbb{Z}, |\omega|\ge\frac{1}{\sqrt{2}}e^{\ell-1}} |\mathcal{F}f|\right)^2 \\
&\le\; \left(\sum_{\omega\in\mathbb{Z}} \omega^2 |\mathcal{F}f|^2\right)\left(\sum_{\omega\in\mathbb{Z}, |\omega|\ge\frac{1}{\sqrt{2}}e^{\ell-1}} \frac{1}{\omega^2}\right) \\
&\lesssim\; e^{-\ell} \sum_{\omega\in\mathbb{Z}} \omega^2 |\mathcal{F}f|^2 \\
&\sim\; e^{-\ell} \int_0^{2\pi} |\frac{df}{dz}|^2\,dz.
\end{aligned}$$

This concludes the proof of (65).

We finally address the long-range estimate (66). We infer from (62) that $\mathcal{F}(\phi_0 * w)(k,\omega) = 0$ for $|k|^2 + \omega^2 \ge 1$. Since $\omega \in \mathbb{Z}$, the only surviving mode is $\omega = 0$. Hence $\phi_0 * w$ does not depend on z so that as desired

$$\begin{aligned}
\sup_z \langle |\nabla^2 \phi_0 * w|^2 \rangle \;&\lesssim\; \int_0^{2\pi} \langle |\nabla^2 \phi_0 * w|^2 \rangle\,dz \\
&\overset{(62)}{\le}\; \int_0^{2\pi} \langle |\nabla^2 w|^2 \rangle\,dz \\
&\overset{(61)}{\le}\; \int_0^{2\pi} \langle \theta^2 \rangle\,dz \\
&\lesssim\; \sup_z \langle \theta^2 \rangle.
\end{aligned}$$

To conclude, we argue how (64) & (65) & (66) implies (60). Indeed, we have for any

$N \in \mathbb{N}$ by the triangle inequality

$$
\begin{aligned}
&\sup_z \langle |\nabla^2 w|^2\rangle^{1/2} \\
&\le \quad \sup_z \langle |\nabla^2 (w - \phi_N * w)|^2\rangle^{1/2} + \sum_{\ell=0}^{N-1} \sup_z \langle |\nabla^2 (\phi_\ell - \phi_{\ell+1}) * w|^2\rangle^{1/2} \\
&\qquad + \sup_z \langle |\nabla^2 \phi_0 * w|^2\rangle^{1/2} \\
&\le \quad \sup_z \langle |\nabla^2 (w - \phi_N * w)|^2\rangle^{1/2} + \sum_{\ell=0}^{N-1} \langle \sup_x |\nabla^2 (\phi_\ell - \phi_{\ell+1}) * w|^2\rangle^{1/2} \\
&\qquad + \sup_z \langle |\nabla^2 \phi_0 * w|^2\rangle^{1/2} \\
&\overset{(64),(65),(66)}{\lesssim} \quad \left(e^{-N} \int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz \right)^{1/2} + N \langle \sup_x \theta^2\rangle^{1/2} + \sup_z \langle \theta^2\rangle^{1/2} \\
&\le \quad \left(e^{-N} \int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz \right)^{1/2} + (N+1) \langle \sup_x \theta^2\rangle^{1/2}. \qquad (74)
\end{aligned}
$$

We now optimize in $N \in \mathbb{N}$ by choosing it such that,

$$
N \;\le\; \ln\left(\frac{\int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz}{\langle \sup_x \theta^2\rangle} + e \right) \;<\; N+1,
$$

to the effect of

$$
e^{-N} \;\sim\; \frac{\langle \sup_x \theta^2\rangle}{\int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz + \langle \sup_x \theta^2\rangle} \quad \text{and} \quad N \;\sim\; \ln\left(\frac{\int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz}{\langle \sup_x \theta^2\rangle} + e \right).
$$

Hence (74) turns as desired into

$$
\begin{aligned}
&\sup_z \langle |\nabla^2 w|\rangle^{1/2} \\
&\lesssim \left(\frac{\langle \sup_x \theta^2\rangle \int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz}{\langle \sup_x \theta^2\rangle + \int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz} \right)^{1/2} + \ln\left(\frac{\int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz}{\langle \sup_x \theta^2\rangle} + e \right) \langle \sup_x \theta^2\rangle^{1/2} \\
&\le \left(1 + \ln\left(\frac{\int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz}{\langle \sup_x \theta^2\rangle} + e \right) \right) \langle \sup_x \theta^2\rangle^{1/2} \\
&\lesssim \ln\left(\frac{\int_0^{2\pi} \langle |\nabla\theta|^2\rangle \, dz}{\langle \sup_x \theta^2\rangle} + e \right) \langle \sup_x \theta^2\rangle^{1/2}.
\end{aligned}
$$

□

Acknowledgments

FO thanks Charles Doering, Maria Westdickenberg, and Xiaodong Yan for fruitful discussions.

References

[1] Guenter Ahlers, Siegfried Großmann, and Detlef Lohse. Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection. *Rev. Mod. Phys.*, 81(2):503–537, Apr 2009.

[2] Peter Constantin and Charles R. Doering. Infinite Prandtl number convection. *J. Statist. Phys.*, 94(1-2):159–172, 1999.

[3] Charles R. Doering and Peter Constantin. Variational bounds on energy dissipation in incompressible flows. III. Convection. *Phys. Rev. E*, 53(6):5957–5981, Jun 1996.

[4] Charles R. Doering and Peter Constantin. On upper bounds for infinite Prandtl number convection with or without rotation. *J. Math. Phys.*, 42(2):784–795, 2001.

[5] Charles R. Doering, Felix Otto, and Maria G. Reznikoff. Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh-Bénard convection. *J. Fluid Mech.*, 560:229–241, 2006.

[6] Dennis L. Hartmann, Leslie A. Moy, and Qiang Fu. Tropical convection and the energy balance at the top of the atmosphere. *Journal of Climate*, 14(24):4495–4511, 2001.

[7] Eberhard Hopf. Ein allgemeiner Endlichkeitssatz der Hydrodynamik. *Math. Ann.*, 117:764–775, 1941.

[8] Louis N. Howard. Convection at high Rayleigh number. In H. Görtler, editor, *Applied Mechanics, Proc. 11th Int. Congress of Applied Mathematics, Munich*, pages 1109–1115. Springer, 1966.

[9] Glenn R. Ierley, Richard R. Kerswell, and Stephen C. Plasting. Infinite-Prandtl-number convection. Part 2. A singular limit of upper bound theory. *Journal of Fluid Mechanics*, 560:159–227, 2006.

[10] Robert V. Kohn and Felix Otto. Upper bounds on coarsening rates. *Comm. Math. Phys.*, 229(3):375–395, 2002.

[11] Willem V. R. Malkus. The heat transport and spectrum of thermal turbulence. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 225(1161):196–212, 1954.

[12] John Marshall and Friedrich Schott. Open-ocean convection: Observations, theory, and models. *Reviews of Geophysics*, 37(1):1–64, 1999.

[13] Dan P. McKenzie, Jean M. Roberts, and Nigel O. Weiss. Convection in the earth's mantle: towards a numerical simulation. *Journal of Fluid Mechanics*, 62(03):465–538, 1974.

[14] Jesse Otero. Bounds for the heat transport in turbulent convection. *PhD thesis, University of Michigan*, 2002.

[15] Joseph Pedlosky. *Geophysical Fluid Dynamics*. Springer-Verlag, 1987.

[16] Stephen C. Plasting. Turbulence has its limits: a priori estimates of transport properties in turbulent fluid flows. *PhD thesis, University of Bristol*, 2004.

[17] Stephen C. Plasting and Glenn R. Ierley. Infinite-Prandtl-number convection. Part 1. Conservative bounds. *Journal of Fluid Mechanics*, 542(-1):343–363, 2005.

[18] Edward A. Spiegel. Convection in stars.1. Basic Boussinesq convection. *Annual Review of Astronomy and Astrophysics*, 9:323–352, 1971.

[19] Brian Straughan. *The energy method, stability, and nonlinear convection*, volume 91 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 2004.

[20] Peter van Keken. Evolution of starting mantle plumes: a comparison between numerical and laboratory models. *Earth and Planetary Science Letters*, 148(1-2):1–11, 1997.

[21] Xiaoming Wang. Infinite Prandtl number limit of Rayleigh-Bénard convection. *Comm. Pure Appl. Math.*, 57(10):1265–1282, 2004.