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Rayleigh-Bernard convection of non-Newtonian fluid

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Abstract

The main purpose of this work is to present the use of boundary-domain integral method to analyse the flow behaviour of non-Newtonian fluid, ie. power law fluid. The available parametric model is applied representing a non-linear dependence between shear rate and deformation velocity. To evaluate the presented approach the Rayleigh-Benard natural convection of the Newtonian and non-Newtonian fluid has been solved.

1 Introduction

In this study, the boundary-domain integral method (BDIM) for the numerical simulation of unsteady incompressible Newtonian fluid flow is extended to analyse the effects of available non-Newtonian viscosity. The constitutive hypothese used is that of two-parameters viscosity-strain rate approximation model representing power law fluid.

Velocity-vorticity formulation[1] of the conservative equations is employed, where the kinematics is written in its false transient formulation to increase the stability of the proposed solution procedure.

The method is applied to the Rayleigh-Benard natural convection problem and numerical results are obtained for Rayleigh number value 10^4.

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2 Governing Equations

2.1 Primitive variables formulation

With the assumptions of incompressibility within Boussinesq approximation[2], the motion of viscous fluid is governed by conservation laws of mass, momentum and energy written in an indicial notation form for a right-handed Cartesian co-ordinate system

$$\frac{\partial \upsilon_i}{\partial x_i} = 0, \tag{1}$$

$$\rho \frac{D\upsilon_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho F g_i, \tag{2}$$

$$\rho c_p \frac{DT}{Dt} = -\frac{\partial q_i}{\partial t} + I_T + \Phi, \tag{3}$$

where the vector field functions υ_i, g_i, q_i, and x_i are respectively velocity, gravity, heat flux and position. The scalar quantities $P=p-\rho g_i r_i$, T and I_T are modified pressure, temperature and heat source term, while D/Dt represents the Stokes derivative. The material properties such as mass density ρ and specific isobaric heat c_p are assumed to be constant parameters. The normalised density-temperature variation function F and Rayleigh dissipation function Φ can be written as follows

$$F = \frac{\rho - \rho_0}{\rho_0} = -\beta_T (T - T_0), \tag{4}$$

$$\Phi = \tau_{ij} \frac{\partial \upsilon_i}{\partial x_j}, \tag{5}$$

with ρ_0 as a reference mass density at temperature T_0, while β_T is thermal volume expansion coefficient.

The viscous stress tensor τ_{ij} and heat flux vector q_i are defined according to Stokes and Fourier constitutive hypothesis

$$\tau_{ij} = 2\eta s_{ij} \quad , \quad q_i = -\lambda \frac{\partial T}{\partial x_i}, \tag{6}$$

where the material properties η and λ are molecular dynamic viscosity and heat conductivity respectively, and s_{ij} is the strain rate tensor

$$s_{ij} = \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i} \right). \tag{7}$$

Combining eqns (2), (3) through (6) yields the Navier Stokes equations in a conservative form

$$\frac{D\upsilon_i}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu s_{ij}\right) + Fg_i, \tag{8}$$

$$\frac{DT}{Dt} = \frac{\partial}{\partial x_i}\left(\kappa\frac{\partial T}{\partial x_i}\right) + \frac{I_T}{\rho c_p} + \frac{\Phi}{\rho c_p}, \tag{9}$$

where $\nu = \eta/\rho$ is the kinematic viscosity and $\kappa = \lambda/\rho c_p$ is the thermal diffusivity.

The constitutive equation used is that of a two-parameter model[3] for which the viscosity is assumed to be shear strain rate dependent in the following form for power law fluid

$$\eta(\dot{\gamma}) = \eta_0(\dot{\gamma})^{n-1}, \tag{10}$$

which can be rewritten to avoid some numerical problems as

$$\eta(\dot{\gamma}) = \begin{cases} \eta_0 & for \quad \dot{\gamma} \le \dot{\gamma}_0, \\ \eta_0\left(\dfrac{\dot{\gamma}}{\dot{\gamma}_0}\right)^{n-1} & for \quad \dot{\gamma} > \dot{\gamma}_0, \end{cases} \tag{11}$$

where shear strain rate $\dot{\gamma}$ and shear stress τ are expressed by relations

$$\dot{\gamma} = \left(2 s_{ij} s_{ij}\right)^{1/2} \quad , \quad \tau = \eta\dot{\gamma} \tag{12}$$

2.2 Velocity - Vorticity Formulation

With the vorticity vector[1] ω_i the fluid motion computation scheme is partitioned into its kinematics, given by the elliptic velocity vector equation

$$\frac{\partial^2 \upsilon_i}{\partial x_j \partial x_j} + e_{ijk}\frac{\partial \omega_k}{\partial x_j} = 0, \tag{13}$$

and kinetics given by the parabolic - hyperbolic vorticity transport equation obtained as curl of the momentum eqn (8)

$$\frac{D\omega_i}{Dt} = \frac{\partial \omega_j \upsilon_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \omega_i}{\partial x_j}\right) + e_{ijk} g_k \frac{\partial F}{\partial x_j} + \frac{\partial f_{ij}}{\partial x_j}, \tag{14}$$

which is in plane motion case simplified to

$$\frac{D\omega}{Dt} = \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \omega}{\partial x_j}\right) + e_{ij} g_j \frac{\partial F}{\partial x_i} + \frac{\partial f_j}{\partial x_j}, \tag{15}$$

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where the additional term is equated to $f_{ij} = \bar{\nabla} v \times s$.

To accelerate the convergency and stability of the coupled velocity - vorticity iterative numerical scheme, the false transient approach is applied to eqn (13) rendering the following parabolic kinematic statement[4]

$$\frac{\partial^2 \upsilon_i}{\partial x_j \partial x_j} - \frac{1}{\alpha}\frac{\partial \upsilon_i}{\partial t} + e_{ijk}\frac{\partial \omega_k}{\partial x_j} = 0, \tag{16}$$

where α is a relaxation parameter. It is obvious that the governing velocity eqn (13) is exactly satisfied only at the steady state ($t \rightarrow \infty$), when the artificial time derivative term vanish.

2.3 Boundary conditions

The boundary conditions assigned to elliptic kinematic velocity eqn (13) are in general of the first and second kind, e.g.

$$\upsilon_i = \bar{\upsilon}_i \quad on \quad \Gamma_1 \qquad and \qquad \frac{\partial \upsilon_i}{\partial x_j} n_j = \overline{\frac{\partial \upsilon_i}{\partial n}} \quad on \quad \Gamma_2. \tag{17}$$

The most critical computation part of the kinematics is the determination of a new boundary vorticity values, which are the only physical proper boundary condition associated with the parabolic kinetic eqn (14) written for the whole boundary, e.g.

$$e_{ijk}\frac{\partial \upsilon_k}{\partial x_j} = \bar{\omega} \quad on \quad \Gamma, \tag{18}$$

while the vorticity normal flux values are the only unknown boundary values in the vorticity kinetics.

The mathematical description of the energy kinetics is completed by providing suitable natural and essential boundary conditions as well as some initial conditions:

$$T = \bar{T} \; on \; \Gamma_1, \qquad -\lambda \frac{\partial T}{\partial x_j} = \bar{q} \; on \; \Gamma_2, \qquad T = \bar{T}_0 \; in \; \Omega. \tag{19}$$

3 Boundary-Domain Integral Equations

Consider a non-linear time dependent diffusion-convection equation for an arbitrary conservative scalar field function u, e.g.

$$\frac{Du}{Dt} = \frac{\partial}{\partial x_j}\left(a \frac{\partial u}{\partial x_j} \right) + I_u, \tag{20}$$

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here I_u is the source term. Substituting the expression for the diffusivity variation in the form of a constant $\bar{a}$ and variable part $\tilde{a}$, e.g.

$$a = \bar{a} + \tilde{a}(r_j, u)$$

the eqn. (20) can be partitioned into a linear and non-linear part in the following manner

$$\frac{Du}{Dt} = \bar{a}\frac{\partial^2 u}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(\tilde{a}\frac{\partial u}{\partial x_j}\right) + I_u \,. \tag{21}$$

The equation represents a parabolic initial-boundary value problem, thus some boundary and initial conditions have to be known to complete the mathematical description of the problem, e.g. the conditions (19) where T is replaced by u.

By using a finite difference approximation of the field function time derivative for the time increment[2] the eqn (21) can be rewritten in a non-homogenous modified Helmholtz PDE form

$$\frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{u}{a\Delta t} + b = 0 \,, \tag{22}$$

with the following corresponding integral representation

$$c(\xi)u(\xi) + \int_\Gamma u\frac{\partial u^*}{\partial n}d\Gamma = \int_\Gamma \frac{\partial u}{\partial n}u^*\,d\Gamma + \int_\Omega b u^*\,d\Omega \,, \tag{23}$$

where the variable u^* is the modified Helmholtz fundamental solution[1]. Equating the pseudo body force term b with a non-linear diffusion part, convection, source term and initial conditions, e.g.

$$b = \frac{1}{\bar{a}}\frac{\partial}{\partial x_j}\left(\tilde{a}\frac{\partial u}{\partial x_j} - \upsilon_j u\right) + \frac{I_u}{\bar{a}} + \beta u_{F-1} \,, \tag{24}$$

renders equation (24)

$$\begin{aligned} c(\xi)u(\xi) + \int_\Gamma u\frac{\partial u^*}{\partial n}d\Gamma = \frac{1}{\bar{a}}\int_\Gamma \left(a\frac{\partial u}{\partial n} - u\upsilon_n\right)u^*\,d\Gamma + \\ \frac{1}{\bar{a}}\int_\Omega \left(u\upsilon_j - \tilde{a}\frac{\partial u}{\partial x_j}\right)\frac{\partial u^*}{\partial x_j}d\Omega + \frac{1}{\bar{a}}\int_\Omega I_u u^*\,d\Omega + \beta\int_\Omega u_{F-1}u^*\,d\Omega \,, \end{aligned} \tag{25}$$

where $\upsilon_n = \upsilon_j n_j$ and the parameter β is defined as $\beta = 1/\bar{a}\,\Delta t$.

The most adequate and stable integral representation regardless of the Reynolds number values, can be formulated by using the fundamental solution of steady diffusion-convective PDE with a reaction term[5]. Since it only exists for the case of constant coefficients, the velocity field has to be decomposed into an average constant vector $\bar{\upsilon}_i$ and perturbated vector $\tilde{\upsilon}_i$,

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such that $\upsilon_i = \bar{\upsilon}_i + \tilde{\upsilon}_i$. Again a non-symmetric finite difference approximation of the time derivative can be applied. This approach permits to rewrite eqn (20) into the following diffusion-convective non-homogenous PDE form

$$\bar{a}\frac{\partial^2 u}{\partial x_j \partial x_j} - \frac{\partial \bar{\upsilon}_j u}{\partial x_j} - \frac{1}{\Delta t}u + b = 0, \tag{26}$$

with the corresponding integral formulation as follows

$$c(\xi)u(\xi) + \bar{a}\int_\Gamma u\frac{\partial u^*}{\partial n}d\Gamma = \int_\Gamma \left(\bar{a}\frac{\partial u}{\partial n} - u\bar{\upsilon}_n\right)u^* d\Gamma + \int_\Omega bu^* d\Omega. \tag{27}$$

The pseudo-body term b includes the non-linear diffusion flux, convective flux for the perturbated velocity field only, source term and initial conditions, e.g.

$$b = \frac{\partial}{\partial x_j}\left(\tilde{a}\frac{\partial u}{\partial x_j} - \tilde{\upsilon}_j u\right) + I_u + \frac{1}{\Delta t}u_{F-1}, \tag{28}$$

rendering the following integral representation

$$\begin{aligned} c(\xi)u(\xi) + \int_\Gamma u\frac{\partial U^*}{\partial n}d\Gamma &= \frac{1}{\bar{a}}\int_\Gamma \left(a\frac{\partial u}{\partial n} - u\upsilon_n\right)U^* d\Gamma + \\ \frac{1}{\bar{a}}\int_\Omega \left(u\tilde{\upsilon}_j - \tilde{a}\frac{\partial u}{\partial x_j}\right)\frac{\partial U^*}{\partial x_j}d\Omega &+ \frac{1}{\bar{a}}\int_\Omega I_u U^* d\Omega + \beta\int_\Omega u_{F-1}U^* d\Omega, \end{aligned} \tag{29}$$

with $U^* = \bar{a}u^*$.

The integral representations for the vorticity and temperature kinetics can be simply derived using eqn (29), where arbitrary scalar function u is replaced by ω and T respectively.

The boundary-domain integral statement for the plane flow kinematics can be derived from the vector parabolic eqn (13) applying the integral eqn (25), resulting in the following statement

$$\begin{aligned} c(\xi)\upsilon_i(\xi) + \int_\Gamma \upsilon_i\frac{\partial u^*}{\partial n}d\Gamma &= \int_\Gamma \frac{\partial \upsilon_i}{\partial n}u^* d\Gamma + e_{ij}\int_\Gamma \omega n_j u^* d\Gamma \\ &- e_{ij}\int_\Omega \omega\frac{\partial u^*}{\partial x_j}d\Omega + \beta\int_\Omega \upsilon_{i,F-1}u^* d\Omega. \end{aligned} \tag{30}$$

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4 Discretized Boundary-Domain Integral Equations

Searching for an approximate numerical solution the corresponding integral equations are written in a discretized manner[6], e.g. for the modified Helmholtz fundamental solution, eqn (25)

$$c(\xi)u(\xi)+\sum_{e=1}^{E}\{h\}^T\{u\}^n=\frac{1}{\bar{a}}\sum_{e=1}^{E}\{g\}^T\left\{a\frac{\partial u}{\partial n}-u\upsilon_n\right\}^n$$
$$+\frac{1}{\bar{a}}\sum_{c=1}^{C}\{d_j\}^T\left\{u\upsilon_j-\tilde{a}\frac{\partial u}{\partial x_j}\right\}^n+\frac{1}{\bar{a}}\sum_{c=1}^{C}\{d\}^T\{I_u\}^n+\beta\sum_{c=1}^{C}\{d\}^T\{u_{F-1}\}^n, \quad (31)$$

where index n refers to the number of nodes in each boundary element or internal cell and also relates to the degree of the respective interpolation polynomial, and superscript T is used for the transposition.

However, the integral formulation for the diffusion-convective fundamental solution, eqn (29), can be written in a discrete form similar to the eqn (31) where only the velocity component υ_j has to be replaced by the perturbated velocity component $\tilde{\upsilon}_j$.

Applying eqn (31) to all boundary and internal nodes, the following matrix system can be obtained,

$$[H]\{u\}=[G]\left[\frac{a}{\bar{a}}\right]\left\{\frac{\partial u}{\partial n}\right\}-\frac{1}{\bar{a}}[G][\upsilon_n]\{u\}+\frac{1}{\bar{a}}[D_j][\upsilon_j]\{u\}$$
$$-[D_j]\left[\frac{\tilde{a}}{\bar{a}}\right]\left\{\frac{\partial u}{\partial x_j}\right\}+\frac{1}{\bar{a}}[D]\{I_u\}+\beta[D]\{u\}_{F-1}. \quad (32)$$

The matrix siystem corresponding to the discrete form of the integral forlmulation for difusion-convective soulution has the same form as the eqn (32) only υ_j has to be replaced by $\tilde{\upsilon}_j$.

The discretized integral representations for the kinematics and kinetics of the fluid motion can be obtained by following the solution procedure developed for the general conservation field function u. Using discretized eqn (32), through eqn (30), the following implicit matrix system is obtained for the plane kinematic

$$[H]\{\upsilon_i\}=[G]\left\{\frac{\partial \upsilon_i}{\partial n}\right\}+e_{ij}[G]\{\omega n_j\}-e_{ij}[D_j]\{\omega\}+\beta[D]\{\upsilon_i\}_{F-1}, \quad (33)$$

to be solved for unknown boundary velocity components or their normal derivative values respectively, while the computation of all internal domain velocity components, if needed, is done in an explicit manner point by point. Applying eqn (32) (considering the replacement of υ with $\tilde{\upsilon}_j$) through eqns (29) the following implicit matrix systems can be derived, e.g. for the vorticity equation

$$[H]\{\omega\}=[G]\left[\frac{\nu}{\bar{\nu}}\right]\left\{\frac{\partial\omega}{\partial n}\right\}-\frac{1}{\bar{\nu}}[G][\upsilon_n]\{\omega\}+\frac{1}{\bar{\nu}}[G]\{e_{ij}n_i g_j F+f_j n_j\}$$
$$+\frac{1}{\bar{\nu}}[D_j][\tilde{\upsilon}_j]\{\omega\}-[D_j]\left[\frac{\tilde{\nu}}{\bar{\nu}}\right]\left\{\frac{\partial\omega}{\partial x_j}\right\}-\frac{1}{\bar{\nu}}[D_j]\{e_{ij}g_j F+f_i\}+\beta[D]\{\omega\}_{F-1}, \tag{34}$$

while for the temperature equation one can write

$$[H]\{T\}=[G]\left[\frac{\kappa}{\bar{\kappa}}\right]\left\{\frac{\partial T}{\partial n}\right\}-\frac{1}{\bar{\kappa}}[G][\upsilon_n]\{T\}+\frac{1}{\bar{\kappa}}[D_j][\tilde{\upsilon}_j]\{T\}$$
$$-[D_j]\left[\frac{\tilde{\kappa}}{\bar{\kappa}}\right]\left\{\frac{\partial T}{\partial x_j}\right\}+\frac{1}{\bar{\kappa}}[D]\{(I_T+\Phi)/\rho c_p\}+\beta[D]\{T\}_{F-1}. \tag{35}$$

To improve the economics of the computation and thus widen the applicability of the proposed numerical algorithm, the subdomain technique[7] has to be used.

The discrete model used is based on substructure technique derived to its limit version following the concept of finite volume, e.g. that each quadrilateral internal cell represents one subdomain bounded by four boundary elements. The geometrical singularities are overcome by using 3-node discontinuous boundary elements combined with 8-node continuos internal cell.

5 Numerical results and discussion

The physical problem description is shown in Fig. 1. The solution domain consists of a rectangular cavity of aspect ration 2 with the left wall at higher temperature than the right wall.

The compariosn of different flow situation for Newtonian and Non-Newtomnian fluid has been made for Rayleigh number value $Ra = 10^4$, with Prandtl number value $Pr = 0.71$.

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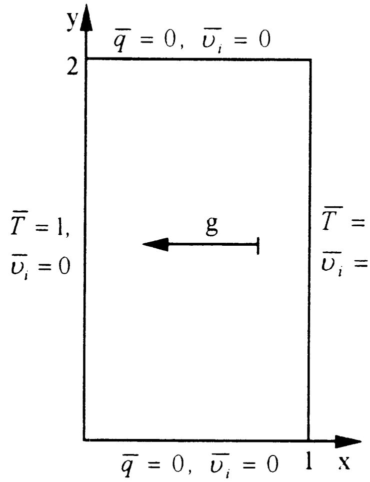

Figure 1. Geometry and boundary conditions for Rayleigh-Benard natural convection.

Time evolution of the velocity field is presented in Fig. 2 for Newtonian fluid. It has to be noted that at this *Ra* number value diffusion transport is dominant what can be clearly seen in Fig. 3 where the time evolution of temperature field at different time steps is presented.

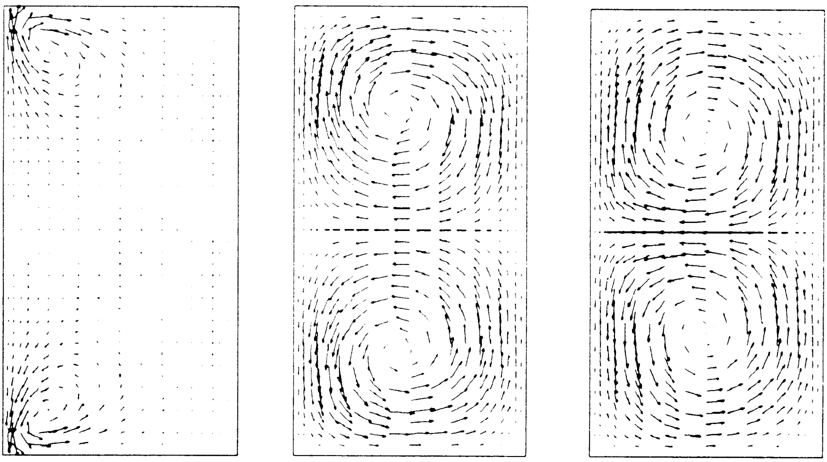

Figure 2. Vector velocity fields at different time steps (0.01 s, 0.1 s and 0.5 s) for Newtonian fluid.

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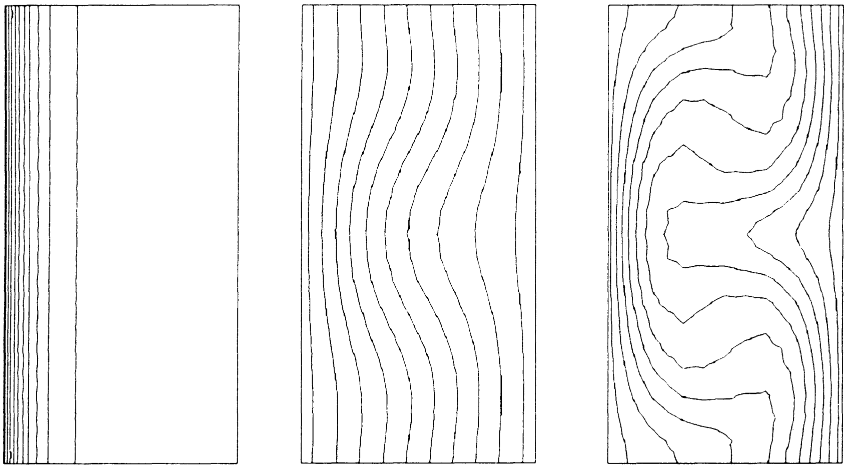

Figure 3. Temperature fields at different time steps 0.01 s, 0.1 s and 0.5 s for Newtonian fluid.

Non-Newtonian viscosity for n=0.7 of the power law fluid in comparison with the Newtonian fluid affects significantly on the velocity field at the same calculation conditions, what can be seen from Fig. 4. After 0.1 s from the beginning of the process four swirls appear which combine to two bigger swirls at the steady state conditions.

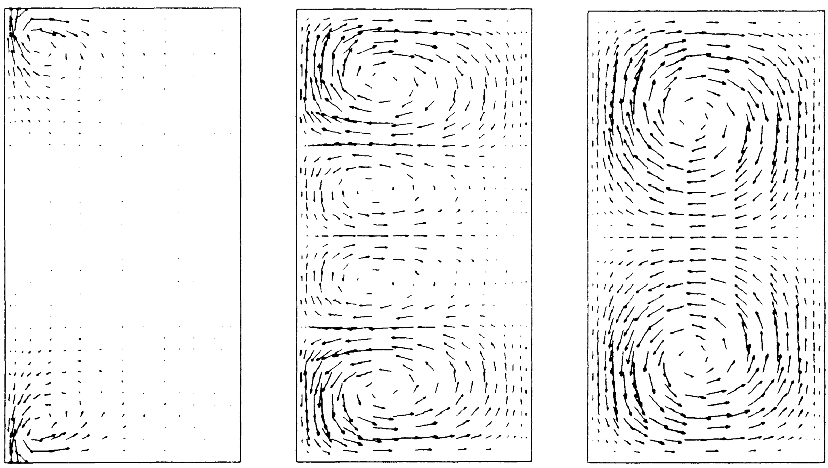

Figure 4. Vector velocity field at different time steps (0.01 s, 0.1 s, 0.5 s) for power law fluid

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The corresponding temperature fields to the velocity fields shown in Fig. 4 are presented in Fig. 5. This is in contrast to the Newtonian fluid (n=1) which shows weak convection for such low Rayleigh number value.

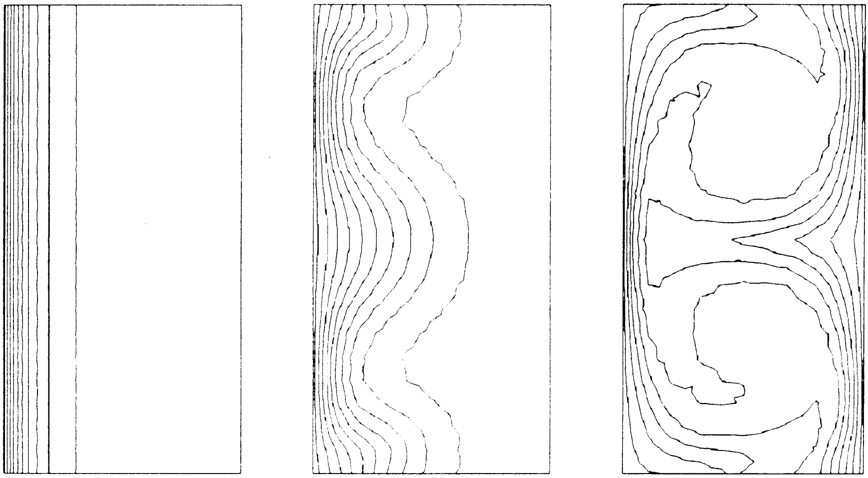

Figure 5. Temperature field at different time steps (0.01 s, 0.1 s and 0.5 s) for power law fluid.

6. Conclusions

A numerical approach based on boundary-domain integral method is applied to solve transport phenomena in an incompressible non-Newtonian fluid motion. The velocity-vorticity formulation is applied, where the false transient approach is used for the kinematic velocity equation. Elliptic modified Helmholtz and elliptic diffusion-convective fundamental solutions are considered for the kinematics and kinetics, respectively. The stability problem associated with Rayleigh-Benard natural convection in a closed cavity for a non-Newtonian fluid is studied.

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